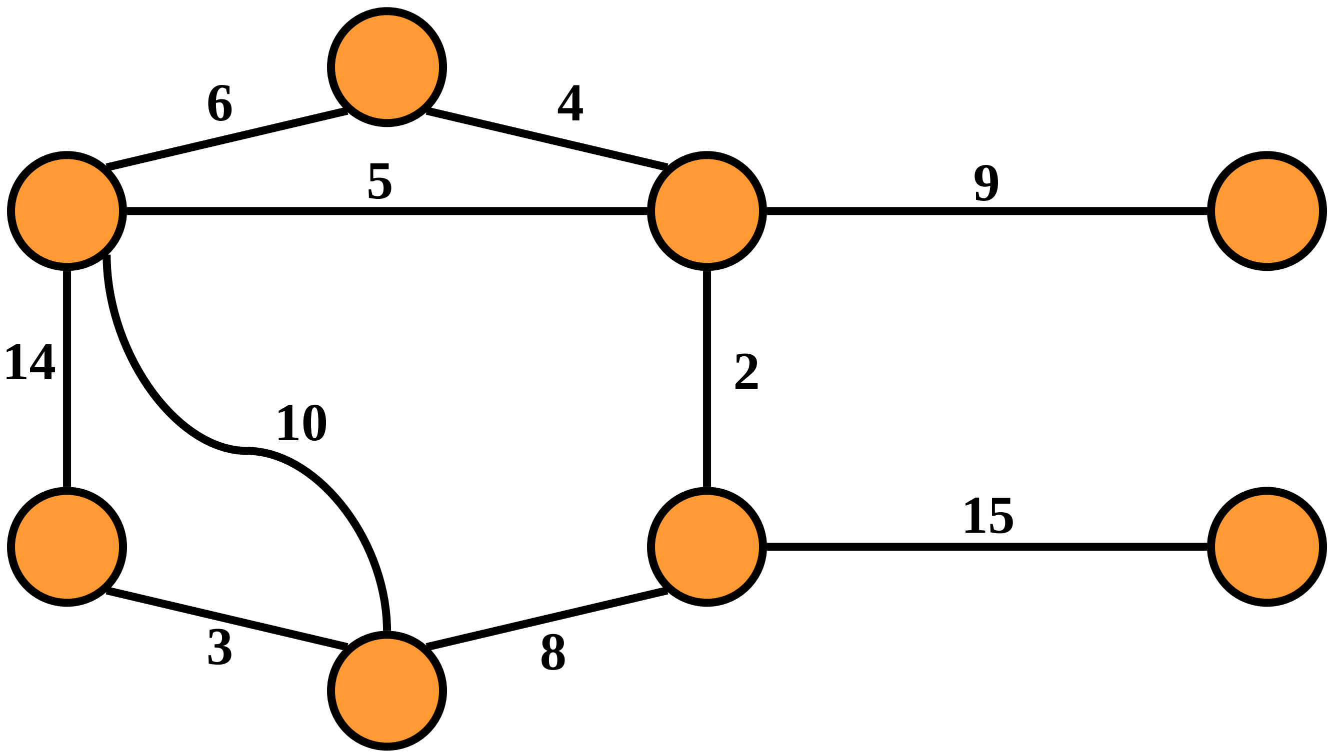
Chapter 23
Minimum Spanning Trees

Algorithm Analysis

School of CSEE

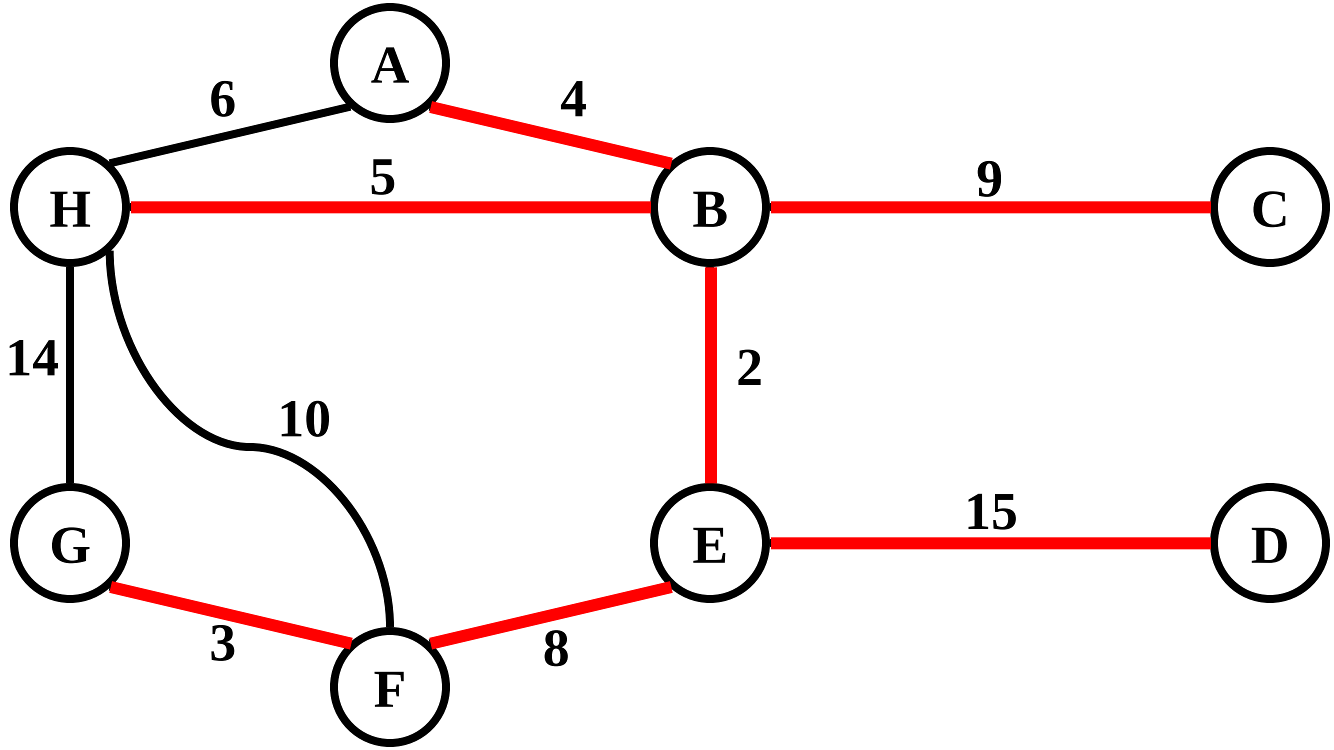
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight.



Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*

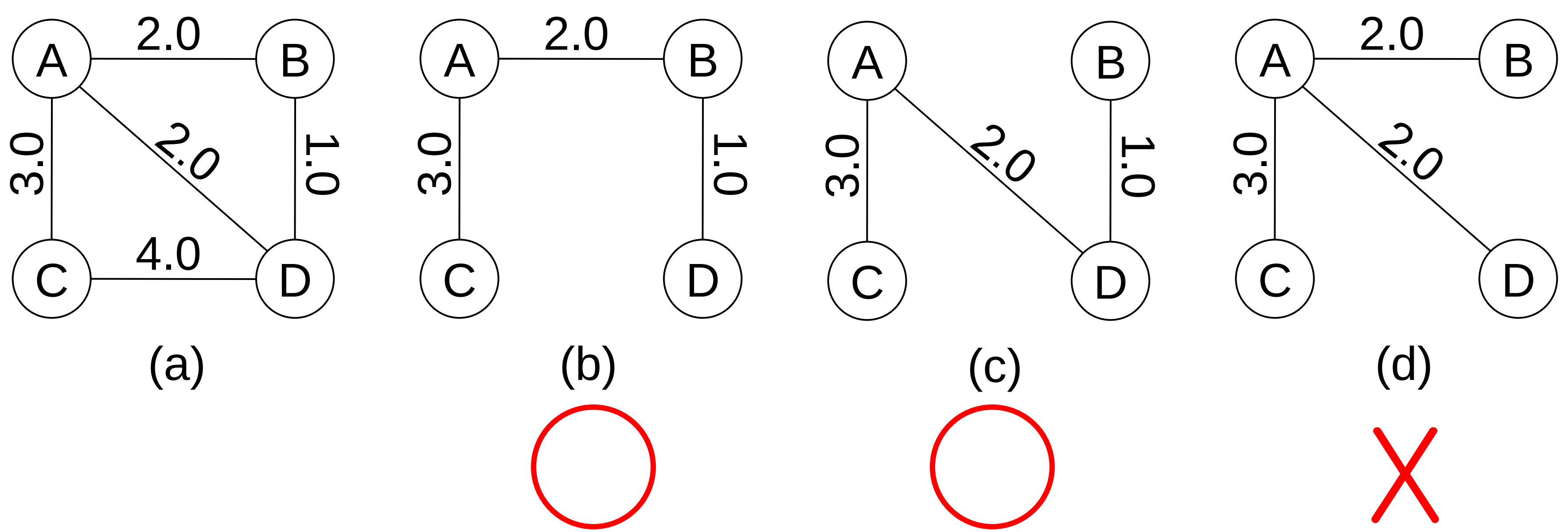


Minimum spanning tree

- Undirected graph $G = (V, E)$
- Weight $w(u,v)$ on each edge $(u,v) \in E$
- Spanning tree of G' is a minimal subgraph of G such that
  - $V(G') = V(G)$ and G' is connected.
  - Any connected graph with n vertices must have at least n-1 edges. All connected graphs with n-1 edges are trees.
- Find $T \subseteq E$ s.t.
  - T connects all vertices (T is a spanning tree), and
  - $w(T) = \sum_{(u,v) \in E} w(u,v)$ is minimized.

Minimum spanning tree

- A spanning tree whose weight is minimum over all spanning trees is called a minimum spanning tree or MST.
  – Has *n* -1 edges, no cycle, might not be unique



Minimum spanning tree

- Example)

  Interconnect *n* pins with *n*-1 wires, each connecting two pins so that we use the least amount of wire.

- We'll look at two greedy algorithms.
  - Kruskal's algorithm
  - Prim's algorithm

Building up the solution

- Build a set *A* of edges.
- Initially, *A* is empty.
- As we add edges to *A*, maintain a loop invariant : *A* is a subset of some MST.
- Add only edges that maintain the invariant.
  - If *A* is a subset of some MST, an edge (*u*,*v*) is *safe* for *A* if and only if *A* U { (*u*,*v*) } is also a subset of some MST.
  - So, we will add only safe edges.

