Assignment

Appendix E: Prove theorems E.3, E.5, E.6, E.7; Section 5.2: 2(bde), 3(adef), 7, 9, 11

Work

Appendix E

3. Let $f(x)$ be a polynomial with coefficients from a field F, and let T be a linear operator on a vector space V over F. Then the following statements are true

   (a) $f(\mathsf{T})$ is a linear operator on V.

   (b) If β is a finite ordered basis for V and $A = [\mathsf{T}]_\beta$, then $[f(\mathsf{T})]_\beta = f(A)$.

   Suppose $f(x)$ is a polynomial with coefficients in F

   Suppose $\mathsf{T} \in \mathcal{V}$ and V is a vector space over F.

   (a) Claim: $f(\mathsf{T})$ is a linear operator on V.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$
$$f(\mathsf{T}) = a_n \mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1 \mathsf{T} + a_0 \mathsf{I_V} \tag{2}$$

Lemma: $\mathsf{T}^n \in \mathcal{L}(\mathsf{V})\ \forall n \in \mathbb{Z}^+$.
Suppose $y, z \in \mathsf{V}$.
Suppose $n = 1$

$$\mathsf{T}(az + y) = a\mathsf{T}(z) + \mathsf{T}(y) \tag{3}$$

Suppose true for $1 \leq n \leq k$.
Suppose $n = k + 1$

$$\mathsf{T}^{k+1}(az + y) = \mathsf{T}(\mathsf{T}^k(az + y)) \tag{4}$$
$$= \mathsf{T}(a\mathsf{T}^k(z) + \mathsf{T}(y)) \tag{5}$$
$$= a\mathsf{T}^{k+1}(z) + \mathsf{T}^{k+1}(y) \tag{6}$$

□

$$\therefore f(\mathsf{T}(az + y)) = a_n \mathsf{T}^n(az + y) + a_{n-1}\mathsf{T}^{n-1}(az + y) + \cdots + + a_1 \mathsf{T}(az + y) + a_0(az + y) \tag{7}$$

$$f(\mathsf{T}(az + y)) = a_n(a\mathsf{T}^n(z) + \mathsf{T}^n(y)) + a_{n-1}(a\mathsf{T}^{n-1}(z) + \mathsf{T}^{n-1}(y)) + \cdots + + a_1(a\mathsf{T}(z) + \mathsf{T}(y)) + a_0(az + y) \tag{8}$$

$$f(\mathsf{T}(az+y)) = a(a_n\mathsf{T}^n(z) + a_{n-1}\mathsf{T}^{n-1}(z) + \cdots + a_1\mathsf{T}(z) + a_o z) + \\ + (a_n\mathsf{T}^n(y) + a_{n-1}\mathsf{T}^{n-1}(y) + \cdots + a_1\mathsf{T}(y) + a_o y) \quad (9)$$

$$f(\mathsf{T}(az+y)) = af(\mathsf{T})(z) + f(\mathsf{T})(y) \in \mathsf{V} \quad (10)$$

$$\Rightarrow f(\mathsf{T}) \in \mathcal{L}(\mathsf{V}) \quad (11)$$

(b) Claim: If β is a finite ordered basis for V and $A = [\mathsf{T}]_\beta$, then $[f(\mathsf{T})]_\beta = f(A)$.

$$f(\mathsf{T}) = a_n\mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1\mathsf{T} + a_0\mathsf{I}_\mathsf{V} \quad (12)$$

$$\begin{aligned} \Rightarrow [f(\mathsf{T})]_\beta &= \left[a_n\mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1\mathsf{T} + a_0\mathsf{I}_\mathsf{V}\right]_\beta && (13) \\ &= a_n\left[\mathsf{T}^n\right]_\beta a_{n-1}\left[\mathsf{T}^{n-1}\right]_\beta + \cdots + a_1\left[\mathsf{T}\right]_\beta + a_0\left[\mathsf{I}_\mathsf{V}\right]_\beta && (14) \\ &= a_n\left(\left[\mathsf{T}\right]_\beta\right)^n + a_{n-1}\left(\left[\mathsf{T}\right]_\beta\right)^{n-1} + \cdots + a_1\left[\mathsf{T}\right]_\beta + a_0\mathsf{I}_\mathsf{V} && (15) \\ &= a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0 && (16) \\ &= f(A) && (17) \end{aligned}$$

5. Let T be a linear operator on a vector space V over a field F, and let A be an $n \times n$ matrix with entries from F. If $f_1(x)$ and $f_2(x)$ are relatively prime polynomials with entries from F, then there exist polynomials $q_1(x)$ and $q_2(x)$ with entries from F such that

   (a) $q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$

   (b) $q_1(A)f_1(A) + q_2(A)f_2(A) = I$.

   Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$, such that V is a vector space over F, and $A \in \mathsf{M}_{n\times n}(F)$

   Suppose $f_1(x), f_2(x) \in \mathsf{P}(F)$ such that they are relatively prime.

   (a) Claim: $\exists q_1(x)$ and $q_2(x)$ such that $q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$

   Because $f_1(x)$ and $f_2(x)$ are relatively prime there exists polynomials $q_1(x)$ and $q_2(x)$ such that

$$f_1(x)q_1(x) + f_2(x)q_2(x) = 1 \quad (18)$$

   It follows that

$$f_1(\mathsf{T})q_1(\mathsf{T}) + f_2(\mathsf{T})q_2(\mathsf{T}) = \mathsf{I}_\mathsf{V} \quad (19)$$

   (b) Claim: $\exists q_1(x)$ and $q_2(x)$ such that $q_1(A)f_1(A) + q_2(A)f_2(A) = I_n$

$$f_1(x)q_1(x) + f_2(x)q_2(x) = 1 \quad (20)$$

$$\Rightarrow f_1(A)q_1(A) + f_2(A)q_2(A) = I_n \quad (21)$$

6. Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

   Claim: Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible, and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

   Because $\phi(x)$ is irreducible, $f(x)$ does not divide $\phi(x)$. Since $\phi(x)$ does not divide $f(x)$ it follows that $\phi(x)$ and $f(x)$ are relatively prime.

7. Any two distinct irreducible monic polynomials are relatively prime.

   Lemma: All factors of an irreducible monic polynomial $\phi(x)$ are either of the form $c \neq 0, c \in F$ of $d\phi(x), d \neq 0, d \in F$

   Suppose $f(x), \phi(x) \in \mathsf{P}(F)$ and $\phi(x)$ is an irreducible polynomial.

   Suppose $f(x)$ divides $\phi(x)$

$$\Rightarrow \phi(x) = f(x)g(x) \quad \text{for some } q(x) \in \mathsf{P}(F) \tag{22}$$

   Case 1 $\deg(f(x)) \notin \mathbb{Z}^+$

$$f(x) \neq 0 \because \phi(x) \neq 0 \tag{23}$$
$$\Rightarrow \deg(f(x)) = 0 \tag{24}$$
$$\Rightarrow f(x) = c \quad \text{for some } c \in F \tag{25}$$

   Case 2 $\deg(f(x)) \in \mathbb{Z}^+$

$$\phi(x) = f(x)q(x) \tag{26}$$

   Because $\phi(x)$ is irreducible, it cannot be expressed as a product of polynomials both possessing positive degree.

$$\Rightarrow \deg(q(x)) \leq 0 \tag{27}$$
$$\Rightarrow g(x) = \frac{1}{d} \quad \text{for some nonzero } d \in F \tag{28}$$
$$\Rightarrow \phi(x) = \frac{f(x)}{d} \tag{29}$$
$$\Rightarrow d\phi(x) = f(x) \tag{30}$$

□

By lemma, suppose the factors of ϕ_1 are c and $d\phi_1$ where $c, d \in F$ and $c, d \neq 0$. suppose the factors of ϕ_2 are e and $g\phi_2$ where $e, g \in F, e, g \neq 0$
Claim: $d\phi_1 \neq g\phi_2$
Suppose $g\phi_2 | \phi_1$

$$\Rightarrow g\phi_2 = d\phi_1 \tag{31}$$

$$\Rightarrow \phi_2 = \frac{d}{g}\phi_1 \Rightarrow \phi_2 = \frac{d}{g}(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \tag{32}$$

$$\Rightarrow \phi_2 = \frac{d}{g}x^n + \frac{d}{g}a_{n-1}x^{n-1} + \cdots + \frac{d}{g}a_1 x + a_0 \tag{33}$$

$$\Rightarrow \frac{d}{g} = 1 \text{ because } \phi_2 \text{ is monic.} \tag{34}$$

$$\Rightarrow \phi_2 = \phi_1 \lightning \text{ Contradiction!} \tag{35}$$

Theorem states ϕ_1 and ϕ_2 are distinct polynomials.

