

## Assignment

Section 4.4: 1, 5, 6; Section 5.1: 3(bc), 4(ceh), 7, 12, 14, 15, 19, 22

## Work

### 4.4

- Label the statements as true or false.

(a)	True	(g)	True
(b)	True	(h)	False
(c)	True	(i)	True
(d)	False	(j)	True
(e)	False	(k)	True
(f)	True		

- Suppose that  $M \in \mathbf{M}_{n \times n}(F)$  can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix}$$

where  $A$  is a square matrix. Prove that  $\det(M) = \det(A)$

Suppose  $A \in \mathbf{M}_{k \times k}(F)$

Perform Type 3 operations such that the partition  $(A \ B)$  becomes an upper triangular matrix.

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & I \end{pmatrix} \quad (1)$$

$M'$  is an upper triangular matrix so the determinant of  $M$  is the product of its diagonal terms.

$$\Rightarrow \det(A) = \prod_{i=1}^n M'_{ii} \quad (2)$$

$$M'_{ii} = 1 \quad \text{if } (k+1 \leq i \leq n) \quad (3)$$

$$\Rightarrow \prod_{i=1}^n M'_{ii} = \prod_{i=1}^k M'_{ii} \quad (4)$$

The first  $k$  diagonal terms of  $M'$  are the diagonal terms of  $A$ . It follows that

$$\prod_{i=1}^k M'_{ii} = \prod_{i=1}^k A'_{ii} \quad (5)$$

Because  $A'$  was obtained from  $A$  using Type 3 operations, and  $M'$  was obtained from  $M$  using Type 3 operations

$$\det(A') = \det(A) \quad (6)$$

$$\det(M') = \det(M) \quad (7)$$

$$\therefore \det(M) = \det(M') = \det(A') = \det(A) \quad (8)$$

6. Prove that if  $M \in \mathbf{M}_{n \times n}(F)$  can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where  $A$  and  $C$  are square matrices, then  $\det(M) = \det(A) \cdot \det(C)$ .

$M$  can be reduced using strictly Type 3 row operations such that the partitions  $(A \ B)$  become an upper triangular matrix  $(A' \ B')$ .  $M$  can also be reduced using strictly Type 3 row operations such that the partition  $(O \ C)$  becomes the matrix  $(O \ C')$  where  $C'$  is an upper triangular matrix in  $\mathbf{M}_{(n-k) \times (n-k)}(F)$  where  $A \in \mathbf{M}_{n \times n}(F)$ .

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & C' \end{pmatrix} \quad (9)$$

$M'$  is upper triangular, so the determinant of  $M'$  is a the product of the diagonal terms.

$$\det(M') = \prod_{i=1}^n M'_{ii} \quad (10)$$

$$= \left( \prod_{i=1}^k M'_{ii} \right) \left( \prod_{i=k+1}^n M'_{ii} \right) \quad (11)$$

$$M'_{ii} = A'_{ii} \quad \forall i \ (1 \leq i \leq k) \quad (12)$$

$$\Rightarrow \prod_{i=1}^k M'_{ii} = \prod_{i=1}^k A'_{ii} \quad (13)$$

$$= \det(A') \quad (14)$$

$$M'_{k+1,k+1} = C'_{ii} \quad \forall i \ (1 \leq i \leq n-k) \quad (15)$$

$$\Rightarrow \prod_{i=k+1}^n M'_{ii} = \prod_{i=1}^{n-k} C'_{ii} \quad (16)$$

$$= \det(C') \quad (17)$$

Matrices  $C'$ ,  $A'$  and  $M'$  were obtained respectively from the matrices  $C$ ,  $A$  and  $M$  strictly using Type 3 row operations. It follows that

$$\det(M') = \det(M) \quad (18)$$

$$\det(A') = \det(A) \quad (19)$$

$$\det(C') = \det(C) \quad (20)$$

$$\therefore \det(M) = \det(A) \cdot \det(C) \quad (21)$$

## 5.1

3. For each of the following matrices  $A \in \mathbf{M}_{n \times n}(F)$ ,

- (i) Determine all the eigenvalues of  $A$ .
- (ii) For each eigenvalue  $\lambda$  of  $A$ , find the set of eigenvectors corresponding to  $\lambda$ .
- (iii) If possible, find a basis for  $F^n$  consisting of eigenvectors of  $A$ .
- (iv) If succesful in finding such a basis, determine and invertible matrix  $Q$  and a diagonal matrix  $D$  such that  $Q^{-1}AQ = D$ .

