1 Notation

- N objects: $O \in \{o_1, o_2, ..., o_N\}$
- I poses: $P \in \{p_1, p_2, ..., p_I\}$
- I actions: $A \in \{a_1, a_2, ..., a_I\}$
- $K = N \cdot I$ sets of object-poses: $(o_k, p_k) = (o_n, p_i)$
- J feature-types: $\mathbf{F} = \{\mathbf{F}^1, \mathbf{F}^2, ..., \mathbf{F}^J\} = \{f_1, ..., f_M\}$
- M_j features of type j: $\mathbf{F}^j = \{f_1^j, f_2^j, ..., f_{M_j}^j\}$
- R traning samples for each object-pose.

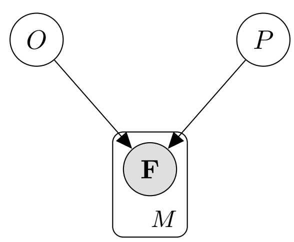
2 Derivation

First we record one set of data to build a database:

$$\mathcal{D} = \left\{ (o_k, p_k, \mathbf{F}_k)_{k=1}^{k=K} \right\}$$

The database is just the set of all unique objects, poses, and features observed in $\mathcal{D}$ to create the structure of our Bayesian Network like so:

$$\begin{aligned} &\mathbf{F} = \{\mathbf{F}_1, ..., \mathbf{F}_K\} = \{f_1, ..., f_M\} \\ &O \in \{o_1, o_2, ..., o_N\} \\ &P \in \{p_1, p_2, ..., p_I\} \end{aligned}$$



Next, we record a set of training data. For each object-pose, k, we record R sets of data:

$$\begin{aligned} \mathcal{T}_k &= \left\{ (o_k^r, p_k^r, \mathbf{F}_k^r)_{r=1}^{r=R} \right\} \\ \mathcal{T} &= \{\mathcal{T}_1, ..., \mathcal{T}_K\} \end{aligned}$$

An error function must be defined for each feature-type in the model which compares 2 features of the same type:

$$\mathcal{E}^j(\cdot, \cdot)$$

For a training sample r, the error with respect to a feature in the model $f^j \in \mathbf{F}$ is the best match with the training features, $\mathbf{F}^r$, defined by:

$$\mathcal{E}_r(f^j) = \min_{f^j_m \in \mathbf{F}^r} \mathcal{E}^j(f^j, f^j_m)$$

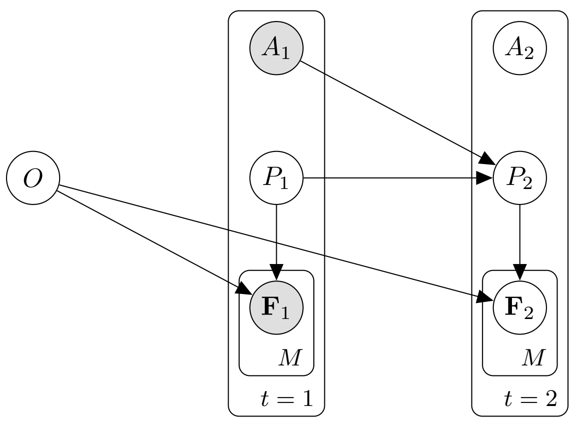
We can then learn a distribution of these errors:

$$\mathrm{p}(f|o,p) \sim \{\mathcal{E}_1(f), ..., \mathcal{E}_R(f)\}$$

After the first observation, we can compute the posterior for all object-poses:

$$\begin{aligned}
\mathrm{p}(o,p|\mathbf{F}) &= \frac{\mathrm{p}(o,p) \cdot \mathrm{p}(\mathbf{F}|o,p)}{\mathrm{p}(\mathbf{F})} \\
\mathrm{p}(o,p) &= \frac{1}{K} \\
\mathrm{p}(\mathbf{F}|o,p) &= \prod_f \mathrm{p}(f|o,p) \\
\mathrm{p}(\mathbf{F}) &= \frac{1}{K} \sum_{n,i} \mathrm{p}(\mathbf{F}|o_n,p_i)
\end{aligned}$$

The next step is to determine the optimal action. After initially observing data $\mathbf{F}_1$, we need to determine the optimal action $a \in A_1$ which will lead to a new pose P_2 and reveal new features $\mathbf{F}_2$.



