1 Assignment

Section 1.2: 7, 20, 21; Section 1.3: 5, 10, 20 ,23, 25; Section 1.4: 2, 4, 11, 13

2 Work

1.2

7. Let $S = \{0, 1\}$ and $F = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1, g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

   Claim: $\forall\ t \in S, f = g$

   - For $t = 0$
   $$f(0) = (2 \times 0) + 1 = g(0) = 1 + (4 \times 0) - \left(2 \times 0^2\right) \tag{1}$$
   $$f(0) = g(0) = 0 \tag{2}$$
   - For $t = 1$
   $$f(1) = (2 \times 1) + 1 = g(1) = 1 + (4 \times 1) - \left(2 \times 1^2\right) \tag{3}$$
   $$f(1) = g(1) = 3 \tag{4}$$

   Claim: $\forall\ t \in S, f + g = h$

   - For $t = 0$
   $$f(0) + g(0) = h(0) = 5^0 + 1 \tag{5}$$
   $$f(0) + g(0) = h(0) = 2 \tag{6}$$
   - For $t = 1$
   $$f(1) + g(1) = h(1) = 5^1 + 1 \tag{7}$$
   $$f(1) + g(1) = h(1) = 6 \tag{8}$$

20. Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}$, $\{b_n\} \in \mathsf{V}$ and real number t define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

$$t\,\{a_n\} = \{ta_n\}$$

Prove that, with these operations, V is a vector space over $\mathbb{R}$.

Definition: $\{a_n\}$ is any sequence where $\sigma\colon \mathbb{Z}^+ \to \mathbb{R}$ given by $\sigma(n) = a_n$

Claim: $\{a_n + b_n\} \in \mathsf{V}$

Since the sequences $\{a_n\}$ and $\{b_n\}$ are defined as sequences whose elements exist in $\mathbb{R}$ and addition is a closed binary operation on $\mathbb{R}$, the sum of any two elements in $\{a_n\}$ and $\{b_n\}$ must also exist in $\mathbb{R}$.

Claim: $\{ta_n\} \in \mathsf{V}$

Since the sequence $\{a_n\}$ is defined as a sequence whose elements exist in $\mathbb{R}$ and multiplication is a closed binary operation on $\mathbb{R}$, the product of an element of $\{a_n\}$ and t an element of $\mathbb{R}$ must also exist in $\mathbb{R}$.

(VS 1) $\forall\, x, y \in \mathsf{V},\ x + y = y + x$

Claim: $\{a_n\} + \{b_n\} = \{b_n\} + \{a_n\}$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} = \{b_n +_{\mathbb{R}} a_n\} = \{b_n\} + \{a_n\} \tag{9}$$

(VS 2) $\forall\, x, y, z \in \mathsf{V},\ (x + y) + z = x + (y + z)$

Claim: $\big(\{a_n\} + \{b_n\}\big) + \{c_n\} = \{a_n\} + \big(\{b_n\} + \{c_n\}\big)$

$$\begin{aligned}
\big(\{a_n\} + \{b_n\}\big) + \{c_n\} &= \{a_n +_{\mathbb{R}} b_n\} + \{c_n\} &(10)\\
&= \{a_n +_{\mathbb{R}} b_n +_{\mathbb{R}} c_n\} &(11)\\
&= \{a_n\} + \{b_n +_{\mathbb{R}} c_n\} &(12)\\
&= \{a_n\} + \big(\{b_n\} + \{c_n\}\big) &(13)
\end{aligned}$$

(VS 3) $\exists\, 0 \in \mathsf{V}$ such that $\forall\, x \in \mathsf{V},\ x + 0 = x$

Suppose $\{b_n\} \in \mathsf{V}$ such that $\forall\, n \in \mathbb{Z}^+,\ \sigma(n) = b_n = 0$

Claim: $\{a_n\} + 0_{\mathsf{V}} = \{a_n\}$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} = \{a_n +_{\mathbb{R}} 0\} = \{a_n\} \tag{14}$$

$$\implies \{b_n\} = 0_{\mathsf{V}} \tag{15}$$

(VS 4) $\forall\, x \in \mathsf{V}\ \exists\, y \in \mathsf{V}$ such that $x + y = 0$