$$(b) \ A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \text{ for } F = \mathbb{R}$$

(i)

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1 - \lambda & -1 \\ 2 & 2 & 5 - \lambda \end{pmatrix} = 0 \quad (22)$$

$$\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1 - \lambda & -1 \\ 2 & 2 & 5 - \lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1 - \lambda & -1 \\ 0 & 4 - 2\lambda & 3 - \lambda \end{pmatrix} \quad (23)$$

$$\det(A - \lambda I) = (-\lambda)(\lambda - 1)(\lambda - 3) + 3(4 - 2\lambda) - \lambda(4 - 2\lambda) - 2(3 - \lambda) \quad (24)$$

$$= \lambda^3 + 6\lambda^2 - 11\lambda + 6 \quad (25)$$

$$= (\lambda - 3)(\lambda - 2)(-\lambda_1) = 0 \quad (26)$$

$$\Rightarrow \lambda = \{3, 2, 1\} \quad (27)$$

(ii) • For  $\lambda = 3$

$$Av = 3v \quad (28)$$

$$(A - 3I)v = 0 \quad (29)$$

$$\left( \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ -0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right) v = 0 \quad (30)$$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} v = 0 \quad (31)$$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (32)$$

$$x_1 = -t \quad (33)$$

$$x_2 = 0 \quad (34)$$

$$x_3 = t \quad (35)$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (36)$$

• For  $\lambda = 2$

$$Av = 2v \quad (37)$$

$$(A - 2I)v = 0 \quad (38)$$

$$\left( \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right) v = 0 \quad (39)$$

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} v = 0 \quad (40)$$

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (41)$$

$$x_1 = -t \quad (42)$$

$$x_2 = t \quad (43)$$

$$x_3 = 0 \quad (44)$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (45)$$

- For  $\lambda = 1$

$$Av = v \quad (46)$$

$$(A - I)V = 0 \quad (47)$$

$$\left( \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & \\ -1 & & \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \quad (48)$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} v = 0 \quad (49)$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (50)$$

$$x_1 = -t \quad (51)$$

$$x_2 = -t \quad (52)$$

$$x_3 = t \quad (53)$$

$$v = \left\{ t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (54)$$

(iii)

$$\beta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \quad (55)$$

(iv)

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \quad (56)$$

$$\left( \begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 & -1 \end{array} \right) \quad (57)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \quad (58)$$

$$D = QAQ^{-1} \quad (59)$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \quad (60)$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (61)$$

(c)  $\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$  for  $F = \mathbb{C}$   
(i)

$$\det(A - \lambda I) = \det \begin{pmatrix} i - \lambda & 1 \\ 2 & -i - \lambda \end{pmatrix} = 0 \quad (62)$$

$$= -(i - \lambda)(i + \lambda) - 2 = 0 \quad (63)$$

$$= i^2 - \lambda^2 + 2 = 0 \quad (64)$$

$$\lambda^2 = 1 \quad (65)$$

$$\lambda = \pm 1 \quad (66)$$

(ii) • For  $\lambda = 1$

$$Av = \lambda v \quad (67)$$

$$(A - I)v = 0 \quad (68)$$

$$\begin{pmatrix} i - 1 & 1 \\ 2 & -i - 1 \end{pmatrix} v = 0 \quad (69)$$

$$\begin{pmatrix} i - 1 & 1 \\ 2 & -i - 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i - 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (70)$$

$$x_1 = -\frac{t}{i - 1} \quad (71)$$

$$x_2 = t \quad (72)$$

$$v = \left\{ t \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \quad (73)$$

