Assignment

Section 6.3: 1, 2, 3, 9; Section 6.4: 1, 2, 9, 11; Section 6.5: 1, 2, 5, 10, 21

Work

6.3

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every linear operator has an adjoint.
   True

   (b) Every linear operator on V has the form $x \to \langle x, y\rangle$ for some $y \in \mathsf{V}$.
   False

   (c) For every linear operator T on V and every ordered basis β for V, we have $[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^*$.
   False

   (d) The adjoint of a linear operator is unique.
   True

   (e) For any linear operators T and U and scalars a and b,

   $$(a\mathsf{T} + b\mathsf{U})^* = a\mathsf{T}^* + b\mathsf{U}^*$$

   False

   (f) For any $n \times n$ matrix A, we have $(\mathsf{L}_A)^* = \mathsf{L}_A$
   True

   (g) For any linear operator T, we have $(\mathsf{T}^*)^* = \mathsf{T}$
   True

2. For each of the following inner product spaces V (over F) and linear transformations $\mathsf{g}\colon \mathsf{V} \to F$, find a vector y such that $\mathsf{g}(x) = \langle x, y\rangle$ for all $x \in \mathsf{V}$.

   (a) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{g}(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$

   $$y = (1, -2, 4) \tag{1}$$

   (b) $\mathsf{V} = \mathsf{C}^2$, $\mathsf{g}(z_1, z_2) = z_1 - 2z_2$

   $$y = (1, -2) \tag{2}$$

(c) $\mathsf{V} = \mathsf{P}^2(\mathbb{R})$ with $\langle f, h\rangle = \int\limits_0^1 f(t)h(t)\ \mathrm{d}t,\ \mathsf{g}(f) = f(0) + f'(1)$

Suppose

$$f(t) = a_2 t^2 + a_1 t + a_0 \tag{3}$$
$$y(t) = b_2 t^2 + b_1 t + b_0 \tag{4}$$

$$\langle f, y\rangle = \mathsf{g}(f) \tag{5}$$

$$f(0) + f'(1) = \int\limits_0^1 f(t)y(t)\ \mathrm{d}t \tag{6}$$

$$\Rightarrow \begin{pmatrix} 12 & 15 & 20 \\ 3 & 4 & 6 \\ 2 & 3 & 6 \end{pmatrix} \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 120 \\ 12 \\ 6 \end{pmatrix} \tag{7}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 33 \\ -204 \\ 210 \end{pmatrix} \tag{8}$$

$$\Rightarrow b_2 = 210 \tag{9}$$
$$b_1 = -204 \tag{10}$$
$$b_0 = 33 \tag{11}$$

$$\Rightarrow y(t) = 210x^2 + -204x + 33 \tag{12}$$

3. For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(a) $\mathsf{V} = \mathsf{R}^2,\ \mathsf{T}(a, b) = (2a + b, a - 3b),\ x = (3, 4)$

$$\langle (a, b), \mathsf{T}^*(x)\rangle = \langle (2a + b, a - 3b), (3, 5)\rangle \tag{13}$$
$$= 3(2a + b) + 5(a - 3b) \tag{14}$$
$$= 11a - 12b \tag{15}$$

$$\Rightarrow \mathsf{T}^*(x) = (11, -12) \tag{16}$$

(b) $\mathsf{V} = \mathsf{C}^2,\ \mathsf{T}(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1),\ x = (3 - i, 1 + 1i)$

$$\langle (z_1, z_2), \mathsf{T}^*(x)\rangle = \langle (2z_1 + iz_2, (1 - i)z, (3 - i, 1 + 2i)\rangle \tag{17}$$
$$= z_1(5 - i) + z_2(-1 + 3i) \tag{18}$$

$$\Rightarrow \mathsf{T}^*(x) = (5 + i, -1 - 3i) \tag{19}$$

(c) $\mathsf{V} = \mathsf{P}_1(\mathbb{R})$ with $\langle f, g\rangle = \int\limits_{-1}^{1} f(t)g(t)\ \mathrm{d}t$, $\mathsf{T}(f) = f' + 3f$, $f(t) = 4 - 2t$

Suppose $f(t) = a_1 t + a_0$

$$\langle (a_1 t + a_o, \mathsf{T}^*(x)\rangle = \langle 3a_1 t + (3a_0 + a_1), 4 - 2t \tag{20}$$
$$\int\limits_{-1}^{1} (a_1 t + a_0)\mathsf{T}^*(x)\ \mathrm{d}x = \int\limits_{-1}^{1} (3a_1 t + 3a_0 + a_1)(4 - 2t)\mathrm{d}t \tag{21}$$
$$\int\limits_{-1}^{1} (a_1 t + a_0)(b_1 t + b_0)\ \mathrm{d}x = \tag{22}$$
$$2\left(\frac{a_1 b_1}{3} + a_0 b_0\right) = 4a_1 + a_0 \tag{23}$$

$$\frac{2a_1 b_1}{3} = 4a_1 \tag{24}$$
$$\Rightarrow b_1 = 6 \tag{25}$$
$$2a_0 b_0 = 24a_0 \tag{26}$$
$$\Rightarrow b_0 = 12 \tag{27}$$
$$\Rightarrow \mathsf{T}^*(x) = 6t + 12 \tag{28}$$

9. Prove that if $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^\perp$ and T is the projection on W along $\mathsf{W}^\perp$, then $\mathsf{T} = \mathsf{T}^*$.

