A SIMULATION STUDY OF A PREFERENTIAL ATTACHMENT MODEL WITH DELAY

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Abstract. Preferential attachment models fit many real-world networks such as follower networks of social media platforms and the citation network of academic papers. Frequently though, newer additions to a network will not be seen by the masses but earlier additions will be. This paper accounts for this by adding delay to preferential attachment model. Through simulation, we compare our model to a model without delay. We make inferences regarding how the delay affects the limiting degree distribution in our model. Our main conjecture is that there is a value such that if the expected value of a delay random variable is greater than that value, the delay has a significant effect of the degree distribution. We also discuss further questions related to dynamic graphs with delay.

1. Introduction

Academic papers, web pages, and social media accounts all have one thing in common. One is more likely to cite, click, or follow a more popular one than a less popular one. This tendency to pick more known things over lesser-known ones gives rise to preferential attachment models. Barabási [1] makes this note on preferential attachment, "In real networks, new nodes tend to link to the more connected nodes. In contrast, in random networks, randomly choose their interaction partners.". Thus, it is practical to study preferential attachment models as they assist our understanding of various growth patterns. Out of preferential attachment study also comes more purely entertaining results. For instance, Edgerton composed the music piece presented in Figure 1[1].

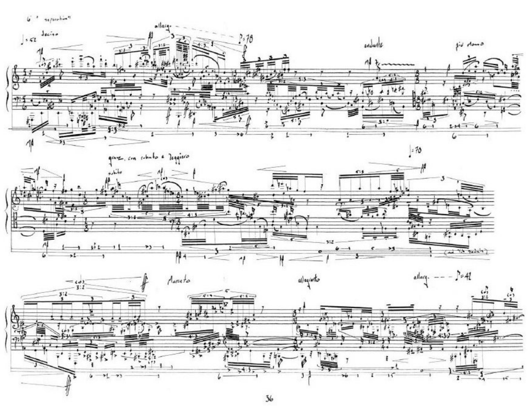

Figure 1. Scale-free sonata by Michel Edward Edgerton.

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[1]Image available at michaeledwardedgerton.wordpress.com/1-sonata/

1.1. **The Original Model.** While preferential attachment is a very general term, in 1999, Barabási and Albert [2] gave a formulation for preferential attachment as follows:

(1) Start with a small number, m_0, of vertices.
(2) At each time step, add a new vertex with m edges that link the new vertex to m different vertices already present in the system with the new vertex forming a connection with probability π defined below.
(3) Let k_i be the connectivity of an existing vertex i. Then define $\pi(k_i) = k_i/\sum_j k_j$.
(4) After t time steps, the model leads to a random network with $t + m_0$ vertices and mt edges.

When $m = 1$, this formulation is clear. One begins with a singular vertex v_0, and at each time step, t, adds a new vertex, v_t, which chooses an existing vertex to form a connection with based on our measure π. This can be easily simulated by choosing a value from the interval $[0, 1]$ uniformly, partitioning the interval $[0, 1]$ into sub-intervals, $[\pi(k_i), \pi(k_{i+1})], i = 0, 1, 2, ...t - 2$, and then selecting a vertex $v_{i<t}$ to connect to v_t based on what sub-interval the value resides in.

This formulation leaves room for interpretation when defining similar models, and we will use it as a basis for our discussion on preferential attachment models.

First, if one only starts with vertices, there is no way to begin the graph generation process. Thus, one must choose an initial starting graph. For instance, the Python NetworkX package sets the initial graph as a star graph of size 1. Second, it is not clear how to choose m edges. Some will resolve this issue by choosing m edges one by one and updating the degrees as each edge get added. Third, one can include an additional additive parameter δ to the model so that each vertex v is chosen with probability proportional to $\deg(v) + \delta$. Fourth, one must choose whether to allow self-loops.