• For  $\lambda = -1$

$$Av = \lambda v \quad (74)$$

$$(A + I)v = 0 \quad (75)$$

$$\begin{pmatrix} i + 1 & 1 \\ 2 & -i + 1 \end{pmatrix} v = 0 \quad (76)$$

$$\begin{pmatrix} i + 1 & 1 \\ 2 & -i + 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i + 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (77)$$

$$x_1 = -\frac{t}{i+1} \quad (78)$$

$$x_2 = t \quad (79)$$

$$v = \left\{ t \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \quad (80)$$

(iii)

$$\beta = \left\{ \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} \right\} \quad (81)$$

(iv)

$$Q = \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \quad (82)$$

$$\left( \begin{array}{cc|cc} \frac{i+1}{2} & \frac{i-1}{2} & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{cc|cc} 1 & 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & -1 & \frac{1+i}{2} \end{array} \right) \quad (83)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \quad (84)$$

$$D = Q^{-1}AQ \quad (85)$$

$$= \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \quad (86)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (87)$$

4. For each linear operator  $T$  on  $V$ , find the eigenvalues of  $T$  and an ordered bases  $\beta$  of  $V$  such that  $[T]_\beta$  is a diagonal matrix.

(c)  $V = \mathbb{R}^3$  and  $T(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (88)$$

$$[T]_\alpha = \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & 6 & 11 \end{pmatrix} \quad (89)$$

$$\det ([\mathbf{T}]_{\alpha} - \lambda I_3) = \det \begin{pmatrix} -4 - \lambda & 3 & -6 \\ 6 & -7 - \lambda & 12 \\ 6 & -6 & 11 - \lambda \end{pmatrix} \quad (90)$$

$$= \det \begin{pmatrix} -4 - \lambda & 4 & -6 \\ 6 & -7 - \lambda & 12 \\ 0 & 1 + \lambda & 1 + \lambda \end{pmatrix} \quad (91)$$

$$= (\lambda + 4)(\lambda + 7)(\lambda + 1) \quad (92)$$

$$- 2(\lambda + 1)(\lambda + 1)(\lambda + 4) - 18(\lambda + 1) \quad (93)$$

$$= -\lambda^3 + 3\lambda + 2 = 0 \quad (94)$$

$$= -(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0 \quad (95)$$

$$\Rightarrow \lambda = \{2, -1\} \quad (96)$$

• For  $\lambda = 2$

$$[\mathbf{T}]_{\alpha} v = 0 \quad (97)$$

$$([\mathbf{T}]_{\alpha} - 2I)v = 0 \quad (98)$$

$$\left( \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) v = 0 \quad (99)$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} v = 0 \quad (100)$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (101)$$

$$x_1 = -\frac{1}{2}t \quad (102)$$

$$x_2 = t \quad (103)$$

$$x_3 = t \quad (104)$$

$$v = \left\{ t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (105)$$



- For  $\lambda = -1$

$$[\mathbf{T}_\alpha]v = -v \quad (106)$$

$$[\mathbf{T}]_\alpha v = 0 \quad (107)$$

$$\left( \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) v = 0 \quad (108)$$

$$\begin{pmatrix} -3 & 3 & -6 \\ 6 & -6 & 12 \\ 6 & -6 & 12 \end{pmatrix} v = 0 \quad (109)$$

$$\begin{pmatrix} -3 & 3 & -6 \\ 6 & -6 & 12 \\ 6 & -6 & 12 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (110)$$

$$x_1 = s - 2t \quad (111)$$

$$x_2 = s \quad (112)$$

$$x_3 = t \quad (113)$$

$$v = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R}^3 \right\} \quad (114)$$

$$\beta = \left\{ \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (115)$$

$$[\mathbf{T}]_\beta = Q^{-1}[\mathbf{T}]_\alpha Q \quad (116)$$

$$Q = \begin{pmatrix} -1/2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (117)$$