Suppose $\{b_n\} \in \mathsf{V}$ such that $\sigma(n) = b_n = -a_n$

Claim: $\{a_n\} + \{b_n\} = 0$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} \tag{16}$$

$$\{a_n +_{\mathbb{R}} (-a_n)\} = 0_{\mathsf{V}} \tag{17}$$

(VS 5) $\forall\, x \in \mathsf{V}, 1 \times x = x$

Claim: $1 \times \{a_n\} = \{a_n\}$

$$1 \times \{a_n\} = \{1 \times_{\mathbb{R}} a_n\} = \{a_n\} \tag{18}$$

(VS 6) $\forall\, a, b \in F, \forall\, x \in \mathsf{V},\ (ab)\, x = a\,(bx)$

Suppose $s, t \in \mathbb{R}$

Claim: $(s \times_{\mathbb{R}} t)\,\{a_n\} = s\,(t\,\{a_n\})$

$$(s \times_{\mathbb{R}} t)\,\{a_n\} = \{(s \times_{\mathbb{R}} t) \times_{\mathbb{R}} a_n\} \tag{19}$$

$$= \{s \times_{\mathbb{R}} (t \times_{\mathbb{R}} a_n)\} \tag{20}$$

$$= s\,\{t \times_{\mathbb{R}} a_n\} \tag{21}$$

$$= s\,(t\,\{a_n\}) \tag{22}$$

(VS 7) $\forall\, a \in F, \forall\, x, y \in \mathsf{V},\ a\,(x+y) = ax + ay$

Suppose $t \in \mathbb{R}$

Claim: $t\big(\,\{a_n\} + \{b_n\}\,\big) = t\,\{a_n\} + t\,\{b_n\}$

$$\begin{aligned} t\,(\{a_n\} + \{b_n\}) &= t\,\{a_n +_{\mathbb{R}} b_n\} && (23)\\ &= \{t \times_{\mathbb{R}} (a_n +_{\mathbb{R}} b_n)\} && (24)\\ &= \{(t \times_{\mathbb{R}} a_n) +_{\mathbb{R}} (t \times_{\mathbb{R}} b_n)\} && (25)\\ &= \{t \times_{\mathbb{R}} a_n\} + \{t \times_{\mathbb{R}} b_n\} && (26)\\ &= t\,\{a_n\} + t\,\{b_n\} && (27) \end{aligned}$$

(VS 8) $\forall\, a, b \in F, \forall\, x \in \mathsf{V},\ (a+b)\,x = ax + bx$

Suppose $s, t \in \mathbb{R}$

Claim: $(s +_{\mathbb{R}} t)\,\{a_n\} = s\,\{a_n\} + t\,\{a_n\}$

$$\begin{aligned} (s +_{\mathbb{R}} t)\,\{a_n\} &= \{(s +_{\mathbb{R}} t)\,a_n\} && (28)\\ &= \{s \times_{\mathbb{R}} a_n + t \times_{\mathbb{R}} a_n\} && (29)\\ &= \{s \times_{\mathbb{R}} a_n\} + \{t \times_{\mathbb{R}} a_n\} && (30)\\ &= s\,\{a_n\} + t\,\{a_n\} && (31) \end{aligned}$$

21. Let V and W be vector spaces over a field F. Let

$$\mathsf{Z} = \{(v, w) : v \in \mathsf{V} \text{ and } w \in \mathsf{W}\}$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c\,(v_1, w_1) = (cv_1, cw_1)$$

Claim: $(v_1 + v_2, w_1 + w_2) \in \mathsf{Z}$

$$v_1, v_2 \in \mathsf{V} \qquad\qquad w_1, w_2 \in \mathsf{W} \tag{32}$$

$$\implies (v_1 + v_2) \in \mathsf{V} \qquad\qquad \implies (w_1 + w_2) \in \mathsf{W} \tag{33}$$

$$\implies (v_1 + v_2, w_1 + w_2) \in \mathsf{Z} \tag{34}$$

$$\therefore\ +\colon \mathsf{Z} \times \mathsf{Z} \to \mathsf{Z} \tag{35}$$

Claim: $(cv_1, cw_1) \in \mathsf{Z}$

$$v_1 \in \mathsf{V} \qquad\qquad w_1 \in \mathsf{W} \tag{36}$$

$$\implies (c \times_{\mathsf{V}} v_1) \in \mathsf{V} \qquad\qquad \implies (c \times_{\mathsf{V}} w_1) \in \mathsf{W} \tag{37}$$

$$\implies (c \times_{\mathsf{V}} v_1, c \times_{\mathsf{W}} w_1) \in \mathsf{Z} \tag{38}$$

$$(cv_1, cw_1) \in \mathsf{Z} \tag{39}$$

$$\therefore\ \times_{\mathsf{Z}}\colon F \times \mathsf{Z} \to \mathsf{Z} \tag{40}$$

