Assignment

2.3: 13,15,16,17; 2.4: 2(bef), 5, 17, 20; 2.5: 3(cd), 6(bc), 10, 13

Work

2.3

13. Let A and B be $n \times n$ matrices. Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.
Claim: $\text{tr}(AB) = \text{tr}(AB)$

$$\begin{aligned}
\text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} && (1)\\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} && (2)\\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} A_{ki}B_{ik} && (3)\\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ik}A_{ki} && (4)\\
&= \sum_{i=1}^{n} (BA)_{ii} && (5)\\
&= \text{tr}(BA) && (6)
\end{aligned}$$

15. Let M and A be matrices for which the product matrix MA is defined. If the jth column of A is a linear combination of a set of columns of A, prove that the jth column MA is linear combination of the corresponding columns of MA with the same corresponding coefficients.
Suppose $M \in \mathsf{M}_{m\times n}(F), A \in \mathsf{M}_{n\times p}, MA \in \mathsf{M}_{m\times p}$
Suppose $v_j = j^{\text{th}}$ column of A and $u_j = j^{\text{th}}$ column of MA

$$v_j = \sum_{i=1}^{p} a_i v_i, \quad a_i \in F \qquad (7)$$

Claim: $u_j = \sum_{i=1}^{p} a_i u_i$

$$Mv_j = M\sum_{i=1}^{p} a_i v_i = \sum_{i=1}^{p} a_i M v_i \qquad (8)$$

$$\Rightarrow u_j = u_j = \sum_{i=1}^{p} a_i u_i \quad \text{(by theorem 2.13)} \qquad (9)$$

16. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear.

(a) If $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$

(b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k

(a) Suppose $x \in R(\mathsf{T}^2)$

$$\exists a \in \mathsf{V} \text{ such that } \mathsf{T}(\mathsf{T}(a)) = x \tag{10}$$
$$\Rightarrow x \in R(\mathsf{T}) \tag{11}$$
$$\Rightarrow R(\mathsf{T}^2) \subseteq R(\mathsf{T}) \tag{12}$$
$$\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2) \tag{13}$$
$$\therefore R(\mathsf{T}) = R(\mathsf{T}^2) \tag{14}$$

Suppose $x \in N(\mathsf{T})$

$$\Rightarrow \mathsf{T}(x) = 0 \tag{15}$$
$$\Rightarrow \mathsf{T}(\mathsf{T}(x)) = 0 \tag{16}$$
$$\Rightarrow x \in N(\mathsf{T}^2) \tag{17}$$

By dimension theory:

$$\dim(R(\mathsf{T})) + \dim(N(\mathsf{T})) = \dim(R(\mathsf{T}^2)) + \dim(N(\mathsf{T}^2)) \tag{18}$$
$$\dim(N(\mathsf{T})) = \dim(N(\mathsf{T}^2)) \quad \text{(by 14)} \tag{19}$$
$$\Rightarrow N(\mathsf{T}) = \mathsf{N}(\mathsf{T}^2) \tag{20}$$

Claim: $R(\mathsf{T} \cap N(\mathsf{T} = \{0\}$
Suppose $u \in R(\mathsf{T}) \cap N(\mathsf{T})$

$$\exists y \in \mathsf{V} \text{ such that } \mathsf{T}(y) = u \tag{21}$$
$$\mathsf{T}^2(y) = 0 \tag{22}$$
$$\therefore y \in N(\mathsf{T}^2) \tag{23}$$
$$\therefore y \in N(\mathsf{T}) \quad \text{(by 20)} \tag{24}$$
$$u = \mathsf{T}(y) = 0 \tag{25}$$

Claim: $V = N(\mathsf{T}) \oplus R(\mathsf{T})$
Suppose β_N is a basis for $N\mathsf{T}$ and β_R is a basis for $R(\mathsf{T})$

$$N(\mathsf{T}) \cap R(\mathsf{T}) = \{0\} \tag{26}$$
$$\Rightarrow \text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0\} \tag{27}$$
$$\therefore \beta_N \cap \beta_R = \{\} \tag{28}$$
$$\Rightarrow \text{card}(\beta_N \cup \beta_R) = n \tag{29}$$
$$\beta_N = \{n_1, n_2, \ldots, n_k\} \Rightarrow \text{card}(\beta_N) = k \tag{30}$$
$$\beta_R = \{r_1, r_2, \ldots, r_m\} \Rightarrow \text{card}(\beta_R) = m \tag{31}$$
$$\tag{32}$$

For i from 1 to k if $n_i \in \text{span}(\beta_R)$

$$\text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0, n_i\} \lightning \text{Contradiction! with 27} \tag{33}$$