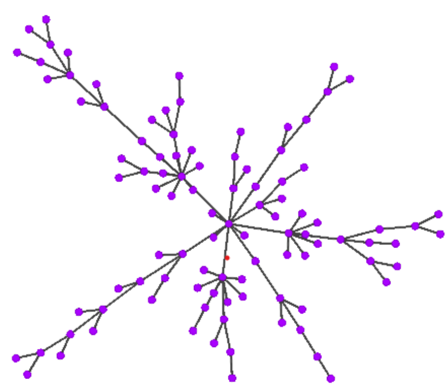
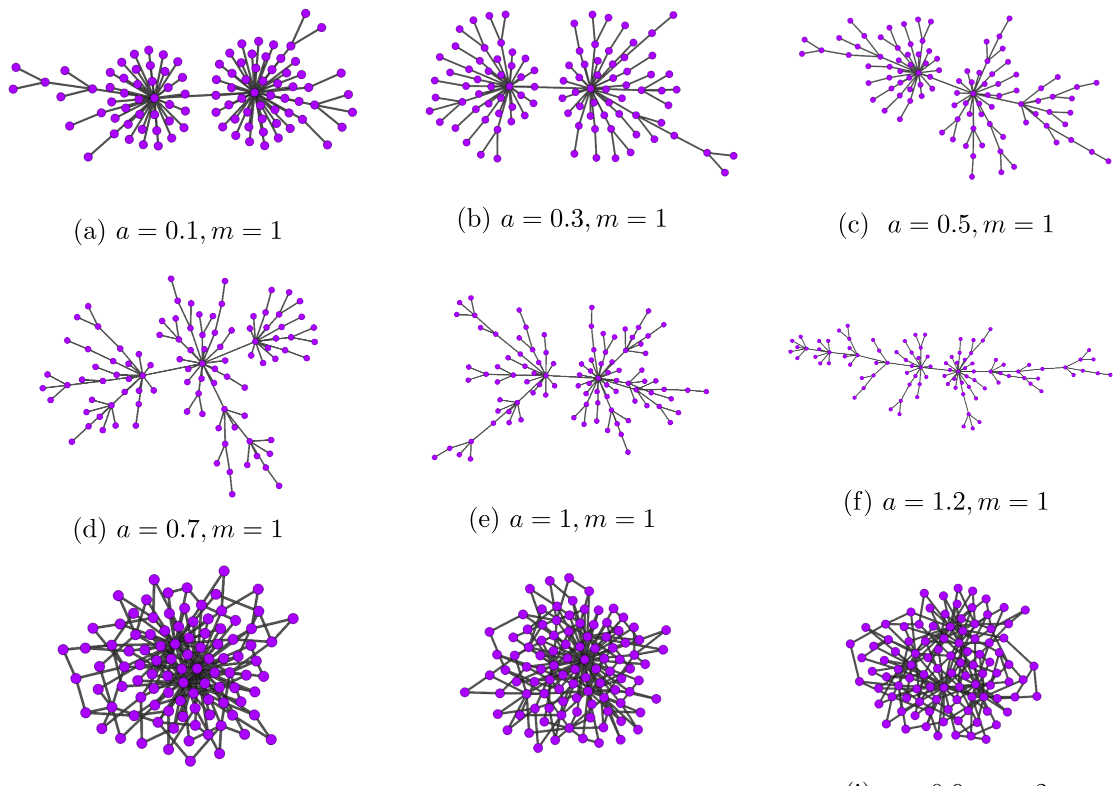


Figure 2. Graph of size 100 generated with $m = 1$, $\delta = 0$

1.2. **Our Model.** Our model takes the notion of preferential attachment and adds a delay factor. In the original model, a node v_t would connect to an existing node of G_t, v_i, with probability proportional to $\deg(v_i)$. In a delay model, we replace G_t with $G_{t'}$ where $t' = t - \xi_t$ where ξ_t is some discrete random variable taking values from $1, 2, ..., t - |V_0|$. Specifically for our model, we define our delay variable in terms of a, with a taking values from $[0, \infty)$. Then, we define $\xi_t \coloneqq min(t - |V_0|, \lceil u^{-\frac{1}{a}} \rceil)$, where u is a random variable on $(0, 1)$ with uniform distribution.



(a) $a = 0.1, m = 1$ (b) $a = 0.3, m = 1$ (c) $a = 0.5, m = 1$

(d) $a = 0.7, m = 1$ (e) $a = 1, m = 1$ (f) $a = 1.2, m = 1$

(g) $a = 0.3, m = 2$ (h) $a = 0.6, m = 2$ (i) $a = 0.9, m = 2$

Figure 3. Sample graph of size $n = 100$ generated from our model with parameter a

Definition 1.1 (Supernode)**.** *Given a graph G with nodes V, if $\exists$ a finite set $S \subseteq V$ such that $\forall v \in S$, $\forall v' \in V \setminus S$, $deg(v) \gg deg(v')$, the members of S are called supernodes. Note that $\mathbb{E}\,|S|$ should not depend on $|V|$, only on a, m, and δ.*

We expect to see supernodes (Definition 1.1) growing more apparent as delay increases, as new nodes will have a higher tendency to connect to the original first few nodes. Comparing the figures above, we can see even on a very small scale, $N = 100$; this change is very apparent.

