Simulation HW1

Problem #1

a)

E = sum of dice is 7, A = first dice lands on 4

for event E to be independent $P(E \cap A) = P(E) \cdot P(A)$

$P(E) = \frac{6}{36} = \frac{1}{6}$ $P(A) = \frac{6}{36} = \frac{1}{6}$ $P(E \cap A) = \frac{1}{36}$

check independence: $\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} \Rightarrow$ true

Therefore E is independent of A

b)

B = second dice lands on 3

$P(E \cap B) = P(E) \cdot P(B)$ for independence to be true

$P(E) = \frac{1}{6}$, $P(B) = \frac{6}{36} = \frac{1}{6}$, $P(E \cap B) = \frac{1}{36}$

check independence: $\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} \Rightarrow$ true

Therefore E is independent of B

Problem #2

$P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.4$

a)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, since mutually exclusive, $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) = 0.2 + 0.3 = \boxed{0.5}$

b)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, since independence, $P(A \cap B) = P(A) \cdot P(B)$

$= P(A) + P(B) - P(A)P(B) = 0.2 + 0.3 - 0.2 \cdot 0.3$

$= \boxed{0.44}$

c)

$P(A \cap B \cap C) = P(A)P(B)P(C)$ if independent

$= 0.2 \cdot 0.3 \cdot 0.4$

$= \boxed{0.024}$

d)

Since mutually exclusive $P(A \cap B \cap C) = 0$

Problem #3

a) $X \in \{-n, -n+2 \cdots n-2, n\}$

b) when $n=3$ $X \in \{-3, -1, 1, 3\}$

possible outcomes $= 2^3 = 8$

$P(X=-3) = \frac{1}{8}$, $P(X=3) = \frac{1}{8}$, $P(X=-1) = \frac{3}{8}$, $P(X=1) = \frac{3}{8}$

c)

$$P_X(x) = \begin{cases} \frac{1}{8}, & x=-3 \\ \frac{1}{8}, & x=3 \\ \frac{3}{8}, & x=-1 \\ \frac{3}{8}, & x=1 \\ 0, & \text{other} \end{cases}$$

d)

$$F(x) = \begin{cases} 0, & x<-3 \\ \frac{1}{8}, & -3 \le x < -1 \\ \frac{1}{2}, & -1 \le x < 1 \\ \frac{7}{8}, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Problem #4

a)

$$P_X(X_1) = \begin{cases} \frac{3}{16}, & X_1 = 0 \\ \frac{1}{8}, & X_1 = 1 \\ \frac{5}{16}, & X_1 = 2 \\ \frac{3}{8}, & X_1 = 3 \\ 0, & \text{other} \end{cases} \qquad P_X(X_2) = \begin{cases} \frac{1}{2}, & X_2 = 2 \\ \frac{1}{2}, & X_2 = 1 \\ 0, & \text{other} \end{cases}$$

b) $E[X_1] = \frac{3}{16} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{5}{16} \cdot 2 + \frac{3}{8} \cdot 3 = \boxed{1.875}$

$E[X_2] = 0.5 \cdot 2 + 0.5 \cdot 1 = \boxed{1.5}$

c) $Var(X_1) = E(X^2) - (E(X))^2$

$(E[X_1])^2 = (1.875)^2 = 3.516$

$E[X^2] = \frac{3}{16} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{5}{16} \cdot 4 + \frac{3}{8} \cdot 9 = 4.75$

$Var(X_1) = 4.75 - 3.516 = \boxed{1.234}$

$Var(X_2) = E[X_2^2] - (E(X_2))^2$

$E(X_2) = 1.5, (E(X_2))^2 = 2.25$

$E(X_2^2) = 0.5 \cdot 4 + 0.5 \cdot 1 = 2.5$

$Var(X_2) = 2.5 - 2.25 = \boxed{0.25}$

$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$

$E(X_1 X_2) = 1 \cdot 1 \cdot \frac{1}{16} + 1 \cdot 2 \cdot \frac{1}{16} + 2 \cdot 1 \cdot \frac{3}{16} + 2 \cdot 2 \cdot \frac{1}{8} + 3 \cdot 1 \cdot \frac{1}{8} + 3 \cdot 2 \cdot \frac{1}{4}$