5.2

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(b)

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{36}$$

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} &= (1-\lambda)^2 - 9 && (37) \\ &= \lambda^2 - 2\lambda - 8 && (38) \\ &= (\lambda-4)(\lambda+2) && (39) \end{aligned}$$

$$\Rightarrow \lambda_1 = 4, \lambda_2 = -2 \tag{40}$$

- For $\lambda_1 = 4$

$$A - 4I_2 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \tag{42}$$

$$\Rightarrow \operatorname{rank}(A - 4I_2) = 1 \tag{43}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

$$\Rightarrow x_1 = x_2 \tag{45}$$

$$S_1 = \left\{ z \begin{pmatrix} 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{46}$$

The eigenvector corresponding to λ_1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- For $\lambda_2 = -2$

$$A + 2I_2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{48}$$

$$\Rightarrow \operatorname{rank}(A + 2I_2) = 1 \tag{49}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{50}$$

$$\Rightarrow x_1 = -x_2 \tag{51}$$

$$S_2 = \left\{ z \begin{pmatrix} -1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{52}$$

The eigenvector corresponding to λ_2 is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{53}$$

$$(Q|I) = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & {}^{1}/_{2} & {}^{1}/_{2} \\ 0 & 1 & {}^{-1}/_{2} & {}^{1}/_{2} \end{array}\right) \tag{54}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} {}^{1}/_{2} & {}^{1}/_{2} \\ {}^{-1}/_{2} & {}^{1}/_{2} \end{pmatrix} \tag{55}$$

$$\Rightarrow Q^{-1}AQ = D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \tag{56}$$

(d)

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \tag{57}$$

$$\det(A - \lambda I_3) = \det((A - \lambda I_3)^t) \tag{58}$$

$$\det((A - \lambda I_3)^t) = \det \begin{pmatrix} 7-\lambda & 8 & 6 \\ -4 & -5-\lambda & -6 \\ 0 & 0 & 3-\lambda \end{pmatrix} \tag{59}$$

$$\det(A - \lambda I_3) = (-1)(\lambda - 3)^3(\lambda + 1) \tag{60}$$

$$\lambda_1 = -1, \text{ multiplicity } 1 \tag{61}$$

$$\lambda_2 = 3, \text{ multiplicity } 2 \tag{62}$$

- For $\lambda_1 = -1$

$$A + I_3 = \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \tag{63}$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{64}$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{65}$$

$$\Rightarrow x_1 = \frac{2}{3} x_3 \tag{66}$$

$$\Rightarrow x_2 = \frac{4}{3} x_3 \tag{67}$$

$$\Rightarrow S_1 = \left\{ z \begin{pmatrix} {}^2\!/_3 \\ {}^4\!/_3 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{68}$$

$$\Rightarrow \text{ an eigenvector is } \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

- For $\lambda_2 = 3$

$$A - 3I_3 = \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{70}$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{71}$$

$$\Rightarrow x_1 = x_2 \tag{72}$$

$$S_2 = \left\{ z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : z_1, z_2 \in \mathbb{R} \right\} \tag{73}$$

$$\Rightarrow \text{ eigenvectors corresponding to } \lambda_2 = 3 \text{ are } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \tag{74}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & {}^{3}/_{2} & {}^{-3}/_{2} & 1 \\ 0 & 0 & 1 & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \end{array}\right) \tag{75}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ {}^{3}/_{2} & {}^{-3}/_{2} & 1 \\ {}^{-1}/_{2} & {}^{1}/_{2} & 0 \end{pmatrix} \tag{76}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{77}$$

(e)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{78}$$

$$A - \lambda I_3 = \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 071 & 1-\lambda & \end{pmatrix} \tag{79}$$

$$\det\left(A - \lambda I_3\right) = (-1)\det\begin{pmatrix} -\lambda & 1 \\ 1 & -1 \end{pmatrix} + (1-\lambda)\det\begin{pmatrix} -\lambda & 0 \\ 1 & -\lambda \end{pmatrix} \tag{80}$$

$$= (-1)(\lambda - 1) + (1-\lambda)(\lambda^2) \tag{81}$$

$$= (1-\lambda)(1+\lambda)^2 \tag{82}$$

This characteristic polynomial does not split over $\mathbb{R}$, thus A is not diagonalizable.