1.3

5. Prove that $A + A^t$ is symmetric for any square matrix A.

   Claim: $A + A^t = (A + A^t)^t$

$$A_{n,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

$$A^t_{n,n} = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{pmatrix}$$

$$A_{n,n} + A^t_{n,n} = \begin{pmatrix} a_{1,1} + a_{1,1} & a_{1,2} + a_{2,1} & \cdots & a_{1,n} + a_{n,1} \\ a_{2,1} + a_{1,2} & a_{2,2} + a_{2,2} & \cdots & a_{2,n} + a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + a_{1,n} & a_{n,2} + a_{2,n} & \cdots & a_{n,n} + a_{n,n} \end{pmatrix}$$

$$\left(A_{n,n} + A^t_{n,n}\right)^t = \begin{pmatrix} a_{1,1} + a_{1,1} & a_{2,1} + a_{1,2} & \cdots & a_{n,1} + a_{1,n} \\ a_{1,2} + a_{2,1} & a_{2,2} + a_{2,2} & \cdots & a_{n,2} + a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} + a_{n,1} & a_{2,n} + a_{n,2} & \cdots & a_{n,n} + a_{n,n} \end{pmatrix}$$

10. Prove that $\mathsf{W}_1 = \left\{ (a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 0 \right\}$ is a subspace of F^n, but $\mathsf{W}_2 = \left\{ (a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 1 \right\}$ is not.

(a) $0_{\mathsf{V}} \in \mathsf{W}$

Suppose $(a_1, a_2, \ldots, a_n) \in \mathsf{W}_1$

Suppose $(b_1, b_2, \ldots, b_n) \in \mathsf{W}_1$ such that $b_i = 0$ for integer $i \in [1, n]$

Claim: $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1, a_2, \ldots, a_n)$

$$\begin{aligned} (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, a_n) &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) && (41) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) && (42) \\ &= (a_1, a_2, \ldots, a_n) && (43) \end{aligned}$$

(b) $\forall\, x, y \in \mathsf{W},\ x + y \in \mathsf{W}$

Suppose $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathsf{W}_1$

Claim: $(a_1 + b_1, a_2 + b_2, \ldots, a_3 + b_3) \in \mathsf{W}_1$

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \tag{44}$$

$$\sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k) = \sum_{k=1}^{n} (a_k + b_k) = 0 \tag{45}$$

$$\therefore (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \in \mathsf{W}_1 \tag{46}$$

(c) $\forall\, a \in F,\ x \in \mathsf{W}\ ax \in \mathsf{W}$

Suppose $c \in F,\ (a_1, a_2, \ldots, a_n) \in \mathsf{W}_1$

Claim: $(ca_1, ca_2, \ldots, ca_n) \in \mathsf{W}_2$

$$c\,(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n) \tag{47}$$

$$\sum_{k=1}^{n} (ca_k) = c \sum_{k=1}^{n} (a_k) = 0 \tag{48}$$

$$\therefore (ca_1, ca_2, \ldots, ca_n +) \in \mathsf{W}_1 \tag{49}$$

Suppose $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathsf{W}_2$

Claim: $(a_1 + b_1, a_1 + b_2, \ldots, a_n + b_n) \notin \mathsf{W}_2$

$$(a_1, a_2, \ldots, a_n + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \tag{50}$$

$$\sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k) = \sum_{k=1}^{n} (a_k + b_k) = 2 \tag{51}$$

$$\therefore (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \notin \mathsf{W}_2 \tag{52}$$

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

    Suppose $a_1, a_2, \ldots, a_n \in F,\ w_1, w_2, \ldots, w_n \in \mathsf{W}$

    Claim: $\forall a_1, a_2, \ldots, a_n \in F$ and $\forall w_1, w_2, \ldots, w_n \in \mathsf{W}, a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$.