Moreover, we expect there to be $|V_0|$ supernodes, as if the delay is high, the chance of connecting to one of the original nodes increases. Moreover, we expect a certain delay value which is a transition point for the degree distribution, *i.e.,* above the transition point, very distinct supernodes emerge. We would like to measure the size of these supernodes, and one way to do that is to measure the maximum degree in our graphs across different delays.

Also, we observe that the structure of the graphs outside the supernodes change as the delay increases, and to quantify this change we want to study the tail behavior of the degree distribution across different delays.

These observations lead us to the formulation of our guiding questions below.

1.3. Guiding Questions.

(1) **Tail Behavior:** What effect will increased delay have on the tail of the degree distribution? How does this affect scale with delay? Does our model align with the proven results for no-delay models? At what point is the effect of delay noticeable?

(2) **Maximum Degree:** How does the graph's maximum degree grow with the graph's size? How does delay affect the maximum degree of our graphs? Will the supernodes eventually take up a constant proportion of the total connections?
(3) **Graph Structure:** When can we see distinct supernodes in our visualizations? Which nodes become supernodes? How many supernodes do we expect there to be given an initial graph?

1.4. **Literature Review.** Research has been done regarding preferential attachment models without delay, and many useful results will aid our understanding of our preferential attachment model with delay.

Throughout the paper, we will use theorems drawn from the work of others. Most of these theorems work for a no-delay model, but we may apply them to our model and explain why they should still hold heuristically. The first relates to the limiting degree distribution power-law exponent, which we have collected data on.

Theorem 1.2 (See [4, Section 8.4])**.** *The limiting graph generated by 1.1 with $m = 1$, $\delta = 0$, has a degree distribution where the number of nodes with degree k is proportional to $k^{-(3+\delta/m)}$.*

This is a general result and holds for most preferential attachment models.

Theorem 1.3 (See [3, Theorem 3.1])**.** *Suppose G is a graph generated by 1.1 with $m = 1$. Then, let M_n be the maximum degree across all nodes in G at time n. Then, with probability 1, we have*

$$\lim_{n\to\infty} n^{\frac{-1}{2+\delta}} M_n = \mu$$

The limit is almost surely positive and finite, and it has an absolutely continuous distribution. The convergence also holds in L_p for all $1 \leqslant p < \infty$.

1.5. **Notations.**

i. m - the number of new connections a node makes when being added to the existing graph
ii. δ - additive probability coefficient
iii. a - delay parameter
iv. N - size of final graph
v. M_t - max degree at time t
vi. G_0 - the initial graph from which we build our model
vii. V_0 - the set of initial nodes of G_0
viii. E_0 - the set of initial edges of G_0

1.6. **Organization.** This paper is organized into four main sections. The first gives an introduction to preferential attachment models and our model and motivation for studying them. The second describes the algorithm used to generate our model and the heuristic description of specific properties of our model. The third presents the results from our simulation study and conjectures drawn from our data. The final section is a discussion on possible similar future studies.

2. Algorithm and Insight

2.1. **Algorithm.** Next, we describe the algorithm used to generate graphs from our proposed model from Section 1.2 in Table 1.

Algorithm 1 Graph Creation

define $repeatedNodes, RN$ - an empty list of nodes
for $v \in V_0$ **do**
    add v to RN $deg(v)$ times
$RN \leftarrow \forall v \in V_0,$ add $v,$ $in\deg(v)$ times
let $source = size(V_0)$
let $G = G_0$
let $position = size(RN)$
while $source \leqslant N$ **do**
    let $targets \leftarrow$ random subset of m nodes from RN
    Add a node to G with value $source$ and create edges from it to the nodes in targets
    Add $targets$ to RN.
    Add node v_{source} $m + \delta$ times to repeated nodes
    let p_l be the last element in $position$
    Add $p + 2 * m * \delta$ to $position$
    let $\hat{\xi}$ be a number sampled from the distribution ξ_t ($t = source$) defined in 1.2
    let $e_{\hat{\xi}} \leftarrow$ the element at position $size(RN) - \hat{\xi}$ in $position$
    $targets \leftarrow$ random subset of nodes from the first $source$ - $e_{\hat{\xi}} * m$ nodes of RN
    $source \mathrel{+}= 1$
return G

2.2. **Heuristic computation for the Original Model.** Let us describe the classical Preferential Attachment model first. At time $t = 0$, we begin with an arbitrary but fixed connected graph G_0. At the time $t > 0$, we recursively add one node v_t. The node v_t forms an edge with node v_i for $i < t$, with probability $\frac{\deg(v_i)+\delta}{2|E_0|+(2m+\delta)t}$, independently m times.

