11 - Public Key Crypto and Discrete Logarithm

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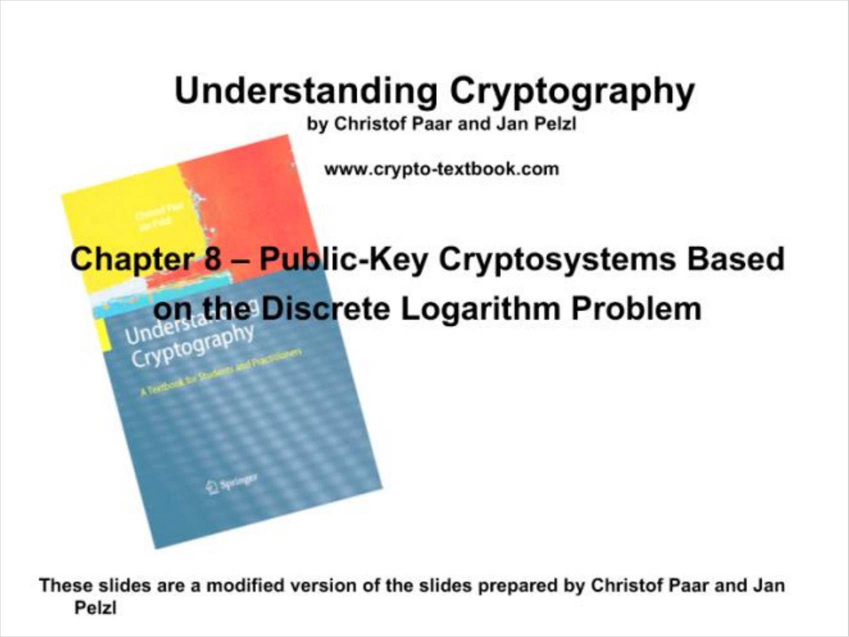

Understanding Cryptography

by Christof Paar and Jan Pelzl

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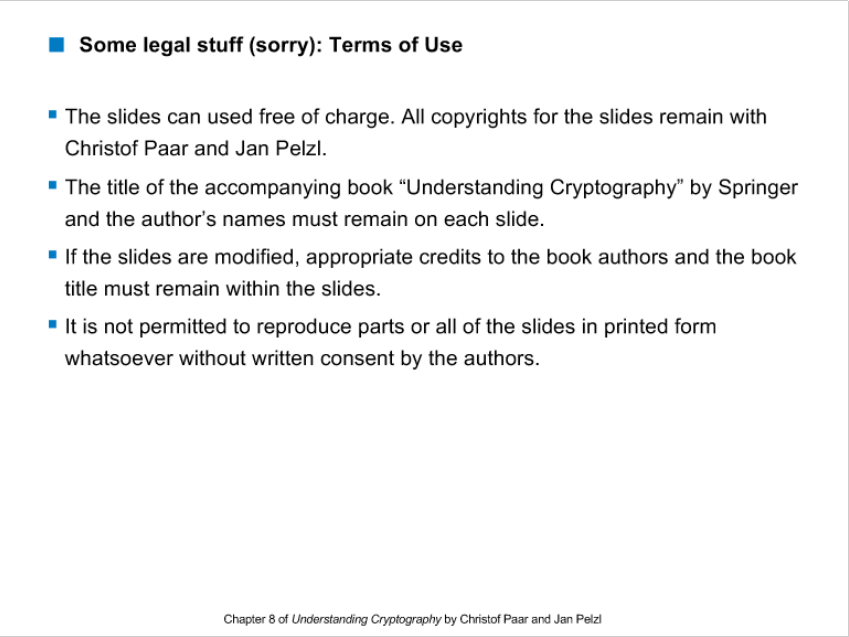
Chapter 8 – Public-Key Cryptosystems Based on the Discrete Logarithm Problem

These slides are a modified version of the slides prepared by Christof Paar and Jan Pelzl



