Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

Work

3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\ x_2 + 2x_4 &= 3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 9 & -4 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \tag{1}$$

$$x_1 = -3 + 4x_4 \tag{2}$$
$$x_2 = 3 - 2x_4 \tag{3}$$
$$x_3 = 1 \tag{4}$$
$$x_4 = x_4 \tag{5}$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{6}$$

(j)

$$2x_1 + 3x_3 - 4x_5 = 5 \tag{7}$$
$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8 \tag{8}$$
$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2 \tag{9}$$
$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8 \tag{10}$$

$$\left(\begin{array}{ccccc|c} 2 & 0 & 3 & 0 & -4 & 5 \\ 3 & -4 & 8 & 3 & 0 & 8 \\ 1 & -1 & 2 & 1 & -1 & 2 \\ -2 & 5 & -9 & -3 & -5 & -8 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \tag{11}$$

$$x_1 = 2x_5 + 1 \tag{12}$$
$$x_2 = 3x_5 \tag{13}$$
$$x_3 = -1 \tag{14}$$
$$x_4 = 2x_5 - 1 \tag{15}$$
$$x_5 = x_5 \tag{16}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{17}$$

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate W. Find a subset $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

$$\mathsf{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \ldots, x_5 \in \mathbb{R} \right\} \tag{18}$$

$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \tag{19}$$

$$\rightsquigarrow \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ & & 1 & -2 & 0 & -2 & 0 & 1 \\ & & & & 1 & -4 & 0 & -2 \\ & & \text{\Large 0} & & & & 1 & -1 \\ & & & & & & & 0 \end{pmatrix} \tag{20}$$

It follows that $\{u_1, u_3, u_5, u_7\}$ is linearly independent by theorem 3.16. Therefore $\{u_1, u_3, u_5, u_7\}$ is a basis for W.

11.

14.

15.

4.1

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]\, I$
(b) $\det(C) = \det(A)$
(c) The classical adjoint of A^t is C^t
(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}\, C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{21}$$
$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{22}$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{23}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{24}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{25}$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{26}$$
$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \tag{27}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \tag{28}$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \tag{29}$$
$$\Rightarrow AC = CA = \det(A)I_2 \tag{30}$$

(b) $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{31}$$
$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \tag{32}$$

It follows that D is the classical adjoint of A^t

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \tag{33}$$

It follows that C^t is the classical adjoint of A^t

(d) Suppose A is invertible

$$\Rightarrow \exists B\colon AB = BA = I \tag{34}$$
$$\Rightarrow \det(A) \neq 0 \tag{35}$$

$$CA = AC = \det(A)I \tag{36}$$
$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \tag{37}$$
$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \tag{38}$$

4.2

23.

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

(a) **Types 1 & 2**

$$E^t = E \quad \text{(by HW.3.1.5)} \tag{39}$$
$$\Rightarrow \det(E^t) = \det(E) \tag{40}$$

(b) **Type 3**

E^t is an type 3 elementary matrix (by HW.3.1.5) $\det(E) = \det(I) = 1$ for any type elementary operation on I_n

$$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \tag{41}$$
$$\Rightarrow \det(E) = \det(E^t) = 1 \tag{42}$$

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$. be $a_1, a_2, \ldots, a_n$ and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

(a) **n is even**

In A, swap

$$a_{n-1} \text{ with } a_1 \tag{43}$$
$$a_{n-2} \text{ with } a_2 \tag{44}$$
$$\vdots$$
$$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \tag{45}$$

From the fact that ${}^{n}/_{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n}{2}} \det(A) \tag{46}$$

(b) **n is odd** In A, swap

$$a_{n-1} \text{ with } a_1 \tag{47}$$
$$a_{n-2} \text{ with } a_2 \tag{48}$$
$$\vdots$$
$$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \tag{49}$$

From the fact that $n - \frac{n+1}{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n-1}{2}} \det(A) \tag{50}$$

4.3

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]\, I$

(b) $\det(C) = \det(A)$

(c) The classical adjoint of A^t is C^t

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}\, C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{51}$$
$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{52}$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{53}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{54}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{55}$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{56}$$
$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \tag{57}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \tag{58}$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \tag{59}$$
$$\Rightarrow AC = CA = \det(A)I_2 \tag{60}$$

(b) $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{61}$$
$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \tag{62}$$

It follows that D is the classical adjoint of A^t

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \tag{63}$$

It follows that C^t is the classical adjoint of A^t

(d) Suppose A is invertible

$$\Rightarrow \exists B\colon AB = BA = I \tag{64}$$
$$\Rightarrow \det(A) \neq 0 \tag{65}$$

$$CA = AC = \det(A)I \tag{66}$$
$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \tag{67}$$
$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \tag{68}$$

12.