Development of a fast algorithm for solving the unsteady incompressible Navier-Stokes equations

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Motivation

In computational fluid dynamics (CFD), high-order (3rd and above) spatially accurate methods used to solve large scale problems require fast convergence. To address this need, we implement a local non-linear implicit LU-SGS time-stepping scheme to accelerate the convergence rate of unsteady incompressible flows[1] in complex geometries, particularly flows that are vortex dominated.

Governing Equations

Consider the unsteady incompressible Navier-Stokes equations with artificial compressibility (AC)

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau} + \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}} = 0 \quad \longrightarrow \quad \nabla\cdot\vec{V}\to 0$$

$$\frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial t} + \frac{\partial (u^2+p-\nu u_x)}{\partial x} + \frac{\partial (vu-\nu u_y)}{\partial y} = 0$$

$$\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} + \frac{\partial (uv-\nu v_x)}{\partial x} + \frac{\partial (v^2+p-\nu v_y)}{\partial y} = 0$$

Pseudo Time Stepping
- implicit
- explicit

Physical Time Stepping
- 2nd order backward differencing

High-order Method
- flux reconstruction

$$\frac{\partial U}{\partial \tau} + \frac{\partial \hat{U}}{\partial t} + \boxed{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}} = 0$$

- Non-linear lower-upper symmetric Gauss-Seidel backward Euler (**LU-SGS**)
- Third-order three-stage Runge-Kutta (**RK33**)

Implicit Time Stepping

Pros
(1) permits a large time step, Δt → quickly establish divergence-free velocity field
(2) utilizes advanced time-stepping techniques for solving hyperbolic/parabolic PDEs
(3) parallel processing and mesh deformation friendly to solve fluid-structure interaction problems

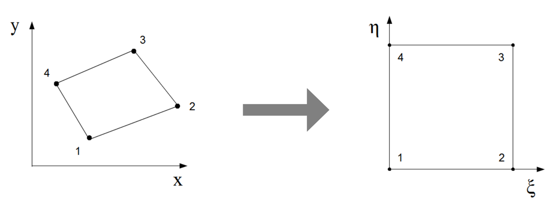
Cons
(4) high memory requirement & implementation difficulty

Mapping to Reference Element

We extend the idea of flux reconstruction[2] to solve incompressible flows with high order accuracy while implementing the following concepts for unstructured linear quadrilateral elements Ω_q:

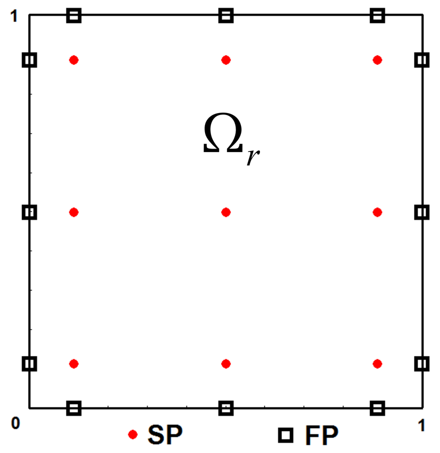
- **isoparametric mapping** of physical element Ω_q to reference element $\Omega_r = \{\xi,\eta \mid 0 \le \xi,\eta \le 1\}$
- **curved boundaries** represented via cubic Bezier curves

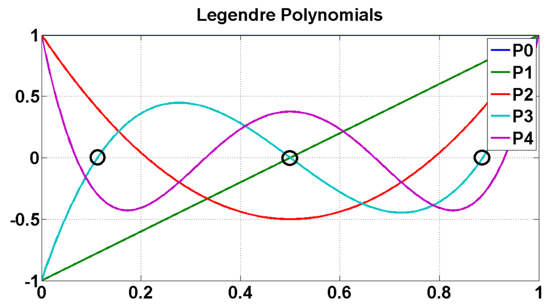
$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{4} \Psi_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$



High-order CFD Method

Each reference cell contains NxN solution points in 2D; with high order accuracy, solution points (SP) located at **Legendre-Gauss** positions, flux points (FP) located along the element boundary, solution is piecewise continuous across domain.