For i from 1 to m, if $r_i \in \text{span}(\beta_N)$

$$\text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0, r_i\} \lightning \text{Contradiction! with 27} \tag{34}$$

$$\therefore \beta_N \cup \beta_R \text{ is linearly independent} \tag{35}$$

$$\begin{align}
\text{card}(\beta_N \cup \beta_R) &= \text{card}(\beta_N) + \text{card}(\beta_R) - \text{card}(\beta_N \cap \beta_R) \tag{36}\\
&= \text{card}(\beta_N) + \text{card}(\beta_R) \tag{37}\\
&= k + m = n \tag{38}
\end{align}$$

From 35 and 38 it follows that

$$\mathsf{V} = \text{span}(\beta_N \cup \beta_R) \tag{39}$$

Claim: $\text{span}(\beta_N \cup \beta_R) = \text{span}(\beta_N) + \text{span}(\beta_R) = N(\mathsf{T}) + R(\mathsf{T})$
($\supseteq$) since $\text{span}(\beta_N)$ and $\text{span}(\beta_R)$ are both contained in V it follows that

$$\text{span}(\beta_N) + \text{span}(\beta_R) \subseteq \mathsf{V} \tag{40}$$

($\subseteq$) Suppose $x \in \text{span}(\beta_N \cup \beta_R)$

$$\begin{align}
x = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} \cdots + a_{k+m} v_{k+m} \tag{41}\\
a_1 v_1 + \cdots + a_k v_k \in \text{span}(\beta_R) \tag{42}\\
a_{k+1} v_{k+1} + \cdots + a_{k+m} \in \text{span}(\beta_N) \tag{43}\\
\Rightarrow x \in \text{span}(\beta_N) + \text{span}(\beta_R) \tag{44}
\end{align}$$

$$\Rightarrow N(\mathsf{T}) \oplus R(\mathsf{T}) = \mathsf{V} \tag{45}$$

(b) Claim: $\text{rank}(\mathsf{T}^k) = \text{rank}(\mathsf{T}^{k+1})$ for some integer k

$$0 \leq \cdots \leq \dim(R(\mathsf{T}^3)) \leq \dim(R(\mathsf{T}^2)) \leq \dim(R(\mathsf{T})) \leq n \tag{46}$$

Suppose $x \in R(\mathsf{T}^{k+1})$

$$\begin{align}
\exists a \in \mathsf{V} \text{ such that } \mathsf{T}(\mathsf{T}(a)) = x \tag{47}\\
\Rightarrow x \in R(\mathsf{T}^k) \tag{48}\\
\Rightarrow R(\mathsf{T}^k) \leq R(\mathsf{T}^{k+1}) \tag{49}
\end{align}$$

Since there is a lower bound 0 and an upper bound n in 46 it follows that for some integer k

$$\begin{align}
\dim(R(\mathsf{T}^{k+1})) = \dim(R(\mathsf{T}^k)) \tag{50}\\
\Rightarrow R(\mathsf{T}^{k+1}) = R(\mathsf{T}^k) \text{ (by 49 and 50)} \tag{51}\\
\Rightarrow \text{rank}(\mathsf{T}^k) = \text{rank}(\mathsf{T}^{k+1}) \quad \text{for some integer } k \tag{52}
\end{align}$$

Claim: $R\left(\mathsf{T}^{k+1}\right) = R(\mathsf{T}^m)\ \forall m \geq 1$
For $m = 1$ by 51
For $m = j \to R(\mathsf{T}^{k+j}) = R(\mathsf{T}^k)$
For $m = j + 1$

$$R\left(\mathsf{T}^{k+j+1}\right) = \mathsf{T}\left(R\left(\mathsf{T}^{k+j}\right)\right) = \mathsf{T}\left(R\left(\mathsf{T}^{k}\right)\right) = R\left(\mathsf{T}^{k+1}\right) = R\left(\mathsf{T}^{k}\right) \tag{53}$$

$$\Rightarrow R\left(\mathsf{T}^{k+j+1}\right) = R\left(\mathsf{T}^{k}\right) \tag{54}$$

$$\Rightarrow \text{rank}\left(\mathsf{T}^{k+m}\right) = \text{rank}\left(\mathsf{T}^{k}\right) \text{ for some integer } k \text{ and for all integer } m \tag{55}$$

$$\Rightarrow \text{rank}(\mathsf{T}^{2k} = \text{rank}(\mathsf{T}^{k}) \text{ for some } k \tag{56}$$

Apply the same method as in part (a) and it follows that

$$\mathsf{V} = R(\mathsf{T}^k) \oplus N(\mathsf{T}^k) \text{ for some } k \tag{57}$$