Suppose $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^\perp$ and T is a projection on W along $\mathsf{W}^\perp$

Suppose $x, y \in \mathsf{V}$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in \mathsf{W}$ and $x_2, y_2 \in \mathsf{W}^\perp$

$$\langle \mathsf{T}(x), y\rangle = \langle x, \mathsf{T}^*(y)\rangle \tag{29}$$
$$= \langle x, y_1 + y_2\rangle \tag{30}$$
$$= \langle x_1, y_2\rangle + \langle x_1, y_2\rangle \tag{31}$$
$$= \langle x_1, y_1\rangle \tag{32}$$

$$\langle x, \mathsf{T}(y)\rangle = \langle \mathsf{T}^*(x), y\rangle \tag{33}$$
$$= \langle x_1 + x_2, y_1\rangle \tag{34}$$
$$= \langle x_1, y_1\rangle + \langle x_2, y_2\rangle \tag{35}$$
$$= \langle x_1, y_1\rangle \tag{36}$$

$$\Rightarrow \langle x, \mathsf{T}(y)\rangle = \langle x, \mathsf{T}^*(y)\rangle \quad \forall x \in \mathsf{V} \tag{37}$$
$$\Rightarrow \mathsf{T}(y) = \mathsf{T}^*(y) \quad \forall y \in \mathsf{V} \tag{38}$$

6.4

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every self-adjoint operator is normal.
   True

   (b) Operators and their adjoints have the same eigenvectors.
   False

   (c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.
   False

   (d) A real or complex matrix A is normal if and only if L_A is normal.
   True

   (e) The eigenvalues of a self-adjoint operator must be real.
   True

   (f) The identity and zero operators are self-adjoint.
   True

   (g) Every normal operator is diagonalizable.
   False

   (h) Every self-adjoint operator is diagonalizable.
   True

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

   (a) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a,b) = (2a - 2b, -2a + 5b)$
   Suppose β is the standard ordered basis for R^2

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{39}$$

$$\Rightarrow ([\mathsf{T}]_\beta)^* = ([\mathsf{T}^*]) = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{40}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{41}$$

$$\det \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = 0 \tag{42}$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \tag{43}$$

$$\Rightarrow \lambda_1 = 6 \tag{44}$$

$$\lambda_2 = 1 \tag{45}$$

- For $\lambda_1 = 6$

$$[\mathsf{T}]_\beta - 6I_2 = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \tag{46}$$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{47}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{48}$$

$$\Rightarrow x_1 = -\frac{1}{2}x_2 \tag{49}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} {}^{-1}/_{2} \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{50}$$

- For $\lambda_2 = 1$

$$[\mathsf{T}]_\beta - I_2 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \tag{51}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{52}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{53}$$

$$\Rightarrow x_1 = 2x_2 \tag{54}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{55}$$

Suppose

$$v_1' = (-\frac{1}{2}, 1) \qquad\qquad v_2' = (2, 1) \tag{56}$$

Let

$$v_1 = v_1' \tag{57}$$

$$v_2 = v_2' - \frac{\langle v_2', v_1 \rangle}{\|v_1\|^2} v_1 \tag{58}$$

$$\langle v_2', v_1 \rangle = 0 \tag{59}$$
$$\Rightarrow v_2 = v_2' \tag{60}$$
$$\|v_1\|^2 = \frac{5}{4} \tag{61}$$
$$\Rightarrow \|v_1\| = \frac{\sqrt{5}}{2} \tag{62}$$
$$\Rightarrow o_1 = \frac{1}{\sqrt{5}}(-1, 2) \tag{63}$$
$$\|v_2\|^2 = 5 \tag{64}$$
$$\Rightarrow \|v_2\| = 5 \tag{65}$$
$$\Rightarrow o_2 = \frac{1}{\sqrt{5}}(2, 1) \tag{66}$$

An orthonormal basis is

$$\gamma = \left\{ \frac{1}{\sqrt{5}}(-1, 2), \frac{1}{\sqrt{3}}(2, 1) \right\} \tag{67}$$

The eigenvector $\frac{1}{\sqrt{5}}(-1, 2)$ corresponds to the eigenvalue 6, and the eigenvector $\frac{1}{\sqrt{3}}(2, 1)$ corresponds to the eigenvalue 1.

(b) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$
Suppose β is the standard ordered basis of R^3

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{pmatrix} \tag{68}$$
$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 5 & -2 \\ 0 & 0 & 5 \end{pmatrix} \tag{69}$$
$$\Rightarrow \mathsf{T}^* \neq \mathsf{T} \tag{70}$$
$$([\mathsf{T}]_\beta)^*[\mathsf{T}]_\beta \neq ([\mathsf{T}]_\beta)^* \tag{71}$$

T is neither normal nor adjoint.

