Assignment

Section 4.4: 1, 5, 6; Section 5.1: 3(bc), 4(ceh), 7, 12, 14, 15, 19, 22

Work

4.4

1. Label the statements as true or false.

(a)	True	(g)	True
(b)	True	(h)	False
(c)	True	(i)	True
(d)	False	(j)	True
(e)	False	(k)	True
(f)	True		

5. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix}$$

where A is a square matrix. Prove that $\det(M) = \det(A)$

Suppose $A \in \mathsf{M}_{k\times k}(F)$

Perform Type 3 operations such that the partition $(A\ B)$ becomes an upper triangular matrix.

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & I \end{pmatrix} \tag{1}$$

M' is an upper triangular matrix so the determinant of M is the product of its diagonal terms.

$$\Rightarrow \det(A) = \prod_{i=1}^{n} M'_{ii} \tag{2}$$

$$M'_{ii} = 1 \quad \text{if } (k+1 \leq i \leq n) \tag{3}$$

$$\Rightarrow \prod_{i=1}^{n} M'_{ii} = \prod_{i=1}^{k} M'_{ii} \tag{4}$$

The first k diagonal terms of M' are the diagonal terms of A. It follows that

$$\prod_{i=1}^{k} M'_{ii} = \prod_{i=1}^{k} A'_{ii} \tag{5}$$

Because A' was obtained from A using Type 3 operations, and M' was obtained from M using Type 3 operations

$$\det(A') = \det(A) \tag{6}$$
$$\det(M') = \det(M) \tag{7}$$
$$\therefore \det(M) = \det(M') = \det(A') = \det(A) \tag{8}$$

6. Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

M can be reduced using strictly Type 3 row operations such that the partitions $(A \; B)$ become an upper triangular matrix $(A' \; B')$. M can also be reduced using strictly Type 3 row operations such that the partition $(O \; C)$ becomes the matrix $(O \; C')$ where C' is an upper triangular matrix in $\mathsf{M}_{(n-k)\times(n-k)}(F)$ where $A \in \mathsf{M}_{n\times n}(F)$.

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & C' \end{pmatrix} \tag{9}$$

M' is upper triangular, so the determinant of M' is a the product of the diagonal terms.

$$\det(M') = \prod_{i=1}^{n} M'_{ii} \tag{10}$$
$$= \left(\prod_{i=1}^{k} M'_{ii}\right)\left(\prod_{i=k+1}^{n} M'_{ii}\right) \tag{11}$$

$$M'_{ii} = A'_{ii} \quad \forall i \; (1 \le i \le k) \tag{12}$$

$$\Rightarrow \prod_{i=1}^{k} M'_{ii} = \prod_{i=1}^{k} A'_{ii} \tag{13}$$
$$= \det(A') \tag{14}$$

$$M'_{k+1,k+1} = C'_{ii} \quad \forall i \; (1 \le i \le n-k) \tag{15}$$

$$\Rightarrow \prod_{i=k+1}^{k} M'_{ii} = \prod_{i=1} n - kC'_{ii} \tag{16}$$
$$= \det(C') \tag{17}$$

Matrices C', A' and M' were obtained respectively from the matrices C, A and M strictly using Type 3 row operations. It follows that

$$\det(M') = \det(M) \tag{18}$$
$$\det(A') = \det(A) \tag{19}$$
$$\det(C') = \det(C) \tag{20}$$

$$\therefore \det(M) = \det(A) \cdot \det(C) \tag{21}$$

5.1

3. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,

   (i) Determine all the eigenvalues of A.

   (ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

   (iii) If possible, find a basis for F^n consisting of eigenvectors of A.