$= \frac{1}{16} + \frac{2}{16} + \frac{6}{16} + \frac{4}{8} + \frac{3}{8} + \frac{3}{2}$

$= 2.9375$

$Cov(X_1, X_2) = 2.9375 - 1.875 \cdot 1.5 = \boxed{0.125}$

d)

$P(X_2 = 1 \mid X_1 = 0) = \frac{\frac{1}{8}}{\frac{3}{16}} = \frac{2}{3}$

$P(X_2 = 2 \mid X_1 = 0) = \frac{\frac{1}{16}}{\frac{3}{16}} = \frac{1}{3}$

Problem #5

For a random sorting of n children, each child is equally as likely to be the tallest, let $I_i = 1$ represent the ith child within k children is the tallest, then $P(I_i = 1) = \frac{1}{k}$, let $X = I_i = 1$

$E(X) = \sum_{k=1}^{n} P(X) = \sum_{k=1}^{n} \frac{1}{k}$

Problem #6

$f(x) = \begin{cases} ce^{-2x}, & 0 \le x < \infty \\ 0, & x < \infty \end{cases}$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$, $\int_0^{\infty} ce^{-2x}\,dx = 1$

$\int_0^{A} e^{-2x}\,dx = 1 \Rightarrow \left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} = 1 \Rightarrow 0 - (-\frac{1}{2}) \Rightarrow \frac{1}{2}$

$c \cdot \frac{1}{2} = 1$, $\boxed{c = 2}$

b) $P(X>2) = \int_2^\infty f(x)dx = \int_2^\infty ce^{-2x}dx = 2\cdot\int_2^\infty e^{-2x}dx$

$= 2\cdot[-\frac{1}{2}e^{-2x}]_2^\infty = 2\cdot[0-(-\frac{1}{2}e^{-4})]$

$= e^{-4} \approx 0.0183$

Problem #7

a)

$f_x(x) = \int_0^{1-x} 3(x+y)\,dy = \int_0^{1-x} 3x+3y = [3xy+\frac{3}{2}y^2]_0^{1-x} = [3x(1-x)+\frac{3}{2}(1-x)^2]-0$

$= 3x-3x^2+\frac{3}{2}(1-2x+x^2) = 3x-3x^2+\frac{3}{2}-3x+\frac{3}{2}x^2 = -\frac{3}{2}x^2+\frac{3}{2} = (1-x^2)\frac{3}{2}$

$\boxed{f_x(x) = \frac{3}{2}(1-x^2) \text{ for } 0\le x\le 1}$

$f_x(y) = \int_0^{1-y} 3(x+y)\,dx = \int_0^{1-y} 3y+3x\,dx = [3xy+\frac{3}{2}x^2]_0^{1-y} = 3y(1-y)+\frac{3}{2}(1-y)^2-0$

$= 3y-3y^2+\frac{3}{2}(1-2y+y^2) = 3y-3y^2+\frac{3}{2}-3y+\frac{3}{2}y^2$

$= -\frac{3}{2}y^2+\frac{3}{2}$

$\boxed{f_x(y) = \frac{3}{2}(1-y^2) \text{ for } 0\le y\le 1}$

b) $E(X) = \int_0^1 x f(x)dx = \int_0^1 x\frac{3}{2}(1-x^2)dx = \int_0^1 \frac{3}{2}x-\frac{3}{2}x^3dx = [\frac{3}{4}x^2-\frac{3}{8}x^4]_0^1 = \frac{3}{4}-\frac{3}{8} = \frac{3}{8}$

$\boxed{E(X) = \frac{3}{8}}$

$Var(X) = E(X^2)-[E(X)]^2$

$E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2(\frac{3}{2}-\frac{3}{2}x^2) = \int_0^1 \frac{3}{2}x^2-\frac{3}{2}x^4dx = [\frac{3}{6}x^3-\frac{3}{10}x^5]_0^1 = \frac{1}{5}$

$E(X^2) = \frac{2}{5}$ $\quad [E(X)]^2 = (\frac{3}{8})^2 = \frac{9}{64}$

$\boxed{Var(X) = \frac{1}{5}-\frac{9}{64} = \frac{19}{320} \approx 0.0594}$

c) $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$

$E(X) = E(Y) = \frac{3}{8}$, $\quad Var(X) = Var(Y) = \frac{19}{320}$

$Cov(X,Y) = E(X,Y) - E(X)E(Y)$

$E(X,Y) = \iint xy\cdot 3(x+y)\,dy\,dx = \int_0^1\int_0^{1-x} xy\cdot 3(x+y)\,dy\,dx$