Generic MST algorithm

GENERIC-MST(*G*, *w*)

$A = \varnothing$

while A is not a spanning tree

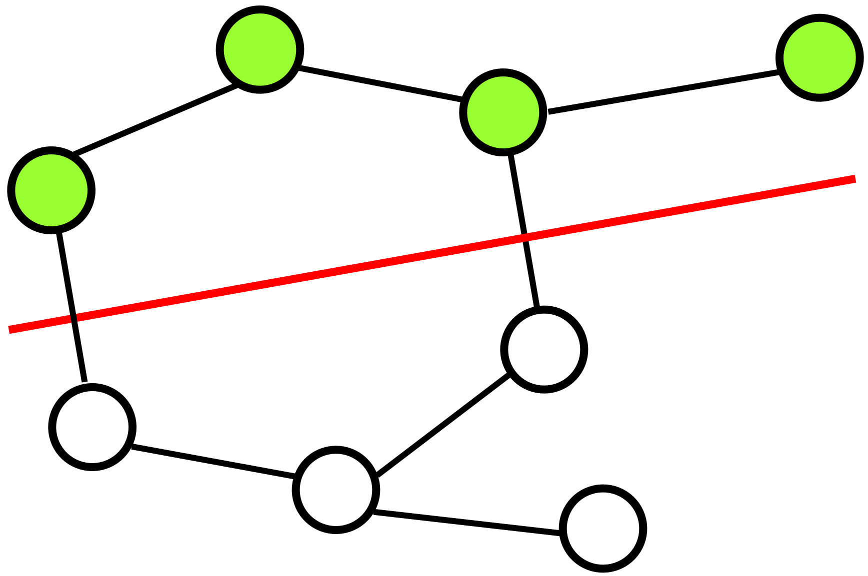
    do find an edge (u,v) that is safe for A

        $A = A \cup \{ (u,v) \}$

return A

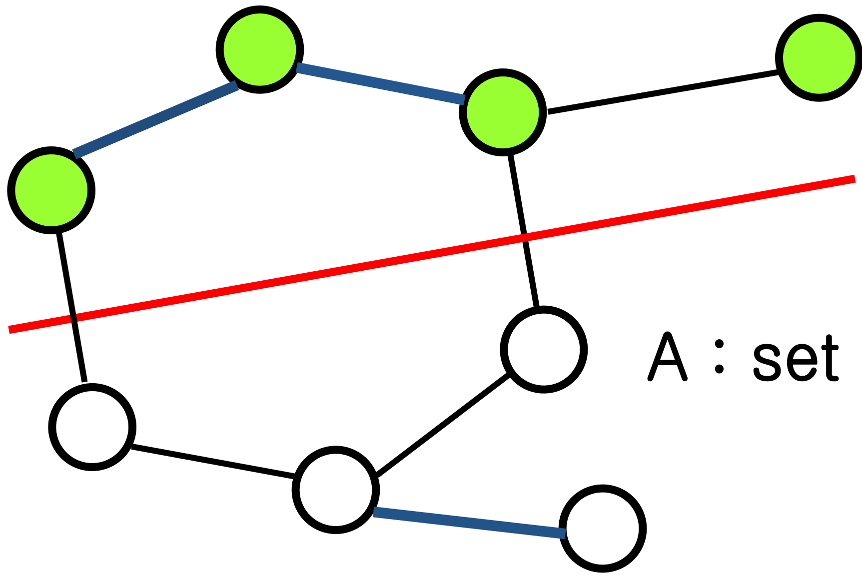
Finding a safe edge

- Let $S \subset V$ and $A \subseteq E$.
  - A *cut* $(S, V - S)$ is a partition of vertices into disjoint sets S and $V - S$.
  - Edge $(u,v) \in E$ *crosses* cut $(S, V - S)$ if one endpoint is in S and the other is in $V - S$.



Finding a safe edge

- Let $S \subset V$ and $A \subseteq E$.
  - A cut *respects* A if and only if no edge in A crosses the cut.
  - An edge is a *light edge* crossing a cut if and only if its weight is minimum over all edges crossing the cut. For a given cut, there can be more than one light edge crossing it.



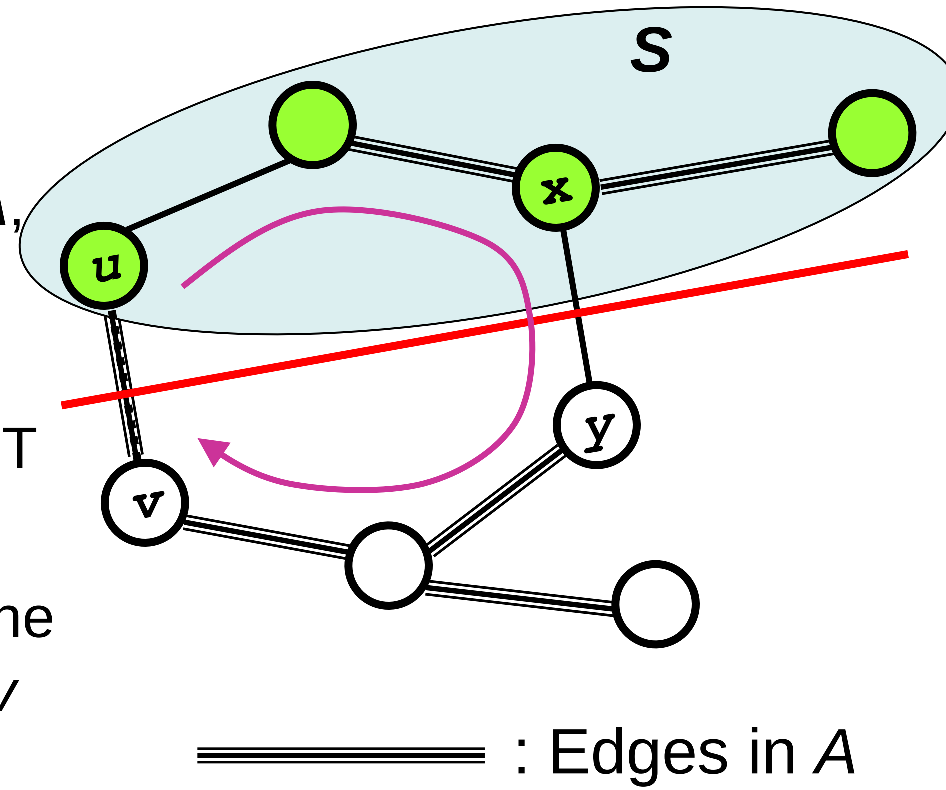
A : set of blue edges

Theorem

Let A be a subset of E in some MST, $(S, V - S)$ be a cut that respects A, and (u, v) be a light edge crossing $(S, V - S)$. Then, (u, v) is safe for A.

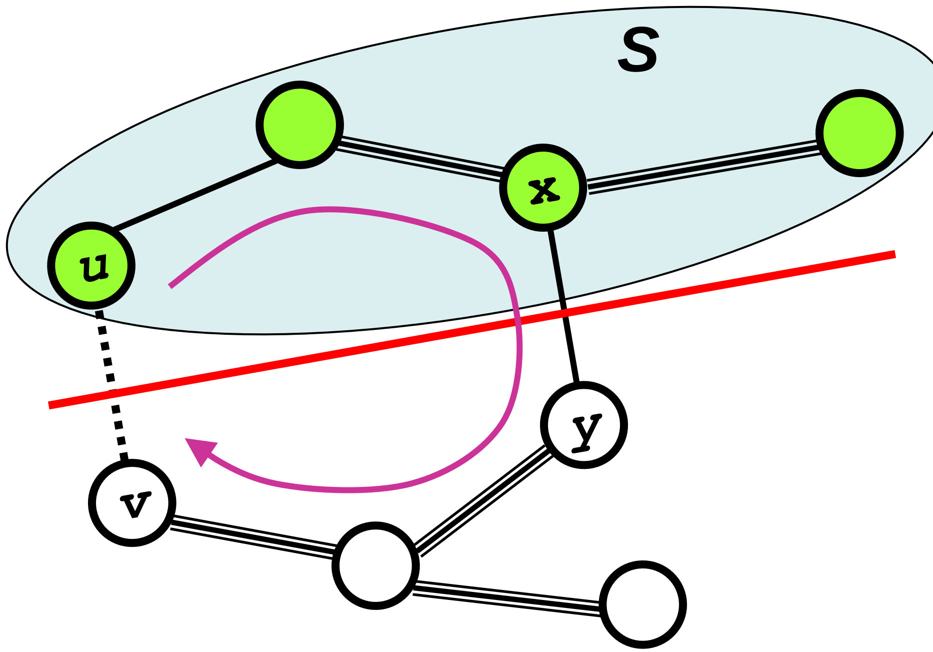
Proof)

1. Let T be a MST that includes A, and assume that T does not contain the light edge (u, v).
2. We shall construct another MST T' that includes $A \cup \{ (u,v) \}$.
3. Edge (u,v) forms a cycle with the edges on the path p from u to v in T.



Theorem

4. Since u and v are on opposite sides of the cut $(S, V - S)$ there is at least one edge in T on the path p that also crosses the cut. Let (x,y) be any such edge.



5. Removing (x,y) breaks T into two components. And adding (u,v) reconnects them to form a new spanning tree T'.

$$w(T') = w(T) - w(x,y) + w(u,v)$$

$$\leq w(T)$$

Thus, T' must be a MST.

And since $A \cup \{ (u,v) \} \subseteq T'$, (u,v) is safe for A.

Corollary

Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_c, E_c)$ be a connected component (tree) in the forest $G_A = (V, A)$. If (u,v) is a light edge connecting C to some other component in G_A, then (u,v) is safe for A.

Proof) The cut $(Vc, V - Vc)$ respects A, and (u,v) is a light edge for this cut. Therefore, (u,v) is safe for A.

Basic idea of Kruskal's algorithm

- Sort edges into nondecreasing order by *w.*
- The algorithm maintains *A*, a forest of trees.
- Repeatedly merges two components into one by choosing the light edge that connects them.

  i.e.,

  1. Choose the light edge crossing the cut between them.

  2. (If it forms a cycle, the edge is discarded.)

- What is the design strategy of Kruskal's algorithm?

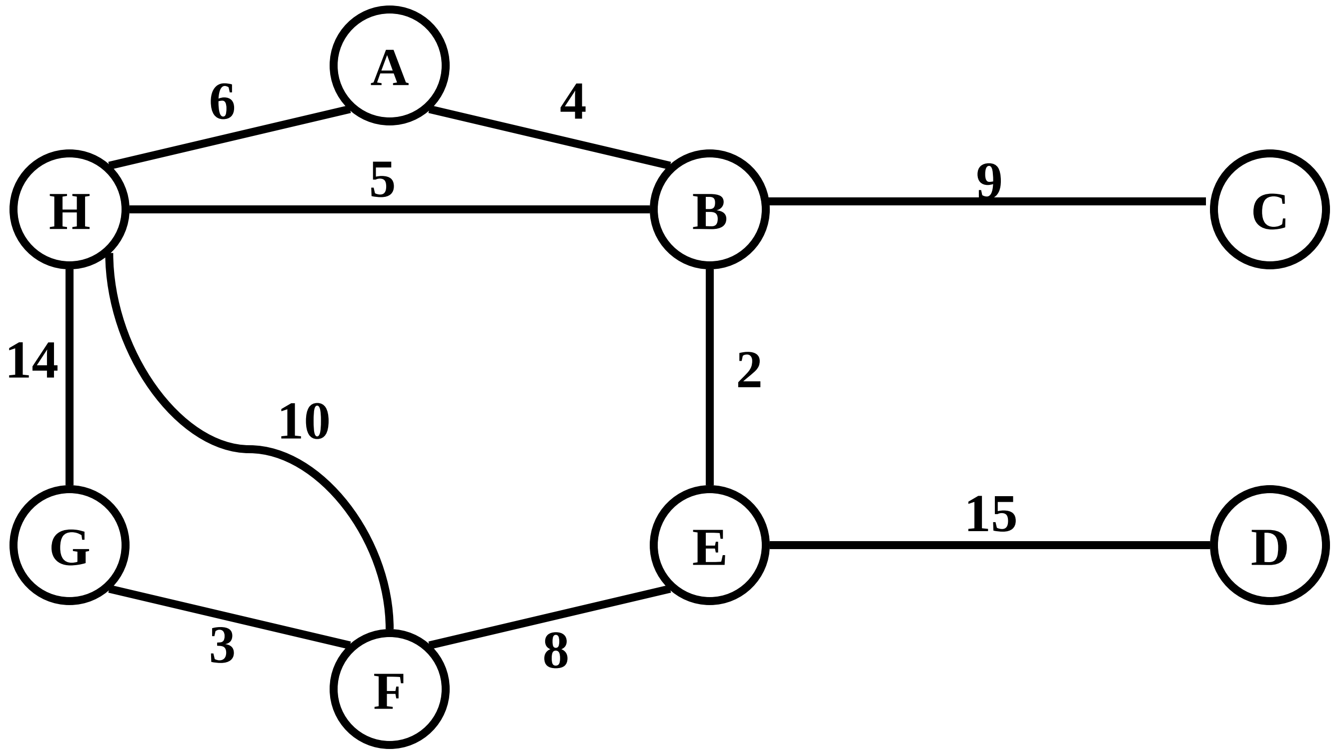
  - Greedy !!