Heuristically, one can calculate the degree distribution in the following manner. Let $N_k(t)$ give the number of vertices in G_t with degree k. Define

$$\widehat{p}_k(t) \coloneqq \frac{N_k(t)}{t + |V_0|} \text{ for } k \geqslant 1, t \geqslant 0.$$

Notice that $(\widehat{p}_k(t))_{k\geqslant 1}$ is a random probability distribution due to the G being randomly generated. Thus, to understand the degree distribution of the limiting graph, we look at the behavior of the expected value of $\widehat{p}_k(t)$ as $t \to \infty$ for all $k \geqslant 1$. At the time t, existing nodes with degree k may attach to v_t and existing nodes with degree $k - 1$ may attach to v_t. This inflow and outflow of new nodes can be represented as follows.

$$\mathbb{E}\, N_k(t) = \mathbb{E}\, N_k(t-1) - \frac{m(k+\delta)}{2|E_0| + (2m+\delta)t} \cdot \mathbb{E}\, N_k(t-1) + \frac{m(k-1+\delta)}{2|E_0| + (2m+\delta)t} \cdot \mathbb{E}\, N_{k-1}(t-1) + \delta_{k=m}. \quad (1)$$

We now assume as $t \to \infty$, $\mathbb{E}\,\widehat{p}_k(t) \to p_k$ with $\sum_{k=1}^{\infty} p_k = 1$. This makes $p_k(t)$ a valid probability distribution with shape estimated by $\mathbb{E}\,\widehat{p}_k(t)$. This requires a proof, but that is outside the scope of this paper. Moreover, we will ignore the last term because we are considering the tail behavior of our distribution, i.e., we will assume $k > m$.

Now we have $p_k(t) \approx \mathbb{E}(\widehat{p}_k(t))$. Reducing (1) yields,

$$p_k \approx -\frac{k+\delta}{2+\frac{\delta}{m}+\frac{2|E_0|}{mt}} \cdot \frac{\mathbb{E}\, N_k(t-1)}{t} + \frac{k-1+\delta}{2+\frac{\delta}{m}+\frac{2|E_0|}{mt}} \cdot \frac{\mathbb{E}\, N_{k-1}(t-1)}{t}$$

Note that equality must include an indicator function for if $k = m$, but in the limiting case, this term is irrelevant. For simplicity, we will take $k > m$. Thus, letting $t \to \infty$, we can simplify the above expression to

$$\begin{aligned} p_k &\approx -\frac{k+\delta}{2+\frac{\delta}{m}+\frac{2|E_0|}{mt}} \cdot p_k + \frac{k-1+\delta}{2+\frac{\delta}{m}+\frac{2|E_0|}{mt}} \cdot p_{k-1} \\ &= -\frac{k+\delta}{2+\frac{\delta}{m}} \cdot p_k + \frac{k-1+\delta}{2+\frac{\delta}{m}} \cdot p_{k-1}. \end{aligned}$$

Further simplification yields

$$p_k \cdot \frac{2+\frac{\delta}{m}+k+\delta}{2+\frac{\delta}{m}} = p_{k-1} \cdot \frac{k-1+\delta}{2+\frac{\delta}{m}}.$$

We then identify the following linear recurrence relation for $k > m$

$$p_k = p_{k-1} \cdot \frac{1}{1+\frac{3+\frac{\delta}{m}}{k-1+\delta}}.$$

Unrolling yields

$$p_k = p_m \cdot \prod_{i=m+1}^{k} \frac{1}{1+\frac{3+\frac{\delta}{m}}{i-1+\delta}}.$$

If $\delta = 0$ and $m = 1$, the simplest case, this equation reduces to

$$p_k = p_1 \cdot \prod_{i=2}^{k} \frac{1}{1+\frac{3}{i-1}} = p_1 \cdot \prod_{i=2}^{k} \frac{i-1}{i+2}$$

