UNIVERSITY OF HELSINKI
DEPARTMENT OF PHYSICS

BASICS OF MONTE CARLO SIMULATIONS

Exercise 3

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Note. In order to run the exercise's source codes, use the command *make all* (see the Makefile for more compilation commands and details). The graphs and plots are available in the ./Figures/ folder.

Problem 1

Please compile and run the **ex3p1** program. See the output below.

COMMAND 1

```
$ ./ex3p1
Ndim                 V
1         2.000000   +/-   0.002001
2         3.142208   +/-   0.003548
3         4.186264   +/-   0.005793
4         4.923648   +/-   0.008873
5         5.248096   +/-   0.012915
6         5.135040   +/-   0.017885
7         4.675072   +/-   0.024012
8         3.998464   +/-   0.031743
9         3.268096   +/-   0.040775
10        2.588672   +/-   0.051421
11        1.818624   +/-   0.061002
12        1.384448   +/-   0.075291
13        0.917504   +/-   0.086691
14        0.671744   +/-   0.104907
15        0.589824   +/-   0.139022
```

We notice here that volume V^{M+1} in all the required dimensions $(1, 2, 3, \ldots, 15)$ can be calculated using the hit-and-miss method without the need of knowing the value of V^M. However, we can see that the uncertainty of the method seems to grow with the number of dimensions.

Problem 2

Please compile and run the **ex3p2** program. See the output below.

COMMAND 2

```
$ ./ex3p2
Ndim              V
2        3.140826   +/-   0.000447
3        4.187182   +/-   0.000836
4        4.933021   +/-   0.001333
5        5.260103   +/-   0.001962
6        5.163138   +/-   0.002748
7        4.728081   +/-   0.003695
8        4.060002   +/-   0.004805
9        3.303517   +/-   0.006078
10       2.593248   +/-   0.007549
11       1.972644   +/-   0.009277
12       1.399622   +/-   0.010908
13       0.959634   +/-   0.013120
14       0.595532   +/-   0.012865
15       0.424484   +/-   0.016036
```

The direct sampling method requires the previous knowledge of the value of V^M in order to calculate the value of V^{M+1}. However, we will not always have the exact value of V^M, so, in my code, using the minimum knowledge of the value of V^M in 1 dimension, I can calculate V^{M+1} for all successive dimensions recursively.

This, however, means I can get cumulative error after each loop, so this is not the best option for higher dimensions. So, in order to avoid that, we need to know the volume of the function in a lower dimension at least not too far from the dimension you want to calculate, so that the cumulative error is kept at a minimum. The best would be using the exact values of V^M for each V^{M+1} but that is not always possible.

Cumulative error aside, this method shows a better uncertainty in higher dimensions if compared to the hit-and-miss.

Notice that this method doesn't calculates the volume in the first dimension, it uses it uses its exact value as a starting point to calculate the volumes in all following dimensions.

To see a version of the problem 2 program that uses the exact value of V^M instead of using recursivity, please compile and run the **ex3p2b** program. See the output below.

COMMAND 3

```
$ ./ex3p2b
Ndim              V
2        3.140826   +/-   0.000447
3        4.188205   +/-   0.000836
4        4.934916   +/-   0.001333
5        5.262002   +/-   0.001963
6        5.166757   +/-   0.002750
7        4.732270   +/-   0.003699
8        4.057156   +/-   0.004801
9        3.302467   +/-   0.006076
10       2.589318   +/-   0.007538
11       1.939870   +/-   0.009123
12       1.336802   +/-   0.010418
13       0.915507   +/-   0.012517
14       0.565120   +/-   0.012208
```

```
15          0.427145     +/-     0.016137
```

We can see the effect of the cumulative error if we compare the results of both functions.

Problem 3

Please compile and run the **ex3p3** program. See the output below.

