

Assignment

Section 6.3: 1, 2, 3, 9; Section 6.4: 1, 2, 9, 11; Section 6.5: 1, 2, 5, 10, 21

Work

6.3

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.
 - (a) Every linear operator has an adjoint.
 - (b) Every linear operator on V has the form $x \rightarrow \langle x, y \rangle$ for some $y \in V$.
 - (c) For every linear operator T on V and every ordered basis β for V , we have $[T^*]_\beta = ([T]_\beta)^*$.
 - (d) The adjoint of a linear operator is unique.
 - (e) For any linear operators T and U and scalars a and b ,

$$(aT + bU)^* = aT^* + bU^*$$

- (f) For any $n \times n$ matrix A , we have $(L_A)^* = L_A$
- (g) For any linear operator T , we have $(T^*)^* = T$

3.

9.

6.4

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.
 - (a) Every self-adjoint operator is normal.
True
 - (b) Operators and their adjoints have the same eigenvectors.
False
 - (c) If T is an operator on an inner product space V , then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V .
False
 - (d) A real or complex matrix A is normal if and only if L_A is normal.
True

(e) The eigenvalues of a self-adjoint operator must be real.

True

(f) The identity and zero operators are self-adjoint.

True

(g) Every normal operator is diagonalizable.

False

(h) Every self-adjoint operator is diagonalizable.

True

2. For each linear operator \mathbf{T} on an inner product space \mathbf{V} , determine whether \mathbf{T} is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of \mathbf{T} for \mathbf{V} and list the corresponding eigenvalues.

(a) $\mathbf{V} = \mathbb{R}^2$ and \mathbf{T} is defined by $\mathbf{T}(a, b) = (2a - 2b, -2a + 5b)$

Suppose β is the standard ordered basis for \mathbb{R}^2

$$[\mathbf{T}]_{\beta} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \quad (1)$$

$$\Rightarrow ([\mathbf{T}]_{\beta})^* = ([\mathbf{T}^*]) = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \quad (2)$$

$$\Rightarrow \mathbf{T} = \mathbf{T}^* \quad (3)$$

$$\det \begin{pmatrix} 2 - \lambda & -2 \\ -2 & 5 - \lambda \end{pmatrix} = 0 \quad (4)$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \quad (5)$$

$$\Rightarrow \lambda_1 = 6 \quad (6)$$

$$\lambda_2 = 1 \quad (7)$$

• For $\lambda_1 = 6$

$$[\mathbf{T}]_{\beta} - 6I_2 = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (9)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (10)$$

$$\Rightarrow x_1 = -\frac{1}{2}x_2 \quad (11)$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (12)$$

- For $\lambda_2 = 1$

$$[\mathbf{T}]_\beta - I_2 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad (13)$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (14)$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (15)$$

$$\Rightarrow x_1 = 2x_2 \quad (16)$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (17)$$

Suppose

$$v'_1 = \left(-\frac{1}{2}, 1\right) \quad v'_2 = (2, 1) \quad (18)$$

Let

$$v_1 = v'_1 \quad (19)$$

$$v_2 = v'_2 - \frac{\langle v'_2, v_1 \rangle}{\|v_1\|^2} v_1 \quad (20)$$

$$\langle v'_2, v_1 \rangle = 0 \quad (21)$$

$$\Rightarrow v_2 = v'_2 \quad (22)$$

$$\|v_1\|^2 = \frac{5}{4} \quad (23)$$

$$\Rightarrow \|v_1\| = \frac{\sqrt{5}}{2} \quad (24)$$

$$\Rightarrow o_1 = \frac{1}{\sqrt{5}}(-1, 2) \quad (25)$$

$$\|v_2\|^2 = 5 \quad (26)$$

$$\Rightarrow \|v_2\| = \sqrt{5} \quad (27)$$

$$\Rightarrow o_2 = \frac{1}{\sqrt{5}}(2, 1) \quad (28)$$

An orthonormal basis is

$$\gamma = \left\{ \frac{1}{\sqrt{5}}(-1, 2), \frac{1}{\sqrt{5}}(2, 1) \right\} \quad (29)$$

The eigenvector $\frac{1}{\sqrt{5}}(-1, 2)$ corresponds to the eigenvalue 6, and the eigenvector $\frac{1}{\sqrt{5}}(2, 1)$ corresponds to the eigenvalue 1.

(b) $V = \mathbb{R}^3$ and T is defined by $T(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$

Suppose β is the standard ordered basis of \mathbb{R}^3

$$\Rightarrow [T]_{\beta} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{pmatrix} \quad (30)$$

$$([T]_{\beta})^* = [T^*]_{\beta} = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 5 & -2 \\ 0 & 0 & 5 \end{pmatrix} \quad (31)$$

$$\Rightarrow T^* \neq T \quad (32)$$

$$([T]_{\beta})^* [T]_{\beta} \neq ([T]_{\beta})^* \quad (33)$$

T is neither normal nor adjoint.