Via telescoping, this further reduces to

$$p_k = \frac{6p_1}{k(k+1)(k+2)}$$

Since we assumed $k > m$ and $\sum_{i=1}^{\infty} p_k = 1$, we calculate p_1 as follows

$$1 = p_1 + \sum_{k=2}^{\infty} \frac{6p_1}{k(k+1)(k+2)} = 3p_1 \cdot \sum_{k=1}^{\infty} \frac{2}{k(k+2)(k+3)}$$

Via a partial fraction decomposition and then telescoping again, we have

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \sum_{k=1}^{\infty} \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} = \frac{1}{2}.$$

Thus $p_1 = 2/3$ which implies for k large

$$p_k = \frac{4}{k(k+2)(k+3)} \approx \frac{4}{k^3}$$

Hence the limiting asymptotic degree sequence follows an inverse power law with an exponent 3.

For the general case, we will not calculate the power law exponent explicitly but explain how one may find it. If there exists a parameter $\gamma_c > 0$ such that $p_k \approx k^{-\gamma_c}$ for k large, then $-\frac{\log p_k}{\log k} \to \gamma_c$ as $k \to \infty$. Thus, we consider

$$\log(p_k/p_m) = -\sum_{i=m+1}^{k} \log\left(1 + \frac{3+\frac{\delta}{m}}{i-1+\delta}\right)$$

Define the constant $\gamma \coloneqq 3 + \delta/m$. Using the fact that $0 \leqslant x - \log(1+x) \leqslant x^2$ for all $x \in [0,1]$, we get that

$$0 \leqslant \gamma \sum_{i=m+1}^{k} \frac{1}{i-1+\delta} - \log(p_k/p_m) \leqslant \frac{\gamma^2}{2} \sum_{i=m+1}^{k} \frac{1}{(i-1+\delta)^2} \leqslant \frac{\gamma^2}{2} \sum_{i=1}^{\infty} \frac{1}{i^2}.$$

Moreover, it is easy to check that

$$\frac{1}{\log k} \sum_{i=m+1}^{k} \frac{1}{i-1+\delta} \to 1 \text{ as } k \to \infty.$$

Thus, $\gamma = \gamma_c$.

2.3. **Heuristic analysis of the proposed Model.** In this section, we will explore the model deeper established in 1.2 further. We will be using the notation and variables defined in that section throughout. In this paper, we will consider what happens when we add delay to this model, i.e., at time t, the new connecting node v_t may only sample from $G_{t-\xi_t}$. For simplicity, we will define $t' \coloneqq t - \xi_t$ and $\psi \coloneqq \lceil u^{-\frac{1}{a}} \rceil$. The following will be a heuristic explanation of why we conjecture that if $\mathbb{E}(\psi) < \infty$, the degree distribution of our new model will be the same as above.

If we keep the same definitions as above, $\mathbb{E}\, N_k(t)$ is similar for our delay model, but the probabilities must use t' instead of t. Moreover a vertex with degree k at time t does not necessarily have degree k at time t'. Since now $\mathbb{E}\, N_k(t)$ is more complex, we will define the following function for readability.

$$f(x) \coloneqq \sum_{v \in G} \mathbb{1}_{\deg_t(v)=x} \cdot \frac{\deg_{t'}(v) + \delta}{2e_0 + (2m+\delta)t}$$

Then we have

$$\frac{1}{t}\, \mathbb{E}\, N_k(t) \approx f(k-1) - f(k)$$

Again this is a slight oversimplification because one would include an indicator function for if $m = k$ to be precise. However, since we are analyzing the tail behavior of our degree distribution, i.e., $k > m$, this term has no effect.

Now onto analyzing $f(x)$, notice that we can separate $f(x)$ into more familiar terms if we add and subtract a $\deg_t(v)$ to the numerator of the inside term.

$$\frac{\deg_{t'}(v) + \delta}{2|E_0| + (2m+\delta)t'} = \frac{\deg_{t'}(v) - \deg_t(v) + \delta + \deg_t(v)}{2|E_0| + (2m+\delta)t'}$$

Since we will consider $f(k)$, we have

$$\frac{\deg_{t'}(v) - \deg_t(v) + \delta + \deg_t(v)}{2|E_0| + (2m+\delta)t'} = \frac{\deg_{t'}(v) - \deg_t(v)}{2|E_0| + (2m+\delta)t'} + \frac{\delta + k}{2|E_0| + (2m+\delta)t'}$$

