

Assignment

Section 5.4: 4, 15, 17, 19, 41; Section 6.1: 3, 8, 12, 17; Section 6.2: 2(a,c,g,j), 6, 7, 15

Work

5.4

4. Let T be a linear operator on a vector space V , and let W be a T -invariant subspace of V . Prove that W is $g(T)$ -invariant for any polynomial $g(t)$.

Suppose $T \in \mathcal{L}(V)$

Let W be a T -invariant subspace of V

Lemma: $T^k(W)$ is T -invariant for all $k \in \mathbb{Z}^+$

Proof by induction.

Base case: Suppose $k = 1$

$$T(W) \subseteq W \quad (1)$$

$$\Rightarrow T^2(W) = T(T(W)) \subseteq W \quad (2)$$

Suppose $T^k(W)$ is T -invariant for $1 \leq k \leq n$.

Suppose $k = n + 1$

$$T^{n+1}(W) \subseteq T(T^n(W)) \quad (3)$$

$$T^n(W) \subseteq W \quad (4)$$

$$\Rightarrow T^{n+1}(W) = T(T^n(W)) \subseteq W \quad (5)$$

$\therefore T^k(W)$ is T -invariant for all $k \in \mathbb{Z}^+$ \square

Suppose $w \in W$ and $g(t) \in P(F)$ such that

$$g(t) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (6)$$

$$\Rightarrow g(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I_V \quad (7)$$

$$\Rightarrow g(T)(W) = a_n T^n(w) + a_{n-1} T^{n-1}(w) + \cdots + a_1 T(w) + a_0 I_V(w) \quad (8)$$

$$\Rightarrow g(T)(w) \in W \quad \forall w \in W \quad (9)$$

$\therefore W$ is $g(T)$ -invariant

15. Use Cayley-Hamilton theorem to prove its corollary for matrices.

Corollary to Cayley-Hamilton theorem for matrices Let $A \in \mathbf{M}_{n \times n}(F)$ and let $f(t)$ be the characteristic polynomial of A . Then $f(A) = O$, the $n \times n$ zero matrix.

Suppose $A \in \mathbf{M}_{n \times n}(F)$ and let $f(t)$ be the characteristic polynomial of A .

Suppose β is the standard ordered basis of F^n

$$\Rightarrow A = [\mathbf{L}_A]_\beta \quad (10)$$

A and \mathbf{L}_A have the same characteristic polynomial by the definition of characteristic polynomial for functions.

$$f(A) = [f(\mathbf{L}_A)]_\beta \text{ by theorem E.3} \quad (11)$$

$$f(\mathbf{L}_A) = 0 \in F^n \text{ by Cayley Hamilton} \quad (12)$$

$$\Rightarrow f(A) = [0]_\beta = O \in \mathbf{M}_{n \times n}(F) \quad (13)$$

17. Let A be an $n \times n$ matrix. Prove that

$$\dim(\text{span}(\{I_n, A, A^2, \dots\})) \leq n$$

The characteristic polynomial of A is

$$f(t) = (-1)t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0 \quad (14)$$

$$\Rightarrow f(A) = (-1)A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I_n = 0 \quad (15)$$

$$\Rightarrow A^n = (-1)^{n+1}a_{n-1}A^{n-1} + \dots + (-1)^{n+1}A + (-1)^{n+1}a_0I_n \quad (16)$$

$$\Rightarrow A^n \in \text{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \quad (17)$$

Claim: $A^k \in \text{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \forall k \in \mathbb{Z}, k \geq n$

Proof by induction.

True for $k = n$

Suppose true for $n \leq k \leq n + i - 1$ for some $i \in \mathbb{Z}^+, i \geq 2$

Suppose $k = n + i$

$$\Rightarrow A^{k+i} = A^n \cdot A^i \quad (18)$$

$$A^k \in \text{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \quad (19)$$

$$i < n + i \quad (20)$$

$$\Rightarrow A^i \in \text{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \quad (21)$$

$$\Rightarrow A^{i+n} = A^i \cdot A^n \in \text{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \quad (22)$$

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where a_0, a_1, \dots, a_{k-1} are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$$

Proof by induction on k .