(c) $V = \mathbb{C}^2$ and T is defined by $T(a, b) = (2a + ib, a + 2b)$

Suppose β is the standard ordered basis of \mathbb{C}^2

$$\Rightarrow [T]_{\beta} = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \quad (34)$$

$$([T]_{\beta})^* = [T^*]_{\beta} = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \quad (35)$$

$$\begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \quad (36)$$

$\Rightarrow T$ is normal.

$$\det \begin{pmatrix} 2 - \lambda & i \\ 1 & 2 - \lambda \end{pmatrix} \quad (37)$$

$$\Rightarrow (2 - \lambda)^2 = i \quad (38)$$

$$\Rightarrow \lambda_1 = \left(2 + \frac{\sqrt{2}}{2}\right) + i \left(\frac{\sqrt{2}}{2}\right) \quad (39)$$

$$\lambda_2 = \left(2 - \frac{\sqrt{2}}{2}\right) + i \left(-\frac{\sqrt{2}}{2}\right) \quad (40)$$

- For $\lambda_1 = \left(2 + \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right)$

$$[\mathbf{T}]_\beta - \lambda_1 I_n = \begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \quad (41)$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (42)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2}(1+i) & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (43)$$

$$\Rightarrow x_1 = \frac{\sqrt{2}}{2}(1+i)x_2 \quad (44)$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \quad (45)$$

- For $\lambda_2 = \left(2 - \frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)$

$$[\mathbf{T}]_\beta - \lambda_2 I_2 = \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \quad (46)$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (47)$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (48)$$

$$\Rightarrow x_1 = -\frac{\sqrt{2}}{2}(1+i)x_2 \quad (49)$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \quad (50)$$

Suppose

$$w_1 = \left(\frac{\sqrt{2}}{2}(1+i), 1 \right) \quad w_2 = \left(-\frac{\sqrt{2}}{2}(1+i), 1 \right) \quad (51)$$

Let

$$v_1 = w_1 \quad (52)$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \quad (53)$$

$$\langle w_2, v_1 \rangle = 0 \quad (54)$$

$$\Rightarrow v_2 = w_2 \quad (55)$$

$$\|v_1\|^2 = 2 \quad (56)$$

$$\Rightarrow \|v_1\| = \sqrt{2} \quad (57)$$

$$\Rightarrow o_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right) \quad (58)$$

$$\|v_2\|^2 = 2 \quad (59)$$

$$\Rightarrow \|v_2\| = \sqrt{2} \quad (60)$$

$$\Rightarrow o_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right) \quad (61)$$

An orthonormal basis is

$$\gamma = \left\{ \left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right), \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right) \right\} \quad (62)$$

The eigenvector $\left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right)$ corresponds to the eigenvalue of $\left(2 + \frac{\sqrt{2}}{2} \right) + i \left(\frac{\sqrt{2}}{2} \right)$. The eigenvector $\left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right)$ corresponds to the eigenvalue of $\left(2 - \frac{\sqrt{2}}{2} \right) + i \left(-\frac{\sqrt{2}}{2} \right)$.

(d) $\mathbf{V} = \mathbf{P}_2(\mathbb{R})$ and \mathbf{T} is defined by $\mathbf{T}(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt$$

Suppose β is the standard ordered basis of $\mathbf{P}_2(\mathbb{R})$

$$[\mathbf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad (63)$$

$$([\mathbf{T}]_\beta)^* = [\mathbf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad (64)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad (65)$$

It follows that \mathbf{T} is neither self-adjoint nor normal.

(e) $\mathbf{V} = \mathbf{M}_{2 \times 2}(\mathbb{R})$ and \mathbf{T} is defined by $\mathbf{T}(A) = A^t$.

Suppose β is the standard ordered basis of $\mathbf{M}_{2 \times 2}(\mathbb{R})$

$$\Rightarrow [\mathbf{T}]_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (66)$$

$$([\mathbf{T}]_{\beta})^* = [\mathbf{T}^*]_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (67)$$

$$\Rightarrow \mathbf{T} = \mathbf{T}^* \quad (68)$$

$$[\mathbf{T}]_{\beta} - \lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \quad (69)$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = 0 \quad (70)$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) = 0 \quad (71)$$

$$\Rightarrow \lambda_1 = 1 \quad (72)$$

$$\lambda_2 = -1 \quad (73)$$

• For $\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (74)$$

$$\Rightarrow x_2 = x_3 \quad (75)$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t, s, r \in \mathbb{R} \right\} \quad (76)$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (77)$$

$$\Rightarrow x_1 = x_4 = 0 \quad (78)$$

$$x_2 = -x_3 \quad (79)$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \quad (80)$$

Suppose

$$w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (81)$$

$$w_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad w_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (82)$$