COMMAND 4

```
$ ./ex3p3
 ITEM (A)

          N              dI                    dt
HM      100        0.2965195178986E+00     0.2100000000000E-04
DS      100        0.7280219174177E-02     0.8999999999999E-05
SS      100        0.6273418478882E-02     0.1100000000000E-04
PSS     100        0.1180224160349E-01     0.1000000000000E-04
IS      100        0.2253781843278E-02     0.1700000000000E-04
-----------------------------------------------------------------
HM      1000       0.4348048210144E-01     0.9000000000000E-04
DS      1000       0.8962901052386E-02     0.8200000000000E-04
SS      1000       0.3158712600770E-02     0.9000000000000E-04
PSS     1000       0.8265733986335E-04     0.8800000000000E-04
IS      1000       0.1088675760484E-03     0.1620000000000E-03
-----------------------------------------------------------------
HM      10000      0.5519517898560E-02     0.8920000000000E-03
DS      10000      0.2246770030228E-02     0.8560000000000E-03
SS      10000      0.2988502493415E-03     0.8130000000000E-03
PSS     10000      0.1363676627197E-03     0.7930000000000E-03
IS      10000      0.9434100424000E-04     0.1440000000000E-02
-----------------------------------------------------------------
HM      100000     0.7195178985596E-03     0.6932000000000E-02
DS      100000     0.6587812739867E-03     0.7561000000000E-02
SS      100000     0.6339618902507E-04     0.7012000000000E-02
PSS     100000     0.4581015277404E-03     0.6816000000000E-02
IS      100000     0.3982838176253E-04     0.1221800000000E-01
-----------------------------------------------------------------
HM      1000000    0.2779517898560E-02     0.7882500000000E-01
DS      1000000    0.1203257861035E-03     0.6250800000000E-01
SS      1000000    0.2505745810066E-05     0.6710300000000E-01
PSS     1000000    0.1493102658567E-04     0.6701100000000E-01
IS      1000000    0.1938042227712E-04     0.1248150000000E+00

 ITEM (B)

HM: Standart Deviation of the Mean =   7.8541245678479E-03
DS: Standart Deviation of the Mean =   9.7675250212641E-04
SS: Standart Deviation of the Mean =   3.9702552725575E-04
PSS: Standart Deviation of the Mean =   3.9824639078413E-04
IS: Standart Deviation of the Mean =   1.3536910605468E-04
```

Considering the results above, if I have limitation in computational time, but I would like to get quite reasonable result in the MC integration, I would probably choose the Stratified Sampling method since it shows a very good accuracy and precision considering its execution time. The Partially Stratified Sampling method is also a good choice in that regard.

We can see that the stratified random numbers (present in the SS and PSS methods) really minimises the uncertainty of the DS algorithm by decreasing its variance (like we saw in class). The IS method also has a smaller uncertainty if compared with the DS method, as expected.

Problem 4

Please compile and run the **ex3p4** program. See the output below.

COMMAND 5

```
$ ./ex3p4
________________________________________
 ITEM (A)

    BASE 7
              0.6010
              0.7439
              0.8867
              0.0500
              0.1928
              0.3357
              0.4786
              0.6214
              0.7643
              0.9071

    BASE 13
              0.6104
              0.6873
              0.7642
              0.8411
              0.9181
              0.9950
              0.0014
              0.0783
              0.1552
              0.2321
________________________________________
 ITEM (B)

         N                  dI
HM       100          0.9651951789856E-01
DS       100          0.2450321191034E-01
SS       100          0.2450321191034E-01
PSS      100          0.5087574992522E-02
IS       100          0.2718523767890E-02
______________________________________
HM       1000         0.2016519517899E+01
DS       1000         0.3143917515744E-02
SS       1000         0.3143917515744E-02
PSS      1000         0.6379274900983E-02
IS       1000         0.3138280167056E-02
______________________________________
HM       10000        0.2003519517899E+01
DS       10000        0.2905700793437E-03
SS       10000        0.2905700793437E-03
PSS      10000        0.1855428348777E-03
IS       10000        0.1229362080436E-03
______________________________________
HM       100000       0.1999519517899E+01
DS       100000       0.6964278297794E-04
SS       100000       0.6964278297794E-04
PSS      100000       0.1047892402082E-03
IS       100000       0.5199779229104E-04
______________________________________
HM       1000000      0.1999879517899E+01
```

```
DS     1000000     0.9976799416500E−05
SS     1000000     0.9976799416500E−05
PSS    1000000     0.1448820238559E−05
IS     1000000     0.1366138803194E−06
```

Let's check the plots of the results of problems 3 and for in the figures 1, 2, 3, and 4.