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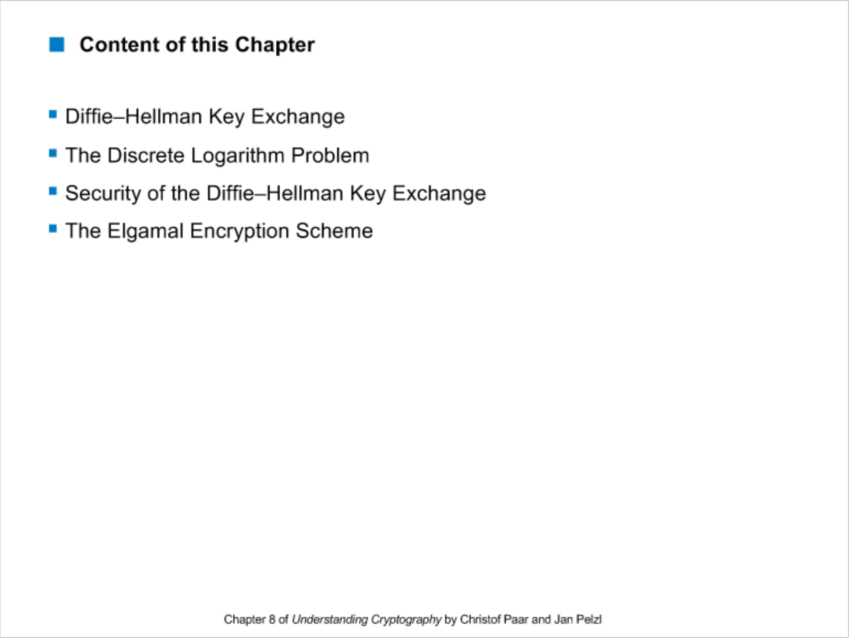
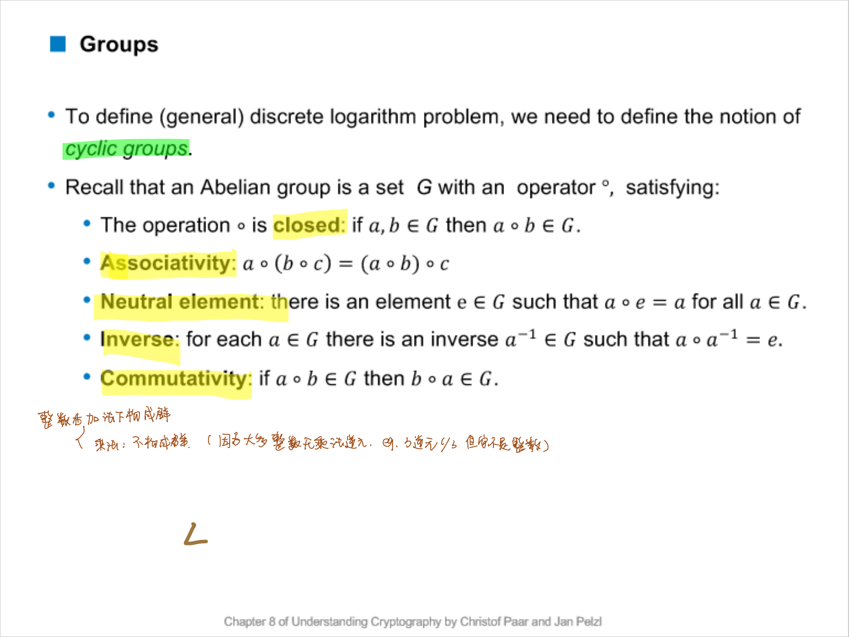


Content of this Chapter

- Diffie–Hellman Key Exchange
- The Discrete Logarithm Problem
- Security of the Diffie–Hellman Key Exchange
- The Elgamal Encryption Scheme

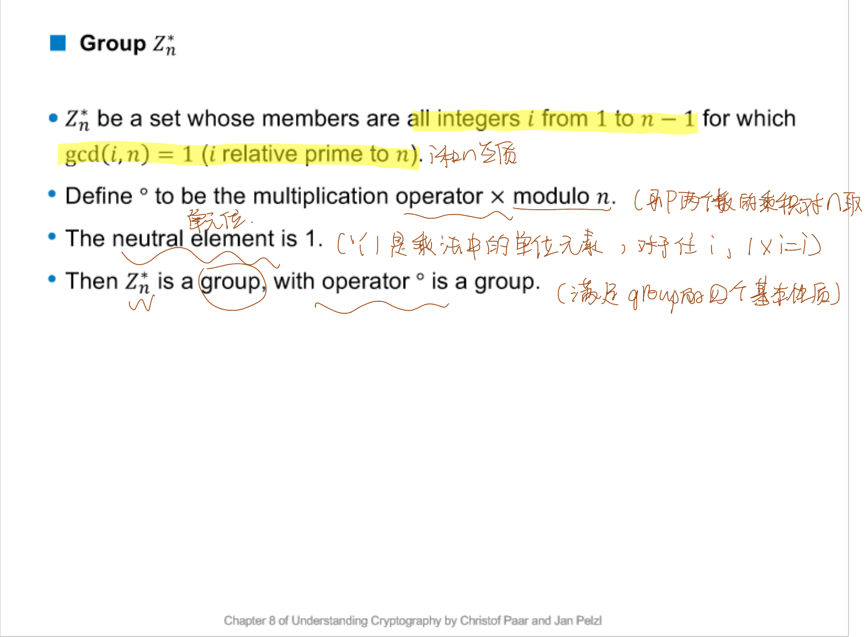


Groups

- To define (general) discrete logarithm problem, we need to define the notion of *cyclic groups*.
- Recall that an Abelian group is a set *G* with an operator °, satisfying:
  - The operation $\circ$ is **closed**: if $a, b \in G$ then $a \circ b \in G$.
  - **Associativity**: $a \circ (b \circ c) = (a \circ b) \circ c$
  - **Neutral element**: there is an element $e \in G$ such that $a \circ e = a$ for all $a \in G$.
  - **Inverse**: for each $a \in G$ there is an inverse $a^{-1} \in G$ such that $a \circ a^{-1} = e$.
  - **Commutativity**: if $a \circ b \in G$ then $b \circ a \in G$.