Moreover, since our indicator function will equal 1 only when $\deg_t v = k$, we have

$$f(k) = \frac{(\delta+k)N_k(t)}{2|E_0| + (2m+\delta)t} - \sum_{v \in G} \mathbb{1}_{\deg_t(v)=x} \cdot \frac{\deg_t(v) - \deg_{t'}(v)}{2|E_0| + (2m+\delta)t}$$

If $\mathbb{E}(\psi) < \infty$, then as $t \to \infty$, $\mathbb{E}(t')/t = (t - \mathbb{E}(\psi))/t \to 1$. Thus, intuitively we can say that

$$f(k) = \frac{(\delta + k)N_k(t)}{2|E_0| + (2m+\delta)t} - \sum_{v \in G} \mathbb{1}_{\deg_t(v)=x} \cdot \frac{\deg_t(v) - \deg_{t'}(v)}{2|E_0| + (2m+\delta)t} \tag{2}$$

However, we cannot say the same for $\mathbb{E}[\deg_{t'}(v)]/\deg_t(v)$, since deg is not necessarily a linear function with respect to t. Thus we must find some other bound for

$$\sum_{v \in G} \mathbb{1}_{\deg_t(v)=x} \cdot \frac{\deg_t(v) - \deg_{t'}(v)}{2|E_0| + (2m+\delta)t}$$

Notice that $\deg_t(v) - \deg_{t'}(v) \leqslant m(t - t')$ because the graph updates $t - t'$ times between t and t' and that maximum increase in degree for a node v every update is m. Now we can find an approximate bound on $\deg_t(v) - \deg_{t'}(v)$ using ψ.

$$\deg_t(v) - \deg_{t'}(v) \leqslant m(t - t') = m(t - (t - \psi)) = M\psi$$

Note that there are at most $t + |V_0|$ vertices in G; hence our expected error is as follows.

$$\mathbb{E}[\sum_{v \in G} \mathbb{1}_{\deg_t(v)=x} \cdot \frac{\deg_t(v) - \deg_{t'}(v)}{2|E_0| + (2m+\delta)t}] \leqslant \frac{m\,\mathbb{E}(\psi)(t+V)}{2|E_0| + (2m+\delta)t}$$

As $t \to \infty$

$$\frac{m\,\mathbb{E}(\psi)(t+V)}{2|E_0| + (2m+\delta)t} = c\,\mathbb{E}(\psi)$$

where c is some constant. Referring to (2) we expect that as $t \to \infty$

$$f(k-1) - f(k) = \frac{(\delta + k - 1)N_{k-1}(t)}{2|E_0| + (2m+\delta)t} - \frac{(\delta + k)N_k(t)}{2|E_0| + (2m+\delta)t} + m\,\mathbb{E}(\psi) - m\,\mathbb{E}(\psi)$$

If $\mathbb{E}(\psi) < \infty$, then this reduces to our expression of $\mathbb{E}\,N_k(t)$ for our no delay case. Thus intuitively, we can say the expected delay distribution for our delay model will be the same as the no delay model if $\mathbb{E}(\psi) < \infty$.

3. Simulation Results and Inference

3.1. **Degree Sequence and Max Degree Data.** Each tail exponent data point is the mean across five graphs generated using the model described in section 1.2, each of size 100,000 and each max node data point is the mean of the maximum degrees of the same graphs.

Table 1. Tail Exponents

(a,δ)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0	4.165	3.555	3.245	3.106	3.067	2.995	2.982	2.898	2.967	2.985	2.955
0.5	4.685	3.969	3.647	3.533	3.459	3.410	3.387	3.311	3.381	3.419	3.409
1	4.916	4.914	3.994	3.767	3.821	3.751	3.750	3.701	3.663	3.687	3.651
-0.5	3.671	3.056	2.771	2.664	2.586	2.554	2.545	2.483	2.518	2.524	2.485

Table 2. Average Max Degrees

(a,δ)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0	27003	13305	6441	3468	1970	1290	921	721	851	756	685
0.5	24562	11118	4784	2401	1190	746	454	362	275	335	302
1	23131	9884	4230	1761	863	521	364	251	214	220	173
-0.5	32671	19144	11281	7426	5847	4650	3913	3527	3310	4353	3191

