Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

Work

1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

   (b)
   $$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } \mathsf{M}_{2\times 2}(\mathbb{R})$$
   $$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (1)$$
   $$a_1 = -a_2 \quad (2)$$
   $$-a_1 = a_2 \quad (3)$$
   $$\implies a_1 = a_2 = 0 \quad (4)$$

   Only the trivial solution exists. The set is linearly independent.

   (d)
   $$\left\{ x^3 -, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6 \right\} \text{ in } \mathsf{P}_3\left(\mathbb{R}\right)$$
   $$a_1 \left(x^3 - x\right) + a_1 \left(2x^2 + 4\right) + a_3 \left(-2x^3 + 3x^2 + 2x + 6\right) = 09 \quad (5)$$
   $$a_1 = 2a_3 \quad (6)$$
   $$2a_2 = -3a_3 \quad (7)$$
   $$4a_2 = -6a_3 \quad (8)$$
   $$a_1 = t \quad (9)$$
   $$a_2 = -\left({}^3/_4\right) t \quad (10)$$
   $$a_3 = -\left({}^1/_2\right) t \quad (11)$$

   The set is linearly dependent.

(f)

$$\{(1,-1,2),(1,-2,1),(1,1,4)\} \text{ in } \mathsf{R}^3$$

$$a\begin{bmatrix}1\\-1\\2\end{bmatrix}+b\begin{bmatrix}2\\0\\1\end{bmatrix}+c\begin{bmatrix}-1\\2\\-1\end{bmatrix}=0 \tag{12}$$

$$1+2b-c=0 \tag{13}$$

$$-a+2c=0 \tag{14}$$

$$2a+b-c=0 \tag{15}$$

$$\begin{pmatrix}1&2&-1&0\\-1&0&2&0\\2&1&-1&0\end{pmatrix}\begin{array}{lll} 1 & -2 & \\ + & & \mid\cdot\frac{1}{2} \quad 3 \\ & + & + \mid\cdot\frac{2}{5}\end{array} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Only the trivial solution exists. This set is linearly independent.

(g)

$$\left\{\begin{pmatrix}1&0\\-2&0\end{pmatrix},\begin{pmatrix}0&-1\\1&1\end{pmatrix},\begin{pmatrix}-1&2\\1&0\end{pmatrix},\begin{pmatrix}2&1\\-4&4\end{pmatrix}\right\} \text{ in } \mathsf{M}_{2\times2}(\mathbb{R})$$

$$a\begin{pmatrix}1&0\\-2&1\end{pmatrix}+b\begin{pmatrix}0&-1\\1&1\end{pmatrix}+c\begin{pmatrix}-1&2\\1&0\end{pmatrix}+d\begin{pmatrix}2&1\\-4&4\end{pmatrix}=0 \tag{16}$$

$$\begin{pmatrix}1&0&-1&2&0\\-2&1&1&-4&0\\0&-1&2&1&0\\1&1&0&4&0\end{pmatrix}\begin{array}{lllll} 2 & -1 & & & \\ + & & 1 & -1 & \\ & & + & & -2 \\ & + & & + & + \end{array} \to \begin{pmatrix}1&0&-1&2&0\\0&1&-1&0&0\\0&0&1&1&0\\0&0&0&0&0\end{pmatrix}$$

$$a=-3t \tag{17}$$

$$b=-t \tag{18}$$

$$c=-t \tag{19}$$

$$d=t \tag{20}$$

There exists a non-trivial solution, therefore the set is linearly dependent.

11. Let $S=\{u_1,u_2,\ldots,u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in $\operatorname{span}(S)$? Justify your answer.

$$\mathbb{Z}_2=\{0,1\} \tag{21}$$

$$c_1u_1+c_2u_2+\cdots+c_nu_n\neq0,\ \forall c_1,\ldots,c_n\in\{0,1\} \tag{22}$$

unless all $c_i=0$.

$$\implies \operatorname{card}(\operatorname{span}(S))=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n} \tag{23}$$

$$=\sum_{i=1}^{n}\binom{n}{i} \tag{24}$$

$$=2^n \tag{25}$$

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for some $k\,(1 \leq k < n)$.