    For every integer $i \in [i, n], a_iw_i \in \mathsf{W}$ by theorem 1.3.c

$$\implies a_1w_1 + a_2w_2 + \cdots + a_nw_n \text{ (by theorem 1.2.c)} \tag{53}$$

23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $\mathsf{W}_1 + \mathsf{W}_2$

Claim: $\mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{V}$

Suppose $x \in \mathsf{W}_1 + \mathsf{W}_2$ such that $z = x + y$ and $x \in \mathsf{W}_1,\ y \in \mathsf{W}_2$

$$\mathsf{W}_1 \in \mathsf{V} \tag{54}$$

$$\mathsf{W}_2 \in \mathsf{V} \tag{55}$$

$$\implies x \in \mathsf{V} \tag{56}$$

$$\implies y \in \mathsf{V} \tag{57}$$

$$\implies x + y \in \mathsf{V} \tag{58}$$

$$\implies \mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{V} \tag{59}$$

(i) $0 \in \mathsf{W}_1$

$$0 \in \mathsf{W}_1 \tag{60}$$

$$0 \in \mathsf{W}_2 \tag{61}$$

Suppose $z \in \mathsf{W}_1 + \mathsf{W}_2$ such that $z = x + y$ and $x = y = 0$

$$\implies z = 0 + 0 = 0 \tag{62}$$

(ii) $x + y \in \mathsf{W}$ when $x \in \mathsf{W}$ and $y \in \mathsf{W}$ Suppose $z_1, z_2 \in \mathsf{W}$ such that $z_1 = x + y, z_2 = a + b$

Claim: $z1_1 + z_2 \in \mathsf{W}_1 + \mathsf{W}_2$

$$z_1 + z_2 = x + y + a + b \tag{63}$$

$$= (x + a) + (y + b) \tag{64}$$

$$x + a \in \mathsf{W}_1 \tag{65}$$

$$y + b \in \mathsf{W}_2 \tag{66}$$

$$\implies z_1 + z_2 \in \mathsf{W}_2 \tag{67}$$

(iii) $cx \in \mathsf{W}$ when $z \in \mathsf{W}_1 + \mathsf{W}_2$ such that $z = x + y$

Suppose $c \in F$ and $z \in W_1 + \mathsf{W}_2$ such that $z = x + y$

$$cz = c\,(x + y) \tag{68}$$

$$= cx + cy \tag{69}$$

$$cx \in \mathsf{W}_1 \tag{70}$$

$$cy \in \mathsf{W}_2 \tag{71}$$

$$\implies cz \in \mathsf{W}_1 + \mathsf{W}_2 \tag{72}$$

Suppose X is a subspace of V and $\mathsf{W}_1 \subseteq \mathsf{X}$, and $\mathsf{W}_2 \subseteq \mathsf{X}$

Claim: $\mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{X}$

Suppose $z \in \mathsf{W}_1 + \mathsf{W}_2$ such that $z = x + y$ for $x \in \mathsf{W}_1$ and $y \in \mathsf{W}_2$

$$x \in \mathsf{W}_1 \implies x \in \mathsf{X} \tag{73}$$

$$y \in \mathsf{W}_2 \implies y \in \mathsf{X} \tag{74}$$

$$\implies x + y \in \mathsf{X} \tag{75}$$

25. Let W_1 denote the set of all polynomials $f(x)$ in $\mathsf{P}(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $\mathsf{P}(F)$ such that in the representation

$$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0,$$

we have $b_i = 0$ whenever i is odd. Prove that $\mathsf{P}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

$$\mathsf{W}_1 = \{x \in \mathsf{P}(F) \colon c_k = 0 \ \forall \text{ integers } k \in [0,2] \text{ such that } 2 \mid (k+1)\} \tag{76}$$

$$\mathsf{W}_2 = \{x \in \mathsf{P}(F) \colon c_k = 0 \ \forall \text{ integers } k \in [0,2] \text{ such that } 2 \mid k\} \tag{77}$$

(a) $0_{\mathsf{V}} \in \mathsf{W}$

Suppose $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0) \in \mathsf{W}_1$, $(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in \mathsf{W}_1$ such that $b_i = 0$ for every integer $i \in [1, m]$

Claim: $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0) + (b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3 x^3 + b_1 x) = (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0)$

$$\big(a_n x^n + \cdots + a_2 x^2 + a_0\big) + \big(b_m x^m + \cdots + b_3 x^3 + b_1 x\big) \tag{78}$$
$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + \big(0x^m + 0x^{m-2} + \cdots + 0x^3 + 0x\big) \tag{79}$$
$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + (0 + 0 + \cdots + 0 + 0) \tag{80}$$

$$(a_n x^n + \cdots + a_0) + 0_{\mathsf{W}_1} = \big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) \tag{81}$$