整数在加法下构成群

乘法：不构成群。（因为大多整数无乘法逆元，例：5逆元1/5 但它不是整数）



Group Z_n^*

- Z_n^* be a set whose members are all integers i from 1 to $n-1$ for which $\gcd(i, n) = 1$ (i relative prime to n). i和n互质
- Define ° to be the multiplication operator × modulo n. （即两个数的乘积对n取模）
- The neutral element is 1. 单位. （∵1是乘法中的单位元素，对于任i，1×i=i）
- Then Z_n^* is a group, with operator ° is a group. （满足group的四个基本性质）

Example

- $\mathbb{Z}_9^* = \{1,2,4,5,7,8\}$. All members are relative prime to 9.
- Multiplication table:

×mod 9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

Example:
$5 \times 7 \text{ mod } 9$
$= 35 \ mod\ 9 = 8$

Order of an element

An order of an element a in a group Z_n^* is the smallest positive integer k such that

$$\underbrace{a \times a \times \cdots \times a}_{k \text{ times}} \equiv 1 \text{ mod } n$$

Example: let $a = 3$ in the group Z_{11}^*

$a^1 = 3$

$a^2 = 3 \times 3 = 9$

$a^3 = 9 \times 3 = 27 \equiv 5 \ (mod\ 11)$

$a^4 = 5 \times 3 = 15 \equiv 4 \ (mod\ 11)$

$a^5 = 4 \times 3 = 12 \equiv 1 \ (mod\ 11)$

So the order of a is 5.

If we multiply a^5 further with a's, we'll cycle through the same numbers.

Cyclic group

- Let $|Z_n^*|$ denote the size of Z_n^*, that is, the number of elements in Z_n^*.
- A group Z_n^* which contains an element a with order $|Z_n^*|$ is called a **cyclic group**.
- Example: $Z_{11}^* = \{1,2,3,4,5,6,7,8,9,10\}$. Then $a = 2$ has order $|Z_{11}^*| = 10$. So $\mathbb{Z}_{11}^*$ is a cyclic group.

$a^1 = 2$

$a^2 = 4$

$a^3 = 8$

$a^4 \equiv 5 \ mod\ 11$

$a^5 \equiv 10 \ mod\ 11$

$a^6 \equiv 9 \ mod\ 11$

$a^7 \equiv 7 \ mod\ 11$

$a^8 \equiv 3 \ mod\ 11$

$a^9 \equiv 6 \ mod\ 11$

$a^{10} \equiv 1 \ mod\ 11$

Primitive element

- An element a of Z_n^* with order $|Z_n^*|$ is called a *primitive element*, or a *generator* of the group.
- Example: $a = 2$ in group Z_{11}^* is a generator of the group: all elements of Z_{11}^* can be generated by powers of a.
- **Fact**: for every prime p, Z_p^* is a cyclic group (i.e., it has a primitive element).

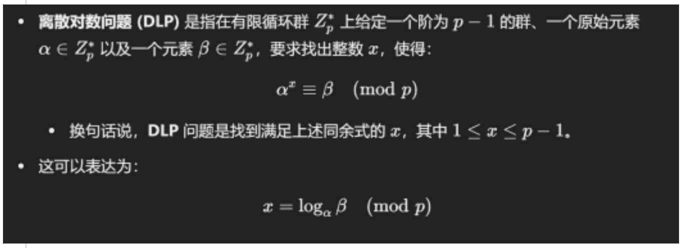
The Discrete Logarithm Problem

Discrete Logarithm Problem (DLP) in Z_p*

- Given is the finite cyclic group Z_p* of *order p−1* and a primitive element *α* ∈ Z_p* and another element *β* ∈ Z_p*.
- The DLP is the problem of determining the integer *1 ≤ x ≤ p−1* such that $\alpha^x \equiv \beta \bmod p$
- This computation is called the **discrete logarithm problem (DLP)**

$$x = \log_\alpha \beta \bmod p$$

Remark: For the coverage of groups and cyclic groups, we refer to Chapter 8 of *Understanding Cryptography*



- **离散对数问题 (DLP)** 是指在有限循环群 Z_p^* 上给定一个阶为 $p-1$ 的群、一个原始元素 $\alpha \in Z_p^*$ 以及一个元素 $\beta \in Z_p^*$，要求找出整数 x，使得：

$$\alpha^x \equiv \beta \pmod p$$

  - 换句话说，**DLP** 问题是找到满足上述同余式的 x，其中 $1 \le x \le p-1$。
- 这可以表达为：

$$x = \log_\alpha \beta \pmod p$$

Example

与11互素的整数.

- $Z_{11}^* = \{1,2,3,4,5,6,7,8,9,10\}$ has a primitive element *a* = 2.

i	1	2	3	4	5	6	7	8	9	10
a^i	2	4	8	5	10	9	7	3	6	1

- Then, $5 = \log_2 10 \bmod 11$ $2^5 = 10 \bmod 11.$ (5是2的离散对数，使得2^5在模11下等于10)
- In general, for large *p*, it is infeasible to construct the table of exponents for all generators of Z_p^* like above. 不切实际.
- Encryption/digital signature based on discrete logarithm relies on the difficulty of finding the logarithm.

The Generalized Discrete Logarithm Problem

广义离散对数问题

The following discrete logarithm problems have been proposed for use in cryptography

1. The multiplicative group of the prime field Z_p *or a subgroup of it*. For instance, the classical DHKE uses this group (cf. next slides), but also Elgamal encryption or the Digital Signature Algorithm (DSA).
2. The cyclic group formed by an elliptic curve (see Chapter 9)
3. The multiplicative group of a Galois field $GF(2^m)$ or a subgroup of it. Schemes such as the DHKE can be realized with them.
4. Hyperelliptic curves or algebraic varieties, which can be viewed as generalization of elliptic curves.

Remark: The groups 1. and 2. are most often used in practice.