Specific pseudo-code

MST-Kruskal(G,w)

```
R = E;
F = 0;
While (R is not empty)
    Remove the light edge, (u,v), from R;
    if ((u,v) does not make a cycle in F)
        Add (u,v) to F;
return F;
```

Example of Kruskal



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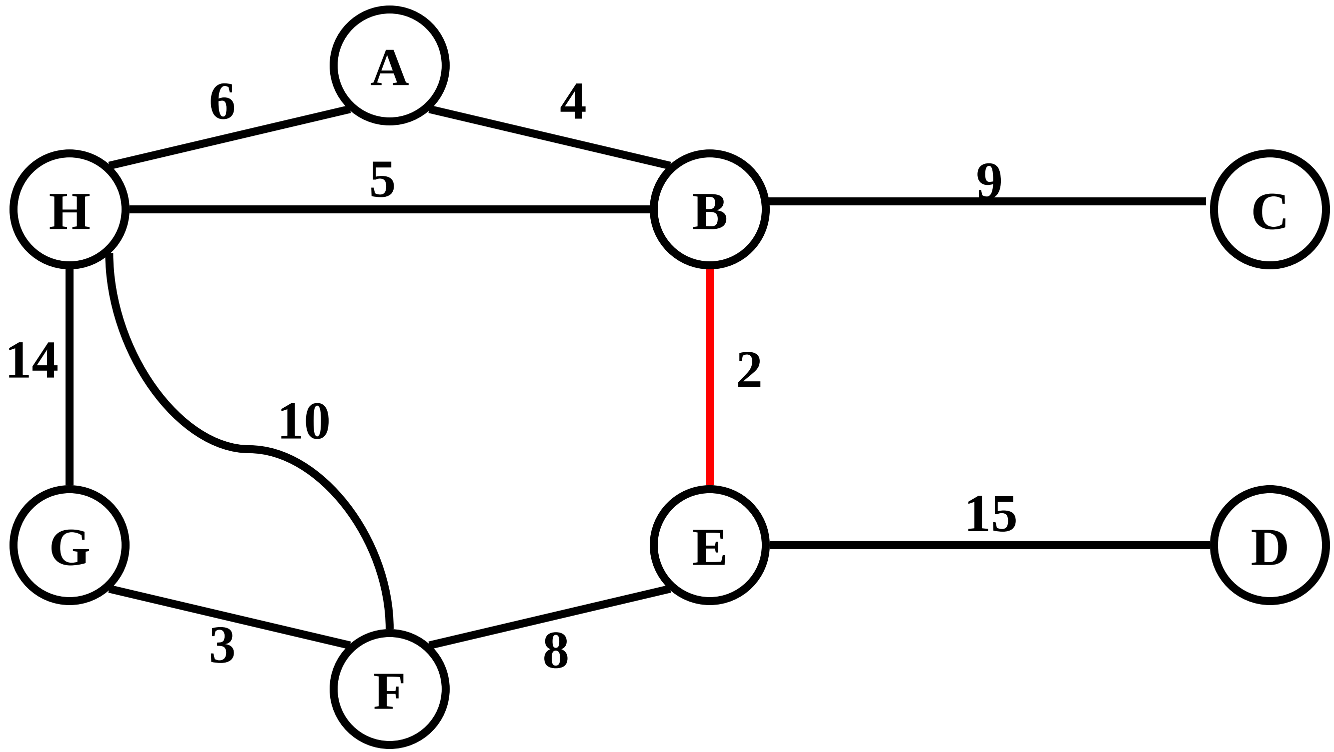

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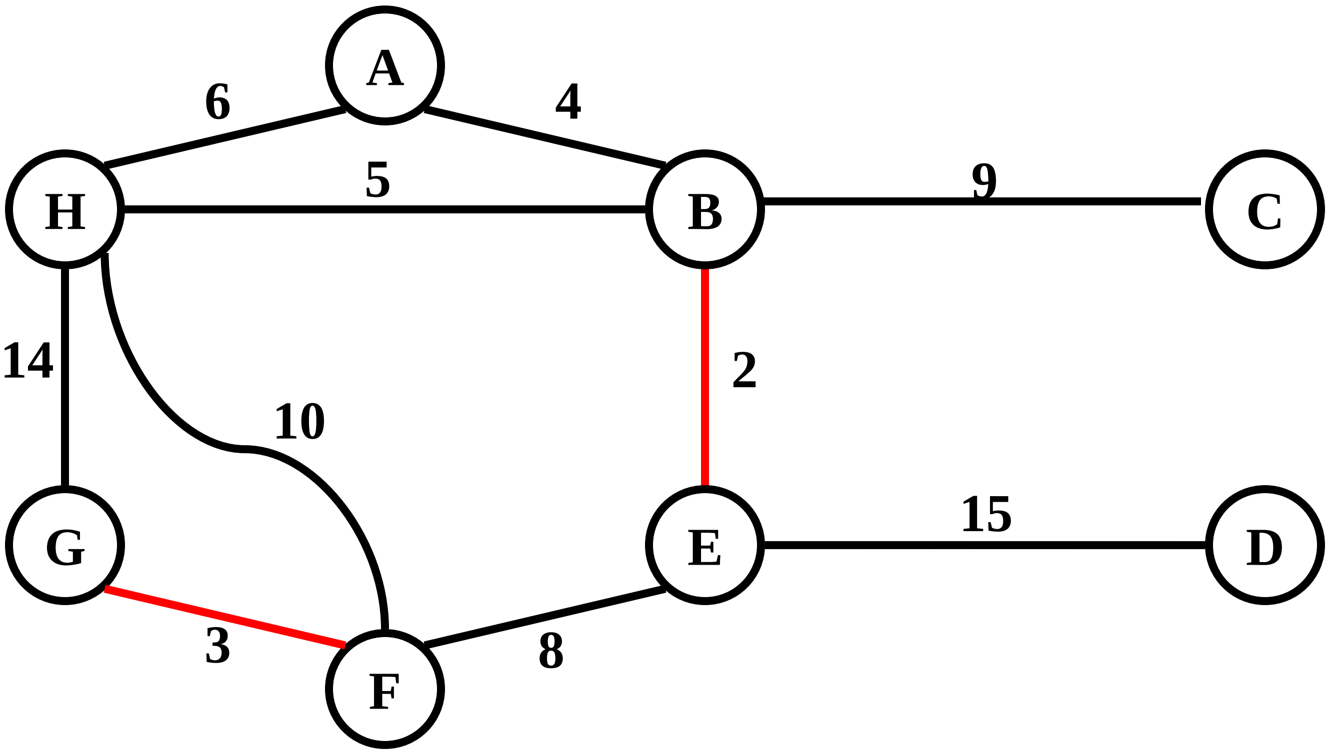

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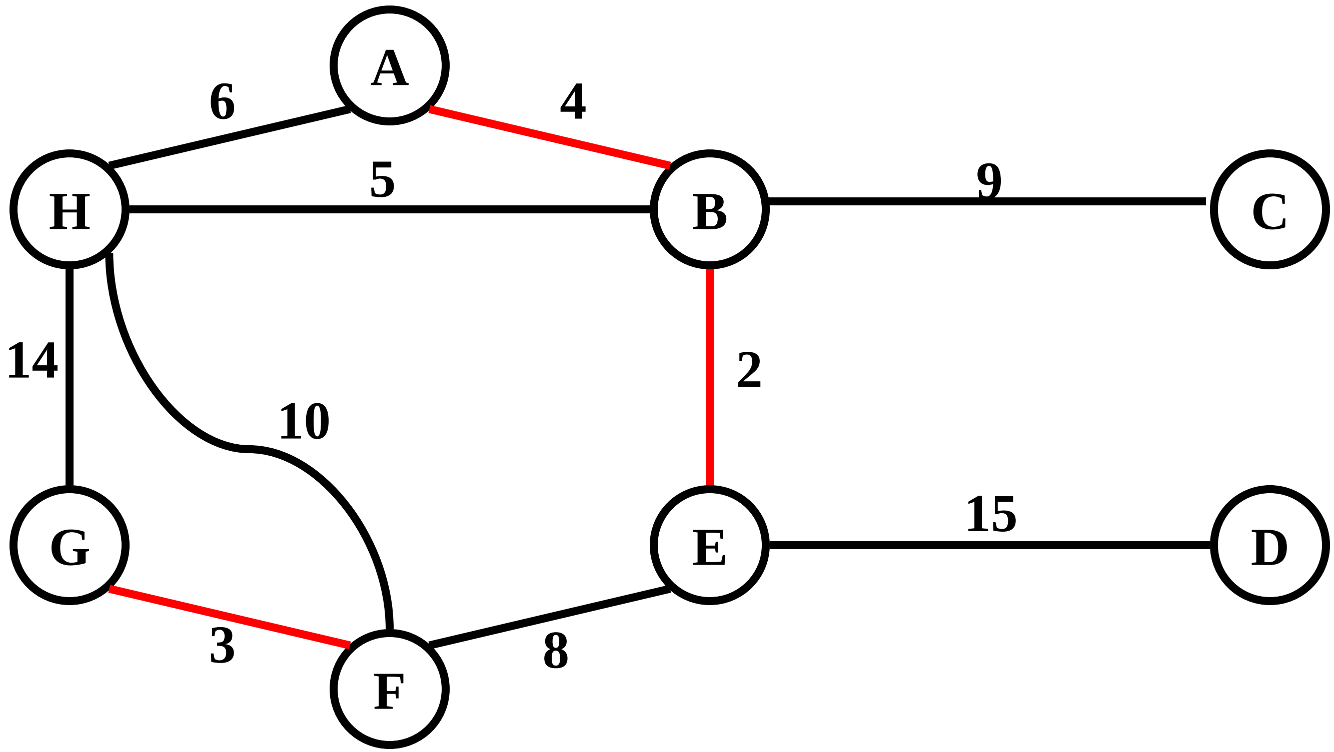

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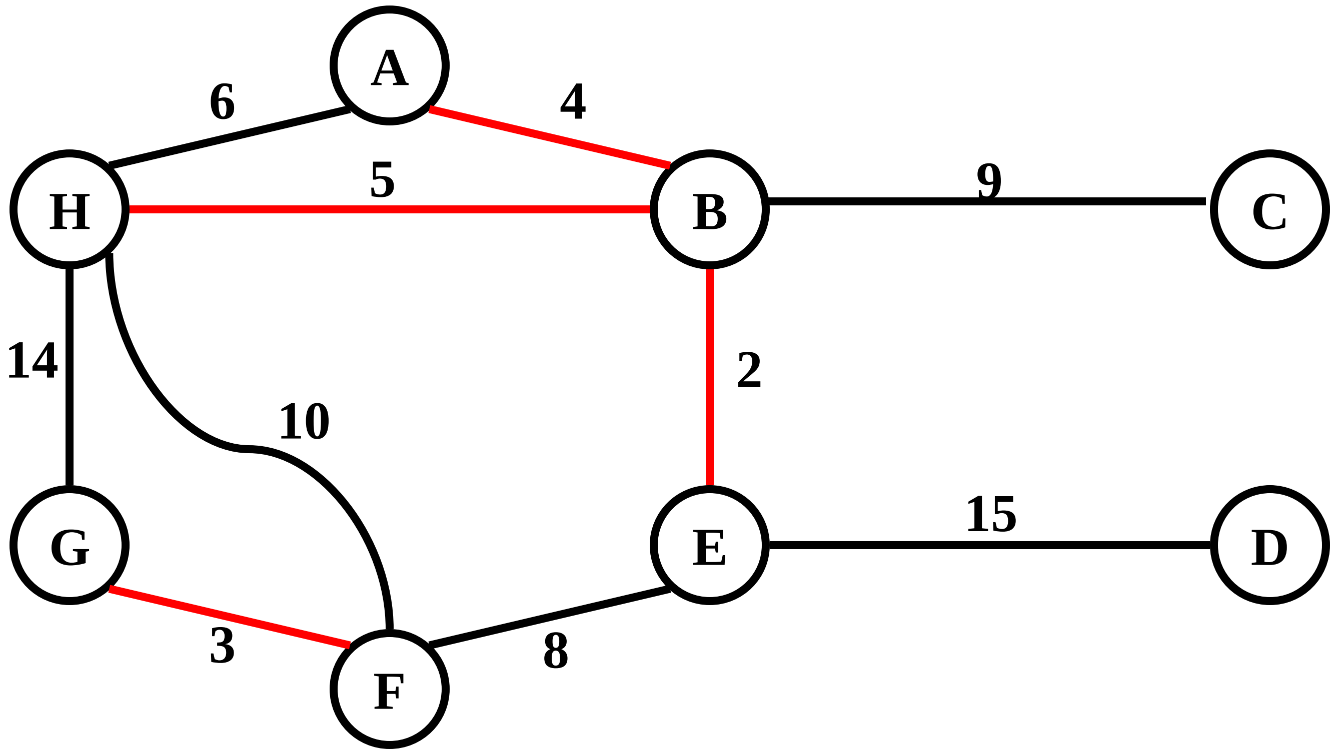