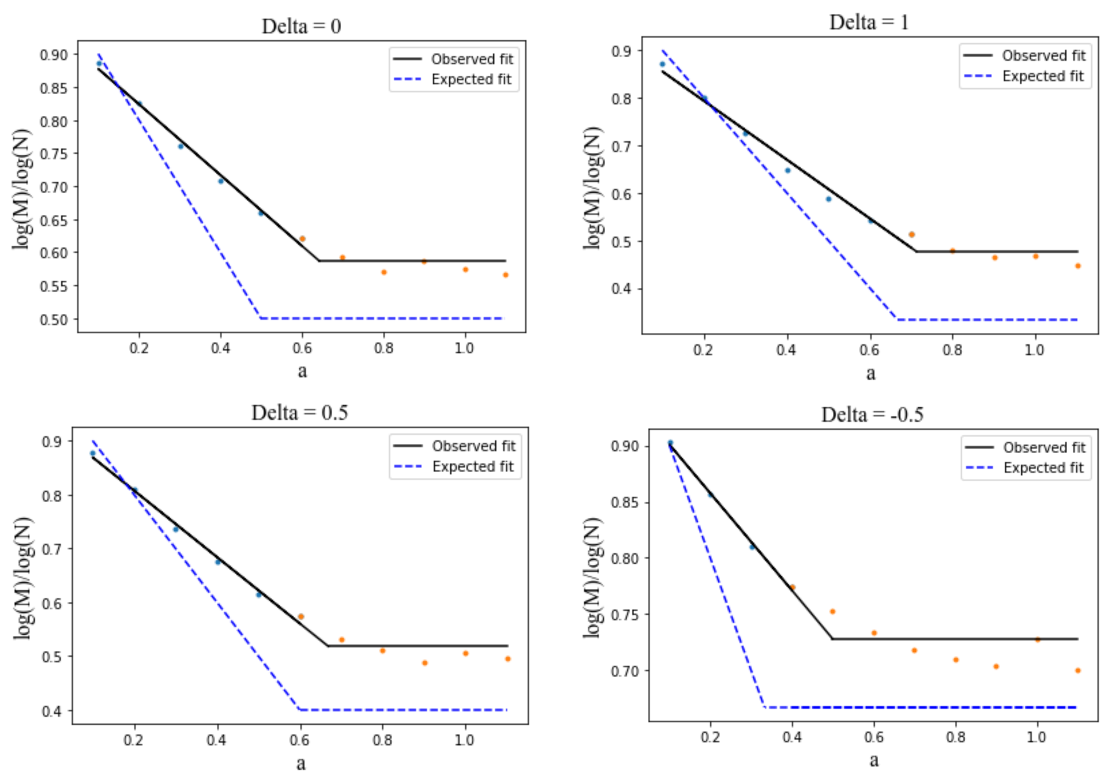


Figure 4. Sample graph of size $n = 100$ generated from our model defined in **1.4** with tail parameter a for the delay distribution.

3.2. **Conjectures.** Examining our graphs, optically there is some transition point where the graphs change shape. We conjecture that each plot is actually a union of two disjoint functions. We will give a heuristic explanation for why this may be the case.

Intuitively, we know if the delay is high, the tail behavior of our degree distribution will be influenced heavily by the development of supernodes, and if there is no delay, the tail behavior will be determined by the preferential attachment mechanism only; see Theorem 1.2 and Theorem 1.3.

Let M_t be the max degree at time t and define $\theta \coloneqq 2 + \frac{\delta}{m}$.

From the literature, we know $M_t \approx t^{\frac{1}{\theta}}$ [1]. We will use this as a lower bound on our maximum degree for now delay. However, adding delay creates another lower bound proportional to $1 - a$. This is because when the delay is involved, we have to account for the higher expectation that a new node connects to an existing supernode. WLOG, we will assume that $m = 1$, and the first node in our graph becomes the node with the highest degree. We will then show that the expected value of the degree of this node is in some cases larger than our no-delay lower bound.

If we let ξ_t be our random variable as defined in 1.2, we then have

$$\mathbb{E}\deg(v_1) = \mathbb{E}\sum_{i=1}^{t} \mathbb{1}_{\xi_t \geqslant i} = \sum_{i=1}^{t} \mathbb{P}(\xi_t \geqslant i) = \sum_{i=1}^{t} i^{-a} \approx \frac{t^{1-a}}{1-a}$$

Hence, we can expect that $M_t \geqslant \frac{t^{1-a}}{1-a}$. Thus, we should find that a plot of a versus $\frac{\log M_N}{\log N}$ should have a transition point where $1 - a = \frac{1}{\theta}$. Moreover, we expect the plot (see Figure 4) to follow the following rule,

$$\frac{\log M_N}{\log N} \approx \begin{cases} 1-a & \text{if } a \leqslant 1 - \frac{1}{\theta} \\ \frac{1}{\theta} & \text{if } a \geqslant 1 - \frac{1}{\theta} \end{cases}$$

It should be noted that in our simulation, since N is relatively small on a logarithmic scale, we observe deviance from the expected graph. The error for the part of the graph after the transition point can be explained by Theorem 1.3. As there we expect the dominant factor influencing the tail of our graph to be the standard preferential attachment mechanism, we can say that beyond the transition point, $\mu_a \to \mu^+$ as $a \to \infty$.