Diffie–Hellman Key Exchange: Overview

- Proposed in 1976 by **Whitfield Diffie and Martin Hellman**
- **Widely used**, e.g. in Secure Shell (SSH), Transport Layer Security (TLS), and Internet Protocol Security (IPSec)
- The Diffie–Hellman Key Exchange (DHKE) is a key exchange protocol and **not** used for encryption

  (For the purpose of encryption based on the DHKE, ElGamal can be used.)

Diffie–Hellman Key Exchange: Set-up

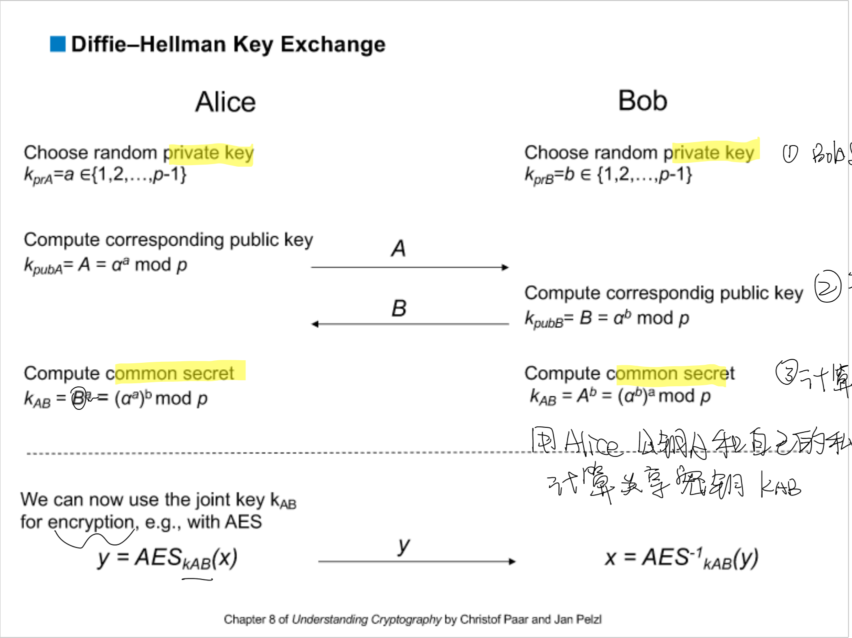
1. Choose a large prime *p*.
2. Choose an integer α ∈ {2,3, . . . , *p*−2}. α must be a primitive element of the cyclic group Z_p^* (意味着 α 可以通过不同的幂次生成群中的所有元素)
3. Publish *p and* α.

[这样双方可以使用这些共享参数来进行后续的密钥交换过程]

eg. $\alpha^1 \bmod 7 = 3$, $\alpha = 3$

$\alpha^2 \bmod 7 = 6$

⋮



Diffie–Hellman Key Exchange

Alice		Bob
Choose random private key $k_{prA}=a \in \{1,2,\ldots,p-1\}$		Choose random private key $k_{prB}=b \in \{1,2,\ldots,p-1\}$
Compute corresponding public key $k_{pubA}= A = \alpha^a \bmod p$	$\xrightarrow{A}$	
	$\xleftarrow{B}$	Compute correspondig public key $k_{pubB}= B = \alpha^b \bmod p$
Compute common secret $k_{AB} = B^a = (\alpha^a)^b \bmod p$		Compute common secret $k_{AB} = A^b = (\alpha^b)^a \bmod p$

① Bob及Alice各自选择随机私钥

②计算各自公钥

③计算共享密钥.

用Alice公钥A和自己的私钥b
计算共享密钥 kAB

We can now use the joint key k_{AB} for encryption, e.g., with AES

$y = AES_{kAB}(x)$ $\xrightarrow{y}$ $x = AES^{-1}_{kAB}(y)$

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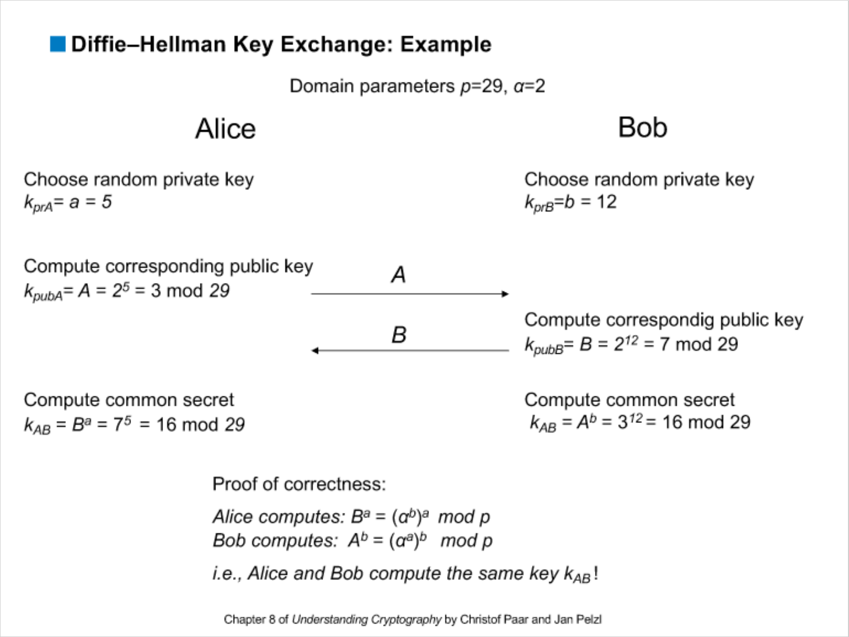

