Assignment

Appendix E: Prove theorems E.3, E.5, E.6, E.7; Section 5.2: 2(bde), 3(adef), 7, 9, 11

Work

Appendix E

3. Let $f(x)$ be a polynomial with coefficients from a field F, and let T be a linear operator on a vector space V over F. Then the following statements are true

   (a) $f(\mathsf{T})$ is a linear operator on V.

   (b) If β is a finite ordered basis for V and $A = [\mathsf{T}]_\beta$, then $[f(\mathsf{T})]_\beta = f(A)$.

   Suppose $f(x)$ is a polynomial with coefficients in F

   Suppose $\mathsf{T} \in \mathcal{V}$ and V is a vector space over F.

   (a) Claim: $f(\mathsf{T})$ is a linear operator on V.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$
$$f(\mathsf{T}) = a_n \mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1 \mathsf{T} + a_0 \mathsf{I}_\mathsf{V} \tag{2}$$

Lemma: $\mathsf{T}^n \in \mathcal{L}(\mathsf{V})\ \forall n \in \mathbb{Z}^+$.
Suppose $y, z \in \mathsf{V}$.
Suppose $n = 1$

$$\mathsf{T}(az + y) = a\mathsf{T}(z) + \mathsf{T}(y) \tag{3}$$

Suppose true for $1 \leq n \leq k$.
Suppose $n = k + 1$

$$\mathsf{T}^{k+1}(az + y) = \mathsf{T}(\mathsf{T}^k(az + y)) \tag{4}$$
$$= \mathsf{T}(a\mathsf{T}^k(z) + \mathsf{T}(y)) \tag{5}$$
$$= a\mathsf{T}^{k+1}(z) + \mathsf{T}^{k+1}(y) \tag{6}$$

$\square$

$$\therefore f(\mathsf{T}(az + y)) = a_n \mathsf{T}^n(az + y) + a_{n-1}\mathsf{T}^{n-1}(az + y) + \cdots + \\ + a_1 \mathsf{T}(az + y) + a_0(az + y) \tag{7}$$

$$f(\mathsf{T}(az + y)) = a_n(a\mathsf{T}^n(z) + \mathsf{T}^n(y)) + a_{n-1}(a\mathsf{T}^{n-1}(z) + \mathsf{T}^{n-1}(y)) + \cdots + \\ + a_1(a\mathsf{T}(z) + \mathsf{T}(y)) + a_0(az + y) \tag{8}$$

$$
\begin{aligned}
f(\mathsf{T}(az+y)) = a(a_n\mathsf{T}^n(z) + a_{n-1}\mathsf{T}^{n-1}(z) + \cdots + a_1\mathsf{T}(z) + a_o z) +& \\
+ (a_n\mathsf{T}^n(y) + a_{n-1}\mathsf{T}^{n-1}(y) + \cdots + a_1\mathsf{T}(y) + a_o y)& \quad (9)
\end{aligned}
$$

$$f(\mathsf{T}(az+y)) = af(\mathsf{T})(z) + f(\mathsf{T})(y) \in \mathsf{V} \quad (10)$$

$$\Rightarrow f(\mathsf{T}) \in \mathcal{L}(\mathsf{V}) \quad (11)$$

(b) Claim: If β is a finite ordered basis for V and $A = [\mathsf{T}]_\beta$, then $[f(\mathsf{T})]_\beta = f(A)$.

$$f(\mathsf{T}) = a_n\mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1\mathsf{T} + a_0\mathsf{I}_\mathsf{V} \quad (12)$$

$$
\begin{aligned}
\Rightarrow [f(\mathsf{T})]_\beta &= \left[a_n\mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1\mathsf{T} + a_0\mathsf{I}_\mathsf{V}\right]_\beta && (13)\\
&= a_n\left[\mathsf{T}^n\right]_\beta a_{n-1}\left[\mathsf{T}^{n-1}\right]_\beta + \cdots + a_1\left[\mathsf{T}\right]_\beta + a_0\left[\mathsf{I}_\mathsf{V}\right]_\beta && (14)\\
&= a_n\left(\left[\mathsf{T}\right]_\beta\right)^n + a_{n-1}\left(\left[\mathsf{T}\right]_\beta\right)^{n-1} + \cdots + a_1\left[\mathsf{T}\right]_\beta + a_0\mathsf{I}_\mathsf{V} && (15)\\
&= a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0 && (16)\\
&= f(A) && (17)
\end{aligned}
$$

5. Let T be a linear operator on a vector space V over a field F, and let A be an $n \times n$ matrix with entries from F. If $f_1(x)$ and $f_2(x)$ are relatively prime polynomials with entries from F, then there exist polynomials $q_1(x)$ and $q_2(x)$ with entries from F such that

   (a) $q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$

   (b) $q_1(A)f_1(A) + q_2(A)f_2(A) = I$.

   Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$, such that V is a vector space over F, and $A \in \mathsf{M}_{n\times n}(F)$

   Suppose $f_1(x), f_2(x) \in \mathsf{P}(F)$ such that they are relatively prime.

   (a) Claim: $\exists q_1(x)$ and $q_2(x)$ such that $q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$

   Because $f_1(x)$ and $f_2(x)$ are relatively prime there exists polynomials $q_1(x)$ and $q_2(x)$ such that

   $$f_1(x)q_1(x) + f_2(x)q_2(x) = 1 \quad (18)$$

   It follows that

   $$f_1(\mathsf{T})q_1(\mathsf{T}) + f_2(\mathsf{T})q_2(\mathsf{T}) = \mathsf{I}_\mathsf{V} \quad (19)$$

   (b) Claim: $\exists q_1(x)$ and $q_2(x)$ such that $q_1(A)f_1(A) + q_2(A)f_2(A) = I_n$

   $$
   \begin{aligned}
   f_1(x)q_1(x) + f_2(x)q_2(x) &= 1 && (20)\\
   \Rightarrow f_1(A)q_1(A) + f_2(A)q_2(A) &= I_n && (21)
   \end{aligned}
   $$

6. Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

   Claim: Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible, and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

   Because $\phi(x)$ is irreducible, $f(x)$ does not divide $\phi(x)$. Since $\phi(x)$ does not divide $f(x)$ it follows that $\phi(x)$ and $f(x)$ are relatively prime.

7. Any two distinct irreducible monic polynomials are relatively prime.

   Lemma: All factors of an irreducible monic polynomial $\phi(x)$ are either of the form $c \neq 0, c \in F$ of $d\phi(x), d \neq 0, d \in F$

   Suppose $f(x), \phi(x) \in \mathsf{P}(F)$ and $\phi(x)$ is an irreducible polynomial.

   Suppose $f(x)$ divides $\phi(x)$

$$\Rightarrow \phi(x) = f(x)g(x) \quad \text{for some } q(x) \in \mathsf{P}(F) \tag{22}$$

   Case 1 $\deg(f(x)) \notin \mathbb{Z}^+$

$$f(x) \neq 0 \because \phi(x) \neq 0 \tag{23}$$
$$\Rightarrow \deg(f(x)) = 0 \tag{24}$$
$$\Rightarrow f(x) = c \quad \text{for some } c \in F \tag{25}$$

   Case 2 $\deg(f(x)) \in \mathbb{Z}^+$

$$\phi(x) = f(x)q(x) \tag{26}$$

   Because $\phi(x)$ is irreducible, it cannot be expressed as a product of polynomials both possessing positive degree.

$$\Rightarrow \deg(q(x)) \leq 0 \tag{27}$$
$$\Rightarrow g(x) = \frac{1}{d} \quad \text{for some nonzero } d \in F \tag{28}$$
$$\Rightarrow \phi(x) = \frac{f(x)}{d} \tag{29}$$
$$\Rightarrow d\phi(x) = f(x) \tag{30}$$

$\square$

By lemma, suppose the factors of ϕ_1 are c and $d\phi_1$ where $c, d \in F$ and $c, d \neq 0$. suppose the factors of ϕ_2 are e and $g\phi_2$ where $e, g \in F, e, g \neq 0$
Claim: $d\phi_1 \neq g\phi_2$
Suppose $g\phi_2 | \phi_1$

$$\Rightarrow g\phi_2 = d\phi_1 \tag{31}$$

$$\Rightarrow \phi_2 = \frac{d}{g}\phi_1 \Rightarrow \phi_2 = \frac{d}{g}(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) \tag{32}$$

$$\Rightarrow \phi_2 = \frac{d}{g}x^n + \frac{d}{g}a_{n-1}x^{n-1} + \cdots + \frac{d}{g}a_1x + a_0 \tag{33}$$

$$\Rightarrow \frac{d}{g} = 1 \text{ because } \phi_2 \text{ is monic.} \tag{34}$$

$$\Rightarrow \phi_2 = \phi_1 \lightning \text{ Contradiction!} \tag{35}$$

Theorem states ϕ_1 and ϕ_2 are distinct polynomials.