(c) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(a, b) = (2a + ib, a + 2b)$
Suppose β is the standard ordered basis of C^2

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \tag{72}$$
$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \tag{73}$$
$$\begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \tag{74}$$
$$\Rightarrow \mathsf{T} \text{ is normal.}$$

$$\det \begin{pmatrix} 2-\lambda & i \\ 1 & 2-\lambda \end{pmatrix} \tag{75}$$

$$\Rightarrow (2-\lambda)^2 = i \tag{76}$$

$$\Rightarrow \lambda_1 = \left(2+\frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right) \tag{77}$$

$$\lambda_2 = \left(2-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right) \tag{78}$$

- For $\lambda_1 = \left(2+\frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right)$

$$[\mathsf{T}]_\beta - \lambda_1 I_n = \begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \tag{79}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{80}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2}(1+i) & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{81}$$

$$\Rightarrow x_1 = \frac{\sqrt{2}}{2}(1+i)x_2 \tag{82}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t\begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{83}$$

- For $\lambda_2 = \left(2-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)$

$$[\mathsf{T}]_\beta - \lambda_2 I_2 = \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \tag{84}$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{85}$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{86}$$

$$\Rightarrow x_1 = -\frac{\sqrt{2}}{2}(1+i)x_2 \tag{87}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t\begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{88}$$

Suppose

$$w_1 = \left(\frac{\sqrt{2}}{2}(1+i), 1\right) \qquad w_2 = \left(-\frac{\sqrt{2}}{2}(1+i), 1\right) \tag{89}$$

Let

$$v_1 = w_1 \tag{90}$$
$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 \tag{91}$$

$$\langle w_2, v_1\rangle = 0 \tag{92}$$
$$\Rightarrow v_2 = w_2 \tag{93}$$
$$\|v_1\|^2 = 2 \tag{94}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{95}$$
$$\Rightarrow o_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right) \tag{96}$$
$$\|v_2\|^2 = 2 \tag{97}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{98}$$
$$\Rightarrow o_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right) \tag{99}$$

An orthonormal basis is

$$\gamma = \left\{\left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right)\right\} \tag{100}$$

The eigenvector $\left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right)$ corresponds to the eigenvalue of $\left(2+\frac{\sqrt{2}}{2}\right)+ i\left(\frac{\sqrt{2}}{2}\right)$. The eigenvector $\left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right)$ corresponds to the eigenvalue of $\left(2-\frac{\sqrt{2}}{2}\right)+i\left(-\frac{\sqrt{2}}{2}\right)$.

(d) $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\, \mathrm{d}t$$

Suppose β is the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{101}$$
$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{102}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{103}$$

It follows that T is neither self-adjoint nor normal.

(e) $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$ and T is defined by $\mathsf{T}(A) = A^t$.
Suppose β is the standard ordered basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{104}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{105}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{106}$$

$$[\mathsf{T}]_\beta - \lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \tag{107}$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{108}$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) = 0 \tag{109}$$

$$\Rightarrow \lambda_1 = 1 \tag{110}$$

$$\lambda_2 = -1 \tag{111}$$

- For $\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{112}$$

$$\Rightarrow x_2 = x_3 \tag{113}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t, s, r \in \mathbb{R} \right\} \tag{114}$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{115}$$

$$\Rightarrow x_1 = x_4 = 0 \tag{116}$$

$$x_2 = -x_3 \tag{117}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{118}$$

Suppose

$$w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{119}$$

$$w_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{120}$$

Let

$$v_1 = w_1 \tag{121}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{122}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{123}$$

$$v_4 = w_4 - \left(\sum_{i=1}^{3} \frac{\langle w_4, v_i \rangle}{\|v_1\|^2} v_i \right) \tag{124}$$

$$\langle w_2, v_1 \rangle = 0 \tag{125}$$
$$\Rightarrow v_2 = w_2 \tag{126}$$
$$\langle w_3, v_1 \rangle = 0 \tag{127}$$
$$\langle w_3, v_2 \rangle = 0 \tag{128}$$
$$\Rightarrow v_3 = w_3 \tag{129}$$
$$\langle w_4, v_1 \rangle = 0 \tag{130}$$
$$\langle w_4, v_2 \rangle = 0 \tag{131}$$
$$\langle w_4, v_3 \rangle = 0 \tag{132}$$
$$\Rightarrow v_4 = w_4 \tag{133}$$
$$\|v_1\|^2 = 2 \tag{134}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{135}$$
$$\Rightarrow o_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{136}$$
$$\|v_2\|^2 = 1 \tag{137}$$
$$\Rightarrow \|v_2\| = 1 \tag{138}$$
$$\Rightarrow o_2 = v_2 \tag{139}$$
$$\|v_3\|^2 = 1 \tag{140}$$
$$\Rightarrow \|v_3\| = 1 \tag{141}$$
$$\Rightarrow o_3 = v_3 \tag{142}$$
$$\|v_4\|^2 = 2 \tag{143}$$
$$\Rightarrow \|v_4\| = \sqrt{2} \tag{144}$$
$$\Rightarrow o_4 = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{145}$$

An orthonormal basis is

$$\gamma = \left\{ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right\} \tag{146}$$

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$ and T is defined by $\mathsf{T}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$
Suppose β is the standard ordered basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{147}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{148}$$

$$\Rightarrow \mathsf{T} \text{ is self adjoint.}$$

$$[\mathsf{T}]_\beta - \lambda I_4 = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix} \tag{149}$$

$$\det \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix} = 0 \tag{150}$$

$$(\lambda - 1)^2(\lambda + 1) \tag{151}$$

$$\Rightarrow \lambda_1 = 1 \tag{152}$$

$$\lambda_2 = -1 \tag{153}$$

- For $\lambda_1 = 1$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{154}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{155}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R} \right\} \tag{156}$$

$$\tag{157}$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{158}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{159}$$

$$\Rightarrow x_1 = -x_3 \tag{160}$$

$$x_2 = -x_4 \tag{161}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\} \tag{162}$$