$$\left( \begin{array}{ccc|ccc} -1/2 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & -2/5 & 4/5 \\ 0 & 1 & 0 & 2/5 & 3/5 & 4/5 \\ 0 & 0 & 1 & -2/5 & 2/5 & 1/5 \end{array} \right) \quad (118)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 2/5 & -2/5 & 4/5 \\ 2/5 & 3/5 & 4/5 \\ -2/5 & 2/5 & 1/5 \end{pmatrix} \quad (119)$$

$$[\mathbf{T}]_\beta = \begin{pmatrix} 2/5 & -2/5 & 4/5 \\ 2/5 & 3/5 & 4/5 \\ -2/5 & 2/5 & 1/5 \end{pmatrix} \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix} \begin{pmatrix} -1/2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (120)$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (121)$$

(e)  $V = P_3(\mathbb{R})$  and  $T(f(x)) = xf'(x) + f(2)x + f(3)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (122)$$

$$[T]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad (123)$$

$$\det([T]_\alpha - \lambda I) = 0 \quad (124)$$

$$= \det \begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} \quad (125)$$

$$= (1-\lambda)(3-\lambda)(2-\lambda) - 6(2-\lambda) \quad (126)$$

$$= \lambda(2-\lambda)(\lambda-4) \quad (127)$$

$$\lambda = \{0, 2, 4\} \quad (128)$$

• For  $\lambda = 0$

$$Av = 0v \quad (129)$$

$$Av = 0 \quad (130)$$

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (131)$$

$$x_1 = -3t \quad (132)$$

$$x_2 = t \quad (133)$$

$$x_3 = 0 \quad (134)$$

$$v = \left\{ t \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (135)$$

• For  $\lambda = 2$

$$Av = 2v \quad (136)$$

$$(A - 2I)v = 0 \quad (137)$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} v = 0 \quad (138)$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 3 & 4 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{pmatrix} \quad (139)$$

$$x_1 = -\frac{39}{4}t + 9t \quad (140)$$

$$x_2 = -\frac{12}{4}t \quad (141)$$

$$x_3 = t \quad (142)$$

$$v = \left\{ t \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (143)$$

• For  $\lambda = 4$

$$Av = 4v \quad (144)$$

$$(A - 4I)v = 0 \quad (145)$$

$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} v = 0 \quad (146)$$

$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 21 \\ 1 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad (147)$$

$$x_1 = t \quad (148)$$

$$x_2 = t \quad (149)$$

$$x_3 = 0 \quad (150)$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (151)$$

$$\beta = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -12 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad (152)$$

$$\Rightarrow \{(-3+x), (-3-13x+4x^2), (1+x)\} \quad (153)$$

$$Q = \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad (154)$$

$$\left( \begin{array}{ccc|ccc} -3 & -3 & 1 & 1 & 0 & 0 \\ 1 & -13 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 1/4 & 5/8 \\ 0 & 1 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 1/4 & 3/4 & 21/8 \end{array} \right) \quad (155)$$

$$Q^{-1} = \begin{pmatrix} -1/4 & 1/4 & 5/8 \\ 0 & 0 & 1/4 \\ 1/4 & 3/4 & 21/8 \end{pmatrix} \quad (156)$$

$$[\mathbf{T}]_\beta = Q^{-1}[\mathbf{T}]_\alpha Q \quad (157)$$

$$= \begin{pmatrix} -1/4 & 1/4 & 5/8 \\ 0 & 0 & 1/4 \\ 1/4 & 3/4 & 21/8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad (158)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad (159)$$

$$(h) \quad V = M_{n \times n}(\mathbb{R}) \text{ and } \mathbf{T} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (160)$$

$$[\mathbf{T}]_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (161)$$

$$\det([\mathbf{T}]_\alpha - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \quad (162)$$