5.2

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(b)

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{36}$$

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} &= (1-\lambda)^2 - 9 && (37)\\ &= \lambda^2 - 2\lambda - 8 && (38)\\ &= (\lambda - 4)(\lambda + 2) && (39)\end{aligned}$$

$$\Rightarrow \lambda_1 = 4, \lambda_2 = -2 \tag{40}$$

- For $\lambda_1 = 4$

$$A - 4I_2 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \tag{42}$$

$$\Rightarrow \operatorname{rank}(A - 4I_2) = 1 \tag{43}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

$$\Rightarrow x_1 = x_2 \tag{45}$$

$$S_1 = \left\{ z \begin{pmatrix} 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{46}$$

The eigenvector corresponding to λ_1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- For $\lambda_2 = -2$

$$A + 2I_2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{48}$$

$$\Rightarrow \operatorname{rank}(A + 2I_2) = 1 \tag{49}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{50}$$

$$\Rightarrow x_1 = -x_2 \tag{51}$$

$$S_2 = \left\{ z \begin{pmatrix} -1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{52}$$

The eigenvector corresponding to λ_2 is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{53}$$

$$(Q|I) = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right) \tag{54}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \tag{55}$$

$$\Rightarrow Q^{-1}AQ = D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \tag{56}$$

(d)

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \tag{57}$$

$$\det(A - \lambda I_3) = \det((A - \lambda I_3)^t) \tag{58}$$

$$\det((A - \lambda I_3)^t) = \det \begin{pmatrix} 7-\lambda & 8 & 6 \\ -4 & -5-\lambda & -6 \\ 0 & 0 & 3-\lambda \end{pmatrix} \tag{59}$$

$$\det(A - \lambda I_3) = (-1)(\lambda - 3)^3(\lambda + 1) \tag{60}$$

$$\lambda_1 = -1, \text{ multiplicity } 1 \tag{61}$$

$$\lambda_2 = 3, \text{ multiplicity } 2 \tag{62}$$

- For $\lambda_1 = -1$

$$A + I_3 = \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \tag{63}$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{64}$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{65}$$

$$\Rightarrow x_1 = \frac{2}{3} x_3 \tag{66}$$

$$\Rightarrow x_2 = \frac{4}{3} x_3 \tag{67}$$

$$\Rightarrow S_1 = \left\{ z \begin{pmatrix} {}^2/_3 \\ {}^4/_3 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{68}$$

$$\Rightarrow \text{ an eigenvector is } \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

- For $\lambda_2 = 3$

$$A - 3I_3 = \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{70}$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{71}$$

$$\Rightarrow x_1 = x_2 \tag{72}$$

$$S_2 = \left\{ z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : z_1, z_2 \in \mathbb{R} \right\} \tag{73}$$

$$\Rightarrow \text{ eigenvectors corresponding to } \lambda_2 = 3 \text{ are } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \tag{74}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & {}^{3}/_{2} & {}^{-3}/_{2} & 1 \\ 0 & 0 & 1 & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \end{array}\right) \tag{75}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ {}^{3}/_{2} & {}^{-3}/_{2} & 1 \\ {}^{-1}/_{2} & {}^{1}/_{2} & 0 \end{pmatrix} \tag{76}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{77}$$

(e)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{78}$$

$$A - \lambda I_3 = \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 071 & 1-\lambda & \end{pmatrix} \tag{79}$$

$$\begin{aligned} \det(A - \lambda I_3) &= (-1)\det\begin{pmatrix} -\lambda & 1 \\ 1 & -1 \end{pmatrix} + (1-\lambda)\det\begin{pmatrix} -\lambda & 0 \\ 1 & -\lambda \end{pmatrix} && (80) \\ &= (-1)(\lambda - 1) + (1-\lambda)(\lambda^2) && (81) \\ &= (1-\lambda)(1+\lambda)^2 && (82) \end{aligned}$$

This characteristic polynomial does not split over $\mathbb{R}$, thus A is not diagonalizable.

7.

9.

11.