Suppose

$$w_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{163}$$

$$w_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \tag{164}$$

Let

$$v_1 = w_1 \tag{165}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{166}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{167}$$

$$v_4 = w_4 - \left(\sum_{i=1}^{3} \frac{\langle w_4, v_i \rangle}{\|v_1\|^2} v_i \right) \tag{168}$$

$$\langle w_2, v_a \rangle = 0 \tag{169}$$

$$\Rightarrow v_2 = w_2 \tag{170}$$

$$\langle w_3, v_2 \rangle = 0 \tag{171}$$

$$\langle w_3, v_1 \rangle = 0 \tag{172}$$

$$\Rightarrow v_3 = w_3 \tag{173}$$

$$\langle w_4, v_1 \rangle = 0 \tag{174}$$

$$\langle w_4, v_2 \rangle = 0 \tag{175}$$

$$\langle w_4, v_3 \rangle = 0 \tag{176}$$

$$\Rightarrow v_4 = w_4 \tag{177}$$

$$\|v_1\|^2 = 2 \tag{178}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{179}$$
$$\|v_2\|^2 = 2 \tag{180}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{181}$$
$$\|v_3\|^2 = 2 \tag{182}$$
$$\Rightarrow \|v_3\| = \sqrt{2} \tag{183}$$
$$\|v_4\|^2 = 2 \tag{184}$$
$$\Rightarrow \|v_4\| = \sqrt{2} \tag{185}$$

An orthonormal basis is

$$\gamma = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\} \tag{186}$$

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(\mathsf{T}) = N(\mathsf{T}^*)$ and $R(\mathsf{T}) = R(\mathsf{T}^*)$.

   Claim: $N(\mathsf{T}) = N(\mathsf{T}^*)$

   ($\subseteq$) Suppose $x \in N(\mathsf{T})$

$$\Rightarrow \mathsf{T}(x) = 0 \cdot x \tag{187}$$
$$\Rightarrow \mathsf{T}^*(x) = \bar{0} \cdot x = 0 \quad \text{(Theorem 6.15.C)} \tag{188}$$
$$\Rightarrow x \in N(\mathsf{T}^*) \tag{189}$$

   ($\supseteq$) Suppose $x \in N(\mathsf{T}^*)$

$$\Rightarrow \mathsf{T}^*(x) = 0 \cdot x \tag{190}$$
$$\Rightarrow (\mathsf{T}^*)^*(x) = \bar{0} \cdot x = x \quad \text{(Theorem 6.15.C)} \tag{191}$$
$$(\mathsf{T}^*)^*(x) = \mathsf{T} \tag{192}$$
$$\Rightarrow \mathsf{T}(x) = 0 \tag{193}$$
$$\Rightarrow x \in N(\mathsf{T}) \tag{194}$$

   Claim: $R(\mathsf{T}) = R(\mathsf{T}^*)$

$$N(\mathsf{T}) = N(\mathsf{T}^*) \tag{195}$$
$$N(\mathsf{T}^*) = R(\mathsf{T}^{**})^\perp = R(\mathsf{T})^\perp \quad \text{(Problem 6.3.12)} \tag{196}$$
$$N(\mathsf{T}) = R(\mathsf{T}^*)^\perp \quad \text{(Problem 6.3.12)} \tag{197}$$
$$\Rightarrow R(\mathsf{T}^*)^\perp = R(\mathsf{T})^\perp \tag{198}$$
$$\mathsf{V} = N(\mathsf{T}) \oplus R(\mathsf{T}) = N(\mathsf{T}^*) \oplus R(\mathsf{T}^*) \tag{199}$$
$$\therefore \mathsf{V} = R(\mathsf{T}^*)^\perp \oplus R(\mathsf{T}^*) = R(\mathsf{T})^\perp \oplus R(\mathsf{T}) \tag{200}$$

($\subseteq$) Suppose $x \in R(\mathsf{T})$

Case 1 $x \neq 0$

$$\Rightarrow x \in R(\mathsf{T})^{\perp} \oplus R(\mathsf{T}) \tag{201}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \oplus R(\mathsf{T}^*) \tag{202}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \text{ or } x \in R(\mathsf{T}^*) \tag{203}$$
$$R(\mathsf{T}^*)^{\perp} = N(\mathsf{T}) \text{ and } x \notin N(\mathsf{T}) \tag{204}$$
$$\Rightarrow x \in R(\mathsf{T}^*) \tag{205}$$

Case 2 $x = 0$

$$0 \in R(\mathsf{T}^*) \tag{206}$$

($\supseteq$) Suppose $x \in R(\mathsf{T}^*)$

Case 1 $x \neq 0$

$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \oplus R(\mathsf{T}^*) \tag{207}$$
$$\Rightarrow x \in R(\mathsf{T})^{\perp} \oplus R(\mathsf{T}) \tag{208}$$
$$\Rightarrow x \in R(\mathsf{T})^{\perp} \text{ or } x \in R(\mathsf{T}) \tag{209}$$
$$R(\mathsf{T})^{\perp} = N(\mathsf{T}^*) \text{ and } x \notin N(\mathsf{T}^*) \tag{210}$$
$$\Rightarrow x \in R(\mathsf{T}) \tag{211}$$

Case 2 $x = 0$

$$0 \in R(\mathsf{T}) \tag{212}$$
$$\therefore R(\mathsf{T}) = R(\mathsf{T}^*) \tag{213}$$

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

    (a) If T is self-adjoint, then $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$.

