Assignment

3.1: 5, 12; 3.2: 5(beg), 6(adf), 14, 20; 3.3: 2(ad), 3(ad), 7(bd), 9, 10

Work

3.1

5. Prove that E is an elementary matrix if and only if E^t is.
   Claim: $E \rightsquigarrow E^t$
   $$I_n = \begin{bmatrix} e_1 & e_2 & \cdots & e_i & \cdots & e_j & \cdots & e_n \end{bmatrix} \tag{1}$$
   (a) Claim: The interchange of any two rows i and j is equivalent to interchanging any two columns i and j
   By applying the interchange to E it follows that:
   $$E = \begin{bmatrix} e_1 & e_2 & \cdots & e_j & \cdots & e_i & \cdots & e_n \end{bmatrix} \tag{2}$$
   $$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & e_j & \cdots & e_i & \cdots & e_n \end{bmatrix} = E \tag{3}$$
   (b) Claim: Multiplying any row i with nonzero scalar c is equivalent to multiplying any column j with the same scalar c.
   By applying the scaling to E is follows that:
   $$E = \begin{bmatrix} e_1 & e_2 & \cdots & ce_j & \cdots & e_n \end{bmatrix} \tag{4}$$
   $$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & ce_i & \cdots & e_n \end{bmatrix} = E \tag{5}$$
   (c) Claim: Adding any scalar multiple of row i to row j is equivalent to adding any scalar multiple of column i it column j
   By applying the replacement to E it follows that:
   $$E = \begin{bmatrix} e_1 & e_2 & \cdots & ce_i + e_j & \cdots & e_n \end{bmatrix} \tag{6}$$
   $$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & ce_j + e_i & \cdots & e_n \end{bmatrix} \tag{7}$$
   $$\therefore E^t \text{ is elementary} \tag{8}$$

12. Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.
   (a) For $m = 2$
      i. If $a_{11} = 0$ and $a_{21} \neq 0$ interchanging rows 1 and 2 creates an upper triangular matrix.
      ii. If $a_{11} \neq 0$ adding the row 1 scaled by a_{21}/a_{11} and subtracted from row 2 creates an upper triangular matrix.

(b) For $m = k$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \tag{9}$$

i. If $m > n$

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \\ & \text{\huge 0} & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \tag{10}$$

ii. If $m < n$

$$A \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} \cdots & a_{1n} \\ & a_{22} & & \vdots & & \vdots \\ & \text{\huge 0} & \ddots & \vdots & & \vdots \\ & & & a_{mm} & a_{m,m+1} \cdots & a_{mn} \end{pmatrix} \tag{11}$$

iii. If $m = n$

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & \text{\huge 0} & \ddots & \vdots \\ & & & a_{mn} \end{pmatrix} \tag{12}$$

(c) For $m = k + 1$

i. If $m > n$, assume the $m = k$ case holds

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \\ & \text{\huge 0} & & 0 \\ & & & \vdots \\ & & & 0 \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,n} \end{pmatrix} \tag{13}$$

Using row operations of type 3 on row $m+1$ from row 1 to row n in order and make $a_{m+1,i}=0$ for i from 1 to n.

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \\ & \text{\huge 0} & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \tag{14}$$

ii. If $m<n$, assume the $m=k$ case holds

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ & a_{22} & & \vdots & & & \vdots \\ \text{\huge 0} & & \ddots & \vdots & & & \vdots \\ & & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,n} & a_{m+1,m+1} & \cdots & a_{m+1,n} \end{pmatrix} \tag{15}$$

Using row operations of type 3 on row $m+1$ from row 1 to row m in order and make $a_{m+1,i}=0$ for i from 1 to m

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ & a_{22} & & \vdots & & & \vdots \\ & & \ddots & \vdots & & & \vdots \\ & \text{\huge 0} & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \\ & & & & a_{m+1,m+1} & \cdots & a_{m+1,n} \end{pmatrix} \tag{16}$$

iii. If $m=n$, assume the $m=k$ case hold

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & \text{\huge 0} & \ddots & \vdots \\ & & & a_{mn} \\ a_{m+1,i} & \cdots & \cdots & a_{m+1,n} \end{pmatrix} \tag{17}$$

Using row operations of type 3 on row $m+1$ from row 1 to row m in order to make $a_{m+1,i}=0$ fro i from 1 to n

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & \text{\huge 0} & \ddots & \vdots \\ & & & a_{mn} \\ & & & 0 \end{pmatrix} \tag{18}$$

3.2

5. For each of the following matrices, compute the rank and the inverse if it exists.

(b) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{19}$$

The rank of the matrix is 1, and it is not invertible.