(b) $\forall\, x, y \in \mathsf{W},\ x + y \in \mathsf{W}$

Suppose $\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right)$,

$$\left(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3x^3 + b_1x\right) \in \mathsf{W}_1$$

Claim: $\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) +$

$$\left(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3x^3 + b_1x\right) \in \mathsf{W}_1$$

Without loss of generality assume $n \leq m \implies \exists\, k \in \left[0, \frac{n-1}{2}\right]$ such that $m = n - 2k$

$$\begin{multline}\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) + \\ \left(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3x^3 + b_1x\right) \tag{82}\end{multline}$$

$$\begin{multline}\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + \right. \\ \left(a_{n-2k} + b_{n-2k}\right)\left(x^{n-2k}\right) + \left(a_{n-2k-2} + b_{n-2k-2}\right)\left(x^{n-2k-2}\right) + \cdots + \\ \left(a_3 + b_3\right)x^3 + \left(a_1 + b_1\right)x \tag{83}\end{multline}$$

$$\begin{multline}\implies \left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) + \\ \left(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3x^3 + b_1x\right) \in \mathsf{W}_1 \tag{84}\end{multline}$$

(c) $\forall\, a \in F, x \in \mathsf{W},\ ax \in \mathsf{W}$

Suppose $c \in F$ and $\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) \in \mathsf{W}_1$

Claim: $c\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) \in \mathsf{W}_1$

$$\begin{multline}c\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) \\ = \left(ca_n x^n + ca_{n-2}x^{n-2} + \cdots + ca_3x^3 + ca_1x\right) \tag{85}\end{multline}$$

$$ca_n, ca_{n-2}, \ldots, ca_1 \in F \tag{86}$$

$$\implies c\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x\right) \in \mathsf{W}_1 \tag{87}$$

(a) $0_{\mathsf{V}} \in \mathsf{W}$

Suppose $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0) \in \mathsf{W}_2$,

$(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_2^2 + b_0) \in \mathsf{W}_2$ such that $b_i = 0$ for every integer $i \in [1, m]$

Claim: $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0) + (b_m x^m + b_{m-2}x^{m-2} + \cdots + b_2 x^2 + b_0) = (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0)$

$$\big(a_n x^n + \cdots + a_2 x^2 + a_0\big) + \big(b_m x^m + \cdots + b_2 x^2 + b_0\big) \tag{88}$$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + \big(0x^m + 0x^{m-2} + \cdots + 0x^2 + 0\big) \tag{89}$$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + (0 + 0 + \cdots + 0 + 0) \tag{90}$$

$$(a_n x^n + \cdots + a_0) + 0_{\mathsf{W}_1} = \big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) \tag{91}$$

(b) $\forall\, x, y \in \mathsf{W},\ x + y \in \mathsf{W}$

Suppose $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3 x^3 + a_1 x)\,,$

$(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in \mathsf{W}_2$

Claim: $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0) +$

$(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_2 x^2 + b_0) \in \mathsf{W}_2$

Without loss of generality assume $n \le m \implies \exists\, k \in \big[0, \frac{n}{2}\big]$ such that $m = n - 2k$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + \big(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_2 x^2 + b_0\big) \tag{92}$$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + (a_{n-2k} + b_{n-2k})\big(x^{n-2k}\big) + (a_{n-2k-2} + b_{n-2k-2})\big(x^{n-2k-2}\big) + \cdots + (a_2 + b_2)\, x^2 + (a_0 + b_0) \tag{93}$$

$$\implies \left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0\right) + \left(b_m x^m + b_{m-2}x^{m-2} + \cdots + b_2x^2 + b_0\right) \in \mathsf{W}_2 \quad (94)$$

(c) $\forall\, a \in F, x \in \mathsf{W},\ ax \in \mathsf{W}$

Suppose $c \in F$ and $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0) \in \mathsf{W}_2$

Claim: $c\,(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0) \in \mathsf{W}_2$

$$c\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0\right) = \left(ca_n x^n + ca_{n-2}x^{n-2} + \cdots + ca_2x^2 + ca_0\right) \quad (95)$$

$$ca_n, ca_{n-2}, \ldots, ca_0 \in F \quad (96)$$

$$\implies c\left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0\right) \in \mathsf{W}_1 \quad (97)$$