$$\begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} \quad (163)$$

$$\det \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} = 0 \quad (164)$$

$$\Rightarrow (1-\lambda)^2(1-\lambda^2) = 0 \quad (165)$$

$$\Rightarrow \lambda = \pm 1 \quad (166)$$

- For  $\lambda = 1$

$$[\mathbf{T}]_\alpha v = v \quad (167)$$

$$([[\mathbf{T}]_\alpha - I]v = 0 \quad (168)$$

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} v = 0 \quad (169)$$

$$x_1 = t \quad (170)$$

$$x_2 = k \quad (171)$$

$$x_3 = s \quad (172)$$

$$x_4 = t \quad (173)$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t, k, s \in \mathbb{R} \right\} \quad (174)$$

• For  $\lambda = -1$

$$[\mathbf{T}]_{\alpha} v = v \quad (175)$$

$$([\mathbf{T}]_{\alpha} + I)v = 0 \quad (176)$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} v = 0 \quad (177)$$

$$x_1 = -t \quad (178)$$

$$x_2 = 0 \quad (179)$$

$$x_3 = 0 \quad (180)$$

$$x_4 = t \quad (181)$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (182)$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (183)$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (184)$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad (185)$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 0 & 1/2 \end{array} \right) \quad (186)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix} \quad (187)$$

$$[T]_{\beta} = Q^{-1}[T]_{\alpha}Q \quad (188)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (189)$$

7. Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ . We define the **determinant** of  $T$ , denoted  $\det(T)$ , as follows: Choose any ordered basis  $\beta$  for  $V$ , and define  $\det(T) = \det([T]_{\beta})$ .

- (a) Prove that the preceding definition is independent of the choice of an ordered basis for  $V$ . That is, prove that if  $\beta$  and  $\gamma$  are two ordered bases for  $V$ , then  $\det([T]_{\beta}) = \det([T]_{\gamma})$ .
- (b) Prove that  $T$  is invertible if and only if  $\det T \neq 0$ .
- (c) Prove that if  $T$  is invertible, then  $\det(T^{-1}) = [\det(T)]^{-1}$ .
- (d) Prove that if  $U$  is also a linear operator on  $V$ , then  $\det(TU) = \det(T) \cdot \det(U)$ .
- (e) Prove that  $\det(T - \lambda I_V) = \det([T]_{\beta} - \lambda I)$  for any scalar  $\lambda$  and any ordered basis  $\beta$  for  $V$ .

(a) Suppose  $Q$  is the change of coordinates matrix from  $\gamma$  to  $\beta$ .

$$Q = [I_V]_{\gamma}^{\beta} \Rightarrow Q^{-1} = [I_V]_{\beta}^{\gamma} \quad (190)$$

$$[I_V]_{\gamma}^{\beta} [T]_{\gamma} [I_V]_{\beta}^{\gamma} = [T]_{\beta} \quad (191)$$

$$\Rightarrow \det([I_V]_{\gamma}^{\beta} [T]_{\gamma} [I_V]_{\beta}^{\gamma}) = \det[T]_{\beta} \quad (192)$$

$$\Rightarrow \det[I_V]_{\gamma}^{\beta} \det[T]_{\gamma} \det[I_V]_{\beta}^{\gamma} \quad (193)$$

$$[I_V]_{\gamma}^{\beta} = ([I_V]_{\beta}^{\gamma})^{-1} \quad (194)$$

$$\det[I_V]_{\gamma}^{\beta} = \det([I_V]_{\beta}^{\gamma})^{-1} \quad (195)$$

$$\Rightarrow \det [\mathbf{l}_V]_\gamma^\beta \det [\mathbf{T}]_\gamma \det [\mathbf{l}_V]_\beta^\gamma = \det ([\mathbf{l}_V]_\beta^\gamma)^{-1} \det [\mathbf{T}]_\gamma \det [\mathbf{l}_V]_\beta^\gamma \quad (196)$$

$$= \det [\mathbf{T}]_\gamma = \det [\mathbf{T}]_\beta \quad (197)$$

(b) ( $\Rightarrow$ )

