

Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

Work

1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

(b)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$
$$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (1)$$

$$a_1 = -a_2 \quad (2)$$

$$-a_1 = a_2 \quad (3)$$

$$\implies a_1 = a_2 = 0 \quad (4)$$

Only the trivial solution exists. The set is linearly independent.

(d)

$$\{x^3 - 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\} \text{ in } P_3(\mathbb{R})$$
$$a_1(x^3 - x) + a_1(2x^2 + 4) + a_3(-2x^3 + 3x^2 + 2x + 6) = 0 \quad (5)$$

$$a_1 = 2a_3 \quad (6)$$

$$2a_2 = -3a_3 \quad (7)$$

$$4a_2 = -6a_3 \quad (8)$$

$$a_1 = t \quad (9)$$

$$a_2 = -\left(\frac{3}{4}\right)t \quad (10)$$

$$a_3 = -\left(\frac{1}{2}\right)t \quad (11)$$

The set is linearly dependent.

(f)

$$\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\} \text{ in } \mathbb{R}^3$$

$$a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 0 \quad (12)$$

$$1 + 2b - c = 0 \quad (13)$$

$$-a + 2c = 0 \quad (14)$$

$$2a + b - c = 0 \quad (15)$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 0 \end{pmatrix} \begin{array}{c} \boxed{1} \\ \leftarrow + \\ \boxed{2} \\ \leftarrow + \end{array}^{-2} \mid \cdot \frac{1}{2} \begin{array}{c} \boxed{3} \\ \leftarrow + \\ \boxed{5} \\ \leftarrow + \end{array} \mid \cdot \frac{2}{5} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Only the trivial solution exists. This set is linearly independent.

(g)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\} \text{ in } \mathbf{M}_{2 \times 2}(\mathbb{R})$$

$$a \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = 0 \quad (16)$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ -2 & 1 & 1 & -4 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 4 & 0 \end{pmatrix} \begin{array}{c} \boxed{2} \\ \leftarrow + \\ \boxed{1} \\ \leftarrow + \end{array}^{-1} \begin{array}{c} \boxed{1} \\ \leftarrow + \\ \boxed{2} \\ \leftarrow + \end{array}^{-1} \begin{array}{c} \boxed{2} \\ \leftarrow + \end{array}^{-2} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a = -3t \quad (17)$$

$$b = -t \quad (18)$$

$$c = -t \quad (19)$$

$$d = t \quad (20)$$

There exists a non-trivial solution, therefore the set is linearly dependent.

11. Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space \mathbf{V} over the field \mathbb{Z}_2 . How many vectors are there in $\text{span}(S)$? Justify your answer.

$$\mathbb{Z}_2 = \{0, 1\} \quad (21)$$

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n \neq 0, \forall c_1, \dots, c_n \in \{0, 1\} \quad (22)$$

unless all $c_i = 0$.

$$\implies \text{card}(\text{span}(S)) = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n} \quad (23)$$

$$= \sum_{i=1}^n \binom{n}{i} \quad (24)$$

$$= 2^n \quad (25)$$

15. Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$ for some k ($1 \leq k < n$).

Forward Direction:

Claim: S is linearly dependent

Suppose $u_1 = 0$

Assume S is linearly independent.

Take the linear combination:

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0 \text{ such that } c_1 \neq 0, c_2, \dots, c_n = 0 \quad (26)$$

$$\implies c_1 u_1 = 0 \not\leq \text{Contradiction!} \quad (27)$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Suppose $u_{k+1} \in \text{span}(\{u_1, \dots, u_k\})$

Assume S is linearly independent.

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \dots + a_k u_k \quad (28)$$

Take the linearly combination:

$$c_1 u_1 + c_2 u_2 + \dots + c_k u_k + c_{k+1} + \dots + c_n u_n = 0 \quad (29)$$

Choose $c_i = (-a_i)$ for $i(1 \leq i \leq k)$ and $c_{k+1} = 1$

$$\implies u_{k+1} = 0 \not\leq \text{Contradiction!} \quad (30)$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Reverse Direction:

Suppose $u_{k+1} \in \text{span}(\{u_1, \dots, u_k\})$

Claim: S is linearly dependent

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \dots + a_k u_k \quad (31)$$

$$-u_{k+1} = (-1)(a_1 u_1 + a_2 u_2 + \dots + a_k u_k) \quad (32)$$

$$= (-a_1) u_1 + (-a_2) u_2 + \dots + (-a_k) u_k \quad (33)$$

Take the linear combination of all u_1, \dots, u_n

$$((-a_1) u_1 + (-a_2) u_2 + \dots + (-a_k) u_k) + (1u_{k+1} + 0u_{k+2} + \dots + 0u_n) = 0 \quad (34)$$

1.6

20. Let V be a vector space having dimension n , and let S be a subset of V that generates V .
- (a) Prove that there is subset of S that is a basis for V . (Be careful not to assume that S is finite.)
 - (b) Prove that S contains at least n vectors.