Case I

Suppose $A \in \mathsf{P}(F)$, $A \neq 0_{\mathsf{P}}$

Claim: $A \notin \mathsf{W}_1 \cap \mathsf{W}_2$

Case (i) Suppose $A \in \mathsf{W}_2, A = (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0)$ such that $a_n \neq 0$

By definition of W_2 $2 \nmid (n+1) \implies 2 \mid n$

Claim: $A \notin \mathsf{W}_1$

Suppose $A \in \mathsf{W}_1$

$$2 \mid n \implies a_n = 0 \ \lightning \text{ Contradiction!} \quad (98)$$

$$\implies A \notin \mathsf{W}_1 \quad (99)$$

Case (ii) Suppose $A \in \mathsf{W}_1, A = (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x)$ such that $a_n \neq 0$ By definition of W_1 $2 \nmid n \implies 2 \mid (n+1)$

Claim: $A \notin \mathsf{W}_2$

Suppose $A \in \mathsf{W}_2$

$$2 \mid (n+) \implies a_n = 0 \ \lightning \text{ Contradiction!} \tag{100}$$

$$\implies A \notin \mathsf{W}_2 \tag{101}$$

Case II

Suppose $A \in \mathsf{P}(F), A = 0$

Claim: $A \in \mathsf{W}_1 \cap \mathsf{W}_2$

$$0_{\mathsf{W}_1} \in \mathsf{W}_1 \tag{102}$$

$$0_{\mathsf{W}_2} \in \mathsf{W}_2 \tag{103}$$

$$0_{\mathsf{W}_1} = 0_F = 0_{\mathsf{W}_2} \tag{104}$$

$$0_{\mathsf{W}_1} = 0_{\mathsf{W}_2} \tag{105}$$

$$\implies 0 \in \mathsf{W}_1 \cap \mathsf{W}_2 \tag{106}$$

$$\implies \mathsf{W}_1 \cap \mathsf{W}_2 = \{0\} \tag{107}$$

Claim: $\mathsf{P}(F) \supseteq \mathsf{W}_1 + \mathsf{W}_2$

$$\mathsf{W}_1 \subseteq \mathsf{P}(F) \tag{108}$$

$$\mathsf{W}_2 \subseteq \mathsf{P}(F) \tag{109}$$

Suppose $x \in \mathsf{W}_1, y \in \mathsf{W}_2$

$$x + y \in \mathsf{P}(F) \tag{110}$$

$$\implies \mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{P}(F) \tag{111}$$

Claim: $\mathsf{P}(F) \subseteq \mathsf{W}_{+}\mathsf{W}_2$

Suppose $h \in \mathsf{P}(F)$, $h = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$

Case (i) n is odd

$$h = \big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3 x^3 + a_1 x\big) + \big(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_2 x^2 + a_0\big) \quad (112)$$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3 x^3 + a_1 x\big) \in \mathsf{W}_2 \quad (113)$$

$$\big(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_2 x^2 + a_0\big) \in \mathsf{W}_1 \quad (114)$$

Case (ii) n is even

$$h = \big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) + \big(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_3 x^3 + a_1 x\big) \quad (115)$$

$$\big(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\big) \in \mathsf{W}_2 \quad (116)$$

$$\big(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_3 x^3 + a_1 x\big) \in \mathsf{W}_1 \quad (117)$$

Claim:

$\mathsf{P}(F) = \big\{ \big(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + x_0\big) : c_i \in F \text{ for integer } i \in [1, n]\big\} = \mathsf{W}_1 \oplus \mathsf{W}_2$

$$\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\} \quad (118)$$

$$\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{P}(F) \quad (119)$$

$$\implies \mathsf{P}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2 \quad (120)$$

1.4

2.

(a)

$$\begin{pmatrix} 2 & -2 & -3 & 0 & -2 \\ 3 & -3 & -3 & 5 & 7 \\ 1 & -1 & -2 & -1 & -3 \end{pmatrix} \quad | \cdot 1/4 \;\to\; \begin{pmatrix} 1 & -1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 2 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $x_2 = s, x_4 = t$

$$x_1 = -35 + s + 3t \tag{121}$$

$$x_2 = s \tag{122}$$

$$x_3 = 16 - 2t \tag{123}$$

$$x_4 = t \tag{124}$$

(b)

$$\begin{pmatrix} 3 & -7 & 4 & 10 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{pmatrix} \quad | \cdot -1 \quad \to \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$x_1 = -2 \tag{125}$$

$$x_2 = -4 \tag{126}$$

$$x_3 = -3 \tag{127}$$

(c)

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & 1 & 5 \\ 2 & 3 & -1 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 0 & -1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Inconsistent; not solvable.

