Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

Work

3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\ x_2 + 2x_4 &= 3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 9 & -4 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \tag{1}$$

$$x_1 = -3 + 4x_4 \tag{2}$$
$$x_2 = 3 - 2x_4 \tag{3}$$
$$x_3 = 1 \tag{4}$$
$$x_4 = x_4 \tag{5}$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{6}$$

(j)

$$2x_1 + 3x_3 - 4x_5 = 5 \tag{7}$$
$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8 \tag{8}$$
$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2 \tag{9}$$
$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8 \tag{10}$$

$$\left(\begin{array}{ccccc|c} 2 & 0 & 3 & 0 & -4 & 5 \\ 3 & -4 & 8 & 3 & 0 & 8 \\ 1 & -1 & 2 & 1 & -1 & 2 \\ -2 & 5 & -9 & -3 & -5 & -8 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \tag{11}$$

$$x_1 = 2x_5 + 1 \tag{12}$$
$$x_2 = 3x_5 \tag{13}$$
$$x_3 = -1 \tag{14}$$
$$x_4 = 2x_5 - 1 \tag{15}$$
$$x_5 = x_5 \tag{16}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{17}$$

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate W. Find a subset $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

$$\mathsf{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \ldots, x_5 \in \mathbb{R} \right\} \tag{18}$$

$$
\begin{pmatrix}
2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\
-3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\
4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\
-5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\
2 & -6 & 1 & 6 & -3 & 12 & -2 & 7
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
1 & -3 & {}^{3}/_{2} & 1 & {}^{-1}/_{2} & 0 & {}^{1}/_{2} & 1 \\
0 & 0 & 1 & -2 & {}^{-1}/_{5} & {}^{-6}/_{5} & {}^{3}/_{5} & {}^{4}/_{5} \\
0 & 0 & 0 & 0 & {}^{21}/_{5} & {}^{-84}/_{5} & {}^{-23}/_{5} & {}^{-19}/_{5} \\
0 & 0 & 0 & 0 & {}^{-9}/_{5} & {}^{36}/_{5} & {}^{32}/_{5} & {}^{-14}/_{5} \\
0 & 0 & 0 & 0 & {}^{-12}/_{5} & {}^{48}/_{5} & {}^{-9}/_{5} & {}^{33}/_{5}
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
1 & -3 & {}^{3}/_{2} & 1 & {}^{-1}/_{2} & 0 & {}^{1}/_{2} & 1 \\
0 & 0 & 1 & -2 & {}^{-1}/_{5} & {}^{-6}/_{5} & {}^{3}/_{5} & {}^{4}/_{5} \\
0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\
 & & 1 & -2 & 0 & -2 & 0 & 1 \\
 & & & & 1 & -4 & 0 & -2 \\
 & \text{\Large 0} & & & & & 1 & -1 \\
 & & & & & & & 0
\end{pmatrix}
\tag{19}
$$

It follows that $\{u_1, u_3, u_5, u_7\}$ is linearly independent by theorem 3.16. Therefore $\{u_1, u_3, u_5, u_7\}$ is a basis for W.

11. Let V be as in Exercise 10.

 (a) Show that $S = \{(1, 2, 1, 0, 0)\}$ is a linearly independent subset of V.

 (b) Extend S to a basis for V.

 (a) Claim: S is a linearly independent subset of V.
 For $x \in S, x = (1, 2, 1, 0, 0)$

$$
\begin{aligned}
1 + (-2)(2) + 3(1) + (-1)(0) + (2)(0) = 0 && (20)\\
\Rightarrow x \in \mathsf{V} && (21)\\
\Rightarrow S \subseteq \mathsf{V} && (22)
\end{aligned}
$$

 Suppose $cx - 0$

$$
\begin{aligned}
c(1, 2, 1, 0, 0) = 0 && (23)\\
\Rightarrow (c, 2c, c, 0, 0) = 0 && (24)\\
\Rightarrow c = 0 && (25)
\end{aligned}
$$

 It follows that S is linearly independent.

(b) Suppose $x_1, x_2, \ldots, x_5 \in F \colon x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 + x_4 - 2x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{26}$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{27}$$

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{28}$$

Where β is a basis V.

$$(S \,|\, B) = \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow (S' \,|\, B') = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

Choose the linearly independent columns of $(S' \,|\, B')$. The entries of these columns provide the coefficients for the linearly combinations of columns of $(S \,|\, B)$ that determine a basis for S. Thus β is a basis for S.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{30}$$

14. If $(A\,|b)$ is in reduced row echelon form, prove that A is also in reduced row echelon form.

    Suppose $(A\,|b)$ is an element of $\mathsf{M}_{m\times(n+1)}(F)$ such that. $(A\,|b)$ is in row reduced echelon form. To complete this proof, it is necessary to show that A satisfies the definition of reduced row echelon form.