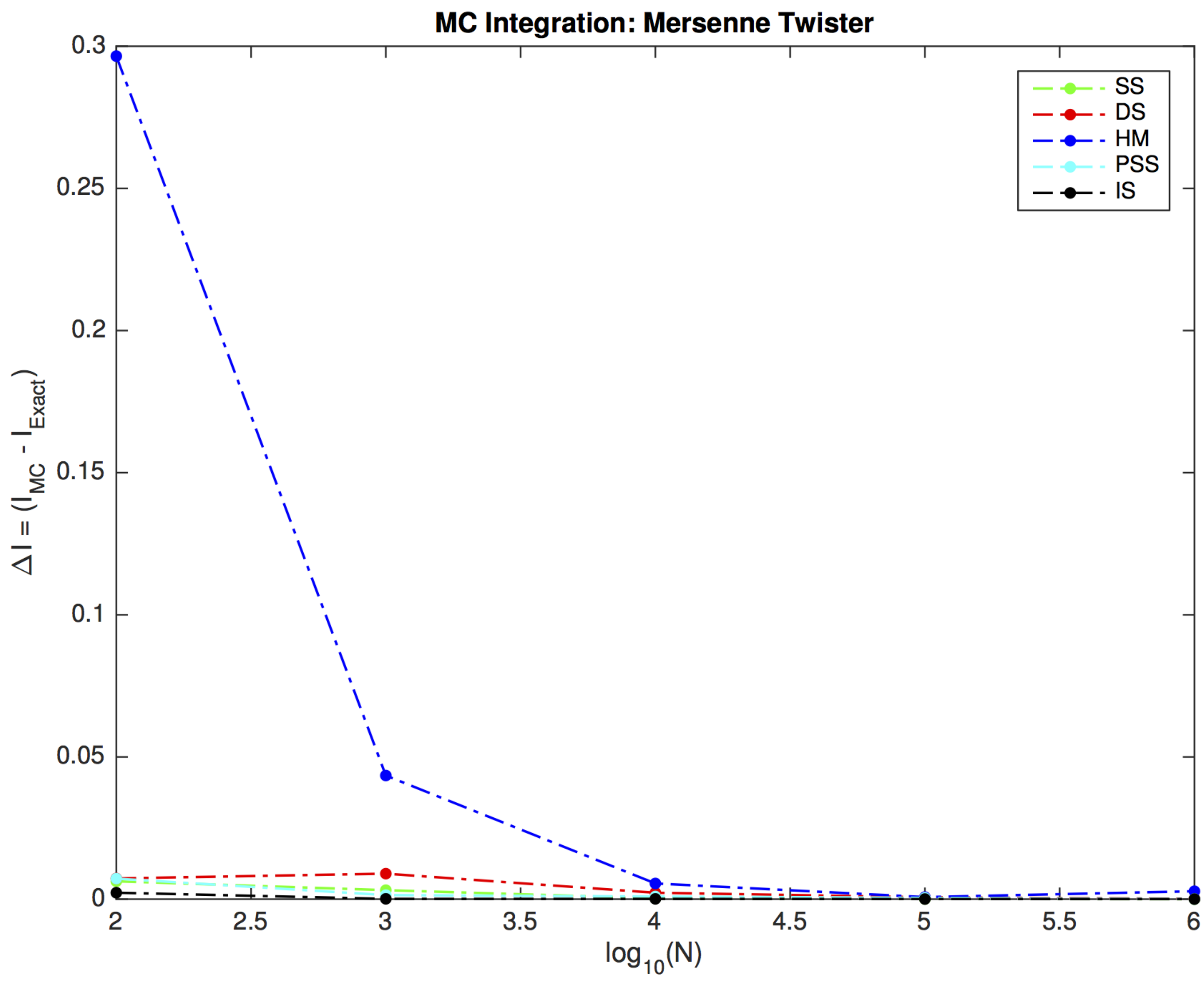

Figure 1: Accuracy of the MC integration methods using the Mersenne Twister in function of the number of points.

We can see that the Halton sequence has a good effect in the MC integration methods except for the hit-and-miss. If we are using the hit-and-miss method we should avoid using the Halton Sequence and use the Mersenne Twister. In higher values of N, we can see that the other integration methods have pretty close results when using the Halton sequence (except for the hit-and-miss method, of course).