Suppose $k = 1$

$$\Rightarrow A = -a_0 \tag{23}$$

$$\det(A - tI_1) = \det(-a_0 - t) \tag{24}$$

$$= -a_0 - t \tag{25}$$

$$= (-1)^1(a_0 + t^1) \tag{26}$$

Suppose true for $2 \leq k \leq n - 1$

Suppose $k = n$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \tag{27}$$

$$\Rightarrow A - tI_n = \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} - t \end{pmatrix} \tag{28}$$

$$\Rightarrow \det(A - tI_n) = (-t)(-1)^2 \det \tilde{A}_{11} + (a_0)(-1)^{n+1} \det \tilde{A}_{1n} \tag{29}$$

$$\det \tilde{A}_{11} = \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} - t \end{pmatrix} \quad (30)$$

$$= (-a)^{n-1}(a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) \quad (31)$$

$$\det \tilde{A}_{1n} = \det \begin{pmatrix} 1 & -t & 0 & \cdots & 0 \\ 0 & 1 & -t & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -t \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (32)$$

$$= 1 \because (\tilde{A}_{1n})_{ii} = 1 \forall i (1 \leq i \leq n-1) \quad (33)$$

$$\Rightarrow \det(A - tI_n) = (-1)(t)(-1)^{n-1}(a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) + (a_0)(-1)^n \quad (34)$$

$$= (-1)^n(a_0 + a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) \quad (35)$$

41. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2 - n + 1 & n^2 - n + 2 & \cdots & n^2 \end{pmatrix}$$

Find the characteristic polynomial of A . *Hint:* First prove that A has rank 2 and that $\text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\})$ is \mathbb{L}_A -invariant.

Hint #2 Show that $\text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\})$ is \mathbb{L}_A -invariant.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} \quad (36)$$

Let

$$b = \frac{(n)(n+1)}{2} \quad (37)$$

$$Av_1 = \begin{pmatrix} b \\ n^2 + b \\ 2n^2 + b \\ \vdots \\ (n-1)(n^2) + b \end{pmatrix} + \begin{pmatrix} 0 \\ n^2 \\ 2n^2 \\ \vdots \\ (n-1)n^2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (38)$$

$$= n^2(v_2 - v_1) + b(v_1) \quad (39)$$

$$= (b - n^2)v_1 + n^2(v_2) \quad (40)$$

$$Av_2 = \begin{pmatrix} \sum_{i=1}^n i^2 \\ \sum_{i=1}^n (n+i)i \\ \sum_{i=1}^n (2n+i)i \\ \vdots \\ \sum_{i=1}^n ((n)(n-1) + i)i \end{pmatrix} \quad (41)$$

$$= v_1 \left(\sum_{i=1}^n i^2 \right) + \begin{pmatrix} 0 \\ \sum_{i=1}^n ni \\ \sum_{i=1}^n 2ni \\ \vdots \\ \sum_{i=1}^n (n-1)(n)i \end{pmatrix} \quad (42)$$

$$Av_2 = \sum_{i=1}^n (i^2)v_1 + nb(v_2 - v_1) \quad (43)$$

$$\sum_{i=1}^n (i^2) = \frac{2n^3 + 2n^2 + n}{6} = a \quad (44)$$

$$\Rightarrow Av_2 = av_1 + nb(v_2 - v_1) \quad (45)$$

$$= (a - nb)v_1 + nb(v_2) \quad (46)$$

Hint #2 Show A has rank 2

Suppose x is a column of A

$$\Rightarrow x = \begin{pmatrix} 0 + k \\ n + k \\ \vdots \\ (n-1)(n) + k \end{pmatrix} \quad (47)$$

$$= kv_1 + n \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \end{pmatrix} \quad (48)$$

$$= kv_1 + n(v_2 - v_1) \quad (49)$$

$$= (k - n)v_1 + nv_2 \quad (50)$$

Every column of A is in $\text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\})$

$$\Rightarrow \text{Col}(A) \subseteq \text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\}) \quad (51)$$

$$\Rightarrow \text{rank}(A) \leq \dim(\text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\})) = 2 \quad (52)$$

$\text{rank}(A) \neq 0$ because $A \neq 0$.