$\Rightarrow \int_0^{1-x} xy\cdot 3(x+y)\,dy = [\frac{3}{2}x^2y^2+xy^3]_0^{1-x} = \frac{3}{2}x^2(1-x)^2+x(1-x)^3$

$$\Rightarrow \int_0^1 \frac{3}{2}x^2(1-x)^2 + x(1-x)^3 = \int_0^1 \frac{3}{2}x^2(1-2x+x^2) + x(1-3x+3x^2-x^3)\,dx$$

$$= \int_0^1 \frac{3}{2}x^2 - 3x^3 + \frac{3}{2}x^4 + x - 3x^2 + 3x^3 - x^4\,dx$$

$$= \int_0^1 x - \frac{3}{2}x^2 + \frac{1}{2}x^4\,dx = \left[\frac{1}{2}x^2 - \frac{3}{6}x^3 + \frac{1}{10}x^5\right]_0^1 = \frac{1}{2} - \frac{3}{6} + \frac{1}{10} = \frac{1}{10}$$

$$Cov(X,Y) = \frac{1}{10} - \left(\frac{3}{8}\cdot\frac{3}{8}\right) = \frac{1}{10} - \frac{9}{64} = -\frac{13}{320}$$

$$Corr(X,Y) = \frac{\frac{-13}{320}}{\sqrt{\frac{19}{320}}\cdot\sqrt{\frac{19}{320}}} = \frac{\frac{-13}{320}}{\frac{19}{320}} = \boxed{\frac{-13}{19} \approx -0.684}$$

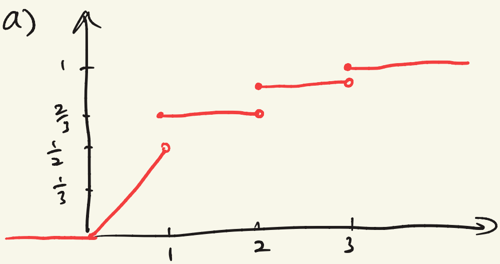
d) $$f_{X|Y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)} \quad y = \frac{1}{2} \Rightarrow \frac{f_{xy}(x,\frac{1}{2})}{f_y(\frac{1}{2})} = \frac{3(x+\frac{1}{2})}{\frac{3}{2}(1-\frac{1}{2}^2)}$$

$$= \frac{3x+\frac{3}{2}}{\frac{3}{2}\cdot\frac{3}{4}} = \frac{(3x+\frac{3}{2})8}{9} = \frac{24}{9}(x+\frac{1}{2}) = \frac{8}{3}(x+\frac{1}{2})$$

$$f_{X|Y}(x|\tfrac{1}{2}) = \begin{cases} \frac{8}{3}(x+\frac{1}{2}), & 0 \le x \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Problem #8

a)



b) $P(X > 1/2) = 1 - F(\frac{1}{2})$

$F(\frac{1}{2}) = \frac{1}{4} \Rightarrow \boxed{P(X > \frac{1}{2}) = \frac{3}{4}}$

c) $P(2 < X \le 4) = F(4) - F(2)$

$F(4) = 1, \quad F(2) = \frac{11}{12}$

$\boxed{P(2 < X \le 4) = \frac{1}{12}}$

d) $\boxed{P(X < 3) = \frac{11}{12}}$

e) $P(X=1) = F(1) - F(1^-)$

$= \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$

Problem #9

if $X \sim Geo(p)$ then $P(X>k) = (1-p)^k$

$$P(X>m+n \mid X>m) = \frac{P(X>m+n)}{P(X>m)} = \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$$

$(1-p)^n = P(X>n)$, Therefore X **is** memoryless

Problem #10

10:00 - 10:30 uniform distribution $T \sim U(0,30)$

$$P(T>10) = \frac{30-10}{30-0} = \frac{20}{30} = \boxed{\frac{2}{3}}$$

$$P(T>25 \mid T>15) = \frac{30-25}{30-15} = \frac{5}{15} = \boxed{\frac{1}{3}}$$