24. Let $f(x)$ be a polynomial of degree n in $P_n(\mathbb{R})$. Prove that for any $g(x) \in P_n(\mathbb{R})$ there exist scalars c_0, c_1, \dots, c_n such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \dots + c_n f^{(n)}(x)$$

where $f^{(n)}(x)$ denotes n th derivative of $f(x)$.

Given vector space $P_n(\mathbb{R})$ where $\dim(P_n(\mathbb{R})) = n + 1$

Suppose S is a subset of $P_n(\mathbb{R})$ and $S = \{f(x), f'(x), f''(x), \dots, f^{(n)}(x)\}$

$$\implies \text{card}(S) = n + 1 \quad (35)$$

If S is linearly independent, then $\text{span}(S) = P(\mathbb{R})$ where $g(x) \in P_n(\mathbb{R})$

Claim: S is linearly independent

$$d_0 f(x) + d_1 f'(x) + d_2 f''(x) + d_3 f'''(x) + \dots + d_n f^{(n)}(x) = 0 \quad (36)$$

$$d_0 = 0 \because x^n \text{ term only exists in } f(x) \quad (37)$$

$$d_1 = 0 \because x^{n-1} \text{ term only exists in } f'(x) \quad (38)$$

$$d_2 = 0 \because x^{n-2} \text{ term only exists in } f''(x) \quad (39)$$

$$\vdots \quad (40)$$

$$d_n = 0 \because x^0 \text{ term only exists in } f^{(n)}(x) \quad (41)$$

31. Let W_1 and W_2 be subspaces of V having dimensions m and n , respectively, where $m \geq n$.

- (a) Prove that $\dim(W_1 \cap W_2) \leq n$
 $W_1 \cap W_2$ is a vector space.

$$W_1 \cap W_2 \subseteq W_2 \quad (42)$$

$$\implies \dim(W_1 \cap W_2) \leq \dim(W_2) \quad (43)$$

$$\dim(W_1 \cap W_2) \leq n \quad (44)$$

- (b) Prove that $\dim(W_1 + W_2) \leq m + n$
 Suppose $W_1 = \{v_1, v_2, \dots, v_m\}$, $W_2 = \{u_1, u_2, \dots, u_n\}$

Lemma: $\text{span}(W_1 \cap W_2) = \text{span}(W_1) + \text{span}(W_2)$

Suppose $x \in W_1 \cap W_2$ such that

$$x = c_1 u_1 + c_2 u_2 + \dots + c_m u_m + c_{m+1} v_1 + \dots + c_{m+n} v_n \quad (45)$$

Let:

$$\sum_{i=1}^m c_i u_i = \alpha \implies \alpha \in \text{span}(W_1) \quad (46)$$

$$\sum_{j=1}^n c_{m+j} v_j = \beta \implies \beta \in \text{span}(W_2) \quad (47)$$

$$\implies x = \alpha + \beta \quad (48)$$

$$\implies x \in \text{span}(W_1) + \text{span}(W_2) \quad (49)$$

2.1

6. $\text{T}: \text{M}_{n \times n}(F) \rightarrow F$ defined by $\text{T}(A) = \text{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\text{tr}(A) = \sum_{i=1}^n A_{ii}$$

12. Is there a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$?

14. Let V and W be vector spaces and $T: V \rightarrow W$ be linear.

- (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W .
- (b) Suppose that T is one-to-one and that S is a subset of V . Prove that S is linearly independent if and only if $T(S)$ is linearly independent.
- (c) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W .

(a) **Forward Direction:**

Suppose $S \subseteq V$ such that S is linearly independent and T is one-to-one.

Let $T(S) = \{T(x) : x \in S\}$

Claim: $T(S)$ is linearly independent.

Suppose $x \in \text{span}(T(S))$ such that $x = c_1u_1 + c_2u_2 + \dots + c_nu_n = 0$ for $c_i \in F$ and $u_i \in T(S)$

$$T \text{ is one-to-one} \implies u_i = T(v_i) \text{ for some } v_i \in S$$

$$\implies x = c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n) \quad (50)$$

$$= T(c_1v_1 + c_2v_2 + \dots + c_nv_n) \quad (51)$$

$$= 0 \quad (52)$$

$$\implies c_1v_1 + c_2v_2 + \dots + c_nv_n = 0 \quad (53)$$

$$\implies c_1 = c_2 = \dots = c_n \quad (54)$$

Therefore there only exists the trivial solution to the linear combination.

Reverse Direction: Suppose $S \subseteq V$ such that S is linearly independent and $T(S)$ is linearly independent.

