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Development of a High-order Incompressible Flow Solver with Implicit Time-Stepping

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Introduction

In computational fluid dynamics (CFD), high-order (3^{rd} and above) spatially accurate methods used to solve large scale problems require fast convergence. To address this need, the current implementation uses an implicit-LU time-stepping scheme to accelerate the convergence rate of steady 2-D incompressible flows on unstructured meshes.

Governing Equations

Consider mass conservation in the steady 2-D Euler equations with Artificial Compressibility (AC)

$$\frac{1}{\beta}\frac{\partial p}{\partial t}+\nabla\cdot V=0$$

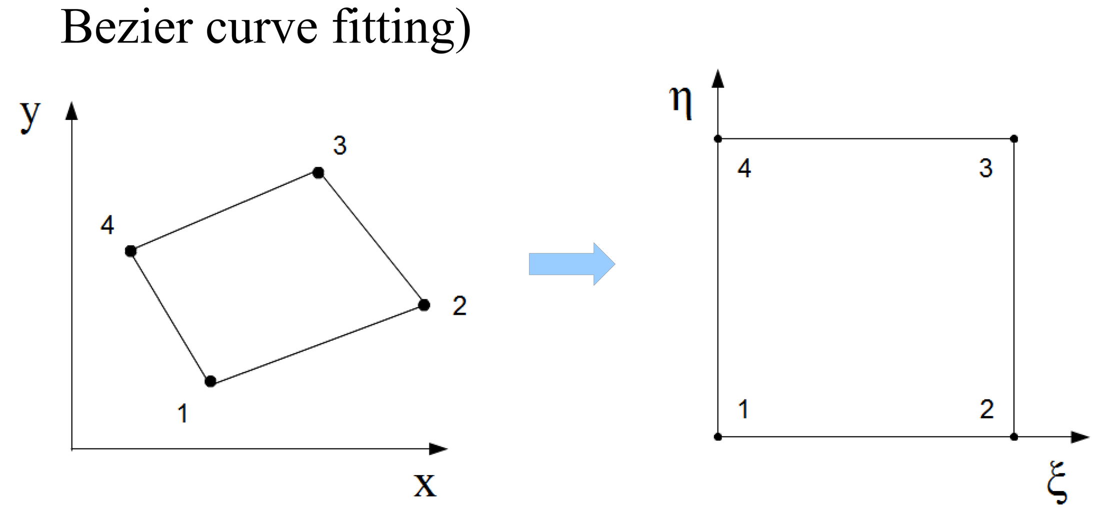
Implicit Time-Stepping

- Permits a large time step, Δt
- Quickly establishes divergence-free velocity field
- Utilizes method of characteristics for solving hyperbolic PDEs
- Parallel processing friendly
- Mesh deformation friendly to solve fluid-structure interaction problems
- High memory requirement
- Implementation difficulty

High-order CFD Method

We extend the idea of flux reconstruction[1,2] to solve incompressible flows and implement the following concepts:

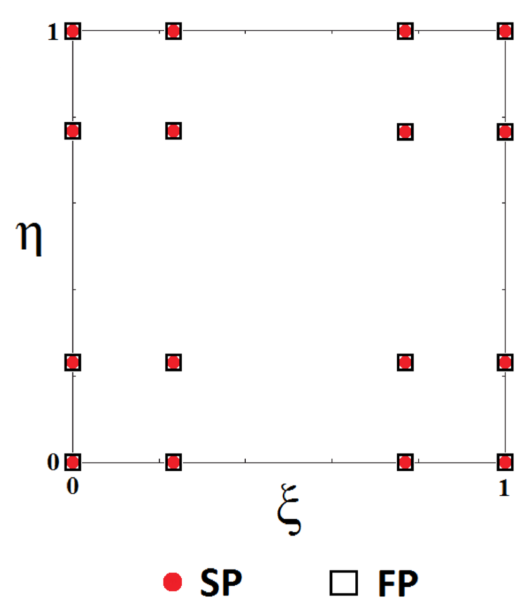
- Isoparametric mapping
- Collocation of solution points (SP) and mass and momentum flux points (FP) at the Legendre-Lobatto positions within a standard element
- High-order curved boundary elements (via cubic Bezier curve fitting)