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

   (a) $\mathsf{V} = \mathsf{P}_3(\mathbb{R})$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$, respectively.
   Let α be the standard ordered basis of $\mathsf{P}_3(\mathbb{R})$.

$$\mathsf{T}(\alpha) = \{0, 1, 2x+2, 3x^2+6x\} \tag{83}$$

$$\Rightarrow [\mathsf{T}]_\alpha = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{84}$$

$$\det([\mathsf{T}]_\beta - \lambda I_4) = \det \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^4 \tag{85}$$

$$\Rightarrow \lambda_1 = 0 \text{ with multiplicity } 4 \tag{86}$$

$$\operatorname{rank}([\mathsf{T}]_\beta - \lambda_1 I_3) = 3 \tag{87}$$

$$4 - 3 \neq 4 \text{ multiplicity of } \lambda_1 \tag{88}$$

$$\Rightarrow \mathsf{T} \text{ is not diagonalizable}$$

   (d) $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.
   Suppose α is the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$.

$$\Rightarrow \mathsf{T}(\alpha) = \{x^2 + x + 1, x + x^2, x + x^2\} \tag{89}$$

$$\Rightarrow [\mathsf{T}]_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{90}$$

$$\det([\mathsf{T}]_\beta - \lambda I_2) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} \tag{91}$$

$$= (\lambda)(1-\lambda)(\lambda - 2) \tag{92}$$

$$\Rightarrow \lambda_1 = 0, \text{ multiplicity } 1 \tag{93}$$

$$\lambda_2 = 1, \text{ multiplicity } 1 \tag{94}$$

$$\lambda_3 = 2, \text{ multiplicity } 1 \tag{95}$$

- For $\lambda_1 = 0$

$$[\mathsf{T}]_\beta - 0I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{96}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{97}$$

$$\Rightarrow \operatorname{rank}([\mathsf{T}]_\beta) = 2 \tag{98}$$

$$\text{multiplicity } \lambda_1 = 1;\ 3 - 2 = 1\checkmark \tag{99}$$

- For $\lambda_2 = 1$

$$[\mathsf{T}]_\beta - I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{100}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{101}$$

$$\Rightarrow \operatorname{rank}([\mathsf{T}]_\beta - I_3) = 2 \tag{102}$$

$$\text{multiplicity } \lambda_2 = 1;\ 3 - 2 = 1\checkmark \tag{103}$$

- For $\lambda_3 = 2$

$$[\mathsf{T}]_\beta - 2I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{pmatrix} \tag{104}$$

$$\Rightarrow \operatorname{rank}([\mathsf{T}]_\beta) = 2 \tag{105}$$

$$\text{multiplicity } \lambda_3 = 1;\ 3 - 2 = 1\checkmark \tag{106}$$

It follows that T is diagonalizable.

- For $\lambda_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{107}$$

$$\Rightarrow x_1 = 0 \tag{108}$$

$$x_2 = -x_3 \tag{109}$$

$$S_1 = \left\{ z \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{110}$$

$$\Rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_2$$

- For $\lambda_2 = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{111}$$

$$\Rightarrow x_1 = -x_3 \tag{112}$$

$$x_2 = x_3 \tag{113}$$

$$S_2 = \left\{ z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{114}$$

$$\Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_2$$

- For $\lambda_3 = 2$

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{115}$$

$$\Rightarrow x_1 = 0 \tag{116}$$

$$x_2 = x_3 \tag{117}$$

$$\tag{118}$$

$$S_3 = \left\{ z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{119}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_2 \tag{120}$$

$$\Rightarrow \beta = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{121}$$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{122}$$

(e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + 2)$
Suppose $\alpha = \{(1, 0), (-, 1)\}$ is a basis for C^2

$$\mathsf{T}(\alpha) = \{(1, i), (i, 1)\} \tag{123}$$

$$\Rightarrow [\mathsf{T}]_\alpha = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{124}$$

$$\det\left([\mathsf{T}]_\beta - \lambda I_2\right) = \det\begin{pmatrix} 1-\lambda & i \\ i & 1-\lambda \end{pmatrix} \tag{125}$$
$$= \lambda^2 - 2\lambda + 2 \tag{126}$$