Diffie–Hellman Key Exchange: Example

Domain parameters p=29, α=2

Alice		Bob
Choose random private key $k_{prA}= a = 5$		Choose random private key $k_{prB}=b = 12$
Compute corresponding public key $k_{pubA}= A = 2^5 = 3 \bmod 29$	$\xrightarrow{A}$	
	$\xleftarrow{B}$	Compute correspondig public key $k_{pubB}= B = 2^{12} = 7 \bmod 29$
Compute common secret $k_{AB} = B^a = 7^5 = 16 \bmod 29$		Compute common secret $k_{AB} = A^b = 3^{12} = 16 \bmod 29$

Proof of correctness:

Alice computes: $B^a = (\alpha^b)^a \bmod p$
Bob computes: $A^b = (\alpha^a)^b \bmod p$

i.e., Alice and Bob compute the same key k_{AB}!

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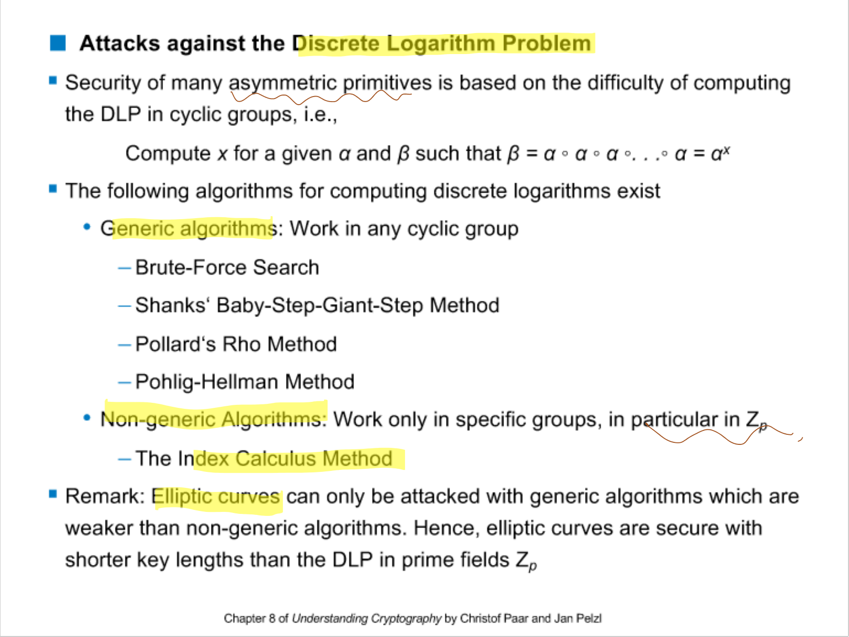

Attacks against the Discrete Logarithm Problem

- Security of many asymmetric primitives is based on the difficulty of computing the DLP in cyclic groups, i.e.,

  Compute x for a given α and β such that $\beta = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha = \alpha^x$

- The following algorithms for computing discrete logarithms exist
  - Generic algorithms: Work in any cyclic group
    - Brute-Force Search
    - Shanks' Baby-Step-Giant-Step Method
    - Pollard's Rho Method
    - Pohlig-Hellman Method
  - Non-generic Algorithms: Work only in specific groups, in particular in Z_p
    - The Index Calculus Method
- Remark: Elliptic curves can only be attacked with generic algorithms which are weaker than non-generic algorithms. Hence, elliptic curves are secure with shorter key lengths than the DLP in prime fields Z_p

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Attacks against the Discrete Logarithm Problem

Summary of records for computing discrete logarithms in Z_p*

Decimal digits	Bit length	Date
58	193	1991
68	216	1996
85	282	1998
100	332	1999
120	399	2001
135	448	2006
160	532	2007

In order to prevent attacks that compute the DLP, it is recommended to use primes with a length of at least 1024 bits for schemes such as Diffie-Hellman in Z_p*

Security of the classical Diffie–Hellman Key Exchange

- Which information does Oscar have?
  - α, p
  - $k_{pubA} = A = \alpha^a \bmod p$
  - $k_{pubB} = B = \alpha^b \bmod p$
- Which information does Oscar want to have?
  - $k_{AB} = \alpha^{ba} = \alpha^{ab} \bmod p$
  - This is kown as Diffie-Hellman Problem (DHP)

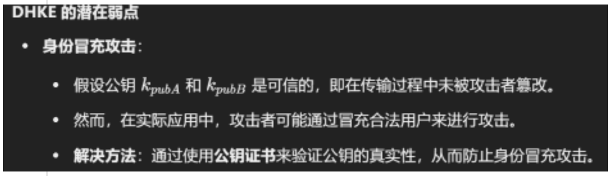
Security of the classical Diffie–Hellman Key Exchange

- The only known way to solve the DHP is to solve the DLP, i.e.
  1. Compute $a = \log_\alpha A \bmod p$
  2. Compute $k_{AB} = B^a = \alpha^{ba} = \bmod p$

  It is conjectured that the DHP and the DLP are equivalent, i.e., solving the DHP implies solving the DLP.
- To prevent attacks, i.e., to prevent that the DLP can be solved, choose $p > 2^{1024}$