Consider actions as pariwise *relative* actions between poses. For example, an action could be flip upside-down.

$$\mathrm{p}(o, P_2|\mathbf{F}_1, \mathbf{F}_2, a) = \frac{\sum_{P_1} \mathrm{p}(o, P_1|\mathbf{F}_1)\ \mathrm{p}(\mathbf{F}_2|P_2, o)\ \mathrm{p}(P_2|P_1, a)}{\sum_{P_1,P_2,O} \mathrm{p}(O, P_1|\mathbf{F}_1)\ \mathrm{p}(\mathbf{F}_2|P_2, O)\ \mathrm{p}(P_2|P_1, a)}$$

To determine the optimal action, it we need the probability of the object regardless of pose:

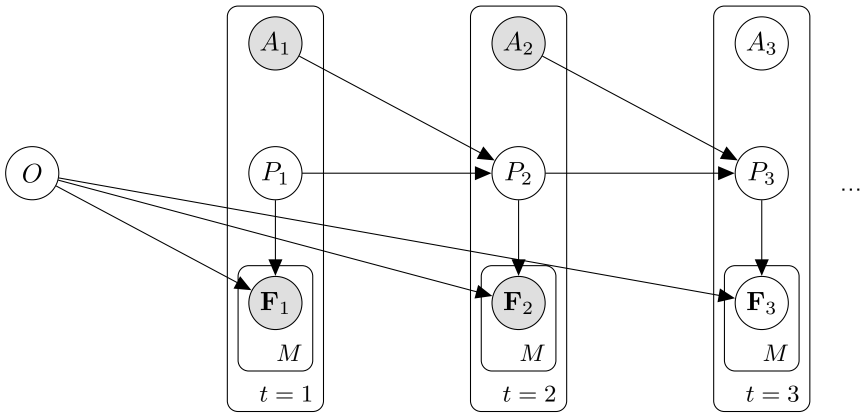
$$\mathrm{p}(o|\mathbf{F}_1, \mathbf{F}_2, a) = \frac{\sum_{P_1,P_2} \mathrm{p}(o, P_1|\mathbf{F}_1)\ \mathrm{p}(\mathbf{F}_2|P_2, o)\ \mathrm{p}(P_2|P_1, a)}{\sum_{P_1,P_2,O} \mathrm{p}(O, P_1|\mathbf{F}_1)\ \mathrm{p}(\mathbf{F}_2|P_2, O)\ \mathrm{p}(P_2|P_1, a)}$$

Note that $\mathrm{p}(P_2|P_1, A_1) \in \{0, 1\}$ is deterministic. Also note that since we have yet to observe $\mathbf{F}_2$, $\mathrm{p}(\mathbf{F}_2|P_2, o)$ is a joint distribution. Thus $\mathrm{p}(o|\mathbf{F}_1, \mathbf{F}_2, a)$ is a mixture of joint distributions divided by a mixture of joint distributions.

Thus, an optimal action would minimize the entropy of *expected* object probabilities.

$$a^* = \underset{A_1}{\operatorname{argmin}} \mathrm{H}\left[\mathbb{E}_{\mathbf{F}_2 \sim \mathrm{p}(\mathbf{F_2}|\mathbf{F}_1, A_1)} \left(O|\mathbf{F}_1, \mathbf{F}_2, A_1\right)\right]$$

Now lets generalize to the nth action.



$$\mathrm{p}(o, P_{n+1}|\mathbf{F}_{1:n+1}, A_{1:n}) = \frac{\sum_{P_n} \mathrm{p}(o, P_n|\mathbf{F}_{1:n}, A_{1:n-1})\mathrm{p}(\mathbf{F}_{n+1}|o, P_{n+1})\mathrm{p}(P_{n+1}|P_n, A_n)}{\sum_{P_n, P_{n+1}, O} \mathrm{p}(o, P_n|\mathbf{F}_{1:n}, A_{1:n-1})\mathrm{p}(\mathbf{F}_{n+1}|o, P_{n+1})\mathrm{p}(P_{n+1}|P_n, A_n)}$$