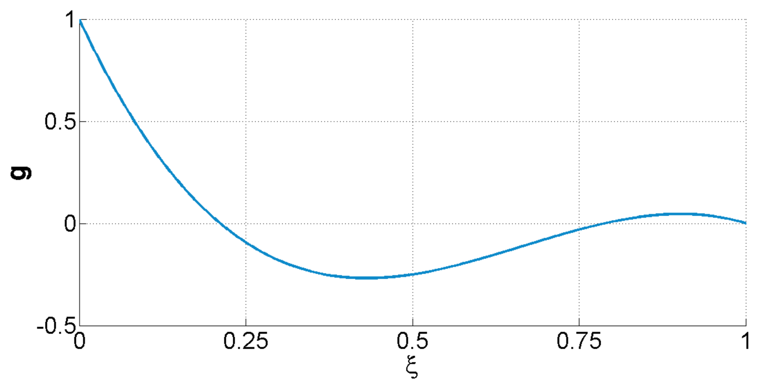




$$l_i(\xi) = \prod_{s=1, s\neq i}^{N} \left(\frac{\xi-\xi_s}{\xi_i-\xi_s} \right)$$

$$U(\xi) = \sum_{i=1}^{N} U_i l_i(\xi) \qquad f_r^D(\xi) = \sum_{i=1}^{N} f_{r|i}^D l_i(\xi)$$

$$\frac{\partial f_r}{\partial \xi} = \frac{d f_r^D}{d\xi} + [f_{r-1/2}^{com} - f_r^D(0)]\frac{d g_r^{LB}}{d\xi} + [f_{r+1/2}^{com} - f_r^D(1)]\frac{d g_r^{RB}}{d\xi}$$



Implicit Time-Stepping

Referring back to our mass conservation equation, we can define the **Residual** for the first equation as

$$Residual: \; R_r = \nabla\cdot\vec{V}_r \to 0$$

and develop an algorithm to drive this residual as close and fast to zero as possible. To solve the governing form with the implicit LU-SGS scheme, a linearization of the governing equations must be performed.

$$\frac{p_r^{n+1,m+1} - p_r^{n+1,m}}{\Delta\tau} + \nabla\cdot\vec{V}_r^{n+1,m+1} = 0$$

$$\downarrow$$

$$R_r^{m+1} - R_r^m \approx \frac{\partial R_r}{\partial U_r} + \sum_{nb\neq r}\frac{\partial R_r}{\partial U_{nb}}\Delta U_{nb}$$

$$\downarrow$$

$$\left[\frac{I}{\Delta t} + \frac{\partial R_r}{\partial U_r}\right]\delta p_r^{k+1} = -R_r^* - \frac{\Delta p^*}{\Delta\tau}$$

$$\downarrow$$

$$A \quad x \quad = \quad b$$

The system of equations is then solved directly using **LU decomposition**. For higher orders of accuracy, the size of matrix A renders the solution of x more computationally expensive.

