HW1

Exercise 1.

Question 1.

$Q_D(p)$ is p^{th} population quantile such that $P(D \leq Q_D(p)) = p$.

In order word:

$$\int_0^{Q_D(p)} \lambda e^{-\lambda D} dD = 1 - e^{-\lambda Q_D(p)} = p$$

$$Q_D(p) = -\frac{1}{\lambda} \ln(1-p)$$

Question 2

First empirical moment of the exponential distribution:

$$\hat{\mu} = \bar{D}_n$$

Population moment of the exponential distribution:

$$E(D_1) = \frac{1}{\lambda}$$

The MOM estimator of λ is : $\hat{\lambda}^{MOM} = \frac{1}{\bar{D}_n}$

Therefore the method of moments-based estimator of $Q_D(p)$:

$$Q_D(p)^{MOM} = -\frac{1}{\hat{\lambda}^{MOM}} \ln(1-p) = -\bar{D}_n \ln(1-p)$$

Question3

From the CLT

$$\sqrt{n}(\bar{D}_n - \frac{1}{\lambda}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{\lambda^2})$$

Hence, by Delta Method we can get, let $g(t) = t * ln(1-p)$ so $g^{'}(t) = ln(1-p)$

$$\sqrt{n}(\ln(1-p)\bar{D}_n + Q_D(p)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

Since $\hat{\lambda}^{MOM} = \frac{1}{\bar{D}_n}$

$$\sqrt{n}(\ln(1-p)\frac{1}{\lambda} + Q_D(p)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

So, the *approximate* $(1-\alpha)$-confidence interval for $Q_D(p)$ is

$$[-\bar{D}_n \ln(1-p) - \frac{z_{1-\alpha/2} \times \ln(1-p)}{\lambda\sqrt{n}}, -\bar{D}_n \ln(1-p) + \frac{z_{1-\alpha/2} \times \ln(1-p)}{\lambda\sqrt{n}}]$$

Question 4.

We know that if $D_1, \ldots, D_n$ are independent exponential random variables with parameter λ, then

$$\sum_{i=1}^{n} D_i \sim \Gamma(n, \lambda)$$

Therefore

$$\lambda \bar{D}_n = \frac{\lambda}{n}\sum_{i=1}^{n} D_i \sim \Gamma(n, n)$$

So, $\lambda \bar{D}_n$ is independent of the parameter λ, which means it is an exact pivot.

From previous question

$$Q_D(p) = -\frac{1}{\lambda}\ln(1-p)$$

$$Q_D(0.5) = \frac{1}{\lambda}\ln(2)$$

To construct 95% confidence interval, let a and b be the 0.025 and 0.975 quantile of $\Gamma(n, n)$

Therefore

$$P(a < \lambda \bar{D}_n < b) = 0.95$$

$$P\left(\frac{\bar{D}_n ln(2)}{b} < \frac{1}{\lambda}\ln(2) < \frac{\bar{D}_n ln(2)}{a}\right) = 0.95$$

The confidence interval is $\left[\frac{\bar{D}_n ln(2)}{b}, \frac{\bar{D}_n ln(2)}{a}\right]$