Suppose $\text{rank}(A) = 1$.

It follows that every column of A could be expressed as a scalar multiple of one vector $z \in \mathbb{F}^n$.

Suppose x, y are distinct columns of A and $c \in \mathbb{F}$

$$\Rightarrow x = (k_1 - n)v_1 + nv_1 \quad y = (k_2 - n)v_1 + nv_2 \quad (53)$$

Suppose $cx = y$.

$$\Rightarrow c(k_1 - n)v_1 + cnv_1 = (k_2 - n)v_1 + nv_2 \quad (54)$$

$$\Rightarrow c(k_1 - n) = (k_2 - n) \text{ and } cn = n \quad (55)$$

$$\Rightarrow n(c - 1) = 0 \quad (56)$$

Case 1 $c = 1$

$$\Rightarrow x = y \not\Leftarrow \text{Contradiction! } x, y \text{ are distinct} \quad (57)$$

Case 2 $n = 0$

Impossible.

Therefore $\text{rank}(A) \neq 1$ and it follows that $\text{rank}(A) = 2$.

By the dimension theorem,

$$\text{rank}(\mathbf{L}_A) + \text{Nullity}(\mathbf{L}_A) = \dim(\mathbf{L}_A) \quad (58)$$

$$\dim(\mathbf{L}_A) = n \quad (59)$$

$$\text{rank}(\mathbf{L}_A) = \text{rank}(A) = 2 \quad (60)$$

$$\Rightarrow \text{Nullity}(\mathbf{L}_A) = n - 2 \quad (61)$$

$$\Rightarrow \text{Nullity}(A) = n - 2 \quad (62)$$

Let the eigenspace of eigenvalue zero be E_0

$$E_0 = N(A = 0 \cdot I_n) = N(A) \quad (63)$$

$$\Rightarrow \dim E_0 = \text{Nullity}(A) = n - 2 \quad (64)$$

Let m be the algebraic multiplicity of eigenvalue zero.

$$m \geq \dim E_0 = n - 2 \text{ by theorem 5.7} \quad (65)$$

Suppose $W = \text{span}(\{(1, 1, \dots, 1), (1, 2, \dots, n)\})$

Suppose $\alpha = \{(1, 1, \dots, 1), (1, 2, \dots, n)\}$

W is \mathbf{L}_A -invariant. It suffices to calculate the characteristic polynomial of the restriction of \mathbf{L}_A to W because its characteristic polynomial divides the characteristic polynomial of \mathbf{L}_A by theorem 5.21. \mathbf{L}_A and A possess the same characteristic polynomial because $[\mathbf{L}_A]_\gamma = A$ for some ordered basis γ of \mathbf{F}^n .

$$Av_1 = (b - n^2)v_1 + (n^2)v_2 \quad (66)$$

$$Av_2 = (a - nb)v_1 + (nb)v_2 \quad (67)$$

$$\Rightarrow [\mathbf{L}_{A_W}]_\alpha = \begin{pmatrix} b - n^2 & a - nb \\ n^2 & nb \end{pmatrix} \quad (68)$$

$$\Rightarrow \det([\mathbf{L}_{A_W}]_\alpha - tI_2) = \det \begin{pmatrix} b - n^2 - t & a - nb \\ n^2 & nb - t \end{pmatrix} \quad (69)$$

$$\Rightarrow t = \frac{1}{2} \left(\frac{n^3 + n}{2} \pm \frac{n}{2\sqrt{2}} \sqrt{3n^4 + 4n^3 + 6n^2 - 4n + 3} \right) \quad (70)$$