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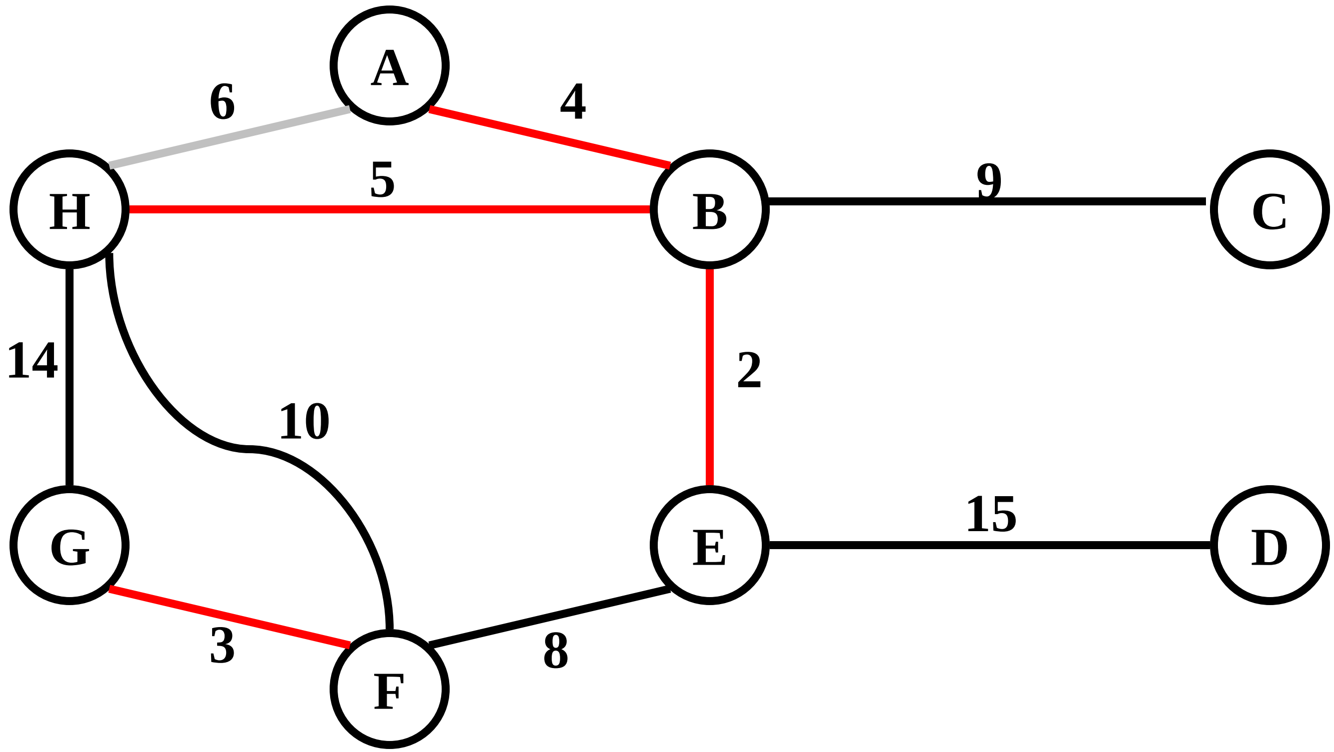

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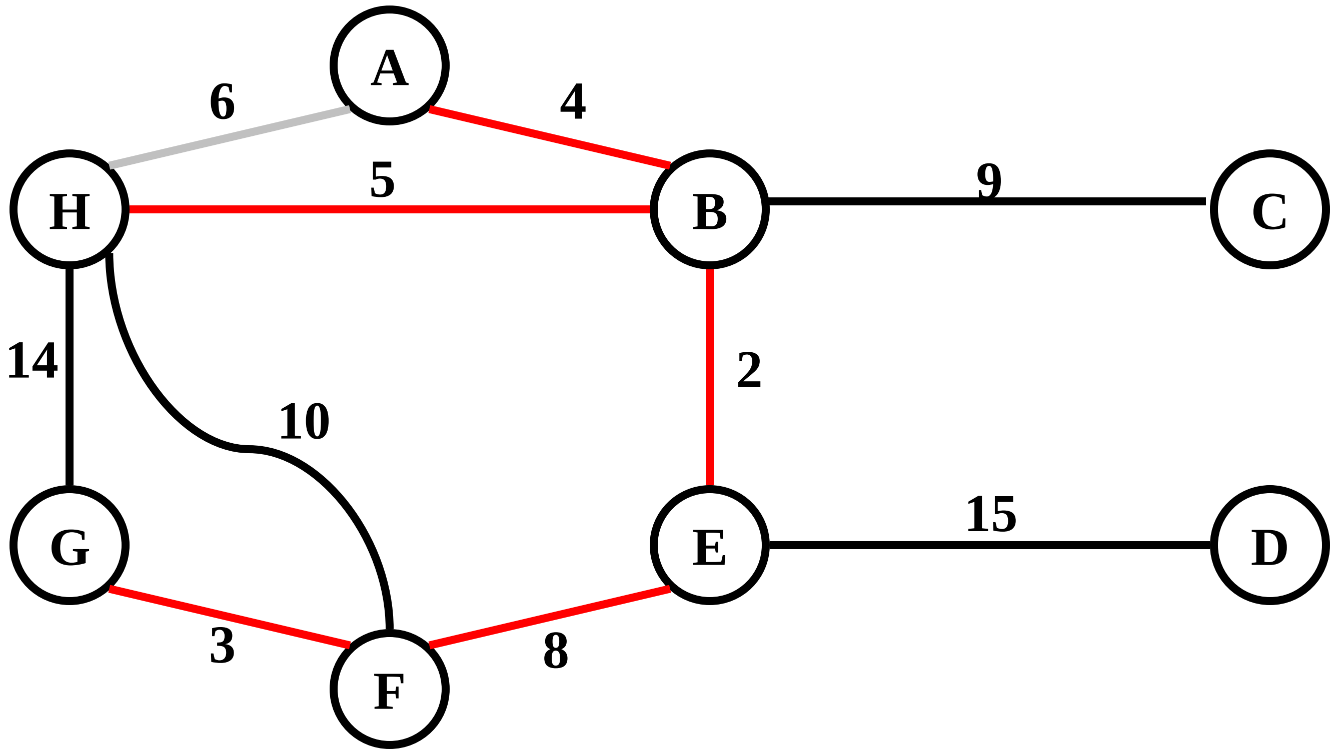

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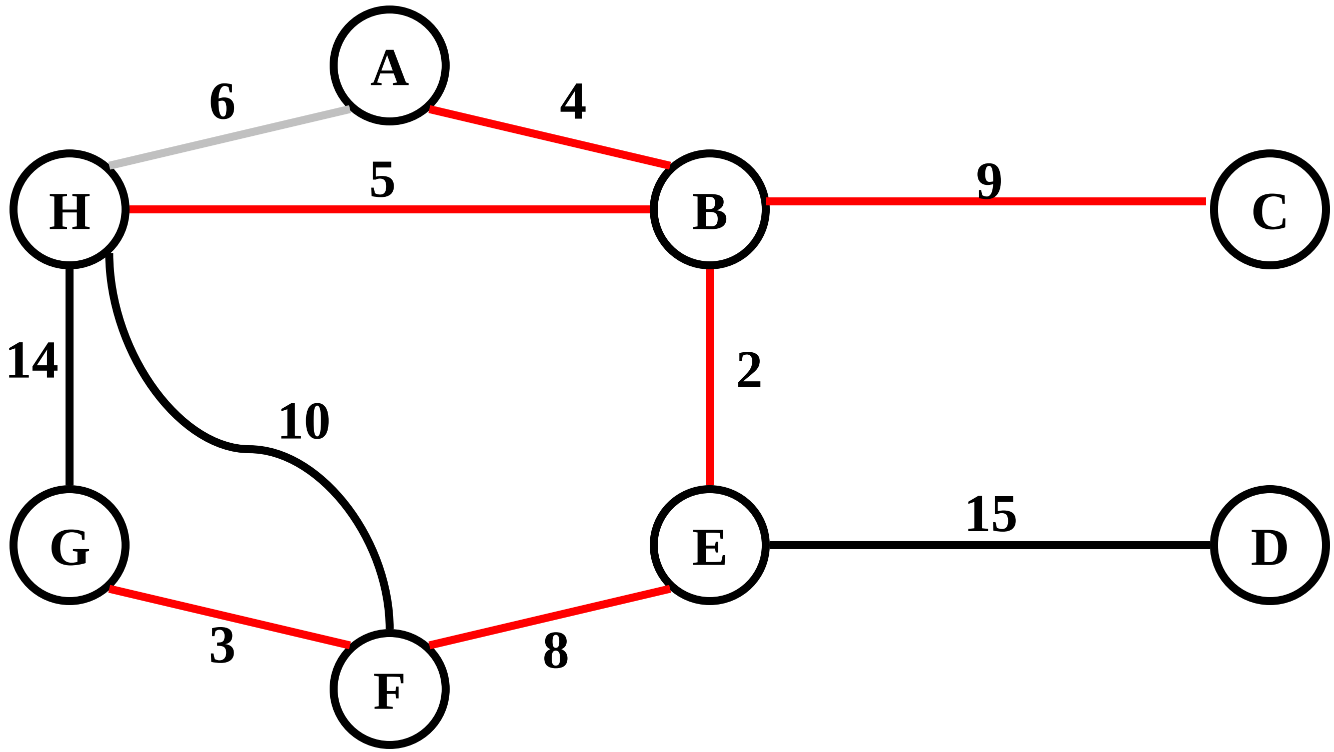

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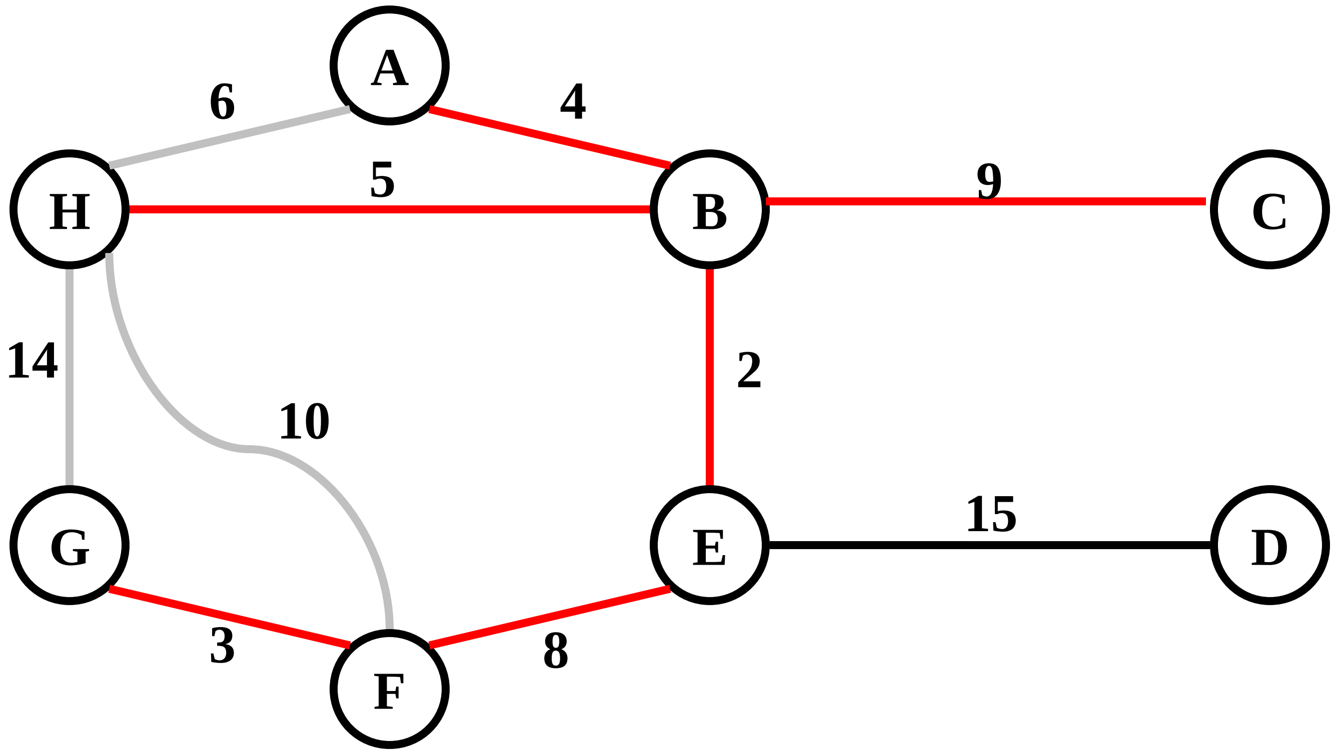