$\mathbb{C}$ is algebraically closed so the characteristic polynomial splits over $\mathbb{C}$

$$\lambda^2 - 2\lambda + 2 = 0 \tag{127}$$
$$\Rightarrow 1 \pm i \tag{128}$$

$$\Rightarrow \lambda_1 = 1 + i, \text{ multiplicity } 1 \tag{129}$$
$$\lambda_2 = 1 - i, \text{ multiplicity } 1 \tag{130}$$

- For $\lambda_1 = 1 + i$

$$[\mathsf{T}]_\beta - (i+1)I_2 = \begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \tag{131}$$
$$\begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \tag{132}$$
$$\operatorname{rank}([\mathsf{T}]_\beta - (i+1)I_2) = 1 \tag{133}$$
$$\text{multiplicity } \lambda_1 = i + 1;\ 2 - 1 = 1\checkmark \tag{134}$$

- For $\lambda_2 = 1 - i$

$$[\mathsf{T}]_\beta - (1-i)I_2 = \begin{pmatrix} i & i \\ i & i \end{pmatrix} \tag{135}$$
$$\begin{pmatrix} i & i \\ i & i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \tag{136}$$
$$\Rightarrow \operatorname{rank}([\mathsf{T}]_\beta - (1-i)I_2) = 1 \tag{137}$$
$$\text{multiplicity } \lambda_2 = 1 - i;\ 2 - 1 = 1\checkmark \tag{138}$$
$$\tag{139}$$

It follows that T is diagonalizable.

- For $\lambda_1 = 1 + i$

$$\begin{pmatrix} -i & i \\ i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{140}$$
$$\Rightarrow x_1 = x_2 \tag{141}$$
$$S_1 = \left\{ z \begin{pmatrix} 1 \\ 1 \end{pmatrix} : x \in \mathbb{C} \right\} \tag{142}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_1$$

- For $\lambda_2 = 1 - i$

$$\begin{pmatrix} i & i \\ i & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{143}$$

$$\Rightarrow x_1 = -x_2 \tag{144}$$

$$S_2 = \left\{ z\begin{pmatrix} -1 \\ 1 \end{pmatrix} : z \in \mathbb{C} \right\} \tag{145}$$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to λ_2

$$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \tag{146}$$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \tag{147}$$

(f) $\mathsf{V} = \mathsf{M}_{n\times n}(\mathbb{R})$ and T is defined by $\mathsf{T}(A) = A^t$
Suppose α is the standard ordered basis of $\mathsf{M}_{n\times n}(\mathbb{R})$

$$\Rightarrow \mathsf{T}(\alpha) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{148}$$

$$\Rightarrow [\mathsf{T}]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{149}$$

$$\Rightarrow \det\left([\mathsf{T}]_\beta - \lambda I_4\right) = \det\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \tag{150}$$

$$= (1-\lambda^2)(\lambda^2 - 1) \tag{151}$$

$$\tag{152}$$

$$\Rightarrow \lambda_1 = 1, \text{ multiplicity } 3 \tag{153}$$

$$\Rightarrow \lambda_2 == 1, \text{ multiplicity } 1 \tag{154}$$

7. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{n\times n}(\mathbb{R}),$$

find an expression for A^n, where n is an arbitrary positive integer.

$$\det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = 0 \tag{155}$$

$$= (\lambda - 5)(\lambda + 1) = 0 \tag{156}$$

$$\Rightarrow \lambda_1 = 5 \tag{157}$$
$$\lambda_2 = -1 \tag{158}$$

- For $\lambda_1 = 5$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{159}$$

$$\Rightarrow x_1 = x_2 \tag{160}$$
$$\Rightarrow S_1 = \left\{ z \begin{pmatrix} 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{161}$$
$$\Rightarrow \text{ eigenvector corresponding to } \lambda_1 = 5 \text{ is } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{162}$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{163}$$

$$\Rightarrow x_1 = -2x_2 \tag{164}$$
$$\Rightarrow S_2 = \left\{ z \begin{pmatrix} -2 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{165}$$
$$\Rightarrow \text{ eigenvector corresponding to } \lambda_2 = -1 \text{ is } \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{166}$$