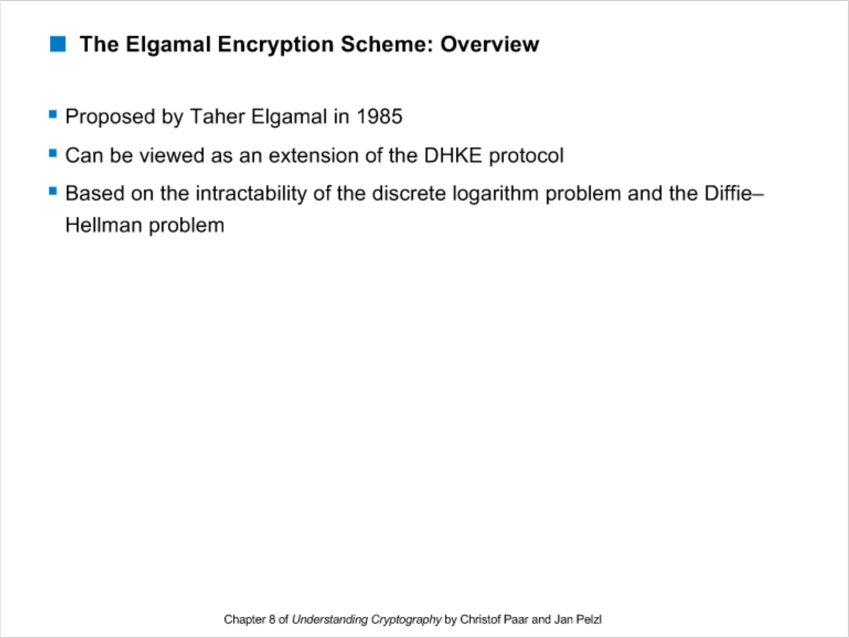
冒充

- However, DHKE is still vulnerable to impersonation attack. The above assumes k_{pub_A} and k_{pub_B} are authentic, i.e., they have not been changed by the attacker during transit.
  - This problem is solved using public key certificates – to be covered later.



DHKE 的潜在弱点

- **身份冒充攻击**：
  - 假设公钥 k_{pubA} 和 k_{pubB} 是可信的，即在传输过程中未被攻击者篡改。
  - 然而，在实际应用中，攻击者可能通过冒充合法用户来进行攻击。
  - **解决方法**：通过使用**公钥证书**来验证公钥的真实性，从而防止身份冒充攻击。



The Elgamal Encryption Scheme: Overview

- Proposed by Taher Elgamal in 1985
- Can be viewed as an extension of the DHKE protocol
- Based on the intractability of the discrete logarithm problem and the Diffie–Hellman problem

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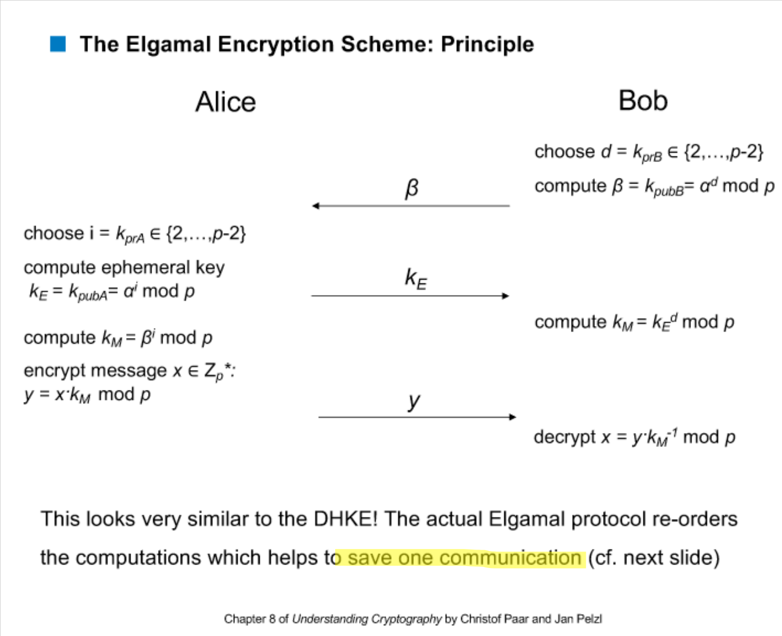

The Elgamal Encryption Scheme: Principle

Alice
Bob
choose $d = k_{prB} \in \{2,\dots,p\text{-}2\}$
compute $\beta = k_{pubB} = \alpha^d \bmod p$
β
choose i = $k_{prA} \in \{2,\dots,p\text{-}2\}$
compute ephemeral key
$k_E = k_{pubA} = \alpha^i \bmod p$
k_E
compute $k_M = k_E^d \bmod p$
compute $k_M = \beta^i \bmod p$
encrypt message $x \in Z_p^*$:
$y = x \cdot k_M \bmod p$
y
decrypt $x = y \cdot k_M^{-1} \bmod p$

This looks very similar to the DHKE! The actual Elgamal protocol re-orders the computations which helps to save one communication (cf. next slide)