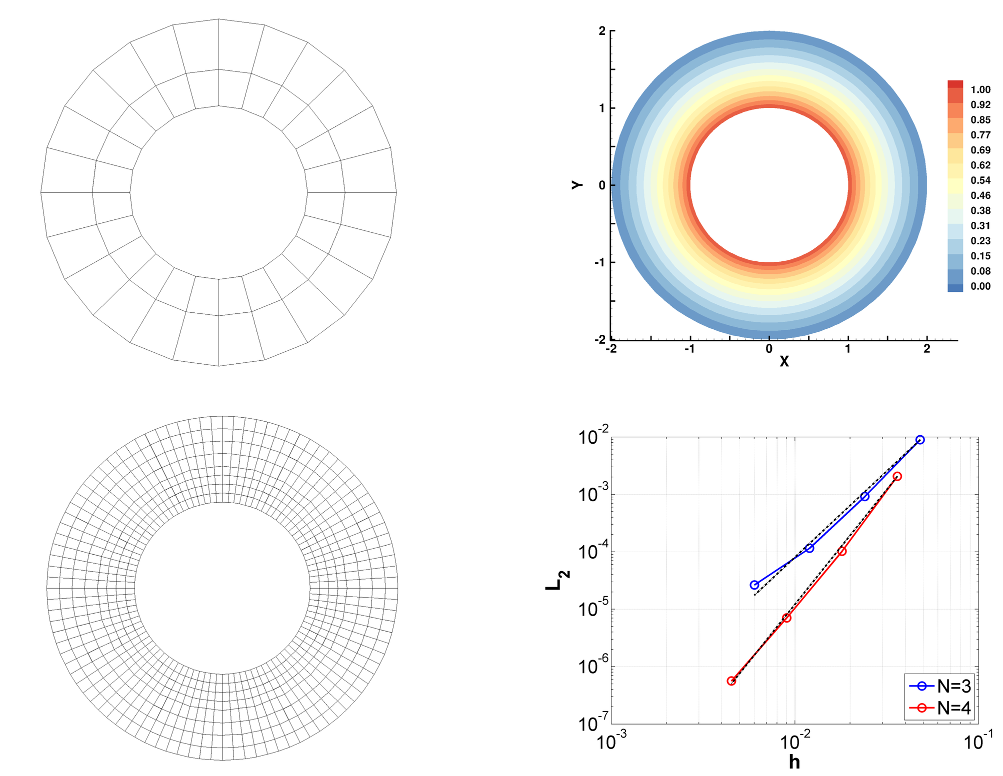
N	A
2	12x12
3	27x27
4	48x48
5	75x75

$$\text{Size of } A = [N_{eq}\cdot N^D] \; x \; [N_{eq}\cdot N^D]$$

Verification

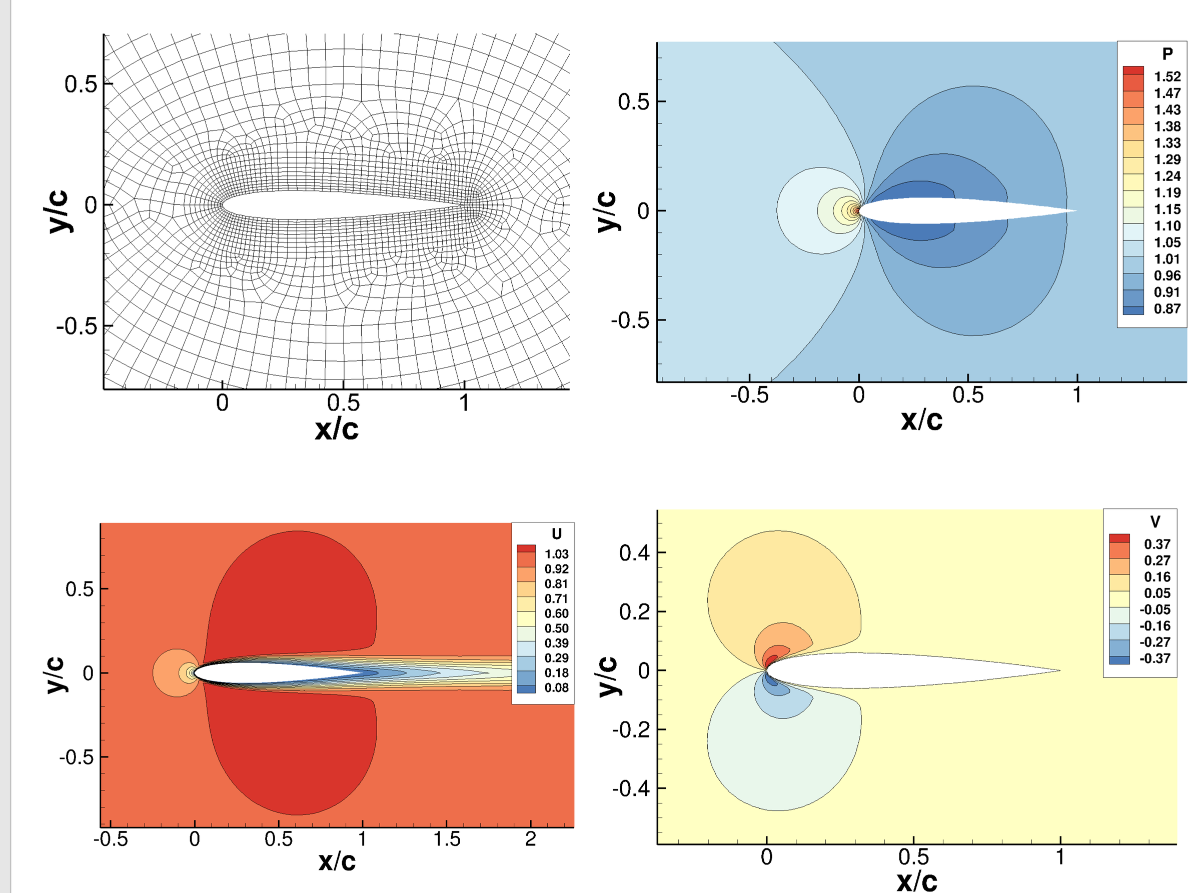
- **Taylor-Couette** flow at *Re=10*

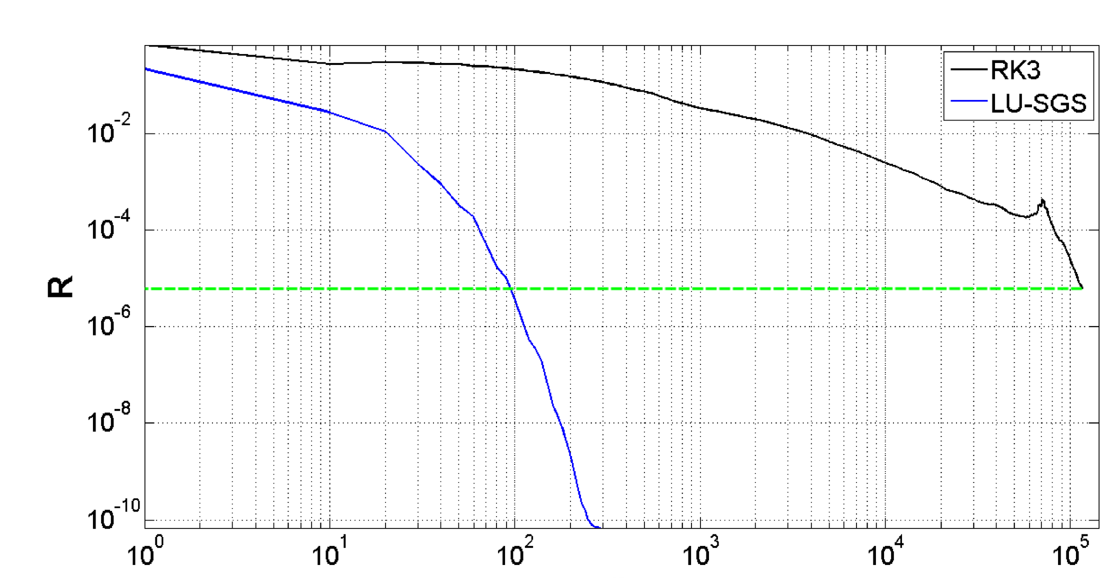
$$V_{exact}(r) = r_i\omega_i \frac{\frac{r_o}{r} - \frac{r}{r_o}}{\frac{r_o}{r_i} - \frac{r_i}{r_o}} + r_o\omega_o \frac{\frac{r}{r_i} - \frac{r_i}{r}}{\frac{r_o}{r_i} - \frac{r_i}{r_o}} \qquad L_2 = \sqrt{\frac{\sum_{i=1}^{nDOF}(V_{numerical} - V_{exact})^2}{nDOF}}$$



Steady Flow

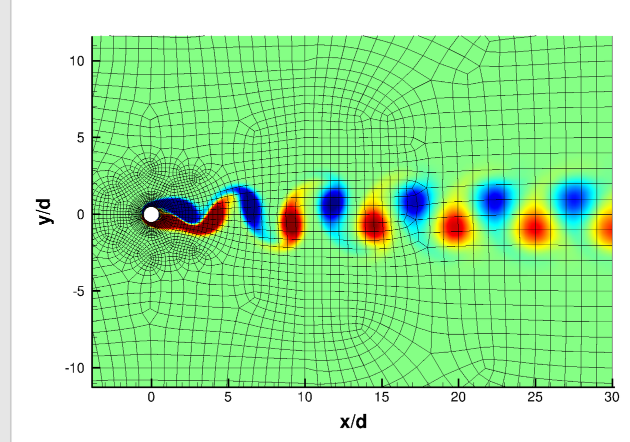
- **NACA-0012** airfoil at *α=0˚* , *Re=1850, N=3*
- Speedup in CPU time to achieve steady state = 55

