Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

Work

1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

(b)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

$$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

$$a_1 = -a_2 \tag{2}$$
$$-a_1 = a_2 \tag{3}$$
$$\implies a_1 = a_2 = 0 \tag{4}$$

Only the trivial solution exists. The set is linearly independent.

(d)

$$\left\{ x^3 -, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6 \right\} \text{ in } \mathsf{P}_3 \left(\mathbb{R} \right)$$
$$a_1 \left(x^3 - x \right) + a_1 \left(2x^2 + 4 \right) + a_3 \left(-2x^3 + 3x^2 + 2x + 6 \right) = 09 \tag{5}$$

$$a_1 = 2a_3 \tag{6}$$
$$2a_2 = -3a_3 \tag{7}$$
$$4a_2 = -6a_3 \tag{8}$$

$$a_1 = t \tag{9}$$
$$a_2 = -\left({}^3/_4 \right) t \tag{10}$$
$$a_3 = -\left({}^1/_2 \right) t \tag{11}$$

The set is linearly dependent.

(f)

$$\{(1,-1,2)\,,(1,-2,1)\,,(1,1,4)\}\ \text{ in } \mathsf{R}^3$$

$$a\begin{bmatrix}1\\-1\\2\end{bmatrix}+b\begin{bmatrix}2\\0\\1\end{bmatrix}+c\begin{bmatrix}-1\\2\\-1\end{bmatrix}=0 \tag{12}$$

$$1+2b-c=0 \tag{13}$$

$$-a+2c=0 \tag{14}$$

$$2a+b-c=0 \tag{15}$$

$$\begin{pmatrix}1&2&-1&0\\-1&0&2&0\\2&1&-1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Only the trivial solution exists. This set is linearly independent.

(g)

$$\left\{\begin{pmatrix}1&0\\-2&0\end{pmatrix},\begin{pmatrix}0&-1\\1&1\end{pmatrix},\begin{pmatrix}-1&2\\1&0\end{pmatrix},\begin{pmatrix}2&1\\-4&4\end{pmatrix}\right\}\ \text{in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

$$a\begin{pmatrix}1&0\\-2&1\end{pmatrix}+b\begin{pmatrix}0&-1\\1&1\end{pmatrix}+c\begin{pmatrix}-1&2\\1&0\end{pmatrix}+d\begin{pmatrix}2&1\\-4&4\end{pmatrix}=0 \tag{16}$$

$$\begin{pmatrix}1&0&-1&2&0\\-2&1&1&-4&0\\0&-1&2&1&0\\1&1&0&4&0\end{pmatrix} \to \begin{pmatrix}1&0&-1&2&0\\0&1&-1&0&0\\0&0&1&1&0\\0&0&0&0&0\end{pmatrix}$$

$$a=-3t \tag{17}$$

$$b=-t \tag{18}$$

$$c=-t \tag{19}$$

$$d=t \tag{20}$$

There exists a non-trivial solution, therefore the set is linearly dependent.

11. Let $S=\{u_1,u_2,\dots,u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in span(S)? Justify your answer.

$$\mathbb{Z}_2=\{0,1\} \tag{21}$$

$$c_1u_1+c_2u_2+\cdots+c_nu_n\neq 0,\ \forall c_1,\dots,c_n\in\{0,1\} \tag{22}$$

unless all $c_i=0$.

$$\implies \text{card}\left(\text{span}\left(S\right)\right)=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n} \tag{23}$$

$$=\sum_{i=1}^{n}\binom{n}{i} \tag{24}$$

$$=2^n \tag{25}$$

15. Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \dots, u_k\})$ for some $k\,(1 \le k < n)$.

Forward Direction:
Claim: S is linearly dependent
Suppose $u_1 = 0$
Assume S is linearly independent.
Take the linear combination:

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0 \text{ such that } c_1 \neq 0, c_2, \dots, c_n = 0 \tag{26}$$

$$\implies c_1u_1 = 0 \lightning \text{ Contradiction!} \tag{27}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \dots, u_k\})$
Assume S is linearly independent.

$$u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k \tag{28}$$

Take the linearly combination:

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k + c_{k+1} + \cdots + c_nu_n = 0 \tag{29}$$

Choose $c_i = (-a_i)$ for $i(1 \le i \le k)$ and $c_{k+1} = 1$

$$\implies u_{k+1} = 0 \lightning \text{ Contradiction!} \tag{30}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Reverse Direction:
Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \dots, u_k\})$
Claim: S is linearly dependent

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k && (31)\\ -u_{k+1} &= (-1)\left(a_1u_1 + a_2u_2 + \cdots + a_ku_k\right) && (32)\\ &= (-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k && (33) \end{aligned}$$

Take the linear combination of all $u_1, \dots, u_n$

$$((-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k) + (1u_{k+1} + 0u_{k+2} + \cdots + 0u_n) = 0 \tag{34}$$

1.6

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

    (a) Prove that there is subset of S that is a basis for V. (Be careful not to assume that S is finite.)

    (b) Prove that S contains at least n vectors.

24. Let $f(x)$ be a polynomial of degree n in $\mathsf{P}_n(\mathbb{R})$. Prove that for any $g(x) \in \mathsf{P}_n(\mathbb{R})$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

    $$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

    where $f^{(n)}(x)$ denotes nth derivative of $f(x)$.

    Given vector space $\mathsf{P}_n(\mathbb{R})$ where $\dim(\mathsf{P}_n(\mathbb{R})) = n + 1$
    Suppose S is a subset of $\mathsf{P}_n(\mathbb{R})$ and $S = \{f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\}$

    $$\implies \text{card}(S) = n + 1 \tag{35}$$

    If S is linearly independent, then $\text{span}(S) = \mathsf{P}(\mathbb{R})$ where $g(x) \in \mathsf{P}_n(\mathbb{R})$
    Claim: S is linearly independent

    $$d_0 f(x) + d_1 f'(x) + d_2 f''(x) + d_3 f'''(x) + \cdots + d_n f^{(n)}(x) = 0 \tag{36}$$

    $$d_0 = 0 \because x^n \text{ term only exists in } f(x) \tag{37}$$
    $$d_1 = 0 \because x^{n-1} \text{ term only exists in } f'(x) \tag{38}$$
    $$d_2 = 0 \because x^{n-2} \text{ term only exists in } f''(x) \tag{39}$$
    $$\vdots \tag{40}$$
    $$d_n = 0 \because x^0 \text{ term only exists in } f^{(n)}(x) \tag{41}$$

31. Let W_1 and W_2 are subspaces of V, and find the dimensions of $\mathsf{W}_1, \mathsf{W}_2, \mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

2.1

6. $\mathsf{T}\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \text{tr}(A)$. Recall (Example 4, Section 1.3) that

    $$\text{tr}(A) = \sum_{i=1}^{n} A_{ij}$$

12. Is there a linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1, 0, 3) = (1, 1)$ and $\mathsf{T}(-2, 0, -6) = (2, 1)$?

14. Let V and W be vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear.

    (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

    (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

    (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

2.2

2. Let β and γ be the standard ordered bases for R^n and R^m respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{R}]_\beta^\gamma$.

    (b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

    (c) $\mathsf{T}\colon \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

    (g) $\mathsf{T}\colon \mathsf{R}^n \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

11. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the exercises of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form
$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$
where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.