It follows that the characteristic polynomial of \mathbf{L}_{A_W} is

$$f_{\mathbf{L}_{A_W}}(t) = (t_1 - t)(t_2 - t) \quad (71)$$

$$f_{\mathbf{L}_{AW}}(t)|f(t) \tag{72}$$

$$\Rightarrow f(t) = g(t)f_{\mathbf{L}_{AW}}(t) = g(t)(t_1 - t)(t_2 - t) \tag{73}$$

Since the degree of $g(t)f_{\mathbf{L}_{AW}}(t)$ is 2, the degree of $g(t)$ is $n - 2$ It follows that the algebraic multiplicity $m = n - 2$.

$$\Rightarrow g(t) = (-t)^{n-2} \tag{74}$$

$$\Rightarrow f(t) = (t_1 - t)(t_2 - t)(-t)^{n-2} \tag{75}$$

$$= (-1)^n(t)^{n-2}(t_1 - t)(t_2 - t) \tag{76}$$

6.1

3. In $C([0, 1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$ (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f + g\|$. Then verify both the Cauchy-Schwartz inequality and the triangle inequality.

$$\langle f, g \rangle := \int_0^1 f(t)g(t) \, dt \quad (77)$$

$$\langle f, g \rangle = \int_0^1 te^t \, dt = 1 \quad (78)$$

$$\|f\| = \sqrt{\langle f, f \rangle} = \int_0^1 t^2 \, dt = \frac{1}{\sqrt{3}} \quad (79)$$

$$\|g\| = \sqrt{\langle g, g \rangle} = \int_0^1 e^{2t} \, dt = \sqrt{\frac{e^2 - 1}{2}} \quad (80)$$

$$\|f + g\| = \sqrt{\langle f + g, f + g \rangle} \quad (81)$$

$$\langle f + g, f + g \rangle = \langle f + g, f \rangle + \langle f + g, g \rangle \quad (82)$$

$$= \langle f, f \rangle + \langle g, f \rangle + \langle f, g \rangle + \langle g, g \rangle \quad (83)$$

$$= \frac{3e^2 + 11}{6} \quad (84)$$

$$\Rightarrow \|f + g\| = \sqrt{\frac{3e^2 + 11}{6}} \quad (85)$$

Cauchy-Schwartz Inequality

$$\|f\| \|g\| \stackrel{?}{\geq} \langle f, g \rangle \quad (86)$$

$$\sqrt{\frac{3e^2 + 11}{6}} \stackrel{?}{\geq} 1 \quad (87)$$

$$\Rightarrow e^2 \geq 7 \checkmark \quad (88)$$

Triangle Inequality

$$\|f\| + \|g\| \stackrel{?}{\geq} \|f + g\| \quad (89)$$

$$\frac{1}{\sqrt{3}} + \sqrt{\frac{e^2 - 1}{6}} \stackrel{?}{\geq} \sqrt{\frac{3e^2 + 11}{6}} \quad (90)$$

$$\Rightarrow \sqrt{\frac{e^2 - 1}{6}} \geq 1 \checkmark \quad (91)$$

8. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a, b), (c, d) \rangle = ac - bd$ on \mathbb{R}^2 .

Suppose $(a, b) = (c, d) = (1, 1)$

$$\langle (1, 1), (1, 1) \rangle = 1 - 1 = 0. \quad (92)$$

This is not an inner product because it fails Property D of an inner product.

(b) $\langle A, B \rangle = \text{tr}(A + B)$ on $\mathbf{M}_{n \times n}(\mathbb{R})$.

Suppose $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle A, A \rangle = \text{tr} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 0 \quad (93)$$

This is not an inner product because it fails Property D of an inner product.

(c) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g'(t) \, dt$ on $\mathbf{P}(\mathbb{R})$, where $'$ denotes differentiation.