(d)

$$\begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 8 & 0 & -16 \\ 0 & 0 & 0 & -5 & -10 \\ 0 & -1 & 3 & 0 & -4 \end{pmatrix}$$

Let $s = x_3$

$$x_1 = -8s - 16 \tag{128}$$

$$x_2 = 3s + 4 \tag{129}$$

$$x_3 = s \tag{130}$$

$$x_4 = 2 \tag{131}$$

(e)

$$\begin{pmatrix} 1 & 2 & -4 & -1 & 1 & 7 \\ -1 & 0 & 10 & -3 & -4 & -16 \\ 2 & 5 & -5 & -4 & -1 & 2 \\ 4 & 11 & -7 & -10 & -2 & 7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -10 & 3 & 0 & -4 \\ 0 & 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}$$

Let $x_3 = s,\ x_4 = t$

$$x_1 = 10s - 3t - 4 \tag{132}$$

$$x_2 = -3s + 2t + 3 \tag{133}$$

$$x_3 = s \tag{134}$$

$$x_4 = t \tag{135}$$

(f)

$$\begin{pmatrix} 1 & 2 & 6 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & -1 & 15 \\ 1 & 3 & 10 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$x_1 = 3 \tag{136}$$

$$x_2 = 4 \tag{137}$$

$$x_3 = -2 \tag{138}$$

4.

(a) $x^3 - 3x + 5 \stackrel{?}{=} c_1\left(x^3 + 2x^2 - x + 1\right) + c_2\left(x^3 + 3x^2 - 1\right)$

$$c_1 + c_2 = 1 \tag{139}$$

$$2c_1 + 3c_2 = 0 \tag{140}$$

$$-c_1 = -3 \tag{141}$$

$$c_1 - c_2 = 5 \tag{142}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & -3 \\ 1 & -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

$$x^3 - 3x + 5 = 3\left(x^3 + 2x^2 - x + 1\right) - 2\left(x^3 + 3x^2 - 1\right) \tag{143}$$

(b) $4x^3 + 2x^2 - 6 \overset{?}{=} c_1\left(x^3 - 2x^3 + 4x + 1\right) + c_2\left(3x^3 - 6x^2 + x + 4\right)$

$$c_1 + 3c_2 = 4 \tag{144}$$

$$-2c_1 + -6c_2 = 2 \tag{145}$$

$$4c_1 + c_2 = 0 \tag{146}$$

$$c_1 + 4c_2 = -6 \tag{147}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ -2 & -6 & 2 \\ 4 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix} \begin{array}{l} {}^{2} \\ \leftarrow {}^{+} \\ \\ \\ \end{array} \rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 10 \\ 4 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix}$$

Inconsistent; no linear combinations.

(c) $-2x^3 - 11x^2 + 3x + 2 \overset{?}{=} c_1\left(x^3 - ex^2 + 3x - 1\right) + c_2\left(2x^3 + x^2 + 3x - 2\right)$

$$c_1 + 2c_2 = -2 \tag{148}$$

$$-2c_1 + c_2 = -11 \tag{149}$$

$$3c_1 + 3c_2 = 3 \tag{150}$$

$$-c_1 = 2 \tag{151}$$

$$\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{array}{lll} \leftarrow {}^{+} & & \\ & \leftarrow {}^{+} & \\ \mid \cdot 1/3 & & \leftarrow {}^{+} \\ {}_{1} & {}_{-2} & {}_{1} \end{array} \rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & -15 \\ 0 & 1 & 5 \\ -1 & 0 & 2 \end{pmatrix}$$

Inconsistent; no linear combinations.

(d) $x^3 + x^2 + 2x + 13 \stackrel{?}{=} c_1\left(2x^3 - 2x^2 + 4x + 1\right) + c_2\left(x^3 - x^2 + 2x + 3\right)$

$$2c_1 + c_2 = 1 \tag{152}$$

$$-3c_1 - c_2 = 1 \tag{153}$$

$$4c_1 + 2c_2 = 2 \tag{154}$$

$$c_1 + 3c_2 = 13 \tag{155}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ -3 & -1 & 1 \\ 4 & 2 & 2 \\ 1 & 3 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix}$$

$$x^3 + x^2 + 2x + 13 = -2\left(2x^3 - 2x^2 + 4x + 1\right) + 5\left(x^3 - x^2 + 2x + 3\right) \tag{156}$$