17. Let V be a vector space. Determine all linear transformations $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T} = \mathsf{T}^2$.
Suppose $\mathsf{W} = \{y\colon \mathsf{T}(y) = y\}$
Claim: $\forall \mathsf{T} \in \mathcal{L}(\mathsf{V}), \mathsf{T} = \mathsf{T}^2 \Leftrightarrow \mathsf{T}$ is a projection of W along $N(\mathsf{T})$
Claim:$\mathsf{W} \cap N(\mathsf{T}) = \{0\}$
Suppose $x \in \mathsf{W} \cap N(\mathsf{T})$

$$\Rightarrow \mathsf{T}(x) = x \tag{58}$$

$$\Rightarrow \mathsf{T}(x) = 0 \tag{59}$$

$$\Rightarrow x = 0 \tag{60}$$

$$\Rightarrow \mathsf{W} \cap N(T) = \{0\} \tag{61}$$

$N(\mathsf{T})$ is a subspace of V (by theorem 2.1)
Claim: W is subspace of V

$$0 \in \mathsf{W} \tag{62}$$

Suppose $x, z \in \mathsf{W}$

$$\mathsf{T}(x+z) = \mathsf{T}(x) + \mathsf{T}(z) = x + z \tag{63}$$

Suppose $x \in \mathsf{W},\ c \in F$

$$\mathsf{T}(cx) = c\mathsf{T}(x) = cx \tag{64}$$

$$\mathsf{W} \subseteq \mathsf{V} \text{ by definition} \tag{65}$$

Claim $\mathsf{V} = \mathsf{W} \oplus N(\mathsf{T})$
($\subseteq$) Suppose $x \in \mathsf{V}$

$$\Rightarrow x = \mathsf{T}(x) + (x - \mathsf{T}(x)) \tag{66}$$

$$\mathsf{T}(x) = \mathsf{T}^2(x) \tag{67}$$

$$\mathsf{T}(x) \in \mathsf{W} \tag{68}$$

$$\mathsf{T}(x - \mathsf{T}(x)) = \mathsf{T}(x) - \mathsf{T}^2(x) = 0 \tag{69}$$

$$\Rightarrow x \in \mathsf{W} \oplus N(\mathsf{T}) \tag{70}$$

$$\Rightarrow \mathsf{V} \subseteq \mathsf{W} \tag{71}$$

($\supseteq$) Suppose $\mathsf{W} \subseteq \mathsf{V}$

$$N(\mathsf{T}) \subseteq \mathsf{V} \tag{72}$$
$$\Rightarrow \mathsf{W} \oplus N(\mathsf{T}) \text{ by closure of } \mathsf{V} \tag{73}$$
$$\forall x \in \mathsf{V}, x = x_1 + x_2 \quad \text{for some } x \in \mathsf{W}, x_2 \in N(\mathsf{T}) \tag{74}$$

$$\mathsf{T}(x) = \mathsf{T}(x_1) + \mathsf{T}(x_2) \tag{75}$$
$$= x_1 + 0 \tag{76}$$
$$= x_1 \tag{77}$$

It follows that T is a projection of W along $N(\mathsf{T})$
Suppose T is a projection of W along $N(\mathsf{T})$
Suppose $x \in \mathsf{W} \oplus N(\mathsf{T})$

$$\Rightarrow x = x_1 + x_2 \quad \text{for some } x_1 \in \mathsf{W}, x_2 \in N(\mathsf{T}) \tag{78}$$
$$\mathsf{T}(x) = \mathsf{T}(x_1 + x_2) = \mathsf{T}(x_1) + \mathsf{T}(x_2) = \mathsf{T}(x_1) + 0 = x_1 \tag{79}$$
$$\Rightarrow \mathsf{T}(x_1) = 0 \tag{80}$$
$$\mathsf{T}^2(x) = \mathsf{T}(x_1) \tag{81}$$
$$\Rightarrow \mathsf{T}^2(x) = x \tag{82}$$
$$\Rightarrow \mathsf{T}^2(x) = \mathsf{T}(x) \; \forall x \in \mathsf{V} \tag{83}$$
$$\therefore \mathsf{T} = \mathsf{T}^2 \tag{84}$$