    Claim $(A\,|b)$ in row reduced echelon form implies A in row reduced echelon form

    It is necessary to show that A satisfies the definition of reduced row echelon form.

    (a) Claim: any row of A containing a nonzero entry precedes any row in which all the entries are zero (if any)

        Case 1: The last column consists entirely of zeroes. $(A\,|b)$ is in row reduced echelon form, so removing the last column still satisfies this requirement

        Case 2: There is a nonzero value in the last column Because there is a nonzero entry, it is the first nonzero entry of its row. However, it is in the last column position, and thus all other values in the column are zero. By property C of the row reduced echelon form, all subsequent rows (if any) must consist entirely of zeroes. Thus, the partition of the row reduced echelon form of $(A\,|b)$ that removes the last column yields matrix A which satisfies this property.

    (b) Claim: The first nonzero entry in each row is the only nonzero entry in its column.

        Removing the last column yields a matrix in which the first nonzero entry in each row is the only nonzero entry in its column.

    (c) Claim: The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

        Case 1: The last column consists entirely of zeroes. Thus, removing the last column from the row reduced echelon form of $(A\,|b)$ yields a matrix in which the first nonzero entry in each row is 1 nd it occurs to the righ tof the first nonzero entry in the preceding row.

        Case 2: There is a non zero value in the last column If there is a nonzero value in the last column, than that row consists entirely of zeroes otherwise and all subsequent rows (if any) consist entire of zeroes. Thus, removing this column from the row reduced echelon form of $(A\,|b)$ yields a matrix that satisfies this property.

15. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Suppose $A \in \mathsf{M}_{m\times n}(F)$ such that $\operatorname{rank}(A) = r \leq \min\{m, n\}$

Proof by induction

Suppose $n = 1$

Case 1

$$A = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{31}$$

A is in reduced row echelon form and this the unique representation.

Case 2

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ A_{m1} \end{pmatrix} \qquad \text{for some } a_{i1} \neq 0 \tag{32}$$

Execute the following sequence of row operations on A. Perform type 1 row operation to move a_{i1} to the first row. Perform type 3 row operations to eliminate all lower terms. Perform type 2 row operation to the change the first term in the column to 1.

$$A \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{33}$$

The reduced row echelon form of a column must be of this form. It follows that this matrix is the unique row reduced echelon form.

Suppose true for $1 \leq n \leq k$

Suppose $n = k + 1$

Suppose $A \in \mathsf{M}_{m\times k+1}(F)$. Suppose $A = (A' \,|b)$ such that $A' \in \mathsf{M}_{m\times k}(F)$ and $b \in \mathsf{M}_{m\times 1}(F)$

Let $(B' \,|b')$ be the reduced row echelon form of $(A' \,|b)$ By HW.3.4.14, B' is in reduced row echelon form. It follows from $B' \in \mathsf{M}_{m\times k}(F)$ that, by the induction hypothesis, that B' is in the unique row reduced echelon form of A'.

It remains to show that the column b' is unique.

Case 1: $b \notin \mathrm{col}(A')$

If b is not in the column space of A', then b' is not in the column space of B'. It follows that b' cannot be replaced as a linear combination of vectors from a basis for the column space of B'. Column b' is replaced as:

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

Where $b'_{r+1,1} = 1$. This is the unique representation. Suppose b' were

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

where $b'_{r+j,1} = 1, j > 1$. B' is in reduced row echelon form and is of rank r . It follows that all rows after the r^{th} are entirely zero. $(B' \,|b')$ is not in reduced row echelon form because there is a row of zeros above a row containing a nonzero value.

Case 2: $b \in \mathrm{col}(A')$

$$\Rightarrow b' \in \mathrm{col}(B') \tag{36}$$

$$b' = \sum_{i=1}^{r} c_i v_i \tag{37}$$

For $c_i \in F$ and $v_i \in \beta$ where β is a basis for $\mathrm{col}(B')$ such that β is a subset of the standard ordered basis for F^m. Because β is linearly independent, the coefficients of the linear combination are unique.

$$b' \in \mathrm{col}(B') \tag{38}$$

$$\Rightarrow \mathrm{rank}(B') = \mathrm{rank}(B' \,|b') \tag{39}$$

It follows that all rows of $(B' \,|b')$ after the r^{th} are entirely zero. Therefore $(B' \,|b')$.