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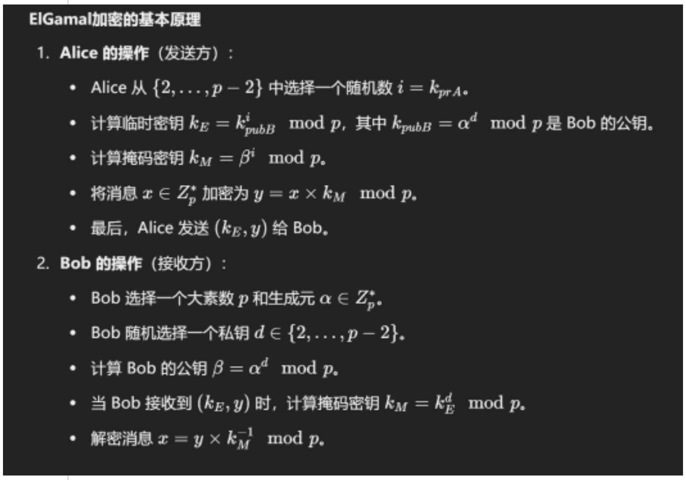
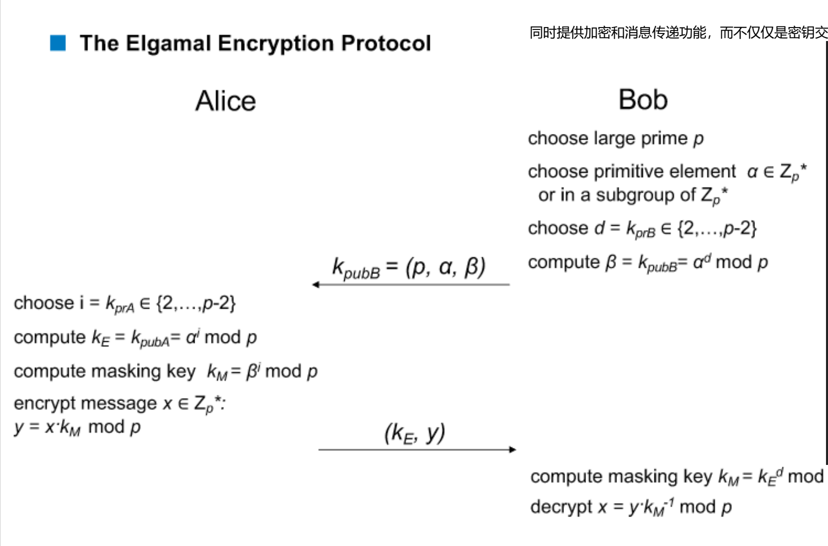


ElGamal加密的基本原理

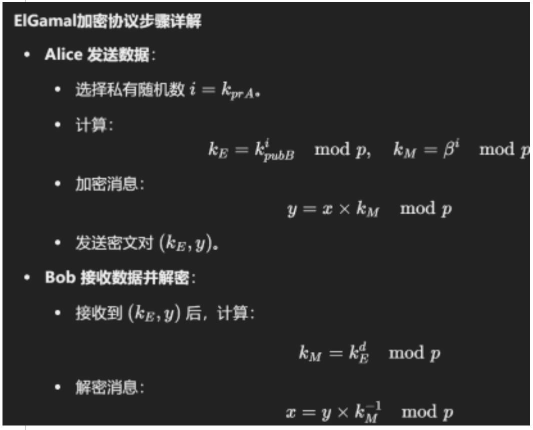
1. **Alice 的操作**（发送方）：
   - Alice 从 $\{2,\dots,p-2\}$ 中选择一个随机数 $i = k_{prA}$。
   - 计算临时密钥 $k_E = k_{pubB}^i \mod p$，其中 $k_{pubB} = \alpha^d \mod p$ 是 Bob 的公钥。
   - 计算掩码密钥 $k_M = \beta^i \mod p$。
   - 将消息 $x \in Z_p^*$ 加密为 $y = x \times k_M \mod p$。
   - 最后，Alice 发送 (k_E, y) 给 Bob。
2. **Bob 的操作**（接收方）：
   - Bob 选择一个大素数 p 和生成元 $\alpha \in Z_p^*$。
   - Bob 随机选择一个私钥 $d \in \{2,\dots,p-2\}$。
   - 计算 Bob 的公钥 $\beta = \alpha^d \mod p$。
   - 当 Bob 接收到 (k_E, y) 时，计算掩码密钥 $k_M = k_E^d \mod p$。
   - 解密消息 $x = y \times k_M^{-1} \mod p$。



The Elgamal Encryption Protocol

同时提供加密和消息传递功能，而不仅仅是密钥交换

Alice
Bob
choose large prime p
choose primitive element $\alpha \in Z_p^*$
or in a subgroup of Z_p^*
choose $d = k_{prB} \in \{2,\dots,p\text{-}2\}$
compute $\beta = k_{pubB} = \alpha^d \bmod p$
$k_{pubB} = (p, \alpha, \beta)$
choose i = $k_{prA} \in \{2,\dots,p\text{-}2\}$
compute $k_E = k_{pubA} = \alpha^i \bmod p$
compute masking key $k_M = \beta^i \bmod p$
encrypt message $x \in Z_p^*$:
$y = x \cdot k_M \bmod p$
(k_E, y)
compute masking key $k_M = k_E^d \bmod p$
decrypt $x = y \cdot k_M^{-1} \bmod p$



ElGamal加密协议步骤详解

- **Alice 发送数据**:
  - 选择私有随机数 $i = k_{prA}$。
  - 计算:

$$k_E = k_{pubB}^i \mod p, \quad k_M = \beta^i \mod p$$

  - 加密消息:

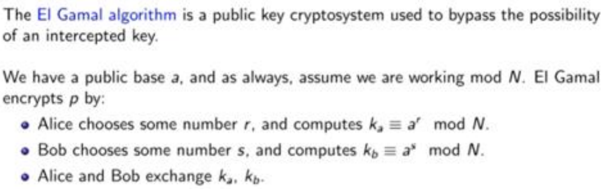
$$y = x \times k_M \mod p$$

  - 发送密文对 (k_E, y)。
- **Bob 接收数据并解密**:
  - 接收到 (k_E, y) 后，计算:

$$k_M = k_E^d \mod p$$

  - 解密消息:

$$x = y \times k_M^{-1} \mod p$$



The El Gamal algorithm is a public key cryptosystem used to bypass the possibility of an intercepted key.