2.4

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

   (b) $\mathsf{T}\colon \mathsf{T}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (3a_1 - 2a_2, a_2, 4a_1)$
   Claim: T is 1-1
   Suppose $x, y \in \mathbb{R}^2$ such that $x = (a_1, a_2), y = (a_3, a_4)$ and $\mathsf{T}(x) = \mathsf{T}(y)$ for $a_i \in \mathbb{R}$

$$(3a_1 - a_2, a_2, 4a_1) = (3a_3 - a_4, a_4, 4a_3) \tag{85}$$

$$\Rightarrow 3a_1 - a_2 = 3a_3 - a_4 \tag{86}$$

$$a_2 = a_4 \tag{87}$$

$$4a_1 = 4a_3 \tag{88}$$

$$\Rightarrow a_1 = a_3 \tag{89}$$

$$a_2 = a_4 \tag{90}$$

$$\Rightarrow x = y \tag{91}$$

   Claim: T is onto
   Suppose $x \in \mathbb{R}^3$ such that $x = (b_1, b_2, b_3)$ for $b_i \in \mathbb{R}$
   Let $b_2 = a_2, b_3 = 4a_1, b_1 = (\frac{3}{4}b_3 - b_2)$

$$\Rightarrow (b_1, b_2, b_3) = (2a_1 - a_2, a_2, 4a_1) \tag{92}$$

$$\Rightarrow x \in R(\mathsf{T}) \tag{93}$$

$$R(\mathsf{T}) \subseteq \mathsf{M}_{n\times n}(\mathbb{R}) \text{ by def of } \mathsf{T} \tag{94}$$

$$\therefore \mathsf{T} \text{ is invertible}$$

   (e) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$
   Claim: T is not 1-1
   Suppose $x, y \in \mathsf{M}_{2\times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} \tag{95}$$

$$\Rightarrow x \neq y \tag{96}$$

$$\mathsf{T}(x) = (0 \cdot 1) + (2 \cdot 0)x + (1+4)x^2 = 5x^2 \tag{97}$$

$$\mathsf{T}(y) = (0 \cdot 1) + (2 \cdot 0)x + (3+2)x^2 = 5x^2 \tag{98}$$

$$\therefore \mathsf{T} \text{ is not invertible}$$

(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Claim: T is 1-1

Suppose $x, y \in \mathsf{M}_{2\times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad y = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{99}$$

Suppose $\mathsf{T}(x) = \mathsf{T}(y)$ for $a, b, c, \ldots, h \in \mathbb{R}$

$$\Rightarrow \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} \tag{100}$$

$$\Rightarrow a+b = e+f \tag{101}$$
$$a = e \tag{102}$$
$$c = g \tag{103}$$
$$c+d = g+h \tag{104}$$

$$\Rightarrow a = e \tag{105}$$
$$b = f \tag{106}$$
$$c = g \tag{107}$$
$$d = h \tag{108}$$

$$\Rightarrow x = y \tag{109}$$

Claim: T is onto

Suppose $x \in \mathsf{M}_{2\times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for } a, b, c, d \in \mathbb{R} \tag{110}$$

Let $e, f, g, h \in \mathbb{R}$ such that

$$e = b \qquad f = e + a \tag{111}$$
$$g = c \qquad h = -g + d \tag{112}$$

$$x = \begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} \tag{113}$$
$$\Rightarrow x \in R(\mathsf{T}) \tag{114}$$
$$R(T) \subseteq \mathsf{M}_{2\times 2}(\mathbb{R}) \text{ by definition of } \mathsf{T} \tag{115}$$
$$\therefore \mathsf{T} \text{ is invertible}$$

5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.
Claim: A^t is invertible

$$\Rightarrow (AB)^t = (BA)^t = I^t = I \tag{116}$$

Lemma: $(AB)^t = B^t A^t$

$$(AB)^t_{ij} = (AB)_{ji} \tag{117}$$
$$= \sum_{k=1}^{n} A_{jk} B_{kj} \tag{118}$$
$$(B^t A^t)_{ij} = \sum_{k=1}^{n} B^t_{ik} A^t_{kj} \tag{119}$$
$$= \sum_{k=1} B_{kj} A_{jk} \tag{120}$$
$$= \sum_{k=1}^{n} A_{jk} B_{kj} \tag{121}$$

$$\Rightarrow B^t A^t = A^t B^t = I \tag{122}$$

Claim: $(A^{-1})^t = (A^t)^{-1}$

$$B = A^{-1} \tag{123}$$
$$(A^t)^{-1} = B^t = (A^{-1})^t \tag{124}$$

17. Let V and W be finite-dimensional vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $\mathsf{T}(\mathsf{V}_o)$ is a subspace of W.

(b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

Let V_0 be a subspace of V

(a) Claim: $0 \in \mathsf{V}_0$

$$0 \in \mathsf{V}_0 \tag{125}$$
$$\Rightarrow \mathsf{T}(0) = 0 \tag{126}$$

Claim: $\forall x, y \in \mathsf{T}(\mathsf{V}_0), x + y \in \mathsf{T}(\mathsf{V}_0)$
Suppose β_{V_0} is basis for V_0