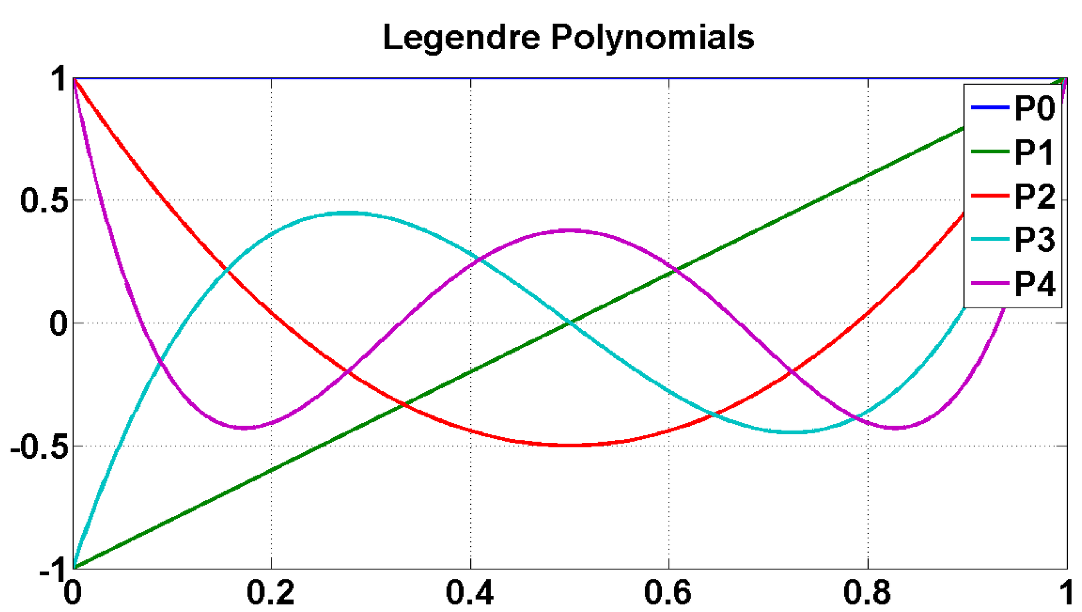


High-order CFD Method

Solutions within each standard cell are approximated by k solution points (in 1-D) that define a polynomial of degree k-1.

- High order accurate
- Discontinuous at interfaces





Implicit Time-Stepping

Referring back to our mass conservation equation, we can define the approximation to the solution

$$Residual=-\beta\nabla\cdot V$$

To solve the governing form with the implicit-LU technique, a linearization of the governing equations must be performed.

$$\left[\frac{I}{\Delta t}+\frac{\partial R_c}{\partial Q_c}\right](Q_c^{n+1}-Q_c^*)=-R_c^*-\frac{\Delta Q_c^*}{\Delta t}$$

$$A \quad x \quad = \quad b$$

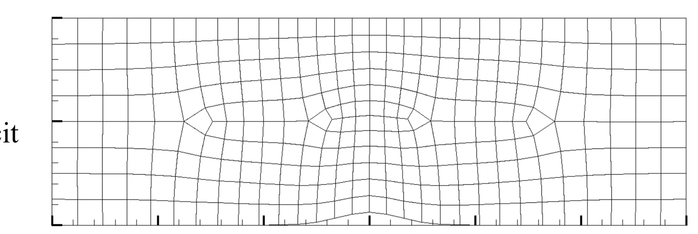
The system of equations is then solved directly using LU decomposition. For higher orders of accuracy, the size of matrix A renders the solution of x computationally expensive.