    (b) If T satisfies $\langle \mathsf{T}(x), x\rangle = 0$ for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}_0$.

    (c) If $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}^*$.

    (a) Claim: If T is self-adjoint then $\langle \mathsf{T}(x), x\rangle$ is real $\forall x \in \mathsf{V}$

$$\langle \mathsf{T}(x), x\rangle = \langle x, \mathsf{T}^*(x)\rangle = \langle x, \mathsf{T}(x)\rangle \tag{214}$$
$$\overline{\langle \mathsf{T}(x), x\rangle} = \langle x, \mathsf{T}(x)\rangle \tag{215}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle = \overline{\langle \mathsf{T}(x), x\rangle} \tag{216}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle \text{ is real}$$

(b) Suppose T satisfies $\langle \mathsf{T}(x), x\rangle = 0 \; \forall x \in \mathsf{V}$
Claim: $\mathsf{T} = \mathsf{T}_0$
Let $z = x + y$

$$\begin{aligned}
\langle \mathsf{T}(z), z\rangle &= \langle \mathsf{T}(x+y), x+y\rangle && (217)\\
&= \langle \mathsf{T}(x+y), x\rangle + \langle \mathsf{T}(x+y), x\rangle && (218)\\
&= \langle \mathsf{T}(x) + \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x) + \mathsf{T}(y), y\rangle && (219)\\
&= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle + \langle \mathsf{T}(y), y\rangle && (220)\\
&= \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle && (221)\\
&= 0 && (222)
\end{aligned}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = -\langle \mathsf{T}(x), y\rangle \tag{223}$$

Let $z = x + iy$

$$\begin{aligned}
\langle \mathsf{T}(z), z\rangle &= \langle \mathsf{T}(x+iy), x+iy\rangle && (224)\\
&= \langle \mathsf{T}(x+iy), x\rangle + \langle \mathsf{T}(x+iy), iy\rangle && (225)\\
&= \langle \mathsf{T}(x) + \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x) + \mathsf{T}(iy), iy\rangle && (226)\\
&= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle + \langle \mathsf{T}(iy), iy\rangle && (227)\\
&= \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle && (228)\\
&= i\langle \mathsf{T}(y), x\rangle + -i\langle \mathsf{T}(x), y\rangle && (229)\\
&= 0 && (230)
\end{aligned}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle \tag{231}$$

$$\begin{aligned}
&\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle = -\langle \mathsf{T}(x), y\rangle && (232)\\
&\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle = 0 \quad \forall x, y \in \mathsf{V} && (233)
\end{aligned}$$

Suppose x, y are nonzero.

$$\begin{aligned}
\langle \mathsf{T}(y), x\rangle = \langle 0, x\rangle &= 0 \quad \forall x \in \mathsf{V} && (234)\\
\Rightarrow \mathsf{T}(y) &= 0 \quad \forall y \in \mathsf{V} && (235)\\
\Rightarrow \mathsf{T} &= \mathsf{T}_0 && (236)
\end{aligned}$$

(c) Suppose $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$
Claim: $\mathsf{T} = \mathsf{T}^*$

$$\begin{aligned}
\langle \mathsf{T}(x), x\rangle &= \overline{\langle \mathsf{T}(x), x\rangle} \quad \forall x \in \mathsf{V} && (237)\\
\overline{\langle \mathsf{T}(x), x\rangle} &= \langle x, \mathsf{T}(x)\rangle && (238)\\
\langle \mathsf{T}(x), x\rangle &= \langle x, \mathsf{T}^*(x)\rangle && (239)\\
\Rightarrow \langle x, \mathsf{T}(x)\rangle &= \langle x, \mathsf{T}^*(x)\rangle \quad \forall x \in \mathsf{T}(x) && (240)\\
\Rightarrow \mathsf{T}(x) &= \mathsf{T}^*(x) \quad \forall x \in \mathsf{V} && (241)\\
\Rightarrow \mathsf{T} &= \mathsf{T}^* && (242)
\end{aligned}$$

6.5

1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

   (a) Every unitary operator is normal.
   True

   (b) Every orthogonal operator is diagonalizable.
   False

   (c) A matrix is unitary if and only if it is invertible.
   False

   (d) If two matrices are unitarily equivalent, then they are also similar.
   True

   (e) The sum of unitary matrices is unitary.
   False

   (f) The adjoint of a unitary operator is unitary.
   True

   (g) If T is an orthogonal operator on V, then $[\mathsf{T}]_\beta$ is an orthogonal matrix for any ordered basis β for V.
   False

   (h) If all the eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.
   False

   (i) A linear operator may preserve the norm, but not the inner product.
   False

2. For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

   (a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (\lambda - 3)(\lambda + 1) = 0 \tag{243}$$

$$\Rightarrow \lambda_1 = -1 \tag{244}$$

$$\lambda_2 = 3 \tag{245}$$

   - For $\lambda_1 = -1$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{246}$$

$$\Rightarrow x_1 = -x_2 \tag{247}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{248}$$

- For $\lambda_2 = 3$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{249}$$
$$\Rightarrow x_1 = x_2 \tag{250}$$
$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{251}$$

Suppose

$$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad\qquad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{252}$$

Let

$$v_1 = w_1 \tag{253}$$
$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|} v_1 \tag{254}$$

$$\langle w_2, v_1 \rangle = 0 \tag{255}$$
$$\Rightarrow v_2 = w_2 \tag{256}$$
$$\|v_1\|^2 = 2 \tag{257}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{258}$$
$$\Rightarrow o_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{259}$$
$$\|v_2\|^2 = 2 \tag{260}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{261}$$
$$\Rightarrow o_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{262}$$