$$\Rightarrow \mathsf{V}_0 = \mathrm{span}(\beta_{\mathsf{V}_0}) \tag{127}$$

Suppose $x, y \in \mathsf{T}(\mathsf{V}_0)$

$$\Rightarrow x = \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \tag{128}$$
$$y = \mathsf{T}(b_1u_1 + b_2u_2 + \cdots + b_nu_n) \quad \text{for } a_i, b_i \in F,\ u_i \in \beta_{\mathsf{V}_0} \tag{129}$$
$$\Rightarrow x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \tag{130}$$
$$y = b_1\mathsf{T}(u_1) + b_2\mathsf{T}(u_2) + \cdots + b_n\mathsf{T}(u_n) \tag{131}$$
$$\Rightarrow x + y = (a_1 + b_1)\mathsf{T}(u_1) + (a_2 + b_2)\mathsf{T}(u_2) + \cdots + (a_n + b_n)\mathsf{T}(u_n) \tag{132}$$
$$= \mathsf{T}((a_1 + b_2)u_1 + (a_2 + b_2)u_2 + \cdots + (a_n + b_n)u_n) \tag{133}$$

$$\Rightarrow x + y \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \tag{134}$$

Claim: $\forall x \in \mathsf{T}(\mathsf{V}_0)$ and $\forall c \in F, cx \in \mathsf{T}(\mathsf{V}_0)$
Suppose $x \in \mathsf{T}(\mathsf{V}_0)$ such that

$$x = \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad \text{for } u_i \in \beta_{\mathsf{V}_0}, a_i \in F \tag{135}$$

$$\Rightarrow x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \tag{136}$$

$$cx = c\,(a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n)) \tag{137}$$
$$= ca_1\mathsf{T}(u_1) + ca_2\mathsf{T}(u_2) + \cdots + ca_n\mathsf{T}(u_n) \tag{138}$$
$$= \mathsf{T}(ca_1u_1 + ca_2u_2 + \cdots + ca_nu_n) \tag{139}$$

$$\Rightarrow cx \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \tag{140}$$

$\mathsf{T}(\mathsf{V}_0) \subseteq \mathsf{W}$ by the definition of T. It follows that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.

(b) Suppose β_{V_0} is a basis of V_0
Claim: $\text{card}(\beta_{\mathsf{V}_0}) = \text{card}(\mathsf{T}(\beta_{\mathsf{V}_0}))$

T is an isomorphism. Equivalently it also invertible and 1-1. Therefore it follows that for every $x \in \mathsf{T}(\beta_{\mathsf{V}_0})$ there exists a unique vector $y \in \beta_{\mathsf{V}_0}$ such that $\mathsf{T}(y) = x$
Claim: $\mathsf{T}(\beta_{\mathsf{V}_0})$ is a basis of $\mathsf{T}(\mathsf{V}_0)$

$$\mathsf{T}(\beta_{\mathsf{V}_0}) \text{ is a basis of } \mathsf{T}(\mathsf{V}_0) \Leftrightarrow \text{span}(\mathsf{T}(\beta_{\mathsf{V}_0})) = \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \tag{141}$$

($\subseteq$) Suppose $x \in \text{span}(\mathsf{T}(\beta_{\mathsf{V}_0}))$

$$x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \tag{142}$$
$$= \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad \text{for } u_i \in \beta_{\mathsf{V}_0}, a_i \in F \tag{143}$$

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n \in \text{span}(\beta_{\mathsf{V}_0}) \tag{144}$$
$$\Rightarrow x \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \tag{145}$$

($\supseteq$) Suppose $x \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0}))$ such that

$$
\begin{aligned}
x &= a_1\mathsf{T}((u_1) + a_1\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n)) \quad \text{for } a_i \in F, u_i \in \beta_{\mathsf{V}_0} && (146)\\
&= a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) && (147)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow x \in \text{span}(\mathsf{T}(\beta_{\mathsf{V}_0})) && (148)\\
&\therefore \dim(\beta_{\mathsf{V}_0}) = \dim(\mathsf{T}(\beta_{\mathsf{V}_0})) && (149)
\end{aligned}
$$

20. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W Let β and γ be ordered bases for V and W respectively. Prove that $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{L}_A)$ and that $\text{nullity}(\mathsf{T} = \text{nullity}(\mathsf{L}_1)$, where $A = [\mathsf{T}]_\beta^\gamma$.
Claim: $\text{nullity}(\mathsf{T}) = \text{nullity}(\mathsf{L}_A)$

$N(\mathsf{T})$ is a subspace of V This implies that $\phi_\beta(N(\mathsf{T}))$ is a subspace of F^n and $\dim(N(\mathsf{T})) = \dim(\phi_\beta(N(\mathsf{T})))$
Claim: $\phi_\beta(N(\mathsf{T})) = N(\mathsf{L}_A)$
($\subseteq$) Suppose $y \in \phi_\beta(N(\mathsf{T}))$

$$
\begin{aligned}
&\Rightarrow y = \phi_\beta(x) && (150)\\
&\Rightarrow \mathsf{L}_A = \mathsf{L}_A\phi_\beta(x) = \phi_\gamma\mathsf{T}(x) = \phi_\gamma(0) = 0 && (151)\\
&y \in N(\mathsf{L}_A) && (152)
\end{aligned}
$$