Note that we are merely updating the previous posterior, $\mathrm{p}(o, P_n|\mathbf{F}_{1:n}, A_{1:n-1})$.

$$a^* = \underset{A_n}{\operatorname{argmin}} \mathrm{H}\left[\mathbb{E}_{\mathbf{F}_{n+1} \sim \mathrm{p}(\mathbf{F_{n+1}}|\mathbf{F}_{1:n}, A_{1:n})} \left(O|\mathbf{F}_{1:n+1}, A_{1:n}\right)\right]$$

3 Algorithm

From the training, we learn $\mathrm{p}(f|o, p)$.

We must keep track of the posterior, evidence, and likelihoods.

Time	Posterior	Evidence	Likelihood
0	$\mathrm{p}(o, p)$		
1	$\mathrm{p}(o, p\|\mathbf{F}_1)$	$\mathrm{p}(\mathbf{F}_1)$	$\mathrm{p}(\mathbf{F}_1\|o, p)$
2	$\mathrm{p}(o, p\|\mathbf{F}_1, \mathbf{F}_2, A_1)$	$\mathrm{p}(\mathbf{F}_2\|\mathbf{F}_1, A_1)$	$\mathrm{p}(\mathbf{F}_2\|o, p)$
...	...	...	...
t	$\mathrm{p}(o, p\|\mathbf{F}_{1:t}, A_{1:t-1})$	$\mathrm{p}(\mathbf{F}_t\|\mathbf{F}_{1:t-1}, A_{1:t-1})$	$\mathrm{p}(\mathbf{F}_t\|o, p)$

We want to keep track of the posterior for each o, p. If we observe $\mathbf{F}_t$, then we can directly compute the likelihood

$$\mathrm{p}(\mathbf{F}_t|o,p) = \prod_{f_t \in \mathbf{F}_t} \mathrm{p}(f_t|o,p)$$

and update the posterior

$$\begin{aligned}\text{posterior}_t &= \frac{\text{posterior}_{t-1} * \text{likelihood}_t}{\text{evidence}_t} \\ &= \frac{\text{posterior}_{t-1} * \text{likelihood}_t}{\sum \text{posterior}_{t-1} * \text{likelihood}_t}\end{aligned}$$

However, when we predict the optimal action, the likelihood is not observed. This likelihood is a very high dimensional joint distribution. To make computation easier, we can sample the distribution. Because $\mathrm{p}(\mathbf{F}_t|o,p) = \prod_{f_t \in \mathbf{F}_t} \mathrm{p}(f_t|o,p)$ is composed of independent features, we can sample from each distribution individually and combine them into a multidimensional sample of the distribution. We can treat these samples as particles.

First compute the evidence distribution,

$$\mathrm{p}(\mathbf{F}_{t+1}|\mathbf{F}_{1:t}, A_{1:t}) = \sum_{O, P_{t+1}, P_t} \mathrm{p}(O, P_t|\mathbf{F}_{1:t}, A_{1:t-1})\mathrm{p}(\mathbf{F}_{t+1}|O, P_{t+1})\mathrm{p}(P_{t+1}|P_t, A_t)$$

Considering actions are deterministic, $A_t * P_t = P_{t+1}$, we can get rid of one of the summations:

$$\mathrm{p}(\mathbf{F}_{t+1}|\mathbf{F}_{1:t}, A_{1:t}) = \sum_{O, P_t} \mathrm{p}(O, P_t|\mathbf{F}_{1:t}, A_{1:t-1})\mathrm{p}(\mathbf{F}_{t+1}|O, P_{t+1})$$

The resulting particles are then used to sample the posterior

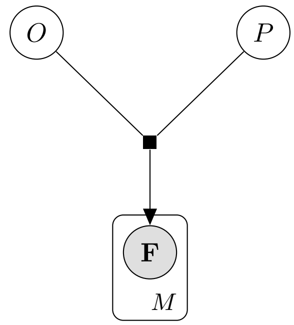
$$\mathrm{p}(o, P_{t+1}|\mathbf{F}_{1:t+1}, A_{1:t}) = \mathrm{p}(o, P_t|\mathbf{F}_{1:t}, A_{1:t-1})\mathrm{p}(\mathbf{F}_{t+1}|o, P_{t+1})_{\mathbf{F}_{t+1} \sim \mathrm{p}(\mathbf{F}_{t+1}|\mathbf{F}_{1:t}, A_{1:t})}$$

Then, taking the average of these particles gives us the expected value.