4.1

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]\, I$

(b) $\det(C) = \det(A)$

(c) The classical adjoint of A^t is C^t

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}\, C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{40}$$

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{41}$$

$$\begin{aligned} AC &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \quad (42)\\ &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \quad (43)\\ &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \quad (44) \end{aligned}$$

$$\begin{aligned} CA &= \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (45)\\ &= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \quad (46)\\ &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \quad (47) \end{aligned}$$

$$\begin{aligned} \det(A) &= A_{11}A_{22} - A_{12}A_{21} \quad (48)\\ \Rightarrow AC &= CA = \det(A) I_2 \quad (49) \end{aligned}$$

(b) $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{50}$$

$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \tag{51}$$

It follows that D is the classical adjoint of A^t

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \tag{52}$$

It follows that C^t is the classical adjoint of A^t

(d) Suppose A is invertible

$$\Rightarrow \exists B\colon AB = BA = I \tag{53}$$
$$\Rightarrow \det(A) \neq 0 \tag{54}$$

$$CA = AC = \det(A)I \tag{55}$$
$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \tag{56}$$
$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \tag{57}$$

4.2

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Suppose $A \in \mathsf{M}_{n\times n}(F)$ and A is upper triangular.

Proof by induction.

For $n = 1$: $A \in \mathsf{M}_{1\times 1}(F) \Rightarrow \det(A) = A_{11}$ by definition.

Suppose $\det(A)$ is the product of its diagonal entries for $1 \leq n \leq k$

For $n = k + 1$: $A \in \mathsf{M}_{k+1\times k+1}(F)$

$$\det(A) = \sum_{j=1}^{k+1} (-1)^{k+1+j} A_{k+1,j} \det(\tilde{A}_{k+1,j}) \tag{58}$$

$$A_{n+1,j} = 0 \quad \text{for } 1 \leq j \leq n \tag{59}$$

$$\Rightarrow \det A = (-1)^{(k+1)+(k+1)} A_{k+1,k+1} \det(\tilde{A}_{k+1,k+1}) \tag{60}$$

$$\tilde{A}_{k+1,k+1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k} \\ & a_{22} & & a_{2,k} \\ & \text{\Large 0} & \ddots & \vdots \\ & & & a_{k,k} \end{pmatrix} \tag{61}$$

$\tilde{A}_{k+1,k+1} \in \mathsf{M}_{n\times n}(F) \Rightarrow \det(\tilde{A}_{n+1,n+1})$ is a product of diagonal entries by induction hypothesis.

$$(-1)^{(n+1)+(n+1)} = (-1)^{2n+2} = (-1)^2(-1)^{2n} = (-1)^{2n} \tag{62}$$

$$n \in \mathbb{Z}^+ \Rightarrow 2n \text{ is even} \tag{63}$$

$$\Rightarrow (-1)^{2n} = 1 \tag{64}$$

$$\Rightarrow \det(A) = A_{n+1,n+1} \prod_{j=1}^{n} A_{jj} = \prod_{j=1}^{n+1} A_{jj} \tag{65}$$

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

   (a) **Types 1 & 2**

$$E^t = E \quad \text{(by HW.3.1.5)} \tag{66}$$
$$\Rightarrow \det(E^t) = \det(E) \tag{67}$$

   (b) **Type 3**

   E^t is an type 3 elementary matrix (by HW.3.1.5) $\det(E) = \det(I) = 1$ for any type elementary operation on I_n

$$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \tag{68}$$
$$\Rightarrow \det(E) = \det(E^t) = 1 \tag{69}$$

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$. be $a_1, a_2, \ldots, a_n$ and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

   (a) **n is even**

   In A, swap

$$a_n \text{ with } a_1 \tag{70}$$
$$a_{n-1} \text{ with } a_2 \tag{71}$$
$$\vdots$$
$$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \tag{72}$$

   From the fact that ${}^{n}/_{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n}{2}} \det(A) \tag{73}$$

   (b) **n is odd** In A, swap

$$a_n \text{ with } a_1 \tag{74}$$
$$a_{n-1} \text{ with } a_2 \tag{75}$$
$$\vdots$$
$$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \tag{76}$$