   (iv) If succesful in finding such a basis, determine and invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   (b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$

   (i)

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} = 0 \tag{22}$$

$$\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 0 & 4-2\lambda & 3-\lambda \end{pmatrix} \tag{23}$$

$$\begin{aligned} \det(A - \lambda I) &= (-\lambda)(\lambda-1)(\lambda-3) + 3(4-2\lambda) \\ &\quad - \lambda(4-2\lambda) - 2(3-\lambda) \end{aligned} \tag{24}$$
$$= \lambda^3 + 6\lambda^2 - 11\lambda + 6 \tag{25}$$
$$= (\lambda-3)(\lambda-2)(-\lambda_1) = 0 \tag{26}$$

$$\Rightarrow \lambda = \{3, 2, 1\} \tag{27}$$

(ii) • For $\lambda = 3$

$$Av = 3v \tag{28}$$

$$(A - 3I)v = 0 \tag{29}$$

$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ -0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right) v = 0 \tag{30}$$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} v = 0 \tag{31}$$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

$$x_1 = -t \tag{33}$$

$$x_2 = 0 \tag{34}$$

$$x_3 = t \tag{35}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{36}$$

• For $\lambda = 2$

$$Av = 2v \tag{37}$$

$$(A - 2I)v = 0 \tag{38}$$

$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right) v = 0 \tag{39}$$

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} v = 0 \tag{40}$$

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

$$x_1 = -t \tag{42}$$

$$x_2 = t \tag{43}$$

$$x_3 = 0 \tag{44}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{45}$$

- For $\lambda = 1$

$$Av = v \tag{46}$$

$$(A - I)V = 0 \tag{47}$$

$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & \\ -1 & & \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \tag{48}$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} v = 0 \tag{49}$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{50}$$

$$x_1 = -t \tag{51}$$

$$x_2 = -t \tag{52}$$

$$x_3 = t \tag{53}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{54}$$

(iii)

$$\beta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \tag{55}$$

(iv)

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \tag{56}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 & -1 \end{array}\right) \tag{57}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \tag{58}$$

$$D = QAQ^{-1} \tag{59}$$
$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \tag{60}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{61}$$

(c) $\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbb{C}$

(i)

$$\det(A - \lambda I) = \det \begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = 0 \tag{62}$$
$$= -(i-\lambda)(i+\lambda) - 2 = 0 \tag{63}$$
$$= i^2 - \lambda^2 + 2 = 0 \tag{64}$$

$$\lambda^2 = 1 \tag{65}$$
$$\lambda = \pm 1 \tag{66}$$

(ii) • For $\lambda = 1$

$$Av = \lambda v \tag{67}$$
$$(A - I)v = 0 \tag{68}$$
$$\begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix} v = 0 \tag{69}$$
$$\begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i-1 & 1 \\ 0 & 0 \end{pmatrix} \tag{70}$$

$$x_1 = -\frac{t}{i-1} \tag{71}$$
$$x_2 = t \tag{72}$$

$$v = \left\{ t \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{73}$$

• For $\lambda = -1$

$$Av = \lambda v \tag{74}$$
$$(A + I)v = 0 \tag{75}$$
$$\begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix} v = 0 \tag{76}$$
$$\begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i+1 & 1 \\ 0 & 0 \end{pmatrix} \tag{77}$$

$$x_1 = -\frac{t}{i+1} \tag{78}$$
$$x_2 = t \tag{79}$$

$$v = \left\{ t \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{80}$$

(iii)

$$\beta = \left\{ \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} \right\} \tag{81}$$

(iv)

$$Q = \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \tag{82}$$

$$\left(\begin{array}{cc|cc} \frac{i+1}{2} & \frac{i-1}{2} & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & -1 & \frac{1+i}{2} \end{array}\right) \tag{83}$$
$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \tag{84}$$

$$D = Q^{-1}AQ \tag{85}$$
$$= \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \tag{86}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{87}$$

4. For each linear operator T on V, find the egienvalues of T and an ordered bases β of V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

   (c) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{88}$$
$$[\mathsf{T}]_\alpha = \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & 6 & 11 \end{pmatrix} \tag{89}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I_3\right) = \det\begin{pmatrix} -4-\lambda & 3 & -6 \\ 6 & -7-\lambda & 12 \\ 6 & -6 & 11-\lambda \end{pmatrix} \tag{90}$$

$$= \det\begin{pmatrix} -4-\lambda & 4 & -6 \\ 6 & -7-\lambda & 12 \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \tag{91}$$