Also, note that while μ is a random variable, it will still accounts for error if N is finite. In our simulation we examine $\frac{\log M_n}{\log N}$. As $\log x$ is a continuous functions, we can derive

$$\lim_{n\to\infty} \log(n^{\frac{-1}{2+\delta}} M_n) = \log\mu \implies \lim_{n\to\infty} \frac{\log M_n}{\log N} = \lim_{n\to\infty} \frac{1}{2+\delta} + \frac{\log\mu}{\log n}$$

from Theorem 1.3.

This gives us a heuristic explanation for the decreasing error we see beyond the transition point in our graphs, as we expect to be able to replace μ with μ_a and have the same result hold.

We also see that the $\frac{\log \mu_a}{\log N}$ decreases as $\log N$ increases. Since our choice of N is relatively small, $\log N \approx 11.5$, we see noticeable difference from the expected graph in our graph. We also see that our graphs are shifted up from expected, which makes sense given our error is positive.

We also expect an upward shift in our delay portion of the graph, but we expect the error to grow as the graph approaches the transition point, and indeed, we see this behavior in our error plot (Figure 5).

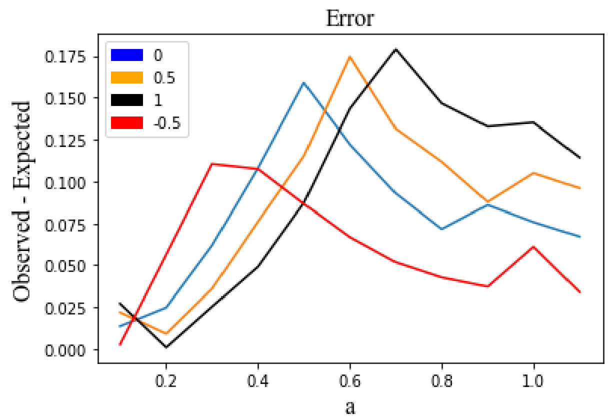


Figure 5. Error in our data plotted against our delay parameter

It is not entirely obvious from the plot that there is a clear transition point, so for $\delta = 0$, we examine the region of a delay from 0.5 to 1.25, more closely, using a sample size of 10 and $N = 200,000$.

Table 3. region inspection

a	0.5	0.6	0.75	0.8	1	1.25
Tail Coefficient	3.169	3.10	3.08	3.08	3.10	3.09
Max Degree	2999	1945	1411	1405	887	1056

We expect the graph structure to change while the expected value of our delay is infinite, but it requires more analysis than we have done here to say exactly what is changing. We know that the change does not affect the tail behavior, so we predict that nodes with degrees in the middle are gaining more degrees.

4. Discussion and Related Questions

Adding delay to dynamic graph models is a general topic that goes far beyond the scope of this paper. One may use another model besides preferential attachment and observe how its growth changes with added delay. Many real-world phenomena that can be represented with dynamic graph models have a delay in some capacity; thus, it is beneficial to investigate the effects of adding delay to said models.

Regarding preferential attachment models, four elements can be changed in future studies. One may try a broader range of δ, different distributions with different moments for the delay variable, a larger m, or other initial graphs. The interesting factor to change is the delay variable, as we have only considered whether the delay variable has a finite first moment in our study.

As preferential attachment models can be used to model systems where inequality arises, citation networks, social-media following, wealth, etc., one may be interested in reducing the effect of delay in the system to reduce inequality. Moreover, the average entity within the system knows only a tiny proportion of the whole system, so it is beneficial to consider that a specific entity can influence the entire system with limited knowledge. This can be represented by introducing different mechanisms to our preferential attachment model, such as the ability for nodes to disconnect from each other and reconnect to other nodes, which were added later. Ultimately, there are many open questions regarding dynamic models and induced delay, and we hope this paper assists with future research in this area.

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