Let

$$P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{263}$$
$$P^t = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{264}$$
$$P^t A P^t = D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \tag{265}$$

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0 \tag{266}$$
$$\lambda^2 + 1 = 0 \tag{267}$$

$$\Rightarrow \lambda_1 = i \tag{268}$$
$$\lambda_2 = -i \tag{269}$$

- For $\lambda_1 = i$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{270}$$
$$\begin{pmatrix} -i & -1 \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{271}$$
$$\Rightarrow x_1 = ix_2 \tag{272}$$
$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} i \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{273}$$

- For $\lambda_2 = -i$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{274}$$
$$\begin{pmatrix} 0 & 0 \\ i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{275}$$
$$\Rightarrow x_1 = -ix_2 \tag{276}$$
$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -i \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{277}$$

Suppose

$$w_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{278}$$

Let

$$v_1 = w_1 \tag{279}$$
$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|} v_1 \tag{280}$$

$$\langle w_2, v_1 \rangle = 0 \tag{281}$$
$$\Rightarrow v_2 = w_2 \tag{282}$$
$$\|v_1\|^2 = 2 \tag{283}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{284}$$
$$\Rightarrow o_1 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{285}$$
$$\|v_2\|^2 = 2 \tag{286}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{287}$$
$$\Rightarrow o_2 = \begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{288}$$

$$\Rightarrow P = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{289}$$

$$\Rightarrow P^* = \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{290}$$

$$P^*AP = D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{291}$$

(c) $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

$$\det \begin{pmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{pmatrix} = 0 \tag{292}$$

$$(\lambda - 8)(\lambda + 1) = 0 \tag{293}$$

$$\Rightarrow \lambda_1 = -1 \tag{294}$$

$$\lambda_2 = 8 \tag{295}$$

- For $\lambda_1 = -1$

$$\begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{296}$$

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{297}$$

$$\Rightarrow x_1 = x_2(i-1) \tag{298}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1+i \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{299}$$

- For $\lambda_2 = 8$

$$\begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{300}$$

$$\begin{pmatrix} 0 & 0 \\ 2 & i-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{301}$$

$$\Rightarrow x_1 = \frac{1-i}{2} x_2 \tag{302}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{303}$$

Suppose

$$w_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix} \tag{304}$$

Let

$$v_1 = w_1 \tag{305}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|} v_1 \tag{306}$$

$$\langle w_2, v_1 \rangle = 0 \tag{307}$$

$$\Rightarrow v_2 = w_2 \tag{308}$$

$$\|v_1\|^2 = 3 \tag{309}$$

$$\Rightarrow \|v_1\| = \sqrt{3} \tag{310}$$

$$\Rightarrow o_1 = \begin{pmatrix} \frac{-1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{311}$$

$$\|v_2\|^2 = \frac{3}{2} \tag{312}$$

$$\|v_2\| = \sqrt{\frac{3}{2}} \tag{313}$$

$$\Rightarrow o_2 = \begin{pmatrix} \frac{1-i}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \tag{314}$$

$$\Rightarrow P = \begin{pmatrix} \frac{-i+1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{3}} \end{pmatrix} \tag{315}$$

$$\Rightarrow P^* = \begin{pmatrix} \frac{-i-i}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1+i}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \tag{316}$$

$$\Rightarrow P^*AP = D = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix} \tag{317}$$

(d) $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{pmatrix} = 0 \tag{318}$$

$$\Rightarrow (z+2)^2(x-4) = 0 \tag{319}$$

$$\Rightarrow \lambda_1 = -2 \tag{320}$$

$$\lambda_2 = 4 \tag{321}$$

- For $\lambda_1 = -2$

$$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{322}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{323}$$

$$\Rightarrow x_1 = -x_2 + -x_3 \tag{324}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R} \right\} \tag{325}$$

- For $\lambda_2 = 4$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{326}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{327}$$

$$\Rightarrow x_1 = x_3 \tag{328}$$

$$x_2 = x_3 \tag{329}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{330}$$

Suppose

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{331}$$

Let

$$v_1 = w_1 \tag{332}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{333}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{334}$$

$$\tag{335}$$

$$\langle w_2, v_1 \rangle = 1 \tag{336}$$
$$\|v_1\|^2 = 2 \tag{337}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{338}$$
$$\Rightarrow o_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{339}$$
$$\Rightarrow v_2 = \begin{pmatrix} {}^{-1}/_{2} \\ {}^{-1}/_{2} \\ 1 \end{pmatrix} \tag{340}$$
$$\|v_2\|^2 = 2 \tag{341}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{342}$$
$$\Rightarrow o_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{343}$$
$$\langle w_3, v_2 \rangle = 0 \tag{344}$$
$$\langle w_3, v_2 \rangle = 0 \tag{345}$$
$$\Rightarrow v_3 = w_3 \tag{346}$$
$$\|v_3\|^2 = 3 \tag{347}$$
$$\Rightarrow \|v_3\| = \sqrt{3} \tag{348}$$
$$\Rightarrow o_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{349}$$

$$\Rightarrow P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{350}$$
$$\Rightarrow P^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{351}$$
$$P^*AP = D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{352}$$