$$= (\lambda+4)(\lambda+7)(\lambda+1) \tag{92}$$

$$- 2(\lambda+1)(\lambda+1)(\lambda+4) - 18(\lambda+1) \tag{93}$$

$$= -\lambda^3 + 3\lambda + 2 = 0 \tag{94}$$

$$= -(\lambda+1)(\lambda-2)(\lambda+1) = 0 \tag{95}$$

$$\Rightarrow \lambda = \{2, -1\} \tag{96}$$

- For $\lambda = 2$

$$[\mathsf{T}]_\alpha v = 0 \tag{97}$$

$$([\mathsf{T}]_\alpha - 2I)v = 0 \tag{98}$$

$$\left(\begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v = 0 \tag{99}$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} v = 0 \tag{100}$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & {}^1/_2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{101}$$

$$x_1 = -\frac{1}{2}t \tag{102}$$

$$x_2 = t \tag{103}$$

$$x_3 = t \tag{104}$$

$$v = \left\{ t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{105}$$

- For $\lambda = -1$

$$[\mathsf{T}_\alpha]v = -v \tag{106}$$

$$[\mathsf{T}]_\alpha]v = 0 \tag{107}$$

$$\left(\begin{pmatrix}-4 & 3 & -6\\ 6 & -7 & 12\\ 6 & -6 & 11\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) v = 0 \tag{108}$$

$$\begin{pmatrix}-3 & 3 & -6\\ 6 & -6 & 12\\ 6 & -6 & 12\end{pmatrix} v = 0 \tag{109}$$

$$\begin{pmatrix}-3 & 3 & -6\\ 6 & -6 & 12\\ 6 & -6 & 12\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & -1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{110}$$

$$x_1 = s - 2t \tag{111}$$

$$x_2 = s \tag{112}$$

$$x_3 = t \tag{113}$$

$$v = \left\{ s\begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\end{pmatrix} : s, t \in \mathbb{R}^3 \right\} \tag{114}$$

$$\beta = \left\{ \begin{pmatrix}-\frac{1}{2}\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\} \tag{115}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{116}$$

$$Q = \begin{pmatrix}{}^{-1}/_{2} & 1 & -2\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \tag{117}$$

$$\left(\begin{array}{ccc|ccc}{}^{-1}/_{2} & 1 & -2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & {}^{2}/_{5} & {}^{-2}/_{5} & {}^{4}/_{5}\\ 0 & 1 & 0 & {}^{2}/_{5} & {}^{3}/_{5} & {}^{4}/_{5}\\ 0 & 0 & 1 & {}^{-2}/_{5} & {}^{2}/_{5} & {}^{1}/_{5}\end{array}\right) \tag{118}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix}{}^{2}/_{5} & {}^{-2}/_{5} & {}^{4}/_{5}\\ {}^{2}/_{5} & {}^{3}/_{5} & {}^{4}/_{5}\\ {}^{-2}/_{5} & {}^{2}/_{5} & {}^{1}/_{5}\end{pmatrix} \tag{119}$$

$$[\mathsf{T}]_\beta = \begin{pmatrix}{}^{2}/_{5} & {}^{-2}/_{5} & {}^{4}/_{5}\\ {}^{2}/_{5} & {}^{3}/_{5} & {}^{4}/_{5}\\ {}^{-2}/_{5} & {}^{2}/_{5} & {}^{1}/_{5}\end{pmatrix} \begin{pmatrix}-4 & 3 & -6\\ 6 & -7 & 12\\ 6 & -6 & 11\end{pmatrix} \begin{pmatrix}{}^{-1}/_{2} & 1 & -2\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \tag{120}$$

$$= \begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{121}$$

(e) $\mathsf{V} = \mathsf{P}_3(\mathbb{R})$ and $\mathsf{T}(f(x)) = xf'(x) + f(2)x + f(3)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{122}$$