(e) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{20}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right)$$

It follows that the rank is 3 and the inverse is

$$\begin{pmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{pmatrix} \tag{21}$$

(g) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \tag{22}$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

It follows that the rank is 4 and the inverse is

$$\begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix} \tag{23}$$

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

   (a) $\mathsf{T}\colon \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $\mathsf{T}(f(x)) = f'' + 2f'(x) - f(x)$

$$\mathsf{T}(1) = -1 \qquad \mathsf{T}(x) = 2 - x \qquad \mathsf{T}(x^2) = 2 + 4x - x^2 \tag{24}$$

$$\Rightarrow [\mathsf{T}]_\alpha^\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \tag{25}$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \mid \cdot -1 \\ \mid \cdot -1 \\ \mid \cdot -1 \end{array} \tag{26}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\Rightarrow [\mathsf{T}^{-1}]_\beta^\alpha = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix} \tag{27}$$

$$\mathsf{T}^{-1}(ax^2 + bx + c) = -ax^2 - (4a + b)x - (10a + 2b + c) \tag{28}$$

   (d) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{P}_2(\mathbb{R})$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1 x^2$$

$$\mathsf{T}(1,0,0) = 1 + x + x^2 \qquad \mathsf{T}(0,1,0) = 1 - x \qquad \mathsf{T}(0,0,1) = 1 + x \tag{29}$$

$$[\mathsf{T}]_\alpha^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{30}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{31}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & {}^1/_2 & {}^{-1}/_2 & 0 \\ 0 & 0 & 1 & {}^1/_2 & {}^1/_2 & -1 \end{array}\right)$$

$$\Rightarrow [\mathsf{T}^{-1}]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 1 \\ {}^1/_2 & {}^{-1}/_2 & 0 \\ {}^1/_2 & {}^1/_2 & -1 \end{pmatrix} \tag{32}$$

$$\mathsf{T}^{-1}(ax^2 + bx + c) = \left(a, \left(\frac{1}{2}\right) c - \left(\frac{1}{2}\right) b, \left(\frac{1}{2}\right) c + \left(\frac{1}{2}\right) b - a\right) \tag{33}$$

(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{R}^4$ defined by

$$\mathsf{T}(A) = (\mathrm{tr}(A), \mathrm{tr}(A^t), \mathrm{tr}(EA), \mathrm{tr}(AE)),$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{34}$$

$$\mathsf{T}(A) = (a + d, a + d, c + b, c + b) \tag{35}$$

$$\mathsf{T}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = (1, 1, 0, 0) \tag{36}$$

$$\mathsf{T}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = (0, 0, 1, 1) \tag{37}$$

$$\mathsf{T}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = (0, 0, 1, 1) \tag{38}$$

$$\mathsf{T}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = (1, 1, 0, 0) \tag{39}$$

$$[\mathsf{T}]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{40}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{41}$$

T is not invertible.

14. Let $\mathsf{T}, \mathsf{U}\colon \mathsf{V} \to \mathsf{W}$ be linear transformations

(a) Prove that $\mathsf{R}(\mathsf{T} + \mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$

(b) Prove that W is finite-dimensional, then $\mathrm{rank}(\mathsf{T} + \mathsf{U}) \leq \mathrm{rank}(\mathsf{T}) + \mathrm{rank}(\mathsf{U})$

(c) Deduce from (b) that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for any $m \times n$ matrices A and B