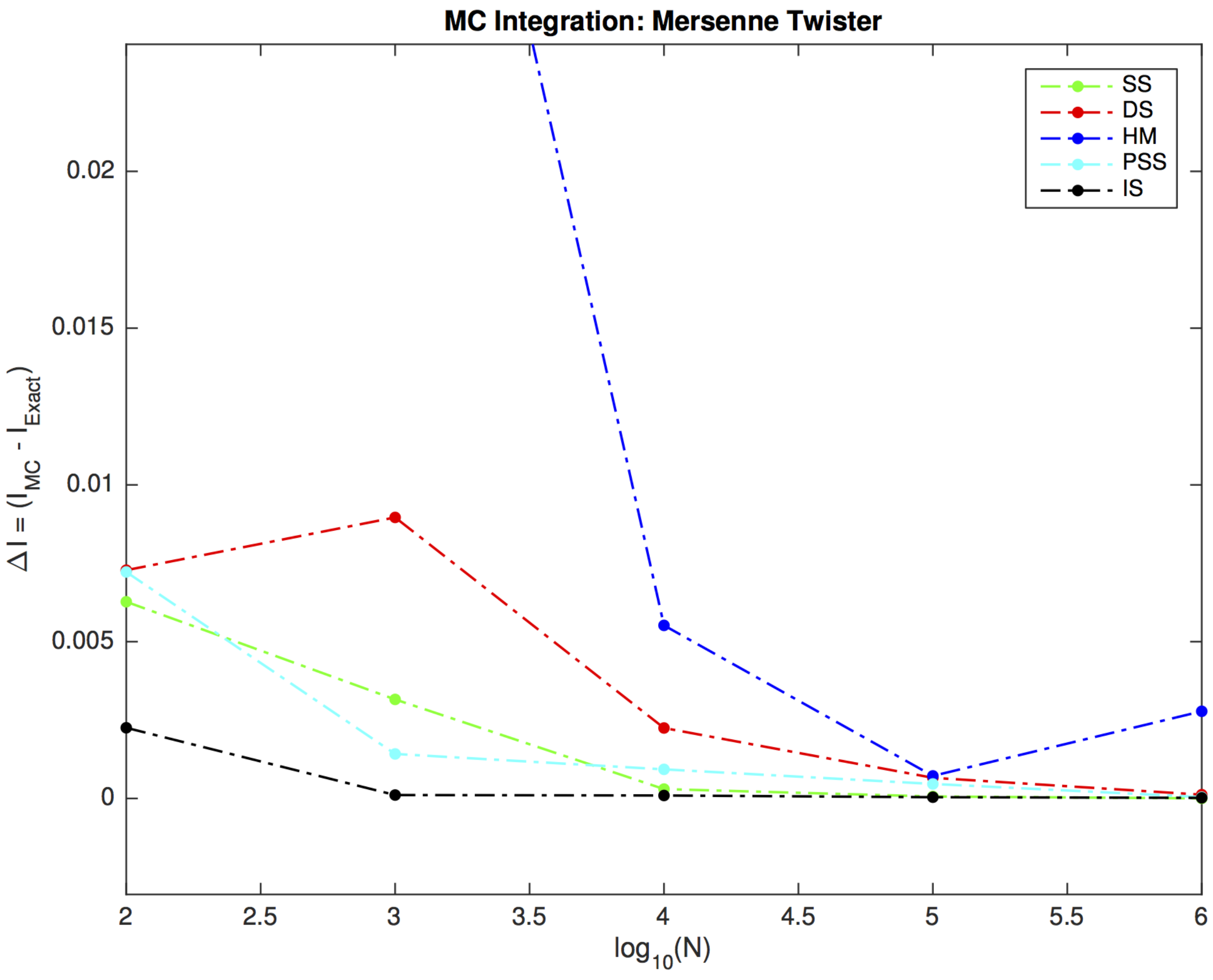

Figure 2: Zoom in the plot of figure 1.