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{123}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I\right) = 0 \tag{124}$$

$$= \det \begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} \tag{125}$$

$$= (1-\lambda)(3-\lambda)(2-\lambda) - 6(2-\lambda) \tag{126}$$

$$= \lambda(2-\lambda)(\lambda-4) \tag{127}$$

$$\lambda = \{0, 2, 4\} \tag{128}$$

- For $\lambda = 0$

$$Av = 0v \tag{129}$$

$$Av = 0 \tag{130}$$

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{131}$$

$$x_1 = -3t \tag{132}$$

$$x_2 = t \tag{133}$$

$$x_3 = 0 \tag{134}$$

$$v = \left\{ t \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{135}$$

- For $\lambda = 2$

$$Av = 2v \tag{136}$$

$$(A - 2I) = v \tag{137}$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} v = 0 \tag{138}$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 3 & 4 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{pmatrix} \tag{139}$$

$$x_1 = -\frac{39}{4}t + 9t \tag{140}$$
$$x_2 = -\frac{12}{4}t \tag{141}$$
$$x_3 = t \tag{142}$$

$$v = \left\{ t \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{143}$$

- For $\lambda = 4$

$$Av = 4v \tag{144}$$
$$(A - 4I) = v \tag{145}$$
$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} v = 0 \tag{146}$$
$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 21 \\ 1 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{147}$$

$$x_1 = t \tag{148}$$
$$x_2 = t \tag{149}$$
$$x_3 = 0 \tag{150}$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{151}$$

$$\beta = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -12 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \tag{152}$$
$$\Rightarrow \left\{ (-3 + x), (-3 - 13x + 4x^2), (1 + x) \right\} \tag{153}$$

$$Q = \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \tag{154}$$
$$\left(\begin{array}{ccc|ccc} -3 & -3 & 1 & 1 & 0 & 0 \\ 1 & -13 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & {}^{-1}/_{4} & {}^{1}/_{4} & {}^{5}/_{8} \\ 0 & 1 & 0 & 0 & 0 & {}^{1}/_{4} \\ 0 & 0 & 1 & {}^{1}/_{4} & {}^{3}/_{4} & {}^{21}/_{8} \end{array}\right) \tag{155}$$
$$Q^{-1} = \begin{pmatrix} {}^{-1}/_{4} & {}^{1}/_{4} & {}^{5}/_{8} \\ 0 & 0 & {}^{1}/_{4} \\ {}^{1}/_{4} & {}^{3}/_{4} & {}^{21}/_{8} \end{pmatrix} \tag{156}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{157}$$

$$= \begin{pmatrix} {}^{-1}/_4 & {}^{1}/_4 & {}^{5}/_8 \\ 0 & 0 & {}^{1}/_4 \\ {}^{1}/_4 & {}^{3}/_4 & {}^{21}/_8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \tag{158}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{159}$$

(h) $\mathsf{V} = \mathsf{M}_{n\times n}(\mathbb{R})$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{160}$$

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{161}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I\right) = \det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \tag{162}$$

$$\begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} \tag{163}$$

$$\det \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} = 0 \tag{164}$$

$$\Rightarrow (1-\lambda)^2(1-\lambda^2) = 0 \tag{165}$$

$$\Rightarrow \lambda = \pm 1 \tag{166}$$

- For $\lambda = 1$

$$[\mathsf{T}]_\alpha v = v \tag{167}$$

$$([[T]_\alpha - I)v = 0 \tag{168}$$

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} v = 0 \tag{169}$$

$$x_1 = t \tag{170}$$
$$x_2 = k \tag{171}$$
$$x_3 = s \tag{172}$$
$$x_4 = t \tag{173}$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t, k, s \in \mathbb{R} \right\} \tag{174}$$

- For $\lambda = -1$

$$[\mathsf{T}]_\alpha v = v \tag{175}$$
$$([\mathsf{T}]_\alpha + I)v = 0 \tag{176}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} v = 0 \tag{177}$$

$$x_1 = -t \tag{178}$$
$$x_2 = 0 \tag{179}$$
$$x_3 = 0 \tag{180}$$
$$x_4 = t \tag{181}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{182}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{183}$$
$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{184}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{185}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & {}^1/_2 & 0 & 0 & {}^1/_2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & {}^{-1}/_2 & 0 & 0 & {}^1/_2 \end{array}\right) \tag{186}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} {}^1/_2 & 0 & 0 & {}^1/_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ {}^{-1}/_2 & 0 & 0 & {}^1/_2 \end{pmatrix} \tag{187}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{188}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{189}$$

7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(\mathsf{T})$, as follows: Choose any ordered basis β for V, and define $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta)$.