Suppose  $\mathbf{T}$  is invertible

Suppose  $\beta$  is an ordered basis of  $V$ .

$$[\mathbf{l}_V]_\beta = [\mathbf{T} \cdot \mathbf{T}^{-1}]_\beta = [\mathbf{T}]_\beta [\mathbf{T}^{-1}]_\beta \quad (198)$$

$$\Rightarrow \det [\mathbf{l}_V]_\beta = \det [\mathbf{T}]_\beta \det [\mathbf{T}^{-1}]_\beta \quad (199)$$

$$\det [\mathbf{l}_V]_\beta = 1 \quad (200)$$

$$\Rightarrow \det [\mathbf{T}]_\beta \det [\mathbf{T}^{-1}]_\beta = 1 \quad (201)$$

$$\Rightarrow \det [\mathbf{T}]_\beta \neq 0 \quad (202)$$

$$\Rightarrow \det \mathbf{T} \neq 0 \quad (203)$$

( $\Leftarrow$ )

Suppose  $\det \mathbf{T} \neq 0$

$$\Rightarrow \det [\mathbf{T}]_\beta \neq 0 \quad \text{for some ordered basis } \beta \text{ of } V \quad (204)$$

$$\det [\mathbf{T}]_\beta \neq 0 \quad (205)$$

$$\Rightarrow \mathbf{T} \text{ is invertible, by corollary to Th. 2.18} \quad (206)$$

(c) Suppose  $\mathbf{T}$  is invertible and  $\beta$  is some ordered basis of  $V$ .

$$\det \mathbf{l}_V = \det \mathbf{T} \cdot \mathbf{T}^{-1} \quad (207)$$

$$= \det [\mathbf{T} \cdot \mathbf{T}^{-1}]_\beta \quad (208)$$

$$= \det [\mathbf{T}]_\beta [\mathbf{T}^{-1}]_\beta \quad (209)$$

$$= \det [\mathbf{T}]_\beta \det [\mathbf{T}^{-1}]_\beta \quad (210)$$

$$= \det \mathbf{T} \det \mathbf{T}^{-1} \quad (211)$$

$$\det \mathbf{l}_V = \det [\mathbf{l}_V]_\beta = \det I_n = 1 \quad (212)$$

$$\Rightarrow \det \mathbf{T} \det \mathbf{T}^{-1} = 1 \quad (213)$$

$$\Rightarrow \det \mathbf{T}^{-1} = (\det \mathbf{T})^{-1} \quad (214)$$

(d) Suppose  $\beta$  is an ordered basis of  $V$ .

$$\det \mathbf{TU} = \det [\mathbf{TU}]_\beta \quad (215)$$

$$= \det [\mathbf{T}]_\beta [\mathbf{U}]_\beta \quad (216)$$

$$= \det [\mathbf{T}]_\beta \det [\mathbf{U}]_\beta \quad (217)$$

$$= \det \mathbf{T} \det \mathbf{U} \quad (218)$$

(e)

$$\det ([T]_\beta - \lambda I) = \det ([T]_\beta - \lambda [I_V]_\beta) \quad (219)$$

$$= \det ([T]_\beta - [\lambda I_V]_\beta) \quad (220)$$

$$= \det [T_\beta - \lambda I_V]_\beta \quad (221)$$

$$= \det (T - \lambda I_V) \quad (222)$$

12. (a) Prove that similar matrices have the same characteristic polynomial.  
 (b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space  $V$  is independent of the choice of basis for  $V$ .