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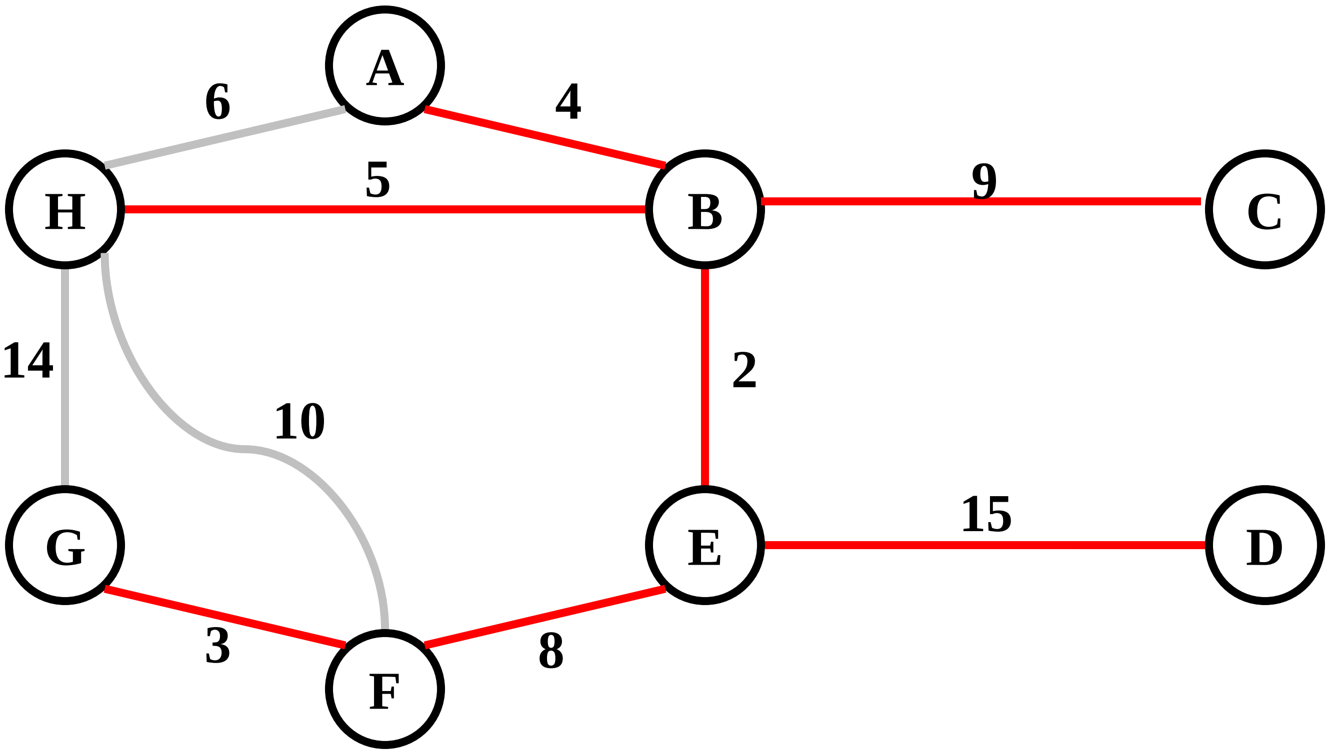


Prim's algorithm

- Builds one tree, so A is always a tree.
- Start from an arbitrary "root" r.
- At each step, find a light edge crossing cut $(V_A, V - V_A)$, where V_A = vertices that A is incident on.
- $\pi[v]$ = parent of v, NIL if it has no parent or $v = r$.
- To find a light edge quickly
  - use a priority queue Q.

Prim's MST: Outline

PrimMST(G,n)

Initialize all vertices as *unseen*.

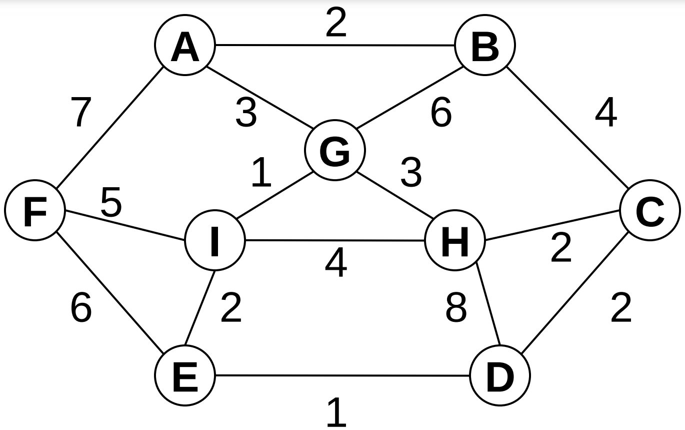
Select an arbitrary vertex *r* to start the tree; reclassify it as *tree*

Reclassify all vertices adjacent to *r* as *fringe*.

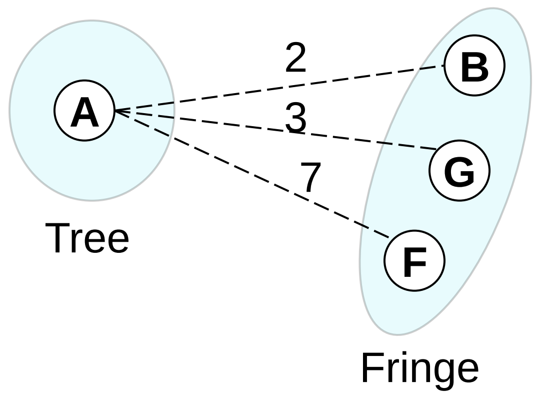
While there are fringe vertices;

- Select an edge of minimum weight between a tree vertex *t* and a fringe vertex *v*;
- Reclassify *v* as *tree*; add edge (*t,v*) to the tree;
- Reclassify all *unseen* vertices adjacent to *v* as *fringe*

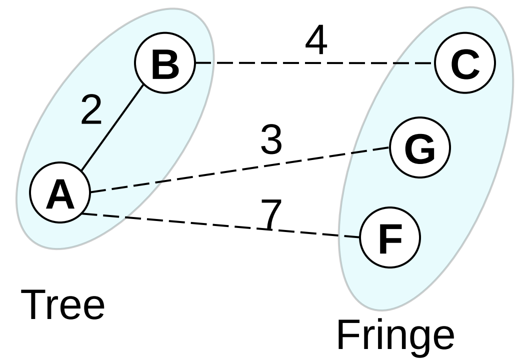
The Algorithm in action, e.g.



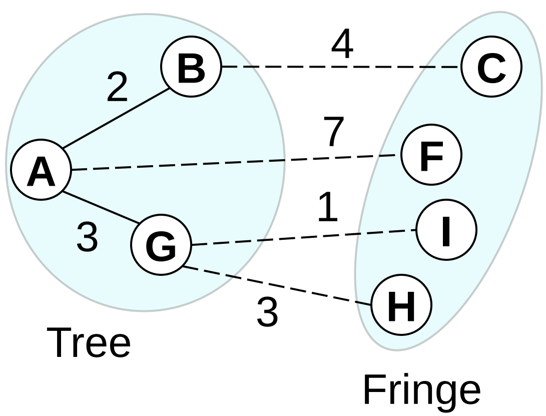
(a) A weighted graph



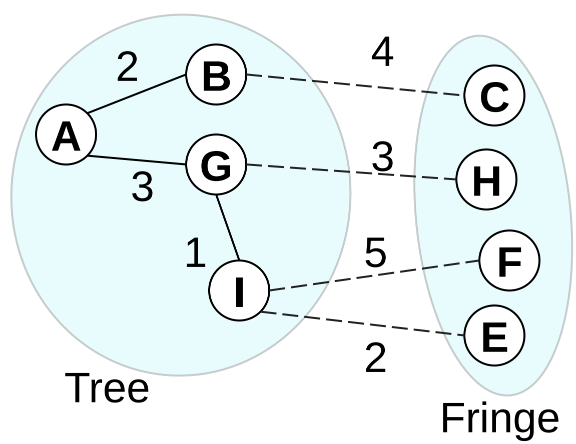
(b) After selection of the starting vertex



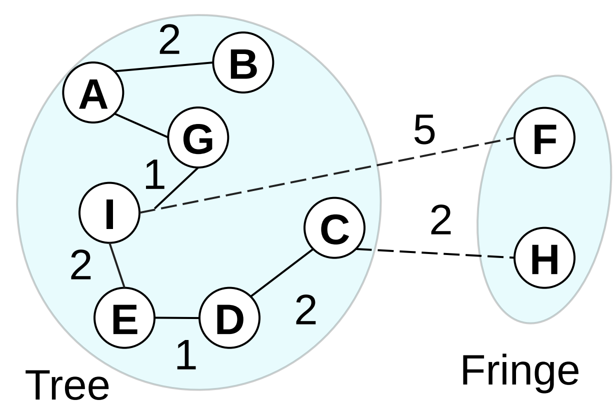
(c) *BG* was considered but did not replace *AG* as a candidate.



(d) After *A G* is selected and fringe and candidates are updated



(e) I F has replaced A F as a candidate.