Claim: T is one-to-one

Suppose $x, y \in \text{span}(S)$ such that $x \neq y$

$$x = a_1u_1 + a_2u_2 + \dots + a_nu_n \quad (55)$$

$$y = b_1u_1 + b_2u_2 + \dots + b_nu_n \quad (56)$$

Because S is linearly independent each linear combination is unique.

$$\implies a_i \neq b_i \text{ for some } i = 1, 2, \dots, n \quad (57)$$

$$\text{Let } T(x) = T(a_1u_1 + a_2u_2 + \dots + a_nu_n) \quad (58)$$

$$\implies T(x) = a_1T(u_1) + a_2T(u_2) + \dots + a_nT(u_n) \quad (59)$$

$$\text{Let } T(y) = T(b_1u_1 + b_2u_2 + \dots + b_nu_n) \quad (60)$$

$$\implies T(y) = b_1T(u_1) + b_2T(u_2) + \dots + b_nT(u_n) \quad (61)$$

$$T(u_i) \in T(S) \forall i = 1, 2, \dots, n \quad (62)$$

$$\implies T(x), T(y) \in \text{span}(T(S)) \quad (63)$$

Because $T(S)$ is linearly independent each linear combination is unique.

$$\implies T(x) \neq T(y) \quad (64)$$

(b) Suppose T is one-to-one, $S \subseteq V$

Forward Direction:

Suppose S is linearly independent.

Claim $T(S)$ is linearly independent.

$T(S)$ is linearly independent by part (a)

Reverse Direction:

Suppose $S = \{v_1, v_2, \dots, v_m\}$

$$\implies T(S) = \{T(v_1) + T(v_2) + \dots + T(v_m)\} \quad (65)$$

Suppose $x \in \text{span}(S)$ such that $x = c_1v_1 + c_2v_2 + \dots + c_mv_m$

$$\implies T(x) = T(c_1v_1 + c_2v_2 + \dots + c_mv_m) \quad (66)$$

$$= T(0) \quad (67)$$

$$= 0 \quad (68)$$

Because T is one-to-one.

$$T(x) = c_1T(v_1) + c_2T(v_2) + \dots + c_mT(v_m) \quad (69)$$

$$T(v_i) \in T(S) \forall i = 1, 2, \dots, m \quad (70)$$

Therefore $T(x) \in \text{span}(T(S))$ and $T(S)$ is linearly independent.

$$c_1 = c_2 = \dots = c_m = 0 \quad (71)$$

(c) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto.

Claim: $T(\beta)$ is a basis for W

$$R(T) = W \quad (72)$$

$$= \text{span}(T(\beta)) \text{ (by theorem 2.2)} \quad (73)$$

$T(\beta)$ is linearly independent by part (a)

2.2

2. Let β and γ be the standard ordered bases for \mathbb{R}^n and \mathbb{R}^m respectively. For each linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, compute $[T]_{\beta}^{\gamma}$.
- (b) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$
- (c) $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$
- (g) $T: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $T(a_1, a_2, \dots, a_n) = a_1 + a_n$

8. Let V be an n -dimensional vector space with an ordered basis β . Define $T: V \rightarrow F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.
 Suppose $\beta = \{v_1, v_2, \dots, v_n\}$
 Suppose $x, y \in V$

$$x = \sum_{i=1}^n a_i v_i \qquad y = \sum_{i=1}^n b_i v_i \qquad (74)$$

$$x + y = \sum_{i=1}^n a_i v_i = \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (a_i v_i + b_i v_i) = \sum_{i=1}^n (a_i + b_i) v_i \qquad (75)$$

$$T(x + y) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = T(x) + T(y) \qquad (76)$$

Suppose $c \in F$ and $x \in V$

$$x = \sum_{i=1}^n a_i v_i \qquad (77)$$

$$cx = c \sum_{i=1}^n a_i v_i = \sum_{i=1}^n ca_i v_i \qquad (78)$$

$$T(cx) = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = cT(x) \qquad (79)$$

11. Let V be an n -dimensional vector space, and let $T: V \rightarrow V$ be a linear transformation. Suppose that W is a T -invariant subspace of V (see the exercises of Section 2.1) having dimension k . Show that there is a basis β for V such that $[T]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where A is a $k \times k$ matrix and O is the $(n - k) \times k$ zero matrix.

Suppose β_W is a basis of W

$$T(\beta_W) \subseteq W \quad (80)$$

Suppose $x \in \beta_W$

$$\implies T(x) \in W \quad (81)$$

Therefore $T(x)$ can be described as a linear combination of vectors in β_W .

Suppose β_W is extended to β (by corollary 2 of theorem 2.2) such that $\beta = \{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_n\}$ and $u_i \in \beta_W$ for $i = 1, 2, \dots, k \implies$ for $u_i, i = 1, 2, \dots, k$;

$$[T(u_i)]_\beta = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (82)$$

$$[T]_\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & & & \\ a_{21} & a_{22} & \cdots & a_{2k} & & & \\ \vdots & \vdots & & \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & [T(u_{k+1})]_\beta & [T(u_{k+2})]_\beta & \cdots & [T(u_n)]_\beta \\ 0 & 0 & \cdots & 0 & & & & \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & \cdots & 0 & & & & \\ 0 & 0 & \cdots & 0 & & & & \end{pmatrix} \quad (83)$$