We have a public base a, and as always, assume we are working mod N. El Gamal encrypts p by:

- Alice chooses some number r, and computes $k_a \equiv a^r \mod N$.
- Bob chooses some number s, and computes $k_b \equiv a^s \mod N$.
- Alice and Bob exchange k_a, k_b.

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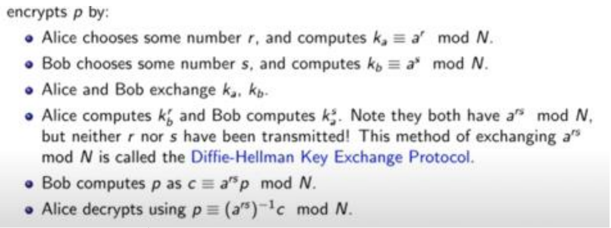
Computational Aspects



encrypts p by:

- Alice chooses some number r, and computes $k_a \equiv a^r \mod N$.
- Bob chooses some number s, and computes $k_b \equiv a^s \mod N$.
- Alice and Bob exchange k_a, k_b.
- Alice computes k_b^r and Bob computes k_a^s. Note they both have $a^{rs} \mod N$, but neither r nor s have been transmitted! This method of exchanging $a^{rs} \mod N$ is called the Diffie-Hellman Key Exchange Protocol.
- Bob computes p as $c \equiv a^{rs}p \mod N$.
- Alice decrypts using $p \equiv (a^{rs})^{-1}c \mod N$.

Computational Aspects

- Key Generation
  - Generation of prime p
  - p has to of size of at least 1024 bits
  - cf. Section 7.6 in *Understanding Cryptography* for prime-finding algorithms
- Encryption

  两个模幂运算 和 **一次模乘运算**
  - Requires two modular exponentiations and a modular multiplication
  - All operands have a bitlength of $\log_2 p$
  - Efficient execution requires methods such as the square-and-multiply algorithm (cf. Chapter 7)
- Decryption
  - Requires one modular exponentiation and one modular inversion
  - As shown *in Understanding Cryptography*, the inversion can be computed from the ephemeral key

Security

- Passive attacks
  - Attacker eavesdrops $p, \alpha, \beta = \alpha^d$, $k_E = \alpha^i$, $y = x \cdot \beta^i$ and wants to recover x
  - Problem relies on the DLP
- Active attacks
  - If the public keys are not authentic, an attacker could send an incorrect public key (cf. Chapter 13)
  - An Attack is also possible if the secret exponent i is being used more than once (cf. *Understanding Cryptography* for more details on the attack)

Lessons Learned

- The Diffie–Hellman protocol is a widely used method for key exchange. It is based on cyclic groups.
- The discrete logarithm problem is one of the most important one-way functions in modern asymmetric cryptography. Many public-key algorithms are based on it.
- For the Diffie–Hellman protocol in Z_p^*, *the prime p should be at least 1024 bits* long. This provides a security roughly equivalent to an 80-bit symmetric cipher.
- For a better long-term security, a prime of length 2048 bits should be chosen.
- The Elgamal scheme is an extension of the DHKE where the derived session key is used as a multiplicative masked to encrypt a message.
- Elgamal is a probabilistic encryption scheme, i.e., encrypting two identical messages does not yield two identical ciphertexts.

Diffie-Hellman 协议 (DH Protocol)

- **用途**：DH 协议是一种广泛使用的密钥交换方法，基于 **循环群**（cyclic groups）。
- **基础**：其安全性依赖于 **离散对数问题**（DLP），这是现代非对称加密中最重要的单向函数之一，许多公钥算法都基于这一问题。

- **密钥长度建议**：
  - 为了达到与 80 **位对称密码** 相当的安全性，使用 Z_pˆ*群时，素数 p 的长度至少应为 1024 **位。**
  - 如果需要更长的安全性（长期使用），建议选择 2048 **位** 的素数。

2. ElGamal 加密方案

- **扩展自 DHKE**：ElGamal 是 Diffie-Hellman **密钥交换**（DHKE）的扩展。生成的会话密钥被用作乘法掩码来加密消息。
- **随机化**：ElGamal 是一种 **概率加密方案**，即对两个相同的消息加密将产生不同的密文。这种特性防止了攻击者通过重复加密结果来推断出原始消息。

3. 总结

- **安全性**：离散对数问题是确保 Diffie-Hellman 和 ElGamal 等协议安全的核心。通过增加素数的长度，可以有效地提高系统的安全性。
- **概率加密**：ElGamal 的随机化特点使其在需要对抗重放攻击和密文分析攻击时特别有用。