(f) After several more passes: The two candidate edges will be put in the tree

Prim's Algorithm

MST-Prim(G, w, r)

$Q = V[G]$;

for each $u \in Q$

  $key[u] = \infty$; $\pi[u]$ = NIL;

$key[r] = 0$;

$\pi[r]$ = NULL;

while (Q not empty)

  u = ExtractMin(Q);

  for each $v \in$ Adj[u]

    if ($v \in Q$ and $w(u,v) < key[v]$)

      $\pi[v] = u$;

      $key[v] = w(u,v)$;

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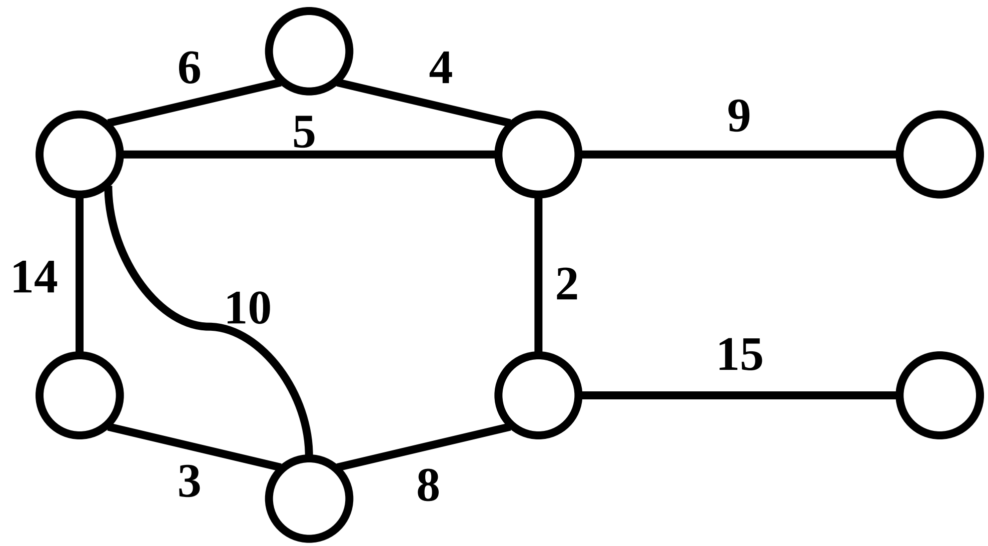

Run on example graph

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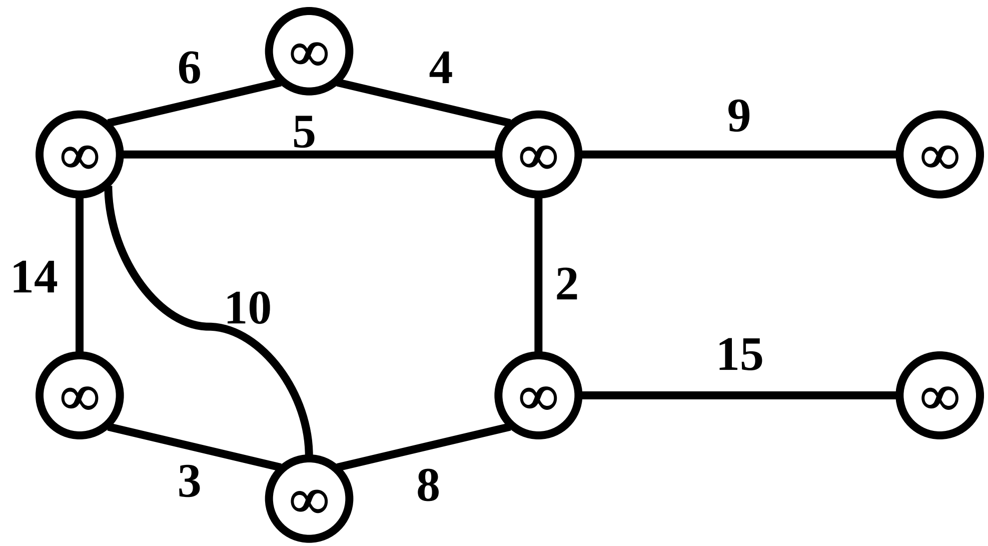

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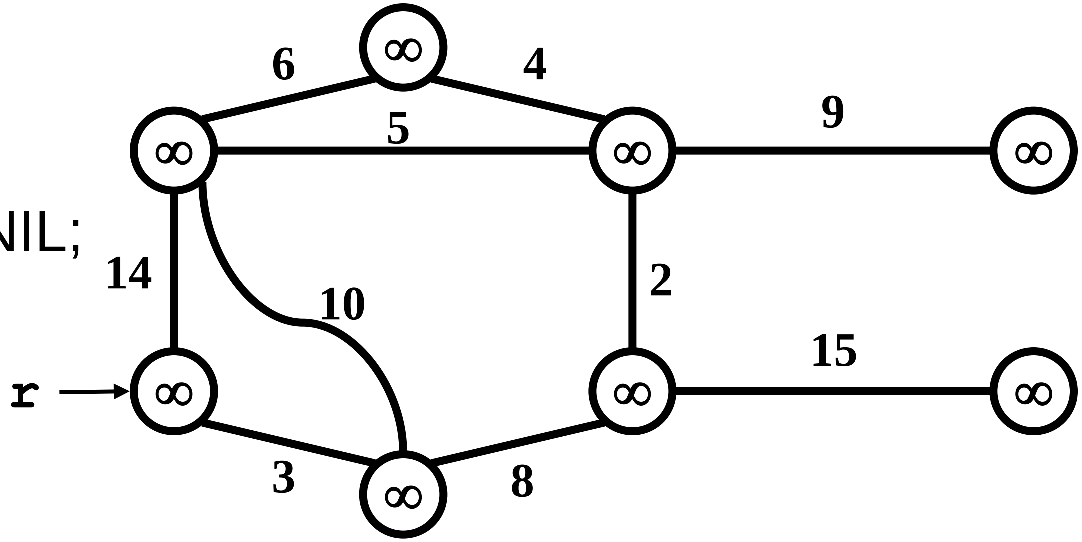

Pick a start vertex r

Prim's Algorithm

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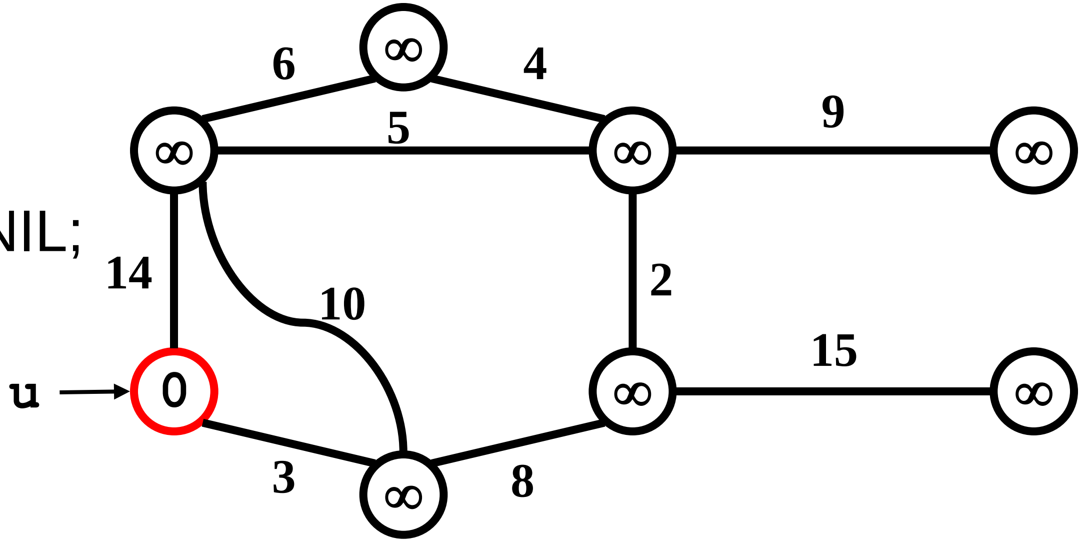

Red vertices have been removed from Q

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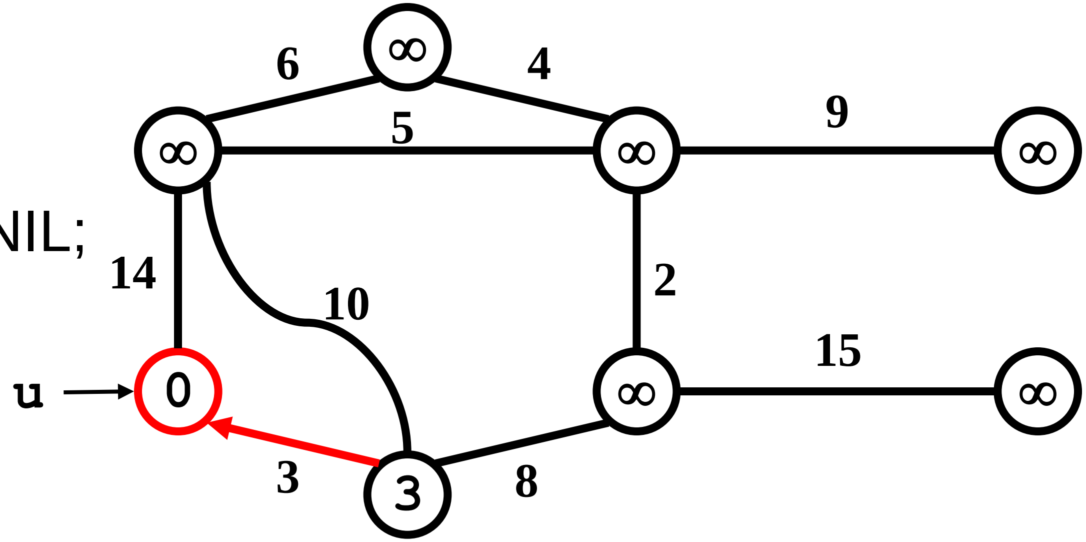

Red arrows indicate parent pointers

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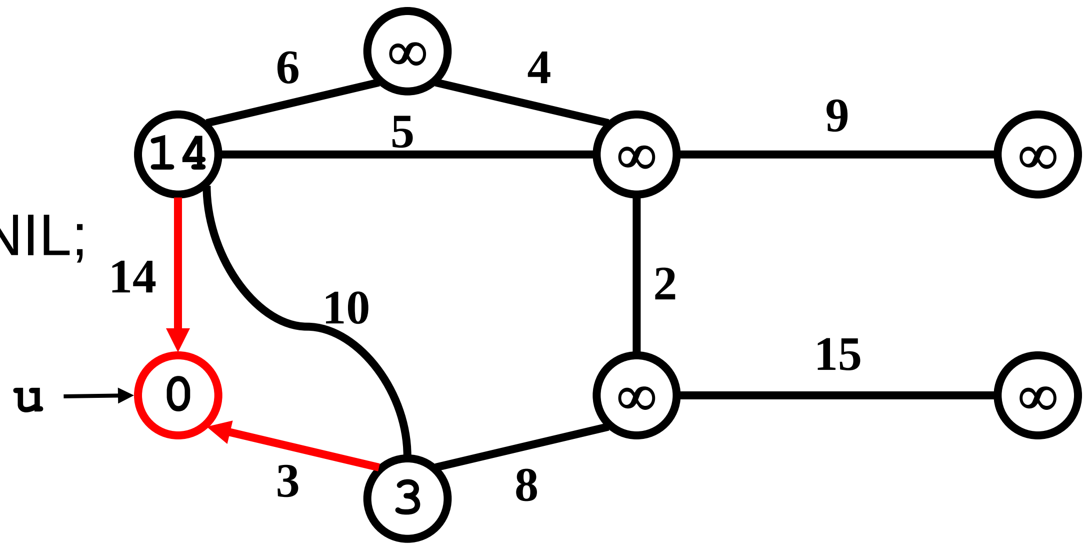

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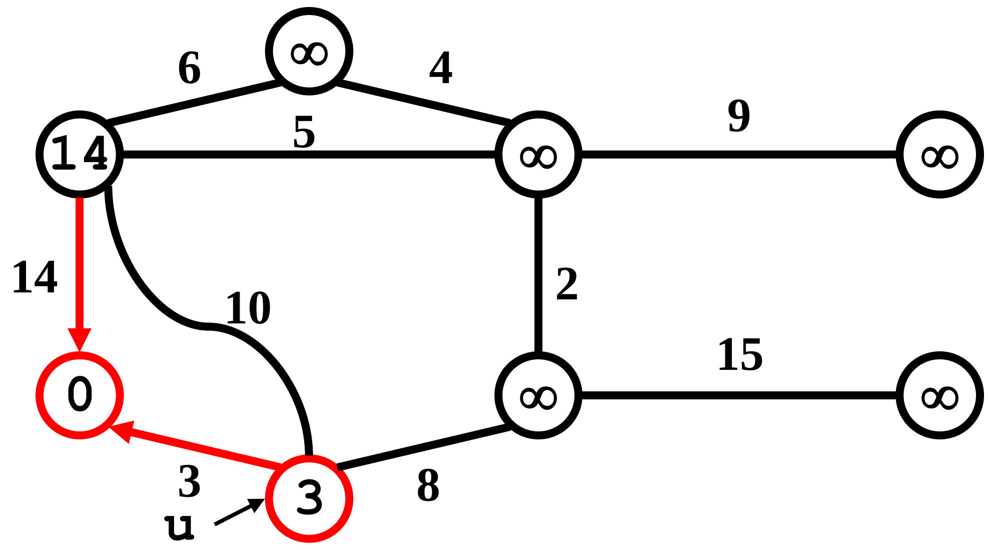