(a) Claim: $\mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$

$$\begin{aligned} \forall x \in \mathsf{V}, (\mathsf{T}+\mathsf{U})(x) = \mathsf{T}(x) + \mathsf{U}(x) \quad \text{where } \mathsf{T}(x) \in \mathsf{R}(\mathsf{T}), \mathsf{U}(x) \in \mathsf{R}(\mathsf{U}) \qquad (42) \\ \Rightarrow \mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U}) \qquad (43) \end{aligned}$$

(b) From (a) it follows that

$$\dim(\mathsf{R}(\mathsf{T}+\mathsf{U})) \leq \dim(\mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})) \tag{44}$$

From 1.6 exercise 31 (b) it follows that

$$\begin{aligned} \dim(\mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})) &\leq \dim(\mathsf{R}(\mathsf{T})) + \dim(\mathsf{R}(\mathsf{U})) && (45) \\ \Rightarrow \dim(\mathsf{R}(\mathsf{T}+\mathsf{U})) &\leq \dim(\mathsf{R}(\mathsf{T})) + \dim(\mathsf{R}(\mathsf{U})) && (46) \\ \Rightarrow \text{rank}(\mathsf{T}+\mathsf{U}) &\leq \text{rank}(\mathsf{T}) + \text{rank}(\mathsf{U}) && (47) \end{aligned}$$

(c) From theorem 3.3 it follows that

$$\begin{aligned} \text{rank}(A+B) &= \text{rank}(\mathsf{L}_{A+B}) && (48) \\ (A+B)x &= Ax + Bx \quad \forall x \in \mathsf{V} && (49) \\ \Rightarrow \mathsf{L}_{A+B} &= \mathsf{L}_A + \mathsf{L}_B && (50) \\ \Rightarrow [\mathsf{T}_{A+B}]_\alpha^\beta &= [\mathsf{T}_A]_\alpha^\beta + [\mathsf{T}_B]_\alpha^\beta && (51) \\ \text{rank}(\mathsf{L}_A + \mathsf{L}_B) &\leq \text{rank}(\mathsf{L}_A) + \text{rank}(\mathsf{L}_B) \quad \text{by 1.6 ex. 31} && (52) \\ \Rightarrow \text{rank}(A+B) &\leq \text{rank}(A) + \text{rank}(B) \quad \text{by theorem 3.3} && (53) \end{aligned}$$

20. Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}$$

(a) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

(b) Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $\text{rank}(B) \leq 2$

(a) Suppose $Ax = 0$, solve for x:

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix} \quad | \cdot \tfrac{1}{2} \\ \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

$$x_1 = s + 3t \qquad x_2 = -2s + t \qquad x_3 = s \qquad x_4 = -2t \qquad x_5 = t \tag{55}$$

$$\Rightarrow x = \left\{ t \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\} \tag{56}$$

It follows that a 5×5 matrix with rank 2 can be made by taking $t = s = 1$ and appended columns of zeros.

$$M = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{57}$$

(b) From part (a) it follows that $\dim(K_H) = 2$ for $Ax = 0$
Claim: $\forall B \in \mathsf{M}_{5\times 5}(\mathbb{R})$ such that $AB = 0, \operatorname{rank}(B) \leq 2$
Suppose AB=0

$$\Rightarrow B_n \in K_H \forall j \tag{58}$$

$$\{B_j \colon j = 1, 2, \ldots, n\} \subseteq K_H \tag{59}$$

$$\operatorname{span}(B_j) \subseteq K_H \forall j \tag{60}$$

$$\Rightarrow \operatorname{col}(B_j) \subseteq K_K \tag{61}$$

$$\Rightarrow \operatorname{rank}(\operatorname{col}(B_j)) \leq \dim(K_H) = 2 \tag{62}$$

$$\Rightarrow \operatorname{rank}(B) \leq 2 \tag{63}$$

3.3

2. For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution set.

   (a)

$$x_1 + 3x_2 = 0 \tag{64}$$
$$2x_2 + 6x_2 = 0 \tag{65}$$
$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \tag{66}$$
$$\Rightarrow x_2 = t \qquad\qquad x_1 = -3t \tag{67}$$
$$\Rightarrow x = \left\{ t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{68}$$
$$\Rightarrow \dim(x) = 1 \tag{69}$$

   Take $t = 1$ it follows that a basis is $\{\left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)\}$