($\supseteq$) Suppose $y \in N(\mathsf{L}_A$

$$
\begin{aligned}
&N(\mathsf{L}_A) \subseteq \mathsf{F}^n \text{ by definition of } N(\mathsf{L}_A) && (153)\\
&\Rightarrow \forall y \in N(\mathsf{L}_A), \exists! x \in \mathsf{V} \text{ such that } \phi_\beta(x) = y && (154)\\
&\because \phi_\beta \text{ is an isomorphism}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{L}_A(y) = \mathsf{L}_A(\phi_\beta(x)) = 0 && (155)\\
&\mathsf{L}_A\phi_\beta = \phi_\gamma\mathsf{T} && (156)\\
&\Rightarrow \phi_\gamma(\mathsf{T}(x)) = 0 \Rightarrow \mathsf{T}(x) = 0 && (157)\\
&\because \phi_\gamma \text{ is an isomorphism}\\
&\Rightarrow y \in \phi_\beta(N(\mathsf{T})) \text{ and } N(\phi_\gamma) = \{0\} && (158)\\
&\Rightarrow \dim(N(\mathsf{T})) = \dim(N(\mathsf{L}_A)) && (159)\\
&\therefore \text{nullity}(\mathsf{T}) = \text{nullity}(\mathsf{L}_A) && (160)
\end{aligned}
$$

Claim: $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{L}_A)$

$R(\mathsf{T}$ is a subspace of W. This implies that $\phi_\gamma(R(\mathsf{T}))$ is subspace of W and $\dim(R(\mathsf{T})) = \dim(\phi_\gamma(R(\mathsf{T}))$
Claim: $\phi_\gamma(R(\mathsf{T})) = R(\mathsf{L}_A)$
($\subseteq$) Suppose $x \in \phi_\gamma(R(\mathsf{T}))$

$$\Rightarrow x \in \phi_\gamma(y) \quad \text{for some } y \in R(\mathsf{T}) \tag{161}$$
$$\Rightarrow y \in \mathsf{T}(z) \quad \text{for some } z \in \mathsf{V} \tag{162}$$

$$\phi_\gamma(\mathsf{T}(z)) = (\phi_\gamma \mathsf{T})(x) \tag{163}$$
$$\phi_\gamma \mathsf{T} = \mathsf{L}_A \phi_\beta \tag{164}$$

$$\Rightarrow x = (\mathsf{L}_A \phi_\beta)(z) \tag{165}$$
$$= \mathsf{L}_A(w) \quad \text{for some } w \in \mathsf{F}^n \tag{166}$$

$$\Rightarrow x \in R(\mathsf{L}_A) \tag{167}$$

($\supseteq$) Suppose $x \in R(\mathsf{L}_A)$

$$\Rightarrow x = [\mathsf{T}]_\beta^\gamma \phi_\beta(z) \quad \text{for some } z \in \mathsf{V} \tag{168}$$
$$\Rightarrow x = \mathsf{L}_A(\phi_\beta(z)) = (\mathsf{L}_A \phi_\beta)(z) = \phi_\gamma \mathsf{T}(z) \tag{169}$$

$$\mathsf{T}(z) \in R(\mathsf{T}) \tag{170}$$
$$\Rightarrow x \in \phi_\gamma(R(\mathsf{T})) \tag{171}$$

2.5

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change or coordinate matrix that changes β'-coordinates into β-coordinates.

   (c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = 1 \tag{172}$$

$$2a + 3b + c = 0 \tag{173}$$
$$-a + c = 0 \tag{174}$$
$$b = 1 \tag{175}$$

$$a = 0 \tag{176}$$
$$b = 1 \tag{177}$$
$$c = -3 \tag{178}$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x \tag{179}$$

$$2a + 3b + c = 0 \tag{180}$$
$$a + 1 \tag{181}$$
$$b = 0 \tag{182}$$

$$a = -1 \tag{183}$$
$$b = 0 \tag{184}$$
$$c = 2 \tag{185}$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x^2 \tag{186}$$

$$2a + 3b + c = 1 \tag{187}$$
$$-a = 0b = 0 \tag{188}$$

$$a = 0 \tag{189}$$
$$b = 0 \tag{190}$$
$$c = 1 \tag{191}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix} \tag{192}$$