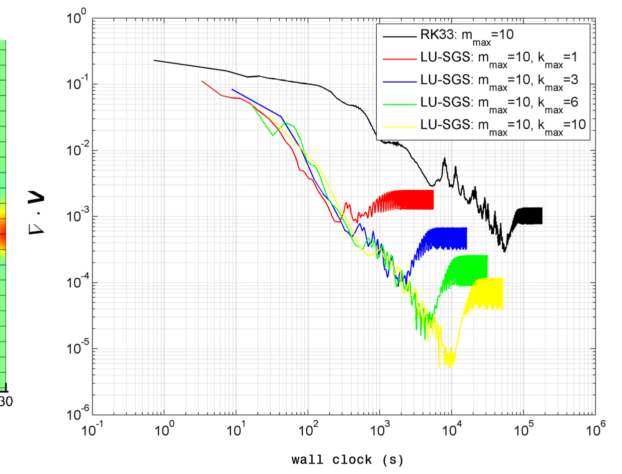




Unsteady Flow

- **Cylinder** at *Re=100, N=3*
- Efficiency speedup measured per shedding cycle

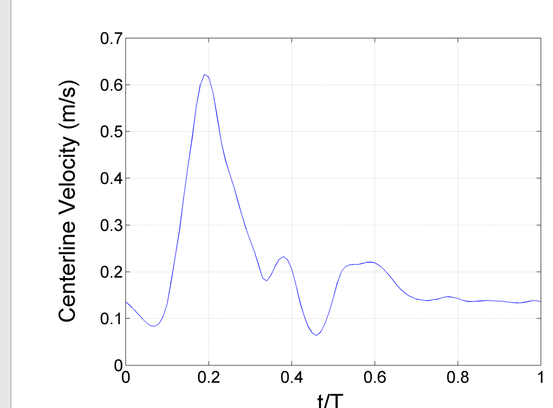




	Explicit	Implicit			
m_{max}	10	10	10	10	10
k_{max}	-	1	3	6	10
$C_{L,rms}$	0.231	0.240	0.230	0.223	0.222
C_D ($C_{D,rms}$)	1.354 (0.006)	1.342 (0.006)	1.338 (0.006)	1.338 (0.006)	1.338 (0.006)
Strouhal	0.162	0.160	0.162	0.163	0.163
Speedup	1	25.2	8.5	4.5	2.7

Pulsatile Flow

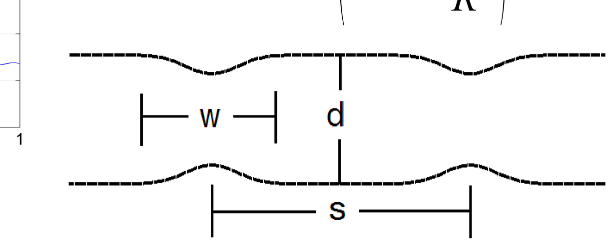
- **Channel with double constriction** at *Re=500, N=4*

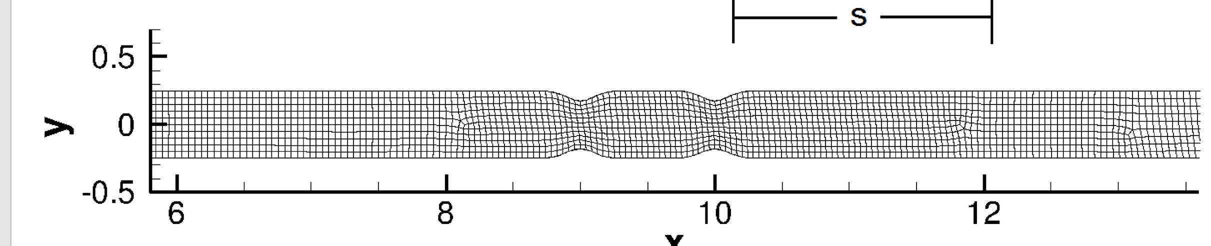


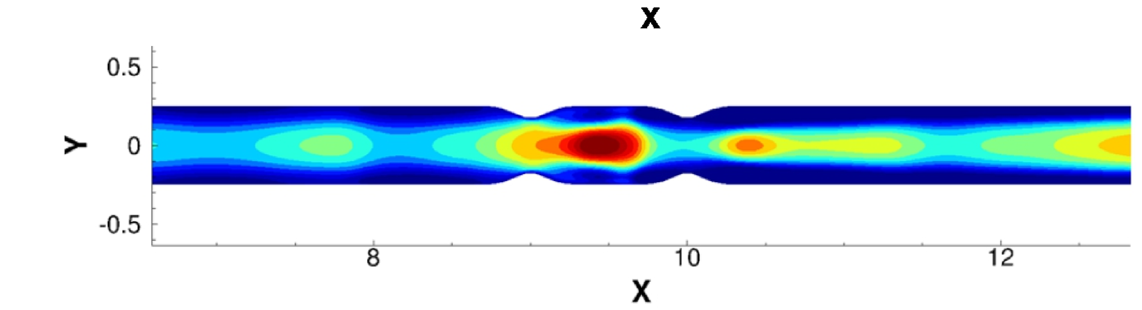
$$L = 20 \qquad w = d$$

$$d = 0.5 \qquad s = 2d$$

$$u = u_c\left(1 - \frac{r^2}{R^2}\right)$$







Future Implementation

(1) Parallel processing
(2) Three-dimensional implementation
(3) Moving and deforming mesh for fluid-structure interaction

References

1) C. Cox, C. Liang and M.W. Plesniak, “A high-order method for solving unsteady incompressible Navier-Stokes equations with implicit tme stepping on unstructured grids,” 53rd AIAA Aerospace Sciences Meeting, January 6, 2015.
2) Huynh, H., “A flux reconstruction approach to high-order schemes including discontinuous Galerkin method,” AIAA Paper AIAA-2007-4079, 2007.
3) C. Liang, C. Cox and M.W. Plesniak, “A comparison of computational efficiencies of spectral difference method and correction procedure via reconstruction,” Journal of Computational Physics, Vol 239, pp 138-146, April 15, 2013.