Forward Direction:
Claim: S is linearly dependent
Suppose $u_1 = 0$
Assume S is linearly independent.
Take the linear combination:

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0 \text{ such that } c_1 \neq 0, c_2, \ldots, c_n = 0 \tag{26}$$

$$\implies c_1u_1 = 0 \text{ ↯ Contradiction!} \tag{27}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \ldots, u_k\})$
Assume S is linearly independent.

$$u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k \tag{28}$$

Take the linearly combination:

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k + c_{k+1} + \cdots + c_nu_n = 0 \tag{29}$$

Choose $c_i = (-a_i)$ for $i(1 \leq i \leq k)$ and $c_{k+1} = 1$

$$\implies u_{k+1} = 0 \text{ ↯ Contradiction!} \tag{30}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Reverse Direction:
Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \ldots, u_k\})$
Claim: S is linearly dependent

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k && (31)\\ -u_{k+1} &= (-1)\left(a_1u_1 + a_2u_2 + \cdots + a_ku_k\right) && (32)\\ &= (-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k && (33) \end{aligned}$$

Take the linear combination of all $u_1, \ldots, u_n$

$$((-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k) + (1u_{k+1} + 0u_{k+2} + \cdots + 0u_n) = 0 \tag{34}$$

1.6

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

    (a) Prove that there is subset of S that is a basis for V. (Be careful not to assume that S is finite.)

    (b) Prove that S contains at least n vectors.

24. Let $f(x)$ be a polynomial of degree n in $\mathsf{P}_n(\mathbb{R})$. Prove that for any $g(x) \in \mathsf{P}_n(\mathbb{R})$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

where $f^{(n)}(x)$ denotes nth derivative of $f(x)$.

Given vector space $\mathsf{P}_n(\mathbb{R})$ where $\dim(\mathsf{P}_n(\mathbb{R})) = n + 1$
Suppose S is a subset of $\mathsf{P}_n(\mathbb{R})$ and $S = \{f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\}$

$$\implies \text{card}(S) = n + 1 \tag{35}$$

If S is linearly independent, then $\text{span}(S) = \mathsf{P}(\mathbb{R})$ where $g(x) \in \mathsf{P}_n(\mathbb{R})$
Claim: S is linearly independent

$$d_0 f(x) + d_1 f'(x) + d_2 f''(x) + d_3 f'''(x) + \cdots + d_n f^{(n)}(x) = 0 \tag{36}$$

$$d_0 = 0 \because x^n \text{ term only exists in } f(x) \tag{37}$$
$$d_1 = 0 \because x^{n-1} \text{ term only exists in } f'(x) \tag{38}$$
$$d_2 = 0 \because x^{n-2} \text{ term only exists in } f''(x) \tag{39}$$
$$\vdots \tag{40}$$
$$d_n = 0 \because x^0 \text{ term only exists in } f^{(n)}(x) \tag{41}$$

31. Let W_1 and W_2 are subspaces of V, and find the dimensions of $\mathsf{W}_1, \mathsf{W}_2, \mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

2.1

6. $\mathsf{T}\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \text{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ij}$$

12. Is there a linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1, 0, 3) = (1, 1)$ and $\mathsf{T}(-2, 0, -6) = (2, 1)$?

14. Let V and W be vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear.

   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

   (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

   (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

   (a) **Forward Direction:**
   Suppose $S \subseteq \mathsf{V}$ such that S is linearly independent and T is one-to-one.
   Let $\mathsf{T}(S) = \{\mathsf{T}(x)\colon x \in S\}$
   Claim: $\mathsf{T}(S)$ is linearly independent.
   Suppose $x \in \text{span}(\mathsf{T}(S))$ such that $x = c_1v_1 + c_2v_2 + \cdots + c_nu_n = 0$ for $c_i \in F$ and $v_i \in \mathsf{T}(S)$

   $$\mathsf{T} \text{ is one-to-one} \implies v_i = \mathsf{T}(v_i) \text{ for some } v_i \in S$$

   # MAKES NO FUCKING SENSE

   Reverse Direction:

2.2

2. Let β and γ be the standard ordered bases for R^n and R^m respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{R}]_\beta^\gamma$.

   (b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

   (c) $\mathsf{T}\colon \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

   (g) $\mathsf{T}\colon \mathsf{R}^n \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.
   Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$
   Suppose $x, y \in \mathsf{V}$

$$x = \sum_{i=1}^{n} a_i v_i \qquad\qquad y = \sum_{i=1}^{n} b_i v_i \tag{42}$$

$$x + y = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n} (a_i v_i + b_i v_i) = \sum_{i=1}^{n} (a_i + b_i)\, v_i \tag{43}$$

$$\mathsf{T}(x+y) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \mathsf{T}(x) + \mathsf{T}(y) \tag{44}$$

   Suppose $c \in F$ and $x \in \mathsf{V}$

$$x = \sum_{i=1}^{n} a_i v_i \tag{45}$$

$$cx = c \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} c a_i v_i \tag{46}$$

$$\mathsf{T}(cx) = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = c\mathsf{T}(x) \tag{47}$$

11. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the exercises of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

   where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.