   (d)

$$x_1 + x_2 - x_3 = 0 \qquad x_1 - x_2 + x_3 = 0 \qquad x_1 + 2x_2 - 2x_3 = 0 \tag{70}$$
$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix} \quad \Big| \cdot \tfrac{1}{3} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{71}$$
$$x_3 = t \qquad\qquad x_2 = t \qquad\qquad x_1 = 0 \tag{72}$$
$$\Rightarrow x = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{73}$$
$$\Rightarrow \dim x = 1 \tag{74}$$

   Take $t = 1$ it follows that a basis is $\left\{ \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right) \right\}$

3. Using the results of Exercise 2, find all solutions to the following systems.

   (a)

$$x_1 + 3x_2 = 5 \qquad\qquad 2x_1 + 6x_2 = 10 \tag{75}$$
$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 10 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right) \tag{76}$$
$$\Rightarrow x_2 = t \qquad\qquad x_1 = 5 - 3t \tag{77}$$
$$x = \left\{ \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{78}$$

(d)

$$2x_1 + x_2 - x_3 = 5 \qquad x_1 - x_2 + x_3 = 1 \qquad x_1 + 2x_2 - 2x_3 = 4 \tag{79}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & -2 & 4 \end{array}\right) \;\; \mid \cdot \tfrac{1}{3} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{80}$$

$$\Rightarrow x_3 = t \qquad x_2 1 + t \qquad x_1 = 2 \tag{81}$$

$$\Rightarrow x = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{82}$$

7. Determine which of the following systems of linear equations has a solution.

(b)

$$x_1 + x_2 - x_2 = 1 \qquad 2x_1 + x_2 + 3x_2 = 2 \tag{83}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 3 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 5 & 0 \end{array}\right) \tag{84}$$

$$\Rightarrow x_3 = t \tag{85}$$

$$x_2 = 5t \tag{86}$$

$$x_1 = 1 - 4t \tag{87}$$

$$\Rightarrow x = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{88}$$

(d)

$$x_1 + x_2 + 3x_3 - x_4 = 0 \tag{89}$$

$$x_1 + x_2 + x_3 + x_3 = 1 \tag{90}$$

$$x_1 - 2x_2 + x_3 - x_4 = 1 \tag{91}$$

$$4x_1 + x_2 + 8x_3 - x_4 = 0 \tag{92}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & 1 \\ 4 & 1 & 8 & -1 & 0 \end{array}\right) \;\; \begin{array}{l} \\ \mid \cdot -\frac{1}{2} \\ \mid \cdot -\frac{1}{2} \\ \\ \end{array} \tag{93}$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & {}^{2}/_{3} & 0 & {}^{-1}/_{3} \\ 0 & 0 & 1 & 1 & {}^{1}/_{2} \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\Rightarrow x_1 = -\frac{5}{2} \qquad x_2 = -2 \qquad x_3 = \frac{5}{2} \qquad x_4 = -2 \tag{94}$$

$$x = \left\{ \begin{pmatrix} -5/2 \\ 2 \\ 5/2 \\ -2 \end{pmatrix} \right\} \tag{95}$$

9. Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in \mathsf{R}(\mathsf{L}_A)$.
(⇒)
Suppose $Ax = b$ has a solution

$$\Rightarrow \exists x \colon \mathsf{L}_A(x) = b \tag{96}$$

$$\Rightarrow b \in \mathsf{R}(\mathsf{L}_A) \tag{97}$$

(⇐)
Suppose $b \in \mathsf{R}(\mathsf{L}_A)$

$$\Rightarrow \exists x \colon Ax = b \tag{98}$$

10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equation in n unknowns has rank m, then the system has a solution.
Suppose $A \in \mathsf{M}_{m\times n}(F)$ and $\operatorname{rank}(A) = m$
Since $\operatorname{rank}(A) = m$ it follows that

$$\operatorname{rank}(A|b) = m \quad \text{for } b \in \mathsf{M}_{m\times 1} \tag{99}$$

It follows that $Ax = b$ is consistent since $\operatorname{rank}(A|b) = \operatorname{rank}(A)$ by theorem 3.11.