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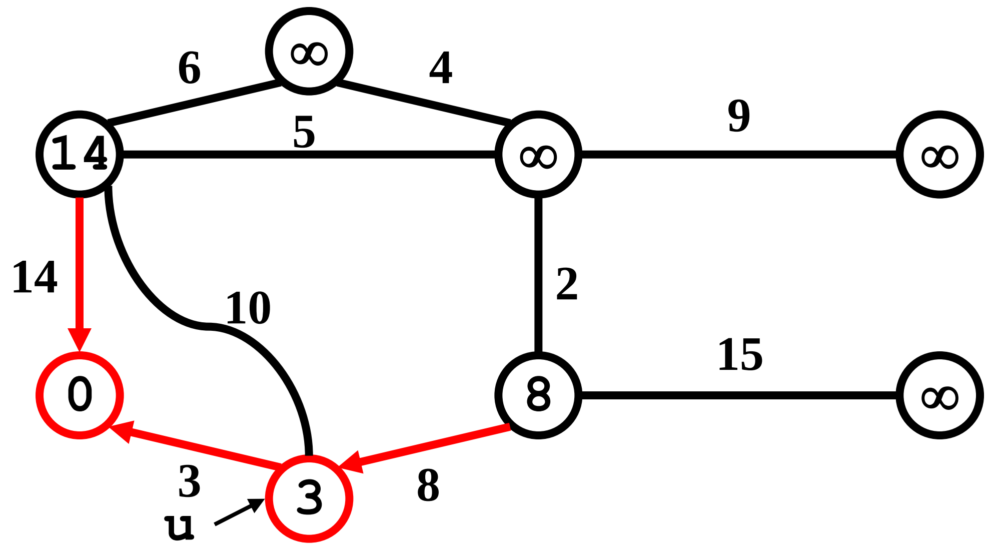

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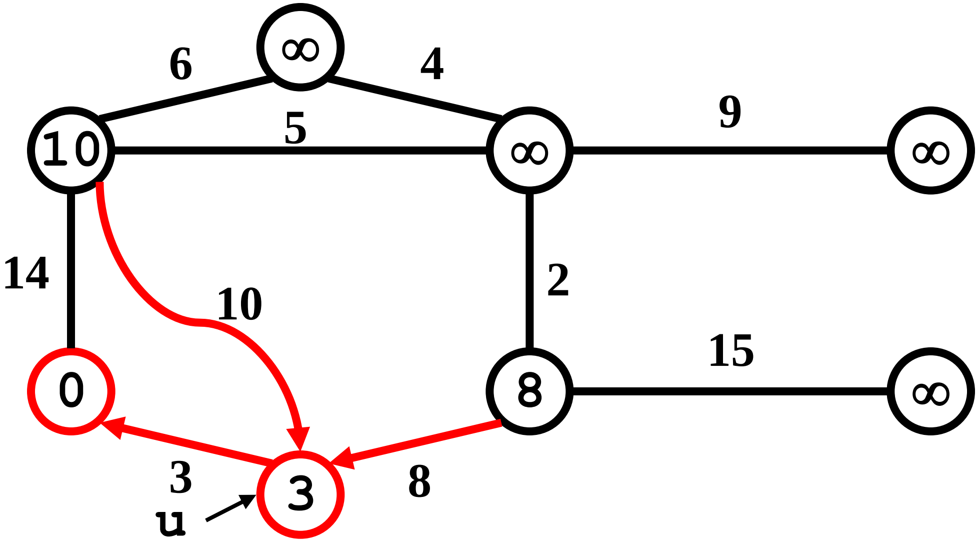

Prim's Algorithm

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for each *u* ∈ *Q*
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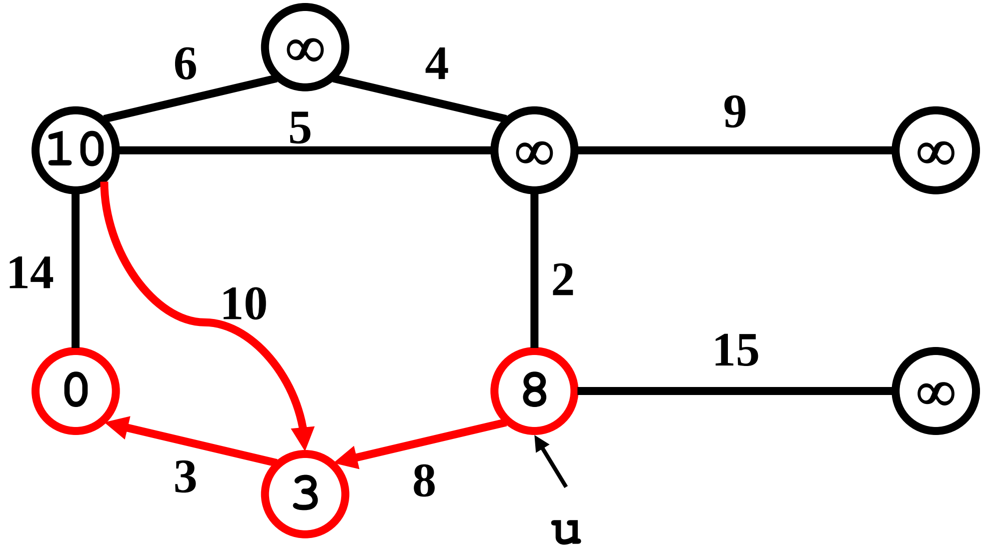
key[*r*] = 0;
π[*r*] = NULL;
while (*Q* not empty)
  u = ExtractMin(*Q*);
  for each *v* ∈ Adj[*u*]
    if (*v* ∈ *Q* and *w*(*u*,*v*) < *key*[*v*])
      π[*v*] = *u*;
      key[*v*] = *w*(*u*,*v*);



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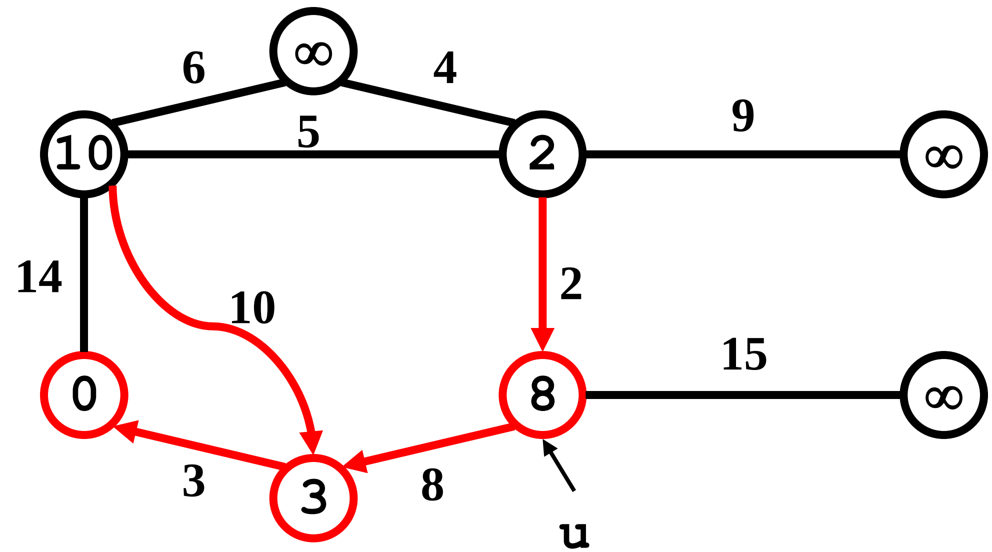

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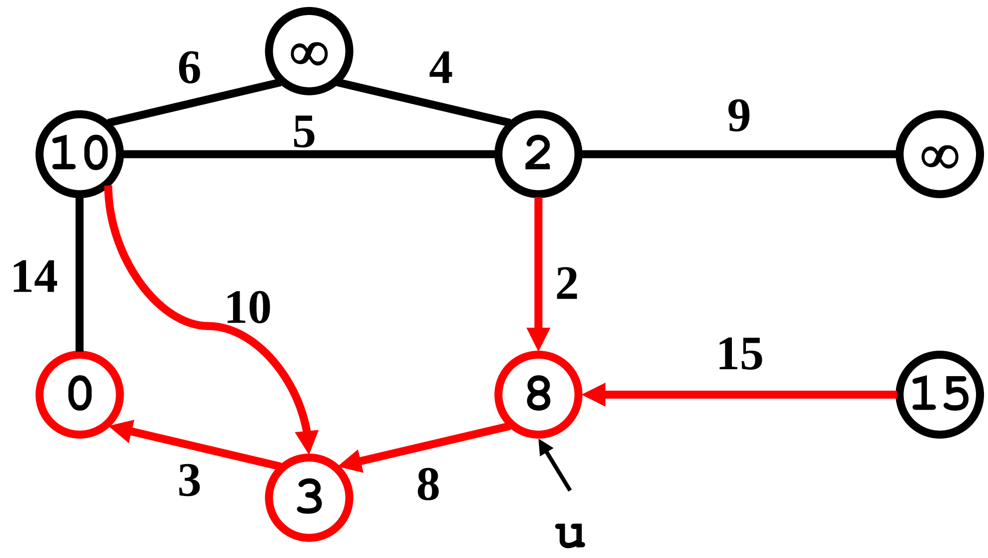

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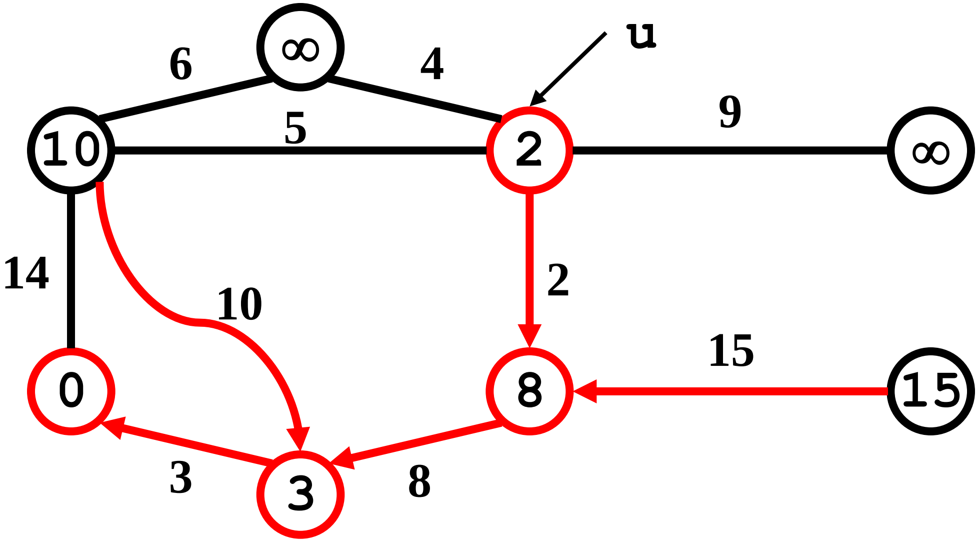