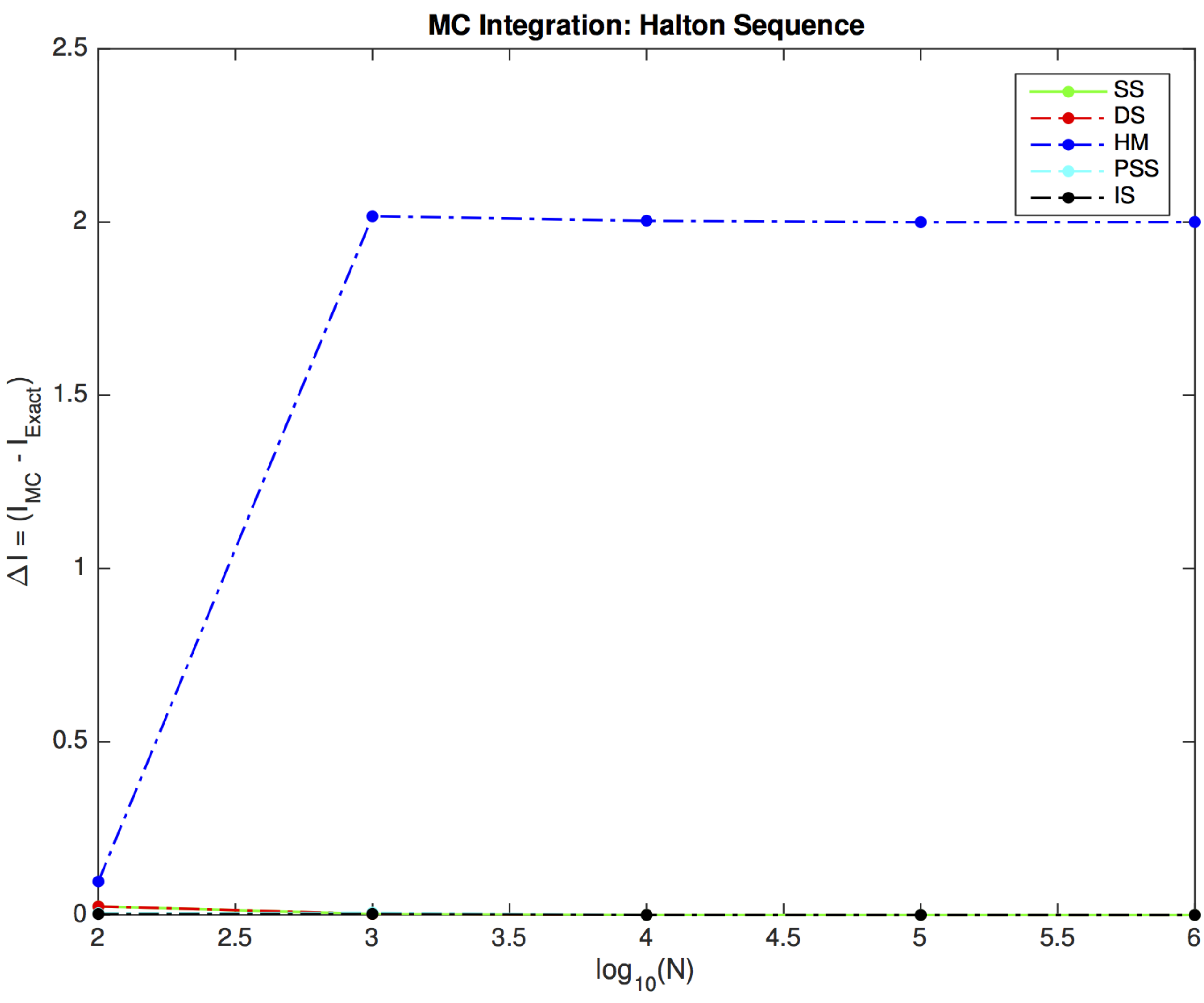

Figure 3: Accuracy of the MC integration methods using the Halton Sequence in function of the number of points.