(e) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = 0 \tag{353}$$

$$(1-\lambda)^2(\lambda-4) \tag{354}$$

$$\Rightarrow \lambda_1 = 1 \tag{355}$$

$$\lambda_2 = 4 \tag{356}$$

- For $\lambda_1 = 1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{357}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{358}$$

$$\Rightarrow x_1 = -x_2 + -x_3 \tag{359}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\} \tag{360}$$

- For $\lambda_2 = 4$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{361}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{362}$$

Suppose

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{363}$$

Let

$$v_1 = w_1 \tag{364}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{365}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{366}$$

$$\tag{367}$$

$$\langle w_2, v_1 \rangle = 1 \tag{368}$$
$$\|v_1\|^2 = 2 \tag{369}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{370}$$
$$\Rightarrow o_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{371}$$
$$\Rightarrow v_2 = \begin{pmatrix} {}^{-1}/_{2} \\ {}^{-1}/_{2} \\ 1 \end{pmatrix} \tag{372}$$
$$\|v_2\|^2 = 2 \tag{373}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{374}$$
$$\Rightarrow o_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{375}$$
$$\langle w_3, v_2 \rangle = 0 \tag{376}$$
$$\langle w_3, v_2 \rangle = 0 \tag{377}$$
$$\Rightarrow v_3 = w_3 \tag{378}$$
$$\|v_3\|^2 = 3 \tag{379}$$
$$\Rightarrow \|v_3\| = \sqrt{3} \tag{380}$$
$$\Rightarrow o_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{381}$$

$$\Rightarrow P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{382}$$
$$\Rightarrow P^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{383}$$
$$P^*AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{384}$$

5. Which of the following pairs of matrices are unitarily equivalent?

   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 \tag{385}$$

$$\det\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix} = \lambda^2 - 1 \tag{386}$$

Not unitarily equivalent because they have different eigenvalues.

(b) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & {}^1/_2\\ {}^1/_2 & 0\end{pmatrix}$

$$\det\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix} = (\lambda - 1)(\lambda + 1) \tag{387}$$

$$\det\begin{pmatrix}-\lambda & {}^1/_2\\ {}^1/_2 & -\lambda\end{pmatrix} = \lambda^2 - \frac{1}{4} \tag{388}$$

Not unitarily equivalent because they have different eigenvalues.

(c) $\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$

$$\det\begin{pmatrix}-\lambda & 1 & 0\\ -1 & -\lambda & 0\\ 0 & 0 & 1-\lambda\end{pmatrix} = (1-\lambda)(\lambda + i)(\lambda - i) \tag{389}$$

$$\det\begin{pmatrix}2-\lambda & 0 & 0\\ 0 & -1-\lambda & 0\\ 0 & 0 & -\lambda\end{pmatrix} = (2-\lambda)(-1-\lambda)(-\lambda) \tag{390}$$

Not unitarily equivalent because they have different eigenvalues.

(d) $\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{pmatrix}$

$$A - I_3 = \begin{pmatrix}-1 & 1 & 0\\ -1 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{391}$$

$$\begin{pmatrix}-1 & 1 & 0\\ -1 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{392}$$

$$\begin{pmatrix}0 & 2 & 0\\ -1 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{393}$$

$$\Rightarrow x_2 = 0 \tag{394}$$

$$x_1 = -x_2 = 0 \tag{395}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} : t \in \mathbb{C}\right\} \tag{396}$$

$$A - iI_3 = \begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 1-i \end{pmatrix} \tag{397}$$

$$\begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 1-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{398}$$

$$\begin{pmatrix} -i & 1 & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{399}$$

$$\Rightarrow x_3 = 0 \tag{400}$$

$$-ix_1 = -x_2 \tag{401}$$

$$\Rightarrow x_1 = -ix_2 \tag{402}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{403}$$

$$A + iI_3 = \begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{pmatrix} \tag{404}$$

$$\begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{405}$$

$$\begin{pmatrix} i & 1 & 0 \\ -i & -1 & 0 \\ 0 & 0 & 1+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{406}$$

$$\Rightarrow ix_1 = x_2 \tag{407}$$

$$x_3 = 0 \tag{408}$$

$$\Rightarrow x_1 = ix_2 \tag{409}$$

$$\Rightarrow E_{\lambda_3} = \left\{ t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{410}$$

Suppose

$$w_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \qquad w_3 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \tag{411}$$

Let

$$v_1 = w_1 \tag{412}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{413}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{414}$$

$$\tag{415}$$

$$\langle w_2, v_1 \rangle = 0 \tag{416}$$

$$\Rightarrow v_2 = w_2 \tag{417}$$

$$\langle w_3, v_1 \rangle = 0 \tag{418}$$

$$\langle w_3, v_2 \rangle = 0 \tag{419}$$

$$\Rightarrow v_3 = w_3 \tag{420}$$

$$\|v_1\|^2 = 1 \tag{421}$$

$$\Rightarrow \|v_1\| = 1 \tag{422}$$

$$\Rightarrow o_1 = v_1 \tag{423}$$

$$\|v_2\|^2 = 2 \tag{424}$$

$$\|v_2\| = \sqrt{2} \tag{425}$$

$$\Rightarrow o_2 = \begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{426}$$

$$\|v_3\|^2 = 2 \tag{427}$$

$$\Rightarrow \|v_3\| = \sqrt{2} \tag{428}$$

$$\Rightarrow o_3 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{429}$$

$$\Rightarrow P_1 = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \tag{430}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{431}$$

$$\Rightarrow P_1^* A P_1 = P_2^* B P_2 \tag{432}$$

It follows that A and B are unitarily equivalent.