(e) $x^3 - 8x^2 + 4x \stackrel{?}{=} c_1\left(x^3 - 2x^2 + 3x - 1\right) + c_2\left(x^3 - 2x + 3\right)$

$$c_1 + c_2 = 1 \tag{157}$$

$$c_1 = 4 \tag{158}$$

$$3c_1 - 2c_2 = 1 \tag{159}$$

$$-c_1 + 3c_2 = 0 \tag{160}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 3 & -2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & {}^{-16}/_{3} \\ 1 & 0 & 4 \\ 3 & -2 & 1 \\ 0 & 1 & {}^{4}/_{3} \end{pmatrix}$$

Inconsistent; no linear combination.

(f) $6x^3 - 3x^2 + x + 2 \stackrel{?}{=} c_1\left(x^3 - x^2 + 2x + 3\right) + c_2\left(2x^3 - 3x + 1\right)$

$$c_1 + c_2 = 6 \tag{161}$$

$$c_1 = 3 \tag{162}$$

$$2c_1 - 3c_2 = 1 \tag{163}$$

$$3c_1 + c_2 = 2 \tag{164}$$

$$\begin{pmatrix} 1 & 1 & 6 \\ 1 & 0 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{array}{l} \leftarrow {}_{+} \quad \leftarrow {}_{+} \\ \mid \cdot -1 \;\; {}_{-1} \;\; {}_{-2} \\ \leftarrow {}_{+} \quad \mid \cdot -1/3 \;\; {}_{-1} \\ \end{array} \rightarrow \begin{pmatrix} 0 & 0 & {}^4/_3 \\ 1 & 0 & 3 \\ 0 & 1 & {}^5/_3 \\ 3 & 1 & 2 \end{pmatrix}$$

Inconsistent; no linear combinations.

11. Prove that $\operatorname{span}(\{x\}) = \{ax : a \in F\}$ for any vector x is a vector space. Interpret this result geometrically in $\mathbb{R}^3$.

Suppose V is a vector space.

Claim: $\operatorname{span}(\{x\}) \subseteq \{ax : a \in F\}$ Suppose $y \in \operatorname{span}(\{x\}$

$$y = a_1x + a_2x + \cdots + a_nx \text{ for } a_1, \ldots, a_n \in F \tag{165}$$

$$= (a_1 + a_2 + \cdots + a_n)\, x \tag{166}$$

$$(a_1 + a_2 + \cdots + a_n) \in F \tag{167}$$

$$\implies y \in \{ax : a \in F\} \tag{168}$$

Claim: $\operatorname{span}(\{x\}) \supseteq \{ax : a \in F\}$

Suppose $x \in \{ax : a \in F\}$

$$z = bx \text{ for } b \in F \tag{169}$$

bx is a linear combination of 1 term.

$$\implies z \in \operatorname{span}(\{x\}) \tag{170}$$

13. Show that is S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = \mathsf{V}$, deduce that $\text{span}(S_2) = \mathsf{V}$.

Claim: $\text{span}(S_1) \subseteq \text{span}(S_2)$

Suppose: $y \in \text{span}(S_1)$

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n \text{ for } a_1, a_2, \ldots, a_n \in F \text{ and } x_1, x_2, \ldots, x_n \in S_1 \tag{171}$$

$$\implies S_1 \subseteq S_2 \; \forall \text{ integers } i \in [1, n], x_i \in S_2 \tag{172}$$

$$\forall\, a_1, a_2, \ldots, a_n \in F,\; a_1x_1 + a_2x_2 + \cdots + a_nx_n \in \text{span}(S_2) \tag{173}$$

$$\implies \text{span}(S_1) \subseteq \text{span}(S_2) \tag{174}$$

Claim: $\text{span}(S_2) \subseteq \mathsf{V}$

Suppose $y \in \text{span}(S_2)$

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n,\; \forall\, a_1x_1, a_2x_2, \ldots, a_n \in F \text{ and } x_1, x_2, \ldots, x_n \in S_2 \tag{175}$$

Given $S_2 \subseteq \mathsf{V}$

$$x_1, x_2, \ldots, x_n \in \mathsf{V} \implies a_1x_1 + a_2x_2 + \cdots + a_nx_n \in \mathsf{V} \tag{176}$$

$$\implies \text{span}(S_2) \subseteq \mathsf{V} \tag{177}$$