# Numerical Analysis homework #1

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# I.

## I-a

The width of interval at the nth step is $2^{1-n}$.

## I-b

The supremum of the distance between the root r and the midpoint of the interval is $2^{-n}$.

# II.

denote $h_n = b_n - a_n$ $(n = 0, 1, 2, 3, \cdots)$, where $[a_n, b_n]$ is the interval of nth step. Let $c_n$ be the midpoint of $[a_n, b_n]$ and let p be zero point. Then

$$\begin{aligned} h_n &= 2^{-(n+1)} h_0 \\ \frac{|c_n - p|}{p} &\le \frac{h_n}{a_0} \le \epsilon \\ i.e. \quad 2^{n+1} &\ge \frac{b_0 - a_0}{a_0 \epsilon} \\ i.e. \quad n &\ge \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1 \end{aligned}$$

# III.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|
| $x$ | -1 | -0.8125 | -0.770804 | -0.768832 | -0.768828 |
| $p(x)$ | -3 | -0.46582 | -0.0201379 | -4.37084e-005 | -2.07412e-010 |

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## IV.

$$
\begin{aligned}
0 = f(p) &= f(x_n - e_n) \\
&= f(x_n) - e_n f'(\xi_n), \text{ where } \xi_n \text{ is between } p \text{ and } x_n \\
e_{n+1} &= (1 - \frac{f'(\xi_n)}{f'(x_0)})e_n
\end{aligned}
$$

so $s = 1$, $c = 1 - \frac{f'(\xi_n)}{f'(x_0)}$.

## V.

Let $g(x) = \arctan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$
g'(x) = \frac{1}{x^2+1} \leq 1
$$

$g'(x) = 1$, iff $x = 0$.

since $g(x) \in \mathcal{C}^1$ and $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-\frac{\pi}{2}, \frac{\pi}{2})$,

thus it converges as

$$
\lim_{n\to\infty} x_n = 0
$$

.

## VI.

Let $g(x) = \frac{1}{p+x}$, $x \in (0, 1)$, then

$$
x_{n+1} = g(x_n)
$$

where $x_1 = 1/p$.

Because $p > 1$, so $0 < x_n < 1$.

$$
|g'(x)| = \frac{1}{(x+p)^2} < 1
$$

since $g(x) \in \mathcal{C}^1$ and $g : (0, 1) \to (0, 1)$, thus it converges.

Let $g(x) = x$, we get

$$
\lim_{n\to\infty} x_n = x = \frac{-p + \sqrt{p^2+4}}{2}
$$

## VII.

The problem is the zero point can be negative which changes the equality so that we cannot measure $n$ correctly.

Actually, we can use absolute error instead of relative error to measure it appropriately.

Denote $h_n = b_n - a_n$ $(n = 0, 1, 2, 3, \cdots)$, where $[a_n, b_n]$ is the interval of nth step. Let $c_n$ be the midpoint of $[a_n, b_n]$ and let p be zero point. Then

$$\begin{aligned} h_n &= 2^{-(n+1)} h_0 \\ |c_n - p| &\le h_n \le \epsilon \\ i.e. \quad 2^{n+1} &\ge \frac{b_0 - a_0}{\epsilon} \\ i.e. \quad n &\ge \frac{\log(b_0 - a_0) - \log \epsilon}{\log 2} - 1 \end{aligned}$$

# VIII.

## VIII-a

Let $p$ be the zero of multiplicity k of the function f, we can denote $f(x) = (x-p)^k q(x)$, and $g(x) = x - \frac{f(x)}{f'(x)}$. So the iteration can be expressed as $x_{n+1} = g(x_n)$.

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)^2 - f(x)f''(x)}{(f'(x))^2} \\ &= \frac{f(x)f''(x)}{(f'(x))^2} \\ &= \frac{(x-p)^k q(x)(k(k-1)(x-p)^{k-2}q(x) + 2k(x-p)^{k-1}q'(x) + q''(x)(x-p)^k)}{(k(x-p)^{k-1}q(x) + q'(x)(x-p)^k)^2} \\ &= \frac{(x-p)^{2k-2}q(x)(k(k-1)q(x) + 2k(x-p)q'(x) + q''(x)(x-p)^2)}{(x-p)^{2k-2}(kq(x) + q'(x)(x-p))^2} \\ &= \frac{q(x)(k(k-1)q(x) + 2k(x-p)q'(x) + q''(x)(x-p)^2)}{(kq(x) + q'(x)(x-p))^2} \end{aligned}$$

since $g'(p) = 1 - \frac{1}{k} < 1$, it converges linearly as[1]

$$\lim_{n\to\infty} \frac{|x_{n+1} - p|}{|x_n - p|} = |g'(p)|$$

## VIII-b

Let $p$ be the zero of multiplicity k of the function f, we can denote $f(x) = (x-p)^k q(x)$, and $g(x) = x - k\frac{f(x)}{f'(x)}$. So the iteration can be expressed as $x_{n+1} = g(x_n)$.

$$\begin{aligned} g'(x) &= 1 - k\frac{f'(x)^2 - f(x)f''(x)}{(f'(x))^2} \\ &= 1 - k + k\frac{(x-p)^k q(x)(k(k-1)(x-p)^{k-2}q(x) + 2k(x-p)^{k-1}q'(x) + q''(x)(x-p)^k)}{(k(x-p)^{k-1}q(x) + q'(x)(x-p)^k)^2} \\ &= 1 - k + k\frac{q(x)(k(k-1)q(x) + 2k(x-p)q'(x) + q''(x)(x-p)^2)}{(kq(x) + q'(x)(x-p))^2} \end{aligned}$$

since $g'(p) = 0$, it converges quadratically as[1]

$$\lim_{n\to\infty} \frac{|x_{n+1} - p|}{|x_n - p|^2} = \frac{|g''(p)|}{2}$$

# References


[1] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9th ed. Richard Stratton, 2010. Chap. 2, pp. 80–81.