   (a) Prove that he preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.

   (b) Prove that T is invertible if and only if $\det \mathsf{T} \neq 0$.

   (c) Prove that if T is invertible, then $\det(\mathsf{T}^{-12}) = [\det(\mathsf{T})]^{-1}$.

   (d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T})\cdot\det(\mathsf{U})$.

   (e) Prove that $\det(\mathsf{T} - \lambda \mathsf{I_V}) = \det[\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

   (a) Suppose Q is the change of coordinates matrix from γ to β.

$$Q = [\mathsf{I_V}]_\gamma^\beta \Rightarrow Q^{-1} = [\mathsf{I_V}]_\beta^\gamma \tag{190}$$

$$[\mathsf{I_V}]_\gamma^\beta [\mathsf{T}]_\gamma [\mathsf{I_V}]_\beta^\gamma = [\mathsf{T}]_\beta \tag{191}$$

$$\Rightarrow \det([\mathsf{I_V}]_\gamma^\beta [\mathsf{T}]_\gamma [\mathsf{I_V}]_\beta^\gamma) = \det[\mathsf{T}]_\beta \tag{192}$$

$$\Rightarrow \det[\mathsf{I_V}]_\gamma^\beta \det[\mathsf{T}]_\gamma \det[\mathsf{I_V}]_\beta^\gamma \tag{193}$$

$$[\mathsf{I_V}]_\gamma^\beta = ([\mathsf{I_V}]_\beta^\gamma)^{-1} \tag{194}$$

$$\det[\mathsf{I_V}]_\gamma^\beta = \det([\mathsf{I_V}]_\beta^\gamma)^{-1} \tag{195}$$

$$\Rightarrow \det\, [\mathsf{I_V}]_\gamma^\beta \det\, [\mathsf{T}]_\gamma \det\, [\mathsf{I_V}]_\beta^\gamma = \det\, ([\mathsf{I_V}]_\beta^\gamma)^{-1} \det\, [\mathsf{T}]_\gamma \det\, [\mathsf{I_V}]_\beta^\gamma \tag{196}$$
$$= \det[\mathsf{T}]_\gamma = \det[\mathsf{T}]_\beta \tag{197}$$

(b) $(\Rightarrow)$

Suppose T is invertible

Suppose β is an ordered basis of V.

$$[\mathsf{I_V}]_\beta = [\mathsf{T}\cdot\mathsf{T}^{-1}]_\beta = [\mathsf{T}]_\beta[\mathsf{T}^{-1}]_\beta \tag{198}$$
$$\Rightarrow \det\, [\mathsf{I_V}]_\beta = \det\, [\mathsf{T}]_\beta \det\, [\mathsf{T}^{-1}]_\beta \tag{199}$$
$$\det\, [\mathsf{I_V}]_\beta = 1 \tag{200}$$
$$\Rightarrow \det\, [\mathsf{T}]_\beta \det\, [\mathsf{T}^{-1}]_\beta = 1 \tag{201}$$
$$\Rightarrow \det\, [\mathsf{T}]_\beta \neq 0 \tag{202}$$
$$\Rightarrow \det \mathsf{T} \neq 0 \tag{203}$$

$(\Leftarrow)$

Suppose $\det \mathsf{T} \neq 0$

$$\Rightarrow \det[\mathsf{T}]_\beta \neq 0 \quad \text{for some ordered basis } \beta \text{ of } \mathsf{V} \tag{204}$$
$$\det\, [\mathsf{T}]_\beta \neq 0 \tag{205}$$
$$\Rightarrow \mathsf{T} \text{ is invertible, by corollary to Th. 2.18} \tag{206}$$

(c) Suppose T is invertible and β is some ordered basis of V.