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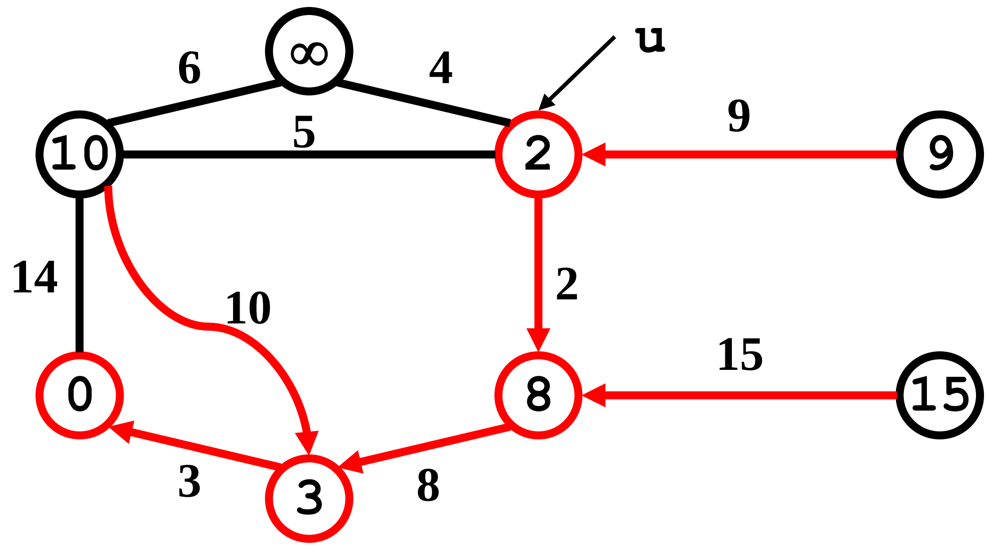
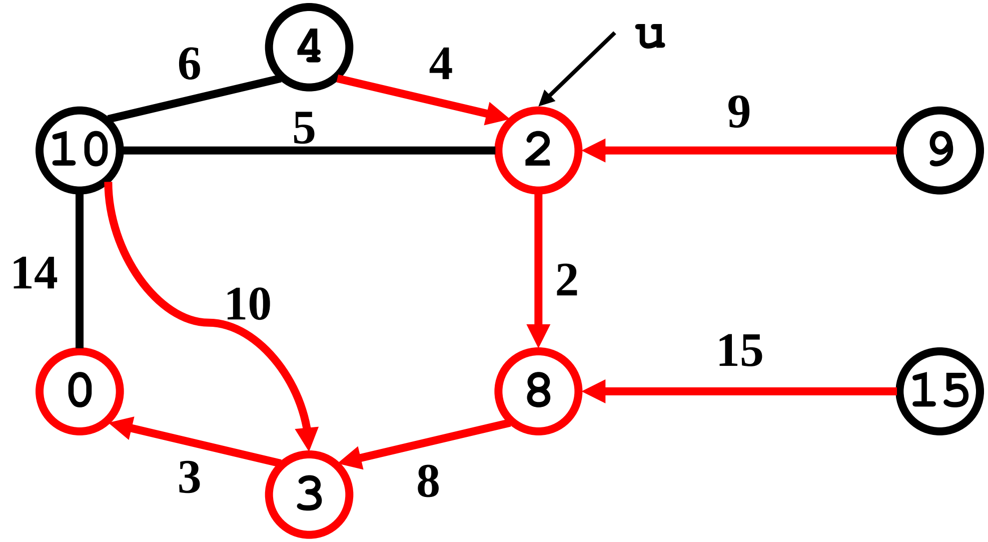
                key[v] = w(u,v);
```



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

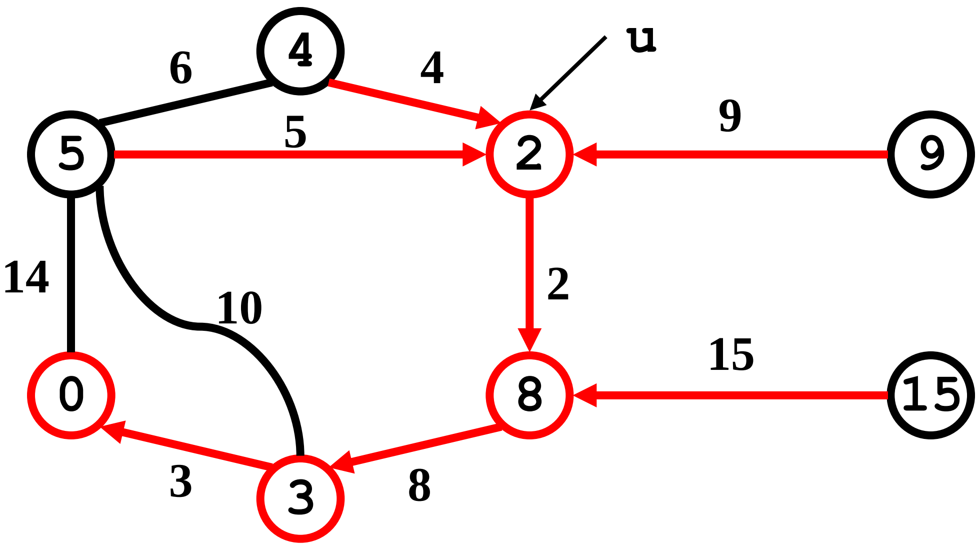
Q = *V*[*G*];
for each *u* ∈ *Q*
    key[*u*] = ∞; π[*u*] = NIL;
key[*r*] = 0;
π[*r*] = NULL;
while (*Q* not empty)
    u = ExtractMin(*Q*);
    for each *v* ∈ Adj[*u*]
        if (*v* ∈ *Q* and *w*(*u*,*v*) < *key*[*v*])
            π[*v*] = *u*;
            key[*v*] = *w*(*u*,*v*);



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

$Q = V[G]$;
for each $u \in Q$
  $key[u] = \infty$; $\pi[u]$ = NIL;
$key[r] = 0$;
$\pi[r]$ = NULL;
while (Q not empty)
  u = ExtractMin(Q);
  for each $v \in$ Adj[u]
    if ($v \in Q$ and $w(u,v) < key[v]$)
      $\pi[v] = u$;
      $key[v] = w(u,v)$;



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

$Q = V[G]$;

for each $u \in Q$

$key[u] = \infty$; $\pi[u]$ = NIL;

$key[r] = 0$;

$\pi[r]$ = NULL;

while (Q not empty)

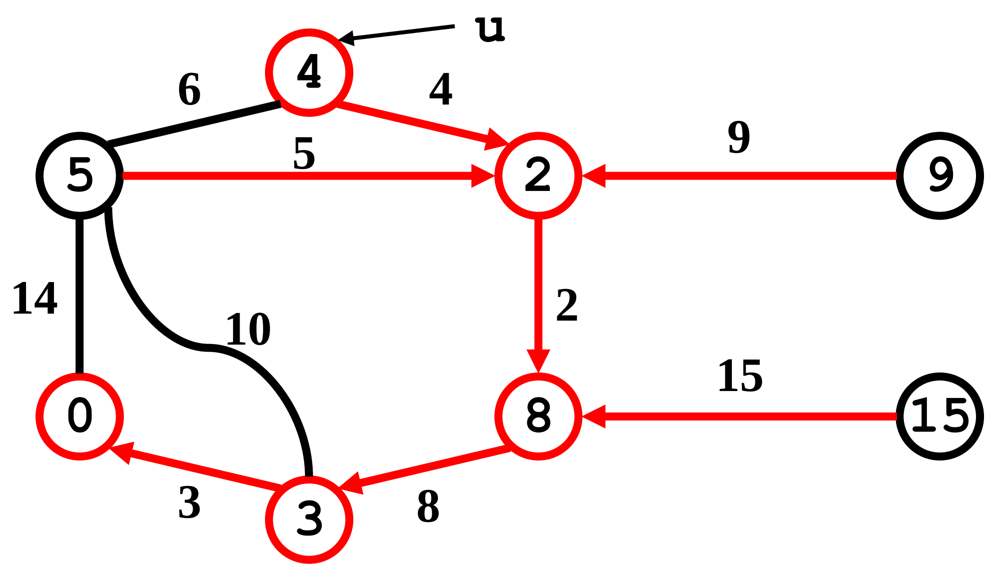
u = ExtractMin(Q);

for each $v \in$ Adj[u]

if ($v \in Q$ and $w(u,v) < key[v]$)

$\pi[v] = u$;

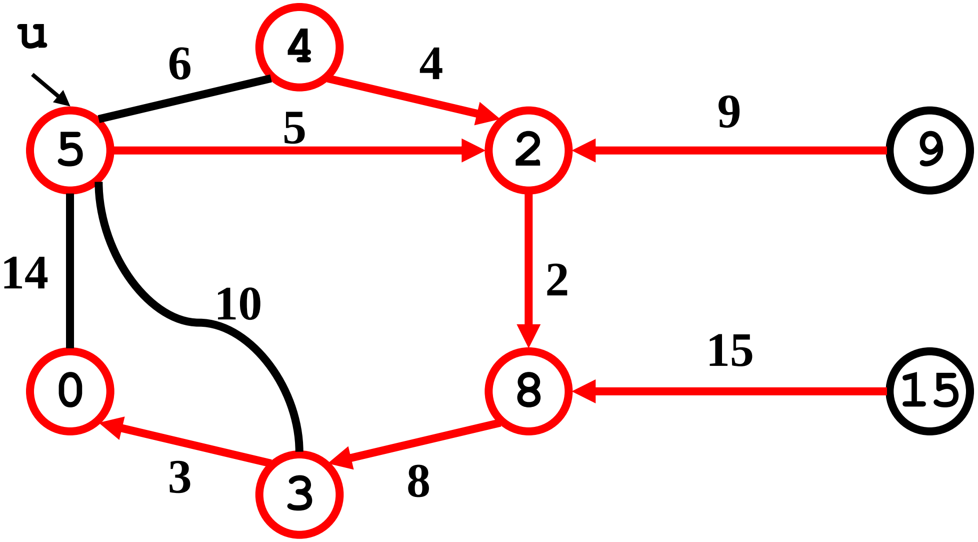
$key[v] = w(u,v)$;



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

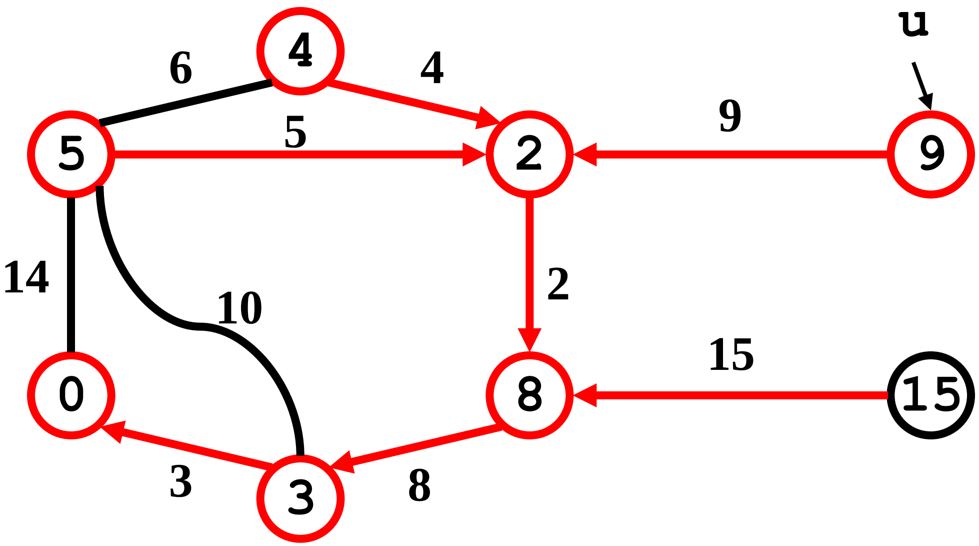
$Q = V[G]$;
for each $u \in Q$
  $key[u] = \infty$; $\pi[u] =$ NIL;
$key[r] = 0$;
$\pi[r] =$ NULL;
while (Q not empty)
  $u =$ ExtractMin(Q);
  for each $v \in$ Adj[u]
    if ($v \in Q$ and $w(u,v) < key[v]$)
      $\pi[v] = u$;
      $key[v] = w(u,v)$;



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

```
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞; π[u] = NIL;
    key[r] = 0;
    π[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                π[v] = u;
                key[v] = w(u,v);
```



Prim's Algorithm

MST-Prim(*G*, *w*, *r*)

- $Q = V[G]$;
- for each $u \in Q$
  - $key[u] = \infty$; $\pi[u]$ = NIL;
- $key[r] = 0$;
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- while (Q not empty)
  - u = ExtractMin(Q);
  - for each $v \in$ Adj[u]
    - if ($v \in Q$ and $w(u,v) < key[v]$)
      - $\pi[v] = u$;
      - $key[v] = w(u,v)$;