- (a) Suppose  $A, B \in M_{n \times n}(F)$  such that  $A = Q^{-1}BQ$  for some invertible  $Q \in M_{n \times n}(F)$

Claim:  $\det (A - \lambda I_n) = \det (B - \lambda I_n)$

$$\det (A - \lambda I_n) = \det (Q^{-1}BQ - \lambda I_n) \quad (223)$$

$$= \det (Q^{-1}BQ - \lambda Q^{-1}Q) \quad (224)$$

$$= \det Q^{-1} \det (BQ - \lambda Q) \quad (225)$$

$$= \det Q^{-1} \det (B - \lambda) \det Q \quad (226)$$

$$= (\det Q)^{-1} \det Q \det (B - \lambda I_n) \quad (227)$$

$$= \det (B - \lambda I_n) \quad (228)$$

- (b) Suppose  $T \in \mathcal{L}(V)$  such that  $V$  is finitely dimensioned and  $\beta$  and  $\gamma$  are ordered bases of  $V$

Let  $A = [T]_\beta$ ,  $B = [T]_\gamma$

Let  $Q$  be the change of coordinates matrix from  $\gamma$  to  $\beta$ .

$$\Rightarrow Q = [I_V]_\gamma^\beta \quad (229)$$

$$A = [T]_\beta = [I_V]_\gamma^\beta [T]_\gamma [I_V]_\beta^\gamma = [I_V]_\gamma^\beta B [I_V]_\beta^\gamma \quad (230)$$

$$[I_V]_\gamma^\beta = ([I_V]_\beta^\gamma)^{-1} \quad (231)$$

It follows that  $A$  is similar to  $B$  and therefore by part (a)

$$\det ([T]_\beta - \lambda I_n) = \det ([T]_\gamma - \lambda I_n) \quad (232)$$

14. For any square matrix  $A$ , prove that  $A$  and  $A^t$  have the same characteristic polynomial.

Suppose  $A \in M_{n \times n}$

Claim:  $\det (A - \lambda I_n) = \det (A^t - \lambda I_n)$ .



$$(A - \lambda I_n)^t = A^t + (-\lambda)(I_n)^t \quad (233)$$

$$= A^t - \lambda I_n \quad (234)$$

$$= A - \lambda I_n \quad (235)$$

$$\det(A - \lambda I_n) = \det(A - \lambda)^t \quad (\text{by theorem 4.8}) \quad (236)$$

$$\Rightarrow \det(A - \lambda I_n) = \det(A^t - \lambda I_n) \quad (237)$$

15. (a) Let  $T$  be a linear operator on a vector space  $V$ , and let  $x$  be an eigenvector of  $T$  corresponding to the eigenvalue  $\lambda$ . For any positive integer  $m$ , prove that  $x$  is an eigenvector of  $T^m$  corresponding to the eigenvalue  $\lambda^m$ .
- (b) State and prove the analogous result for matrices.
- (a) Suppose  $T \in \mathcal{L}(V)$  such that  $x$  is an eigenvector corresponding to  $\lambda$ .

Proof by induction.

Base Case;  $n = 1$

$$T^1(x) = \lambda^1 x \quad (238)$$

Suppose true for  $1 \leq m \leq k$

$$\Rightarrow T^m(x) = \lambda^m x \quad (239)$$

Show True for  $m = k + 1$

$$T^{m+1}(x) = T(T^m(x)) \quad (240)$$

$$= T(\lambda^m x) \quad (241)$$

$$= \lambda^m T(x) \quad (242)$$

$$= \lambda^m \lambda \quad (243)$$

$$= \lambda^{m+1} \quad (244)$$

- (b) Let  $A \in M_{n \times n}$  and let  $x$  be an eigenvector corresponding to the eigenvalue  $\lambda$ . For any positive integer  $m$  prove that  $x$  is an eigenvector of  $A^m$  corresponding to the eigenvalue  $\lambda^m$ .

Suppose  $A \in M_{n \times n}(F)$  and let  $x$  be an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda$ .

Proof by induction.

Base Case;  $n = 1$

$$Ax = \lambda x \tag{245}$$

Suppose true for  $1 \leq m \leq k$

Suppose  $m = k + 1$ .

$$A^{k+1}x = A(A^k x) \tag{246}$$

$$= A(\lambda^k x) \tag{247}$$

$$= \lambda^k (Ax) \tag{248}$$

$$= \lambda^k \lambda \tag{249}$$

$$= \lambda^{k+1} \tag{250}$$

29.