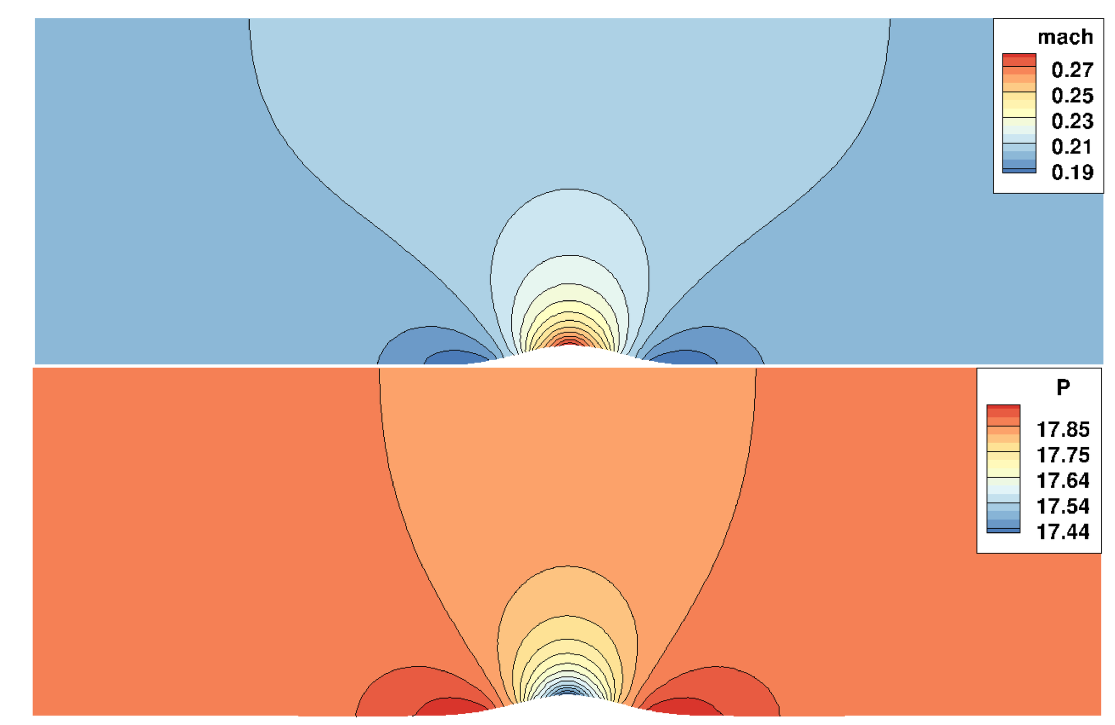
Order	A
2^{nd}	12x12
3^{rd}	27x27
4^{th}	48x48
5^{th}	75x75

Channel w/ Bump

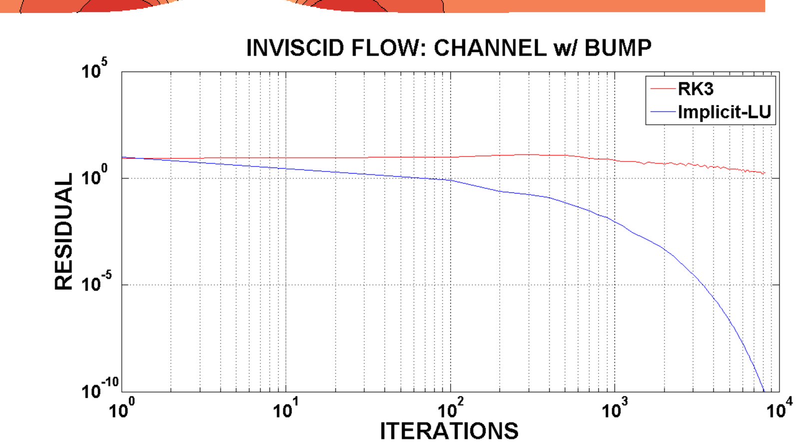
Steady inviscid flow:

- $\Delta t_{implicit} = 500\ \Delta t_{explicit}$
- Reduces residual by 10 orders of magnitude





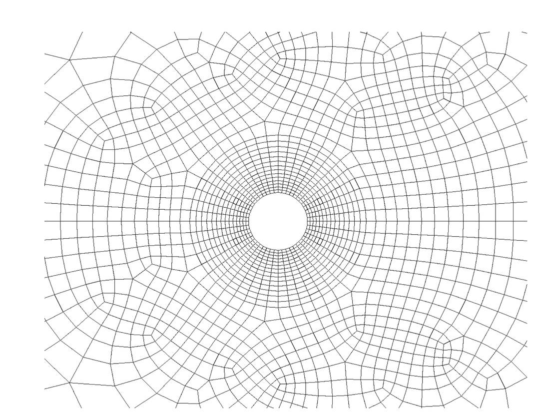
Scheme	Δt
Explicit RK3	2e-4
Implicit LU	1e-1