$$\det \mathsf{I_V} = \det \mathsf{T}\cdot\mathsf{T}^{-1} \tag{207}$$
$$= \det\, [\mathsf{T}\cdot\mathsf{T}^{-1}]_\beta \tag{208}$$
$$= \det\, [\mathsf{T}]_\beta[\mathsf{T}^{-1}]_\beta \tag{209}$$
$$= \det\, [\mathsf{T}]_\beta \det\, [\mathsf{T}^{-1}]_\beta \tag{210}$$
$$= \det \mathsf{T} \det \mathsf{T}^{-1} \tag{211}$$

$$\det \mathsf{I_V} = \det\, [\mathsf{I_V}]_\beta = \det I_n = 1 \tag{212}$$
$$\Rightarrow \det \mathsf{T} \det \mathsf{T}^{-1} = 1 \tag{213}$$
$$\Rightarrow \det \mathsf{T}^{-1} = (\det \mathsf{T})^{-1} \tag{214}$$

(d) Suppose β is an ordered basis of V.

$$\det \mathsf{TU} = \det\, [\mathsf{TU}]_\beta \tag{215}$$
$$= \det\, [\mathsf{T}]_\beta[\mathsf{U}]_\beta \tag{216}$$
$$= \det\, [\mathsf{T}]_\beta \det\, [\mathsf{U}]_\beta \tag{217}$$
$$= \det \mathsf{T} \det \mathsf{U} \tag{218}$$

(e)

$$\det([\mathsf{T}]_\beta - \lambda I) = \det([\mathsf{T}]_\beta - \lambda[\mathsf{I}_\mathsf{V}]_\beta) \tag{219}$$
$$= \det([\mathsf{T}]_\beta - [\lambda\mathsf{I}_\mathsf{V}]_\beta) \tag{220}$$
$$= \det[\mathsf{T}_\beta - \lambda\mathsf{I}_\mathsf{V}]_\beta \tag{221}$$
$$= \det(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V}) \tag{222}$$

12. (a) Prove that similar matrices have the same characteristic polynomial.
    (b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

    (a) Suppose $A, B \in \mathsf{M}_{n\times n}(F)$ such that $A = Q^{-1}BQ$ for some invertible $Q \in \mathsf{M}_{n\times n}(F)$
    Claim: $\det(A = \lambda I_n) = \det(B - \lambda I_n)$

$$\det(A - \lambda I_n) = \det(Q^{-1}BQ - \lambda I_n) \tag{223}$$
$$= \det(Q^{-1}BQ - \lambda Q^{-1}Q) \tag{224}$$
$$= \det Q^{-1} \det(BQ - \lambda Q) \tag{225}$$
$$= \det Q^{-1} \det(B - \lambda) \det Q \tag{226}$$
$$= (\det Q)^{-1} \det Q \det(B - \lambda I_n) \tag{227}$$
$$= \det(B - \lambda I_n) \tag{228}$$

    (b) Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ such that V is finitely dimensioned and β and γ are ordered bases of V
    Let $A = [\mathsf{T}]_\beta$, $B = [\mathsf{T}]_\gamma$
    Let Q be the change of coordinates matrix from γ to β.

$$\Rightarrow Q = [\mathsf{I}_\mathsf{V}]_\gamma^\beta \tag{229}$$

$$A = [\mathsf{T}]_\beta = [\mathsf{I}_\mathsf{V}]_\gamma^\beta[\mathsf{T}]_\gamma[\mathsf{I}_\mathsf{V}]_\beta^\gamma = [\mathsf{I}_\mathsf{V}]_\gamma^\beta B[\mathsf{I}_\mathsf{V}]_\beta^\gamma \tag{230}$$
$$[\mathsf{I}_\mathsf{V}]_\gamma^\beta = ([\mathsf{I}_\mathsf{V}]_\beta^\gamma)^{-1} \tag{231}$$

    It follows that A is similar to B and therefore by part (a)

$$\det([T]_\beta - \lambda I_n) = \det([\mathsf{T}]_\gamma - \lambda I_n) \tag{232}$$

14. For any square matrix A, prove that A and A^t have the same characteristic polynomial.

    Suppose $A \in \mathsf{M}_{n\times n}$

    Claim: $\det(A - \lambda I_n) = \det(A^t - \lambda I_n)$.

$$
\begin{align}
(A - \lambda I_n)^t &= A^t + (-\lambda)(I_n)^t \tag{233}\\
&= A^t - \lambda I_n \tag{234}\\
&= A - \lambda I_n \tag{235}
\end{align}
$$

$$
\begin{align}
\det(A - \lambda I_n) &= \det(A - \lambda)^t \quad \text{(by theorem 4.8)} \tag{236}\\
\Rightarrow \det(A - \lambda I_n) &= \det(A^t - \lambda I_n) \tag{237}
\end{align}
$$

15. (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

    (b) Sate and prove the analogous result for matrices.