Let $Q = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & {}^1/_3 & {}^2/_3 \\ 0 & 1 & {}^{-1}/_3 & {}^1/_3 \end{array}\right) \tag{167}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} {}^1/_3 & {}^2/_3 \\ {}^{-1}/_3 & {}^1/_3 \end{pmatrix} \tag{168}$$
$$D = Q^{-1}AQ = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \tag{169}$$
$$\Rightarrow A = QDQ^{-1} \tag{170}$$

$$\Rightarrow A^n = (QDQ^{-1})^n \tag{171}$$
$$= QD^nQ^{-1} \tag{172}$$
$$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} {}^1/_3 & {}^2/_3 \\ {}^{-1}/_3 & {}^1/_3 \end{pmatrix} \tag{173}$$

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix.

   (a) Prove that the characteristic polynomial for T splits.

   (b) State and prove an analogous results for matrices.

   (a)

$$[\mathsf{T}]_\beta \Rightarrow [\mathsf{T}]_\beta - \lambda I_n \text{ is upper triangular} \tag{174}$$

$$\begin{aligned}\det\left([\mathsf{T}]_\beta - \lambda I_n\right) &= \prod_{i=1}^{n}([\mathsf{T}]_\beta - \lambda I_n) && (175)\\ &= \prod_{i=1}^{n}([\mathsf{T}]_\beta)_{ii} - \lambda && (176)\\ &= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) \text{ for } a_{ii} \in F && (177)\end{aligned}$$

   It follows that V is n dimensional, so the characteristic polynomial splits over F.

   (b) Suppose $A \in \mathsf{M}_{n\times n}(F)$ is similar to an upper triangular matrix. Prove the characteristic polynomial of A splits.
   Suppose $A = Q^{-1}UQ$ for some $U, Q \in \mathsf{M}_{n\times n}(F)$ such that U is upper triangular and Q is invertible.

$$\Rightarrow \det\left(A - \lambda I_n\right) = \det\left(U - \lambda I_n\right) \tag{178}$$

   Because U is upper triangular, $U0\lambda I_n$ is also upper triangular.

$$\begin{aligned}\det\left(U - \lambda I_n\right) &= \prod_{i=1}^{n}(U - \lambda I_n) && (179)\\ &= \prod_{i=1}^{n}(U_{ii} - \lambda) && (180)\end{aligned}$$

   It follows that the characteristic polynomial of U splits. U and A are similar, so A and U have the same characteristic polynomial. Therefore the characteristic polynomial of A splits.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

   (a) $\operatorname{tr}(A) = \sum\limits_{i=1}^{k} m_i\lambda_i$

   (b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$

Suppose $A \in \mathsf{M}_{n\times n}(F)$. A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$.

Suppose $A = Q^{-1}UQ$ for $U, Q \in \mathsf{M}_{n\times n}(F)$ such that U is upper triangular and Q is invertible.

$$\det(A - \lambda I_n) = \det(U - \lambda I_n) = (U_{11} - \lambda)(U_{22} - \lambda)\cdots(U_{nn} - \lambda) \tag{181}$$

A and U have the same characteristic polynomial so they have the same eigenvalues. Thus U has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$.

$$\det(U - \lambda I_n) = (U_{11} - \lambda)(U_{22} - \lambda)\cdots(U_{nn} - \lambda) = 0 \text{ (by HW.5.2.9.b)} \tag{182}$$

It follows that the diagonal entries of U correspond to its eigenvalues.

(a) Claim: $\mathrm{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$

$$\begin{aligned}
\mathrm{tr}(A) &= \mathrm{tr}(Q^{-1}UQ) && (183)\\
&= \mathrm{tr}(Q(Q^{-1})A) \text{ (by HW.2.3.12)} && (184)\\
&= \mathrm{tr}((QQ^{-1})U) && (185)\\
&= \mathrm{tr}(U) && (186)\\
&= \sum_{i=1}^{n} U_{ii} && (187)\\
&= \sum_{i=1}^{k} m_i\lambda_i && (188)
\end{aligned}$$

(b) Claim: $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$

$$\begin{aligned}
\det(A) &= \det(Q^{-1}UQ) && (189)\\
&= (\det Q)^{-1}\det Q \det U && (190)\\
&= \det U && (191)\\
&= \prod_{i=1}^{n} U_{ii} && (192)\\
&= \prod_{i=1}^{k} (\lambda_i)^{m_i} && (193)
\end{aligned}$$