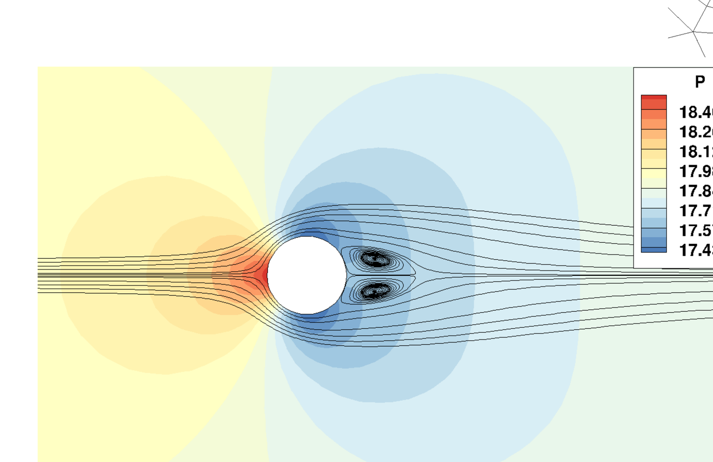


Circle

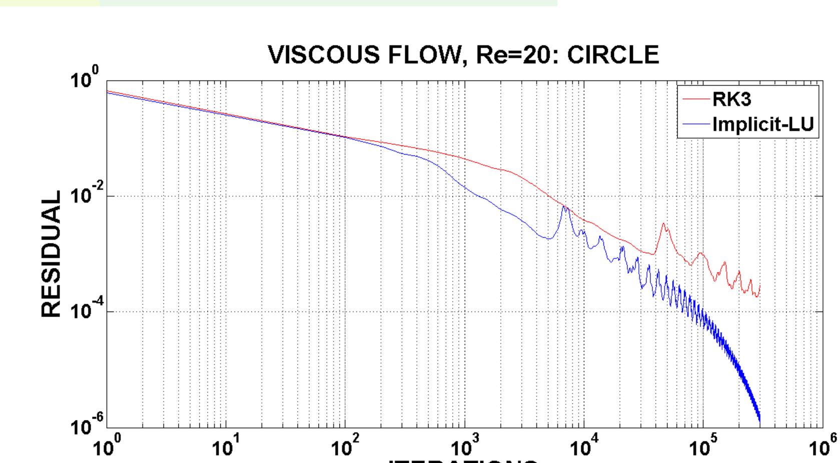
Steady viscous flow, Re=20:

- $\Delta t_{implicit} = 10\ \Delta t_{explicit}$
- Reduces residual by over 2 orders of magnitude





Scheme	Δt
Explicit RK3	1e-3
Implicit LU	1e-2



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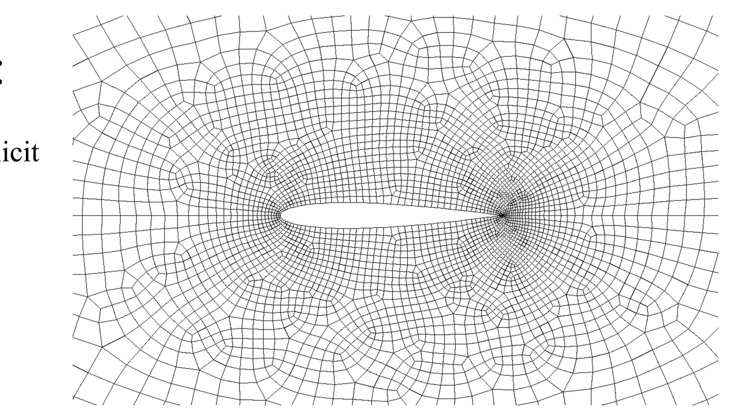
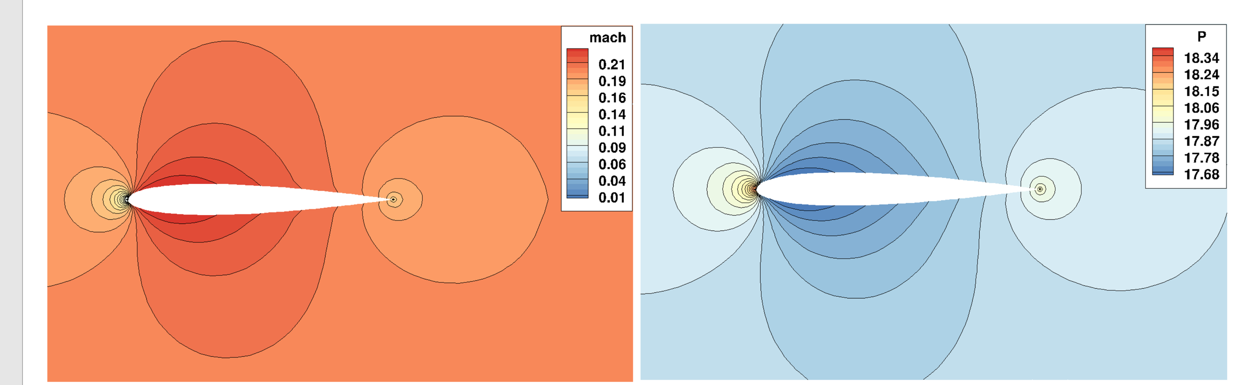
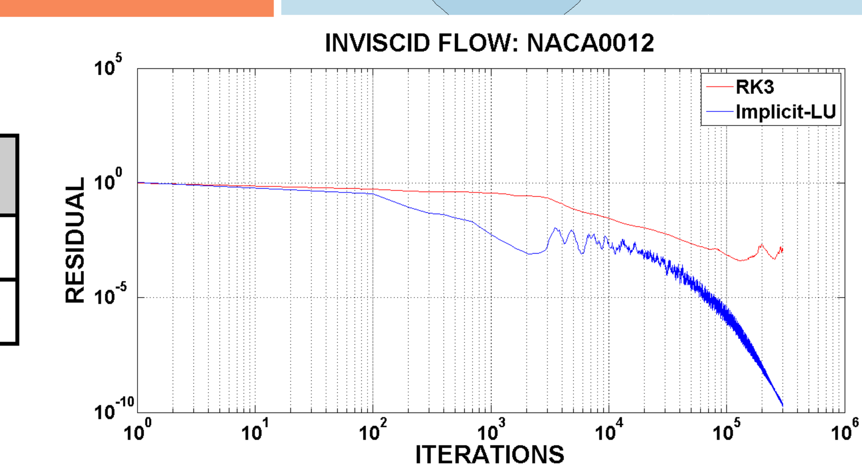
Steady inviscid flow:

- $\Delta t_{implicit} = 100\ \Delta t_{explicit}$
- Reduces residual by 7 orders of magnitude





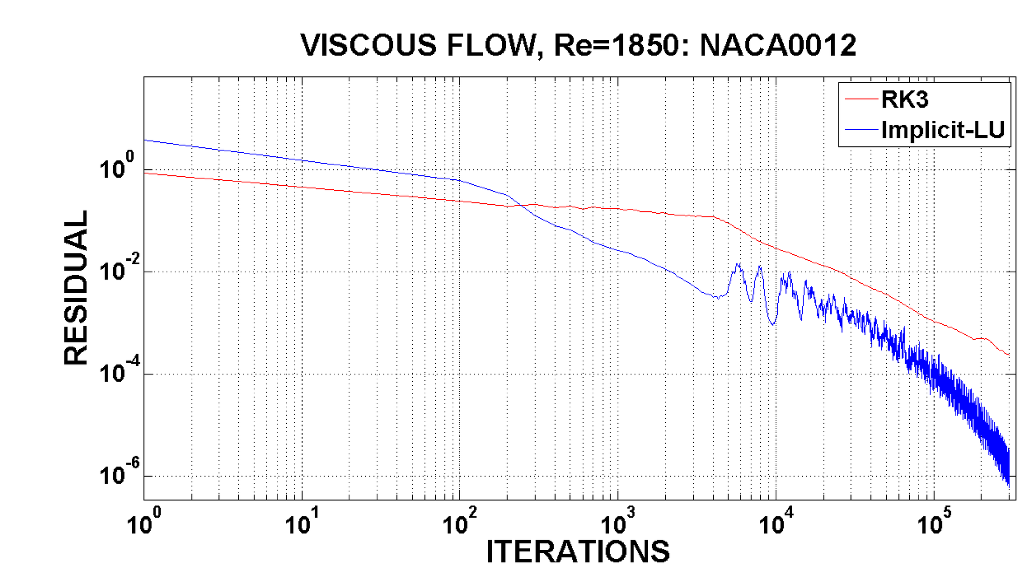
Scheme	Δt
Explicit RK3	1e-4
Implicit LU	1e-2



Steady viscous flow, Re=1850:

- $\Delta t_{implicit} = 40\ \Delta t_{explicit}$
- Reduces residual by 2 orders of magnitude

Scheme	Δt
Explicit RK3	1e-4
Implicit LU	2e-3



Future Work

- Expand the current solver to run on massively parallel computers
- Simulate 3-D incompressible flows

References

[1] Huynh, H., "A flux reconstruction approach to high-order schemes including discontinuous Galerkin method," AIAA Paper AIAA-2007-4079, 2007.

[2] Wang, Z.J., Gao, H., "A unifying lifting collocation penalty formulation including the discontinuous Galerkin, spectral volume/difference methods for conservation laws on mixed grids," Journal of Computational Physics, **228**, 21, pp. 8161-8186, 2009.