(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = x^2 + x + 4 \tag{193}$$

$$a + c = 1 \tag{194}$$
$$-a + b = 1 \tag{195}$$
$$a + b + c = 4 \tag{196}$$

$$a = 2 \tag{197}$$
$$b = 3 \tag{198}$$
$$c = -11 \tag{199}$$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = 4x^2 - 3x + 2 \tag{200}$$

$$a + c = 4 \tag{201}$$
$$-a + b = -3 \tag{202}$$
$$a + b + c = 2 \tag{203}$$

$$a = 1 \tag{204}$$
$$b = -2 \tag{205}$$
$$c = 3 \tag{206}$$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = 2x^2 + 3 \tag{207}$$

$$a + c = 2 \tag{208}$$
$$-a + b = 0 \tag{209}$$
$$a + b + c = 3 \tag{210}$$

$$a = 1 \tag{211}$$
$$b = 1 \tag{212}$$
$$c = 1 \tag{213}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix} \tag{214}$$

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{215}$$

$$\mathsf{L}_A(v_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{216}$$

$$\Rightarrow [\mathsf{L}_A(v_1)]_\beta = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{217}$$

$$\mathsf{L}_A(v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + -1\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{218}$$

$$\Rightarrow [\mathsf{L}_A(v_1)]_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{219}$$

$$\Rightarrow [\mathsf{L}_A] = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \tag{220}$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{221}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} -1 \quad + \\ + \quad \mid \cdot -\frac{1}{2} \quad -1 \end{array} \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & {}^1/_2 & {}^1/_2 \\ 0 & 1 & {}^1/_2 & {}^{-1}/_2 \end{array}\right) \tag{222}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} {}^1/_2 & {}^1/_2 \\ {}^1/_2 & {}^{-1}/_2 \end{pmatrix} \tag{223}$$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{224}$$

$$\mathsf{L}_A(v_1) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \tag{225}$$

$$\mathsf{L}_A(v_2) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \tag{226}$$

$$\mathsf{L}_A(v_3) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \tag{227}$$

$$\begin{pmatrix}1\\3\\2\end{pmatrix} = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\2\end{pmatrix} \tag{228}$$

$$\begin{aligned} a+b+c &= 1 & a+b+c=1 && b &= -2 \\ \Rightarrow a+c &= 3 \Rightarrow & a+c=3 && \Rightarrow a &= 2 \\ a+b+2c &= 2 & c=1 && c &= 1 \end{aligned}$$

$$\Rightarrow [\mathsf{L}_A(v_1)]_\beta = \begin{pmatrix}2\\-2\\1\end{pmatrix} \tag{229}$$

$$\begin{pmatrix}0\\3\\1\end{pmatrix} = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\2\end{pmatrix} \tag{230}$$

$$\begin{aligned} a+b+c &= 0 & a+b+c=0 && b &= -3 \\ \Rightarrow a+c &= 3 \Rightarrow & a=2 && \Rightarrow a &= 2 \\ a+b+2c &= 1 & c=1 && c &= 1 \end{aligned}$$

$$\Rightarrow [\mathsf{L}_A(v_2)]_\beta = \begin{pmatrix}2\\-3\\1\end{pmatrix} \tag{231}$$

$$\begin{pmatrix}0\\4\\2\end{pmatrix} = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\2\end{pmatrix} \tag{232}$$

$$\begin{aligned} a+b+c &= 0 & a+b+c=0 && b &= -4 \\ \Rightarrow a+c &= 4 \Rightarrow & a=2 && \Rightarrow a &= 2 \\ a+b+2c &= 2 & c=2 && c &= 2 \end{aligned}$$

$$\Rightarrow [\mathsf{L}_A(v_3)]_\beta = \begin{pmatrix}2\\-4\\2\end{pmatrix} \tag{233}$$

$$\Rightarrow [\mathsf{L}_A]_\beta = \begin{pmatrix}2 & 2 & 2\\-2 & -2 & -4\\1 & 1 & 2\end{pmatrix} \tag{234}$$

$$Q = \begin{pmatrix}1 & 1 & 1\\1 & 0 & 1\\1 & 1 & 2\end{pmatrix} \tag{235}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\1 & 0 & 1 & 0 & 1 & 0\\1 & 1 & 2 & 0 & 0 & 1\end{array}\right) \quad \mid \cdot -1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \tag{236}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{237}$$

10. Prove that if A and B are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$.

$$\begin{aligned} \text{tr}(B) &= \text{tr}(QAQ^{-1}) && (238) \\ &= \text{tr}((QA)Q^{-1}) && (239) \\ &= \text{tr}(Q^{-1}(QA)) && (240) \\ &= \text{tr}((Q^{-1}Q)A) && (241) \\ &= \text{tr}(A) \text{ (by HW.2.3.13)} && (242) \end{aligned}$$

13. Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \text{ for } 1 \leq j \leq n$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is a coordinate matrix changing β'-coordinates into β-coordinates.
Claim $\text{span}(\beta) = \text{span}(\beta')$

Reverse Direction
Suppose $x' \in \text{span}(\beta')$

$$x' = c_1 \left(\sum_{i=1}^{n} Q_{i1} x_i \right) + c_2 \left(\sum_{i=1}^{n} Q_{i2} x_i \right) + \cdots + c_n \left(\sum_{i=1}^{n} Q_{in} x_i \right) \tag{243}$$

$$\begin{aligned} x' = c_1 \left(Q_{11}x_1 + Q_{21}x_2 + \cdots + Q_{n1}x_n \right) + \\ + c_2 \left(Q_{12}x_1 + Q_{22}x_2 + \cdots + Q_{n2}x_n \right) + \\ + \cdots + c_n \left(Q_{1n}x_1 + Q_{2n}x_2 + \cdots + Q_{nn}x_n \right) \end{aligned} \tag{244}$$

$$\begin{aligned} x' = (c_1Q_{11} + c_2Q_{12} + \cdots + c_nQ_{1n})x_1 + \\ + (c_1Q_{21} + c_2Q_{22} + \cdots + c_nQ_{2n})x_2 + \\ + \cdots + (c_1Q_{n1} + c_2Q_{n2} + \cdots + c_nQ_{nn})x_n \end{aligned} \tag{245}$$

$$\Rightarrow x \in \text{span}(\beta') \tag{246}$$

Forward Direction

Suppose $x \in \text{span}(\beta)$

$$x = c_1x_2 + c_2x_2 + \cdots + c_nx_n \tag{247}$$
$$= \sum_{i=1}^{n} c_ix_i \tag{248}$$

Let $c_i = \sum_{j=1}^{n} a_jQ_{ij}$

$$x = \sum_{i=1}^{n}\left(x_i\sum_{j=1}^{n} a_jQ_{ij}\right) \tag{249}$$

$$x = \sum_{i=1}^{n}\left(\left(a_1Q_{i1} + a_2Q_{i2} + \cdots + a_nQ_{in}\right)x_i\right) \tag{250}$$

$$\begin{aligned} x = (a_1Q_{11} + a_2Q_{12} + \cdots + a_nQ_{1n})x_1 + \\ + (a_1Q_{21} + a_2Q_{22} + \cdots + a_nQ_{2n})x_2 + \\ + \cdots + (a_1Q_{n1} + a_2Q_{n2} + \cdots a_nQ_{nn})x_n \end{aligned} \tag{251}$$

$$\begin{aligned} x = a_1(Q_{11}x_1 + Q_{21}x_2 + \cdots + Q_{n1}x_n) + \\ + a_2(Q_{12}x_1 + Q_{22}x_2 + \cdots + Q_{n2}x_n) + \\ + \cdots + a_n(Q_{1n}x_1 + Q_{2n}x_2 + \cdots + Q_{nn}x_n) \end{aligned} \tag{252}$$

$$x = a_1\sum_{i=1}^{n} Q_{i1}x_i + a_2\sum_{i=1}^{n} Q_{i2}x_i + \cdots + a_n\sum_{i=1}^{n} Q_{in}x_i \tag{253}$$
$$\Rightarrow x \in \text{span}(\beta') \tag{254}$$

Suppose $x \in \text{span}(\beta')$ such that

$$x = a_1Qx_1 + a_2Qx_2 + \cdots + a_nQx_n = 0 \tag{255}$$
$$\Rightarrow Q^{-1}Q(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 0 \tag{256}$$
$$\Rightarrow I_n(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 0 \tag{257}$$
$$\therefore \beta' \text{ is linearly independent} \tag{258}$$

Claim: $Q = [I_{\mathsf{V}}]_{\beta'}^{\beta}$

$$\mathsf{T}_{[I_{\mathsf{V}}]_{\beta'}^{\beta}}(v_i) = v_i' \text{ by definition of } [I_{\mathsf{V}}]_{\beta'}^{\beta} \tag{259}$$
$$\Rightarrow \mathsf{T}_{[I_{\mathsf{V}}]_{\beta'}^{\beta}}(\beta) = \beta' \tag{260}$$

Define $\mathsf{T}_Q \colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}_Q(x) = Qx,\ \forall x \in \mathsf{V}$

$$\mathsf{T}_Q(\beta) = \beta' \tag{261}$$
$$\Rightarrow \mathsf{T}_{[I_{\mathsf{V}}]_{\beta'}^{\beta}} = \mathsf{T}_Q(\text{by theorem 2.6}) \tag{262}$$
$$\Rightarrow [I_{\mathsf{V}}]_{\beta'}^{\beta} = Q \tag{263}$$