    (a) Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ such that x is an eigenvector corresponding to λ.

    Proof by induction.
    Base Case; $n = 1$

$$\mathsf{T}^1(x) = \lambda^1 x \tag{238}$$

    Suppose true for $1 \leq m \leq k$

$$\Rightarrow \mathsf{T}^m(x) = \lambda^m x \tag{239}$$

    Show True for $m = k + 1$

$$
\begin{align}
\mathsf{T}^{m+1}(x) &= \mathsf{T}(\mathsf{T}^m(x)) \tag{240}\\
&= \mathsf{T}(\lambda^m x) \tag{241}\\
&= \lambda^m \mathsf{T}(x) \tag{242}\\
&= \lambda^m \lambda \tag{243}\\
&= \lambda^{m+1} \tag{244}
\end{align}
$$

    (b) Let $A \in \mathsf{M}_{n\times n}$ and let x be an eigenvector corresponding to the eigenvalue λ .For any positive integer m prove that x is an eigenvector of A^m corresponding to the eigenvalue λ^m.

    Suppose $A \in \mathsf{M}_{n\times n}(F)$ and let x be an eigenvector of A corresponding to the eigenvalue λ.

Proof by induction.

Base Case; n= 1

$$Ax = \lambda x \tag{245}$$

Suppose true for $1 \leq m \leq k$

Suppose $m = k + 1$.

$$\begin{aligned} A^{k+1}x &= A(A^k x) &(246)\\ &= A(\lambda^k x) &(247)\\ &= \lambda^k (Ax) &(248)\\ &= \lambda^k \lambda &(249)\\ &= \lambda^{k+1} &(250) \end{aligned}$$

19. Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered bases β and γ for V such that $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_\gamma$.

 Suppose $A, B \in \mathsf{M}_{n\times n}(F)$ such that $A = Q^{-1}BQ$

 Suppose $\mathsf{V} = \mathsf{F}^n$ and $\mathsf{T} = L_A \colon \mathsf{F}^n \to \mathsf{F}^n$

$$\Rightarrow \dim \mathsf{V} = \dim \mathsf{F}^n = n \text{ and } \mathsf{T} \in \mathcal{L}(\mathsf{V}) \tag{251}$$

 Suppose β is the standard ordered basis of F^n and γ is an ordered basis of F^n

$$\Rightarrow A = [L_A]_\beta \tag{252}$$

 Let $Q = [\mathsf{I}_{\mathsf{F}^n}]_\beta^\gamma$

$$\Rightarrow A = [\mathsf{I}_{\mathsf{F}^n}]_\gamma^\beta B [\mathsf{I}_{\mathsf{F}^n}]_\beta^\gamma \Rightarrow [L_A]_\beta = [\mathsf{I}_{\mathsf{F}^n}]_\gamma^\beta B [\mathsf{I}_{\mathsf{F}^n}]_\beta^\gamma \tag{253}$$

$$\begin{aligned} [\mathsf{I}_{\mathsf{F}^n}]_\beta^\gamma [L_A]_\beta [\mathsf{I}_{\mathsf{F}^n}]_\gamma^\beta &= [\mathsf{I}_{\mathsf{F}^n} L_A]_\beta^\gamma &(254)\\ &= [L_A]_\beta^\gamma [\mathsf{I}_{\mathsf{F}^n}] &(255)\\ &= [L_A] \mathsf{I}_{\mathsf{F}^n}]_\gamma &(256)\\ &= [L_A]_\gamma &(257)\\ &= B &(258) \end{aligned}$$

$$\Rightarrow A = [L_A]_\beta = [\mathsf{T}]_\beta \text{ and } B = [L_A]_\gamma = [\mathsf{T}]_\gamma \tag{259}$$

22.