4 Questions

1) We never learned about bayesian factor graphs, but it seems that we could learn a multivatiate distribution to directly learn $\mathrm{p}(\mathbf{F}|o,p)$:

$$\mathrm{p}(\mathbf{F}|o,p) \sim \left\{ \begin{array}{c} \mathcal{E}_1(f_1), ..., \mathcal{E}_R(f_1) \\ \mathcal{E}_1(f_2), ..., \mathcal{E}_R(f_2) \\ ... \\ \mathcal{E}_1(f_M), ..., \mathcal{E}_R(f_M) \end{array} \right\}$$



5 Appendix

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\begin{aligned}
\mathrm{p}(\mathbf{F}) &= \sum_{n,i} \mathrm{p}(o_n, p_i, \mathbf{F}) \\
&= \sum_{n,i} \mathrm{p}(\mathbf{F}|o_n, p_i) \cdot \mathrm{p}(o_n, p_i) \\
&= \frac{1}{K} \sum_{n,i} \mathrm{p}(\mathbf{F}|o_n, p_i)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{p}(o, P_2|\mathbf{F}_1, \mathbf{F}_2, A_1) &= \sum_{P_1} \frac{\mathrm{p}(o, P_1, P_2, \mathbf{F}_1, \mathbf{F}_2, A_1)}{\mathrm{p}(\mathbf{F}_1, \mathbf{F}_2, A_1)} \\
&= \sum_{P_1} \frac{\mathrm{p}(o, P_1)\mathrm{p}(A_1)\mathrm{p}(P_2|P_1, A_1)\mathrm{p}(\mathbf{F}_1|P_1, o)\mathrm{p}(\mathbf{F}_2|P_2, o)}{\mathrm{p}(\mathbf{F}_1, \mathbf{F}_2|A_1)\mathrm{p}(A_1)} \\
&= \sum_{P_1} \frac{\mathrm{p}(o, P_1)\mathrm{p}(\mathbf{F}_1|P_1, o)}{\mathrm{p}(\mathbf{F}_1)} \frac{\mathrm{p}(\mathbf{F}_2|P_2, o)\mathrm{p}(P_2|P_1, A_1)}{\mathrm{p}(\mathbf{F}_2|\mathbf{F}_1, A_1)} \\
&= \sum_{P_1} \frac{\mathrm{p}(o, P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|P_2, o)\mathrm{p}(P_2|P_1, A_1)}{\mathrm{p}(\mathbf{F}_2|\mathbf{F}_1, A_1)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{p}(\mathbf{F}_2|\mathbf{F}_1, A_1) &= \sum_{P_1, P_2, O} \frac{\mathrm{p}(O, P_1, P_2, \mathbf{F}_1, \mathbf{F}_2|A_1)}{\mathrm{p}(\mathbf{F}_1)} \\
&= \sum_{P_1, P_2, O} \frac{\mathrm{p}(O, P_1)\mathrm{p}(\mathbf{F}_1|O, P_1)\mathrm{p}(\mathbf{F}_2|O, P_2)\mathrm{p}(P_2|P_1, A_1)}{\mathrm{p}(\mathbf{F}_1)} \\
&= \sum_{P_1, P_2, O} \mathrm{p}(O, P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|O, P_2)\mathrm{p}(P_2|P_1, A_1)
\end{aligned}
$$

$$
\mathrm{p}(o, P_2|\mathbf{F}_1, \mathbf{F}_2, A_1) = \frac{\sum_{P_1} \mathrm{p}(o, P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|P_2, o)\mathrm{p}(P_2|P_1, A_1)}{\sum_{P_1, P_2, O} \mathrm{p}(O, P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|O, P_2)\mathrm{p}(P_2|P_1, A_1)}
$$