   From the fact that $n - \frac{n+1}{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n-1}{2}} \det(A) \tag{77}$$

4.3

10. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **nilpotent** is for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that M if nilpotent, then $\det(M) = 0$.

$$M^k = 0 \quad \text{for some } k \in \mathbb{Z}^+ \tag{78}$$

$$\Rightarrow \det(M^k) = \det(0) \tag{79}$$
$$\Rightarrow (\det(M))^k = \det(0) \tag{80}$$
$$\Rightarrow (\det(M))^k = 0 \tag{81}$$
$$\Rightarrow \det(M) = 0 \tag{82}$$

11. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

Lemma: $\det(-M) = (-1)^n \det(M)$ for some $M \in \mathsf{M}_{n\times n}(F)$

Proof by induction:

For $n = 1$:

$$M \in \mathsf{M}_{1\times 1}(F) \Rightarrow M = (m_{11}) \Rightarrow -M = (-m_{11}) \tag{83}$$
$$\det(M) = m_{11} \tag{84}$$
$$\det(-M) = -m_{11} = (-1)^1 m_{11} = (-1)^1 \det(M) \tag{85}$$

Suppose true for $1 \le n \le k$

For $n = k+1$, $M \in \mathsf{M}_{k+1\times k+1}(F)$

$$\det(-M) = \sum_{j=1}^{k+1} (-1)^{k+1+j} (-M)_{k+1,j} \det((-M)_{k+1,j}) \tag{86}$$

$$(-\tilde{M})_{k+1,j} \in \mathsf{M}_{k\times k}(F) \tag{87}$$
$$\Rightarrow (-\tilde{M})_{k+1,j} = (-1)(\tilde{(}M)_{k+1,j}) \tag{88}$$

$$\Rightarrow \det(-M) = (-1)^{k+1} \sum_{j=1}^{k+1} (-1)^j (-1) M_{k+1,j} (-1)^k \det(\tilde{M}_{k+1,j}) \tag{89}$$
$$= (-1)^{k+1} \sum_{j=1}^{k+1} (-a)^{k+1+j} M_{k+1,j} \det(\tilde{M}_{k+1,n}) \tag{90}$$
$$= (-1)^{k+1} \det(M) \quad \square \tag{91}$$

$$\det(M^t) = \det(M) \tag{92}$$
$$\Rightarrow \det(-M) = \det(M) \tag{93}$$
$$\Rightarrow (-1)^n \det(M) = \det(M) \quad \text{(by Lemma)} \tag{94}$$
$$\Rightarrow \det(M) - \det(M)(-1)^n = \det(M)(1-(-1)^n) \tag{95}$$

Case 1: n is odd

$$\det(M)(1-(-1)^n) = \det(M)2 = 0 \tag{96}$$
$$\Rightarrow \det(M) = 0 \Rightarrow M \text{ is not invertible}$$

Case 2: n is even

$$\det M(1-(-1)^n) = \det(M)(0) = 0 \tag{97}$$

The determinate of M can either be zero or nonzero. Thus the invertibility of M cannot be determined.

12. A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called **orthogonal** if $QQ^t = I$ Prove that Q is orthogonal, then $\det(Q) = \pm 1$.

    Suppose $QQ^t = I$

$$\Rightarrow \det(QQ^t) = \det(I) \tag{98}$$
$$\Rightarrow \det(Q)\det(Q^t) = 1 \tag{99}$$
$$\Rightarrow (\det(Q))^2 = 1 \tag{100}$$
$$\Rightarrow (\det(Q))^2 - 1 = 0 \tag{101}$$
$$\Rightarrow (\det(Q)+1)(\det(Q)-1) = 0 \tag{102}$$
$$\Rightarrow \det(Q) = 1 \text{ or } \det(Q) = -1 \tag{103}$$

15. Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

    Suppose $A, B \in \mathsf{M}_{n\times n}(F)$ such that A and B are similar.

$$\Rightarrow A = Q^{-1}BQ \text{ for some } Q^{-1} \text{ invertible} \tag{104}$$

$$\begin{aligned}
\Rightarrow \det(A) &= \det(Q^{-1}BQ) && (105)\\
&= \det(Q^{-1})\det(BQ) && (106)\\
&= \det(Q^{-1})\det(B)\det(Q) && (107)\\
&= \det(Q)^{-1}\det(B)\det(Q) && (108)\\
&= \det(B)\det(Q)^{-1}\det(Q) && (109)\\
&= \det(B) && (110)
\end{aligned}$$