Suppose $f(x) \equiv 1$

$$\langle f(x), f(x) \rangle = \int_0^1 0 \cdot 1 \, dt = 0 \quad (94)$$

This is not an inner product because it fails Property D of an inner product.

12.

6.2

2.

6.

7. Let β be a basis for a subspace W of an inner product space V , and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v \rangle = 0$ for every $v \in \beta$.

Claim: $x \in W^\perp \Leftrightarrow \langle z, v \rangle = 0 \quad \forall v \in \beta$

(\Rightarrow)

Suppose $z \in W^\perp$

$$\Rightarrow \langle z, x \rangle = 0 \quad \forall x \in S \quad (95)$$

$$\beta \subseteq S \quad (96)$$

$$\Rightarrow \langle z, y \rangle = 0 \quad \forall y \in \beta \quad (97)$$

(\Leftarrow)

Suppose $\langle z, v \rangle = 0 \quad \forall v \in \beta$

Suppose $w \in W$

$$\text{span}(\beta) = W \quad (98)$$

$$\Rightarrow w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \quad (99)$$

$$\langle z, w \rangle = \left\langle z, \sum_{i=1}^n a_i v_i \right\rangle \quad (100)$$

$$= \sum_{i=1}^n a_i \langle z, v_i \rangle \quad (101)$$

$$= \sum_{i=1}^n a_i \cdot 0 \quad (102)$$

$$= 0 \quad (103)$$

$$\Rightarrow \langle z, w \rangle = 0 \quad \forall w \in W \quad (104)$$

$$\Rightarrow z \in W^\perp \quad (105)$$

15. Let V be a finite-dimensional inner product space over F .

- (a) *Parseval's Identity*. Let $\{v_1, v_2, \dots, v_n\}$ be an orthonormal basis for V . For any $x, y \in V$ prove that

$$\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

- (b) Use (a) to prove that if β is an orthonormal basis for \mathbf{V} with inner product $\langle \cdot, \cdot \rangle$, then for any $x, y \in \mathbf{V}$

$$\langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle [x]_\beta, [y]_\beta \rangle' = \langle x, y \rangle,$$

where $\langle \cdot, \cdot \rangle'$ is the standard inner product on \mathbf{F}^n .

- (a) Suppose $x, y \in \mathbf{V}$

$$\Rightarrow x = \sum_{i=1}^n a_i v_i \quad y = \sum_{i=1}^n b_i v_i \quad (106)$$

Suppose β is an orthonormal basis.

$$\Rightarrow \langle v_i, v_j \rangle \quad \forall v_i, v_j \in \beta \text{ such that } i \neq j \quad (1 \leq i, j \leq n) \quad (107)$$

Fix some k ($1 \leq k \leq n$)

$$\langle x, v_k \rangle = \left\langle \sum_{i=1}^n a_i v_i, v_k \right\rangle \quad (108)$$

$$= \sum_{i=1}^n a_i \langle v_i, v_k \rangle \quad (109)$$

$$= a_k \langle v_k, v_k \rangle \quad (110)$$

$$= a_k \|v_k\|^2 \quad (111)$$

$$= a_k \quad (112)$$

$$\Rightarrow \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle} = \sum_{i=1}^n a_i \bar{b}_i = \langle x, y \rangle \quad (113)$$

- (b) Suppose ϕ_β is the coordinate isomorphism from $\mathbf{V} \rightarrow \mathbf{F}^n$

$$\Rightarrow \phi_\beta(x), \phi_\beta(y) \in \mathbf{F}^n \quad (114)$$

$$\Rightarrow \phi_\beta(x) = [x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \phi_\beta(y) = [y]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad (115)$$

$$\Rightarrow \langle [x]_\beta, [y]_\beta \rangle' = \sum_{i=1}^n a_i \bar{b}_i = \langle x, y \rangle \quad (116)$$

$$\therefore \langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle [x]_\beta, [y]_\beta \rangle' = \langle x, y \rangle \quad (117)$$