$$\mathrm{p}(o|\mathbf{F}_1,\mathbf{F}_2,A_1) = \frac{\sum_{P_1} \mathrm{p}(o,P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|P_2,o)\mathrm{p}(P_2|P_1,A_1)}{\sum_{P_1,P_2,O} \mathrm{p}(O,P_1|\mathbf{F}_1)\mathrm{p}(\mathbf{F}_2|O,P_2)\mathrm{p}(P_2|P_1,A_1)}$$

$$\begin{aligned}
&\mathrm{p}(o,P_3|\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,A_1,A_2)\\
&= \sum_{P_1,P_2} \frac{\mathrm{p}(o,P_1,P_2,P_3,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,A_1,A_2)}{\mathrm{p}(\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,A_1,A_2)}\\
&= \sum_{P_1,P_2} \frac{\mathrm{p}(o,P_1)\mathrm{p}(A_1)\mathrm{p}(A_2)\mathrm{p}(\mathbf{F}_1|o,P_1)\mathrm{p}(\mathbf{F}_2|o,P_2)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_2|P_1,A_1)\mathrm{p}(P_3|P_2,A_2)}{\mathrm{p}(\mathbf{F}_3|\mathbf{F}_1,\mathbf{F}_2,A_1,A_2)\mathrm{p}(\mathbf{F}_2|\mathbf{F}_1,A_1)\mathrm{p}(\mathbf{F}_1)\mathrm{p}(A_1)\mathrm{p}(A_2)}\\
&= \sum_{P_2} \frac{\mathrm{p}(o,P_2|\mathbf{F}_1,\mathbf{F}_2,A_1)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_3|P_2,A_2)}{\mathrm{p}(\mathbf{F}_3|\mathbf{F}_1,\mathbf{F}_2,A_1,A_2)}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{p}(\mathbf{F}_3|\mathbf{F}_1,\mathbf{F}_2,A_1,A_2)\\
&= \sum_{P_1,P_2,P_3,O} \frac{\mathrm{p}(o,P_1,P_2,P_3,\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,A_1,A_2)}{\mathrm{p}(\mathbf{F}_1,\mathbf{F}_2,A_1,A_2)}\\
&= \sum_{P_1,P_2,P_3,O} \frac{\mathrm{p}(o,P_1)\mathrm{p}(A_1)\mathrm{p}(A_2)\mathrm{p}(\mathbf{F}_1|o,P_1)\mathrm{p}(\mathbf{F}_2|o,P_2)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_2|P_1,A_1)\mathrm{p}(P_3|P_2,A_2)}{\mathrm{p}(\mathbf{F}_2|\mathbf{F}_1,A_1)\mathrm{p}(\mathbf{F}_1)\mathrm{p}(A_1)\mathrm{p}(A_2)}\\
&= \sum_{P_2,P_3,O} \mathrm{p}(o,P_2|\mathbf{F}_1,\mathbf{F}_2,A_1)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_3|P_2,A_2)
\end{aligned}$$

$$\mathrm{p}(o,P_3|\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,A_1,A_2) = \frac{\sum_{P_2} \mathrm{p}(o,P_2|\mathbf{F}_1,\mathbf{F}_2,A_1)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_3|P_2,A_2)}{\sum_{P_2,P_3,O} \mathrm{p}(o,P_2|\mathbf{F}_1,\mathbf{F}_2,A_1)\mathrm{p}(\mathbf{F}_3|o,P_3)\mathrm{p}(P_3|P_2,A_2)}$$

$$\mathrm{p}(o,P_{n+1}|\mathbf{F}_{1:n+1},A_{1:n}) = \frac{\sum_{P_n} \mathrm{p}(o,P_n|\mathbf{F}_{1:n},A_{1:n-1})\mathrm{p}(\mathbf{F}_{n+1}|o,P_{n+1})\mathrm{p}(P_{n+1}|P_n,A_n)}{\sum_{P_n,P_{n+1},O} \mathrm{p}(o,P_n|\mathbf{F}_{1:n},A_{1:n-1})\mathrm{p}(\mathbf{F}_{n+1}|o,P_{n+1})\mathrm{p}(P_{n+1}|P_n,A_n)}$$