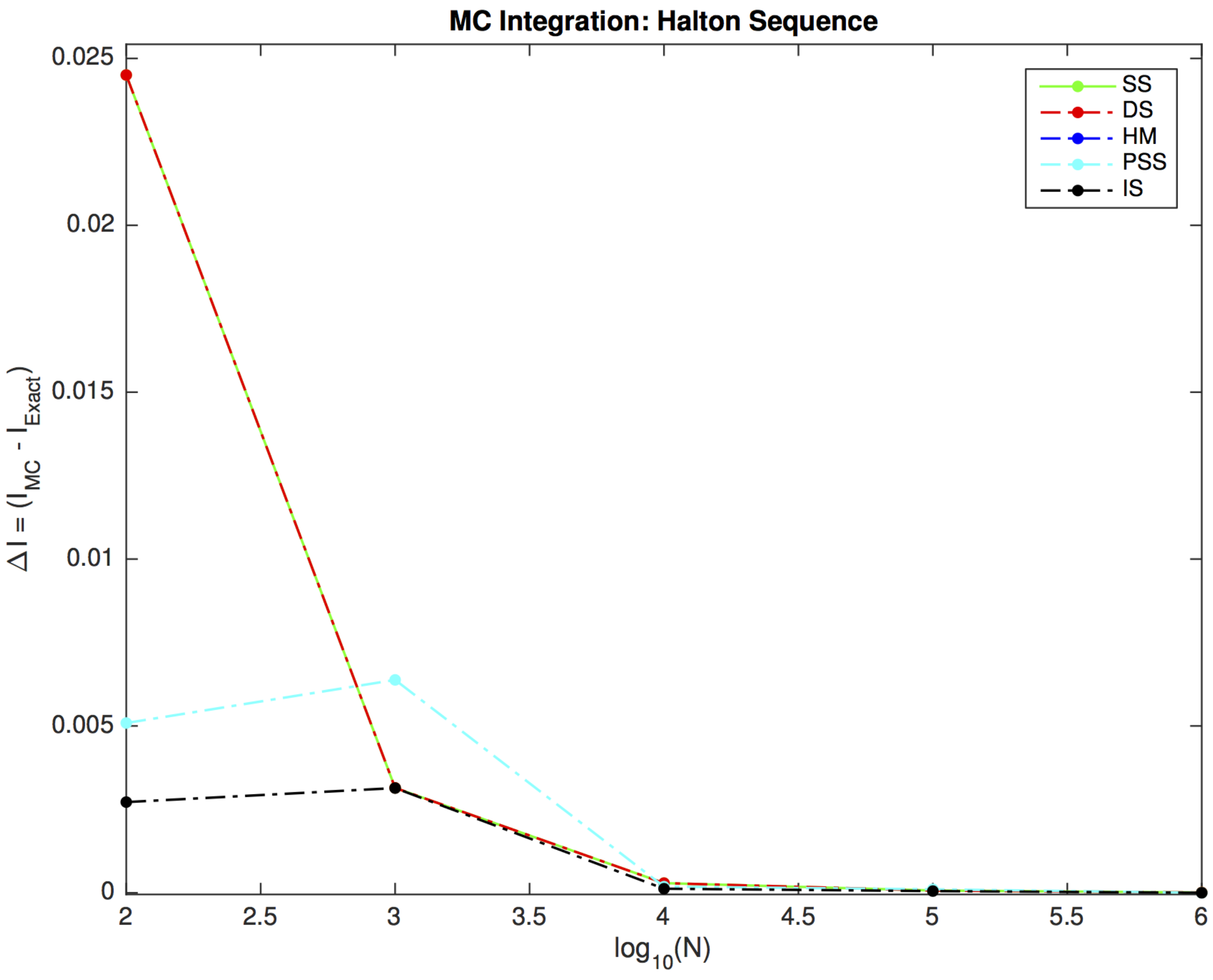

Figure 4: Zoom in the plot of figure 3.

A Importance Sampling in Problems 3 and 4

For the importance sampling method I chose the function:

$$g(x) = x^{-1/3} \tag{1}$$

The assignment defines the function $f(x)$ that must be integrated as:

$$f(x) = \frac{1}{x^{1/3} + x^{1/4}} \tag{2}$$

The flat function $f(x)/g(x)$ is shown plotted alongside with the original functions $f(x)$ and $g(x)$ in figure 5. However, now that we chose a function $g(x)$ we must normalize it in the integration interval [0,1].

