Homework 1 Exercise 3

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Exercise 3

1. Let Y has a standard normal distibution. Let X has a normal distribution $N\left(\mu, \sigma^2\right)$. We have

$$\sigma^2 = E(X^2) - (E(X))^2,$$

$$E(X^2) = \sigma^2 + \mu^2.$$

Then we have some conclusions ahout Y:

$$E(Y) = 0,$$

$$E(Y^2) = 0 + 1 = 0,$$

$$E(Y^3) = \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0.$$

Go back to γ,

$$\begin{aligned}
\gamma &= \mathbb{E}\left[R_1^3\right] \\
&= E[(Y * \sigma + \mu)^3] \\
&= E[Y^3 * \sigma^3 + 3 * y^2 * \sigma^2 * \mu + 3 * y * \sigma * \mu^2 + \mu^3] \\
&= E[3 * y^2 * \mu * \sigma^2 + \mu^3] \\
&= \mu^3 + 3 * \mu * \sigma^2
\end{aligned}$$

2.

(a) $\frac{1}{n}\sum_{i=1}^{n} R_i$ has a normal distribution $\mathbf{N}\left(\mu, \sigma^2/n\right)$, so

$$
\begin{aligned}
E[\hat{\gamma}] - \gamma &= \mu^3 + 3 * \frac{\mu}{n} * \sigma^2 - \mu^3 - 3 * \mu * \sigma^2 \\
&= 3 * \frac{(1-n)\mu}{n} * \sigma^2
\end{aligned}
$$

(b) Yes, it's consistent.

$$
\begin{aligned}
E[\hat{\gamma}^2] - (E[\hat{\gamma}])^2 &= \frac{O(n^6) + 6! * C_6^6 * \sum_{i=1}^{6} R_i}{n^6} - \mu^6 - O(\frac{1}{n}) \\
&\overset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

3. The bias of $(\frac{1}{n}\sum_{i=1}^{n} R_i)^3$ is $3 * \frac{(1-n)\mu}{n} * \sigma^2$. From previous work, we also have

$$\gamma = \mu^3 + 3 * \mu * \sigma^2$$

$$E[\hat{\gamma}] = \mu^3 + 3 * \frac{\mu}{n} * \sigma^2$$

$$E[R_1 R_2 R_3] = \mu^3$$

We can easily have an unbiased estimator $n(\frac{1}{n}\sum_{i=1}^{n} R_i)^3 - (n-1)R_1 R_2 R_3$.

4.

(a)

$$
\begin{aligned}
E[\tilde{\gamma}] - \gamma &= \frac{n}{n} E[R_1^3] - E[R_1^3] \\
&= 0
\end{aligned}
$$

(b) Yes, it's consistent.

$$
\begin{aligned}
E[\tilde{\gamma}^2] - (E[\tilde{\gamma}])^2 &= \frac{(\sum\limits_{i=1}^{n} Y_i)^2}{n^2} - (\mu^3 + 3*\mu*\sigma^2)^2 \\
&= \frac{O(n^2) + n*(n-1)E[R_1^3 R_2^3]}{n^2} - (\mu^3 + 3*\mu*\sigma^2)^2 \\
&= (\mu^3 + 3*\mu*\sigma^2)^2 - (\mu^3 + 3*\mu*\sigma^2)^2 \\
&\overset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

5. Based on previious work, we have two sufficient statistics $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ and $\bar{R^2} = \frac{1}{n}\sum_{i=1}^{n} R_i^2$. We also have unbiased and consistent estimator$\tilde{\gamma} = \frac{1}{n}\sum_{i=1}^{n} R_i^3$.
The $MVUE = E[\tilde{\gamma}|T]$.

$$
\begin{aligned}
E[\tilde{\gamma}|T] &= E[\frac{1}{n}\sum_{i=1}^{n} R_i^3|\bar{R},\bar{R^2}] \\
&= E[\frac{1}{n}\sum_{i=1}^{n} R_i - \bar{R} + \bar{R}^3|\bar{R},\bar{R^2}] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}[(R_i-\bar{R})^3 + 3(R_i-\bar{R})^2\bar{R} + 3(R_i-\bar{R})\bar{R}^2 + \bar{R}^3]|\bar{R},\bar{R^2}]
\end{aligned}
$$

Similar with the derivation of$E(Y) = 0, E(Y^3) = 0,$ in question 1, we get $E(R_i - \bar{R}) = 0, E((R_i-\bar{R})^3) = 0$. So,

$$
\begin{aligned}
E[\tilde{\gamma}|T] &= E[\frac{1}{n}\sum_{i=1}^{n}[3(R_i-\bar{R})^2\bar{R} + \bar{R}^3]|\bar{R},\bar{R^2}] \\
&= E[3R_i^2\bar{R} - 6R_i\bar{R}^2 + 4\bar{R}^3|\bar{R},\bar{R^2}] \\
&= 3\bar{R_i^2}\bar{R} - 2\bar{R}^3
\end{aligned}
$$