(e) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

The two matrices have the same eigenvalues however since the former matrix is asymmetric while the latter is symmetric they cannot be orthogonally equivalent.

10. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i \text{ and } \mathrm{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2$$

where λ_i's are the (not necessarily distinct) eigenvalues of A.

(a) Take the diagonal matrix $D = PAP^*$ where the diagonal terms of D are the eigenvalues of A.

$$\Rightarrow A = P^*DP \tag{433}$$

$$\begin{aligned} \mathrm{tr}(A) &= \mathrm{tr}(P^*DP) && (434) \\ &= \mathrm{tr}(P^*(DP)) && (435) \\ &= \mathrm{tr}(P^*(PD)) && (436) \\ &= \mathrm{tr}((P^*P)D) && (437) \\ &= \mathrm{tr}(ID) && (438) \\ &= \mathrm{tr}(D) && (439) \end{aligned}$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i = \mathrm{tr}(D) = \mathrm{tr}(A) \tag{440}$$

(b) Claim: $\mathrm{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2$

$$A = P^*DP \tag{441}$$

$$\begin{aligned} A^* &= (P^*DP)^* && (442) \\ &= P^*D^*P^{**} && (443) \\ &= P^*D^*P && (444) \end{aligned}$$

$$\begin{aligned} \Rightarrow A^*A &= P^*P^*PP^*DP && (445) \\ &= P^*D^*DP && (446) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathrm{tr}(A^*A) &= \mathrm{tr}(P^*D^*DP) && (447) \\ &= \mathrm{tr}((DP)(P^*D^*)) && (448) \\ &= \mathrm{tr}((D(PP^*)D^*) && (449) \\ &= \mathrm{tr}(D(I)D^*) && (450) \\ &= \mathrm{tr}(DD^*) && (451) \end{aligned}$$

Because $D_{ii} = \lambda_i,\ D^*_{ii} = \bar{\lambda}_i$

$$\Rightarrow (DD^*)_{ii} = \lambda_i \bar{\lambda}_i = |\lambda_i|^2 \quad \forall i\ (1 \leq i \leq n) \tag{452}$$

$$\Rightarrow \operatorname{tr}(A) = \operatorname{tr}(DD^*) = \sum_{i=1}^{n} |\lambda_i|^2 \tag{453}$$

21. Let A and B be $n \times n$ matrices that are unitarily equivalent.

(a) Prove that $\operatorname{tr}(A^*A) = \operatorname{tr}(B * B)$

(b) Use (a) to prove that

$$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2$$

(c) Use (b) to show that the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \text{ and } \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$

(a) Claim: $\operatorname{tr}(A^*A) = \operatorname{tr}(B^*B)$

$$A^* = (P^*BP)^* \tag{454}$$
$$= P^*B^*P \tag{455}$$

$$\Rightarrow A^*A = P^*B^*PP^*BP \tag{456}$$
$$= P^*B^*IBP \tag{457}$$
$$= P^*B^*BP \tag{458}$$

$$\Rightarrow \operatorname{tr}(A^*A) = \operatorname{tr}(P^*B^*BP) \tag{459}$$
$$= \operatorname{tr}((BP)(P^*B)) \tag{460}$$
$$= \operatorname{tr}(BB^*) \tag{461}$$
$$= \operatorname{tr}(B^*B) \tag{462}$$

(b) Claim: $\sum\limits_{i,j=1}^{n} |A_{ij}|^2 = \sum\limits_{i,j=1}^{n} |B_{ij}|^2$

$$\operatorname{tr}(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii} \tag{463}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (A^*_{ij})(A_{ji}) \tag{464}$$

$$= \sum_{i,j=1}^{n} \overline{A_{ji}} A_{ji} \tag{465}$$

$$= \sum_{i,j=1}^{n} |A_{ji}|^2 \tag{466}$$

$$\Rightarrow \sum_{i,j=1}^{n} |B_{ji}|^2 = \operatorname{tr}(B^*B) \tag{467}$$

$$= \operatorname{tr}(A^*A) \tag{468}$$

$$= \sum_{i,j=1}^{n} |A_{ji}|^2 \tag{469}$$

(c) Show that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \text{ and } \qquad B = \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix} \tag{470}$$

are not unitarily equivalent.

$$\sum_{i,j=1}^{n} |A_{ji}|^2 = 10 \tag{471}$$

$$\sum_{i,j=1}^{n} |B_{ji}|^2 = 19 \tag{472}$$

$$\Rightarrow \operatorname{tr}(B^*B) \neq \operatorname{tr}(A^*A) \quad \text{by part (b)} \tag{473}$$

Suppose $A = P^*BP$ for some unitary matrix P.

$$\operatorname{tr}(A^*A) = \operatorname{tr}(P^*B^*PP^*BP) \tag{474}$$

$$= \operatorname{tr}(P^*B^*IBP) \tag{475}$$

$$= \operatorname{tr}((BP)(P^*B^*)) \tag{476}$$

$$= \operatorname{tr}(BIB^*) \tag{477}$$

$$= \operatorname{tr}(BB^*) \tag{478}$$

$$= \operatorname{tr}(B^*B) \ \lightning \ \text{Contradiction!} \tag{479}$$

It follow that $A \neq P^*BP$ and thus A and B are not unitarily equivalent.