$$g(x) = \frac{2x^{-1/3}}{3} \Rightarrow \int_0^1 g(x)\, dx = 1 \tag{3}$$

Then we have G(x) defined as:

$$G(x) = \int_0^x g(t)\, dt = x^{2/3} \tag{4}$$

Finally, we get the inverse function $G^{-1}(u)$:

$$G^{-1}(u) = u^{3/2} \tag{5}$$

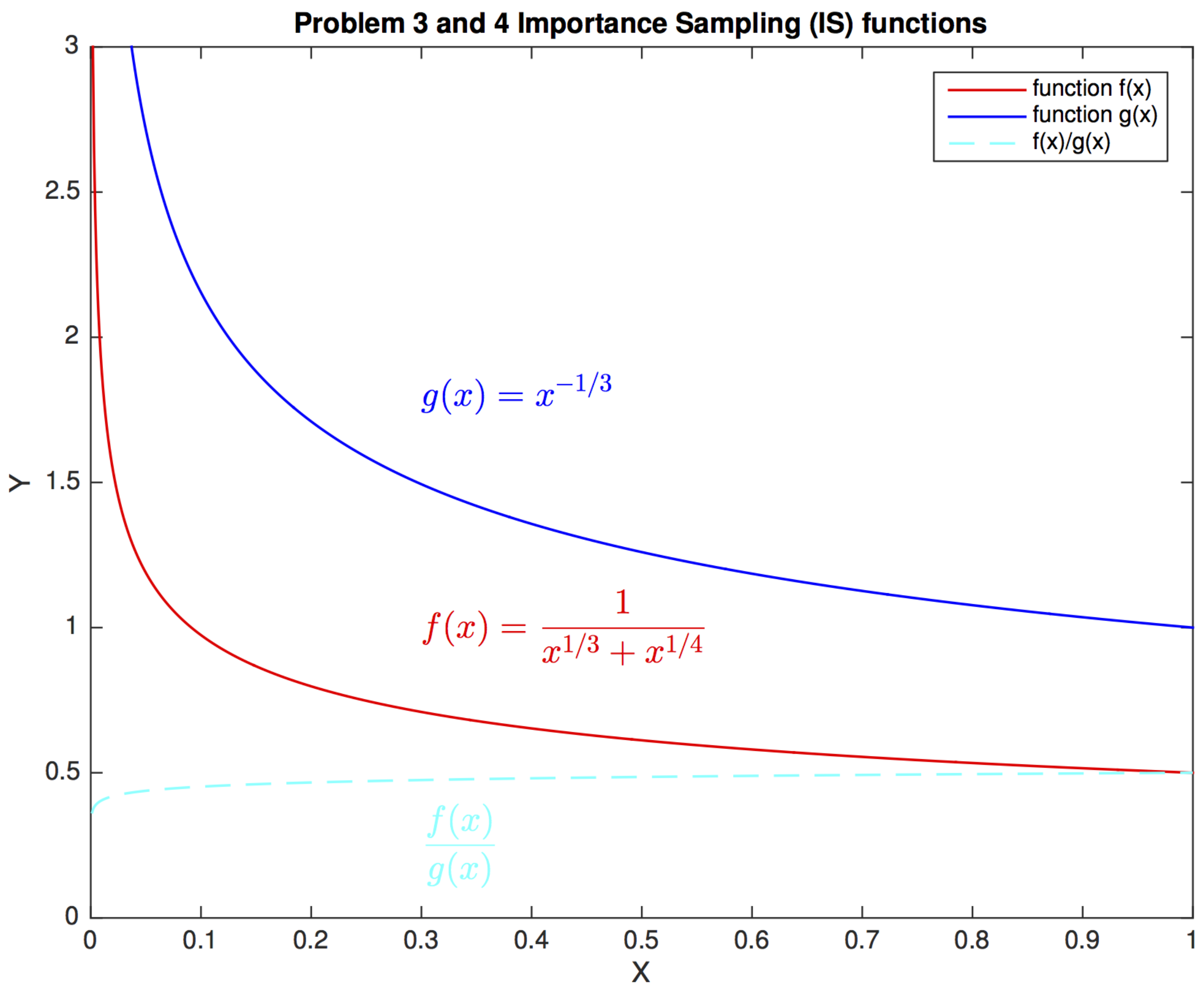

Figure 5: Choosing function $g(x)$ so we can get a flat $f(x)/g(x)$.