Let

$$v_1 = w_1 \quad (83)$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \quad (84)$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \quad (85)$$

$$v_4 = w_4 - \left(\sum_{i=1}^3 \frac{\langle w_4, v_i \rangle}{\|v_i\|^2} v_i \right) \quad (86)$$

$$\langle w_2, v_1 \rangle = 0 \quad (87)$$

$$\Rightarrow v_2 = w_2 \quad (88)$$

$$\langle w_3, v_1 \rangle = 0 \quad (89)$$

$$\langle w_3, v_2 \rangle = 0 \quad (90)$$

$$\Rightarrow v_3 = w_3 \quad (91)$$

$$\langle w_4, v_1 \rangle = 0 \quad (92)$$

$$\langle w_4, v_2 \rangle = 0 \quad (93)$$

$$\langle w_4, v_3 \rangle = 0 \quad (94)$$

$$\Rightarrow v_4 = w_4 \quad (95)$$

$$\|v_1\|^2 = 2 \quad (96)$$

$$\Rightarrow \|v_1\| = \sqrt{2} \quad (97)$$

$$\Rightarrow o_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (98)$$

$$\|v_2\|^2 = 1 \quad (99)$$

$$\Rightarrow \|v_2\| = 1 \quad (100)$$

$$\Rightarrow o_2 = v_2 \|v_3\|^2 = 1 \quad (101)$$

$$\Rightarrow \|v_3\| = 1 \quad (102)$$

$$\Rightarrow o_3 = v_3 \|v_4\|^2 = 2 \quad (103)$$

$$\Rightarrow \|v_4\| = \sqrt{2} \quad (104)$$

$$\Rightarrow o_4 = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (105)$$

An orthonormal basis is

$$\gamma = \left\{ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right\} \quad (106)$$

9. Let T be a normal operator on a finite-dimensional inner product space V . Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

Claim: $N(T) = N(T^*)$

(\subseteq) Suppose $x \in N(T)$

$$\Rightarrow T(x) = 0 \cdot x \quad (107)$$

$$\Rightarrow T^*(x) = \bar{0} \cdot x = 0 \quad (108)$$

$$\Rightarrow x \in N(T^*) \quad (109)$$

(\supseteq) Suppose $x \in N(\mathbf{T}^*)$

$$\Rightarrow \mathbf{T}^*(x) = 0 \cdot x \quad (110)$$

$$\Rightarrow (\mathbf{T}^*)^*(x) = \bar{0} \cdot x = x \quad (111)$$

$$(\mathbf{T}^*)^*(x) = \mathbf{T} \quad (112)$$

$$\Rightarrow \mathbf{T}(x) = 0 \quad (113)$$

$$\Rightarrow x \in N(\mathbf{T}) \quad (114)$$

Claim: $R(\mathbf{T}) = R(\mathbf{T}^*)$

$$N(\mathbf{T}) = N(\mathbf{T}^*) \quad (115)$$

$$N(\mathbf{T}) = R(\mathbf{T}^*)^\perp \quad (\text{Problem 6.3.12}) \quad (116)$$

$$\Rightarrow R(\mathbf{T}^*)^\perp = R(\mathbf{T})^\perp \quad (117)$$

$$\mathbf{V} = R(\mathbf{T}^*)^\perp \oplus R(\mathbf{T}^*) = R(\mathbf{T})^\perp \oplus R(\mathbf{T}) \quad (118)$$

(\subseteq) Suppose $x \in R(\mathbf{T})$

$$\Rightarrow x \in R(\mathbf{T})^\perp \oplus R(\mathbf{T}) \quad (119)$$

$$\Rightarrow x \in R(\mathbf{T}^*)^\perp \oplus R(\mathbf{T}^*) \quad (120)$$

$$\Rightarrow x \in R(\mathbf{T}^*)^\perp \text{ or } x \in R(\mathbf{T}^*) \quad (121)$$

$$R(\mathbf{T}^*) = N(\mathbf{T}) \text{ and } x \notin N(\mathbf{T}) \quad (122)$$

$$\Rightarrow x \in R(\mathbf{T}^*) \quad (123)$$

(\supseteq) Suppose $x \in R(\mathbf{T}^*)$

$$\Rightarrow x \in R(\mathbf{T}^*)^\perp \oplus R(\mathbf{T}^*) \quad (124)$$

$$\Rightarrow x \in (\mathbf{T})^\perp \oplus R(\mathbf{T}) \quad (125)$$

$$\Rightarrow x \in R(\mathbf{T})^\perp \text{ or } x \in R(\mathbf{T}) \quad (126)$$

$$R(\mathbf{T})^\perp = N(\mathbf{T}^*) \text{ and } x \notin N(\mathbf{T}^*) \quad (127)$$

$$\Rightarrow x \in R(\mathbf{T}) \quad (128)$$

11. Assume that \mathbf{T} is a linear operator on a complex (not necessarily finite-dimensional) inner product space \mathbf{V} with an adjoint \mathbf{T}^* . Prove the following results.

(a) If \mathbf{T} is self-adjoint, then $\langle \mathbf{T}(x), x \rangle$ is real for all $x \in \mathbf{V}$.

(b) If \mathbf{T} satisfies $\langle \mathbf{T}(x), x \rangle = 0$ for all $x \in \mathbf{V}$, then $\mathbf{T} = \mathbf{T}_0$.

(c) If $\langle \mathbf{T}(x), x \rangle$ is real for all $x \in \mathbf{V}$, then $\mathbf{T} = \mathbf{T}^*$.

(a) Claim: If T is self-adjoint then $\langle T(x), x \rangle$ is real $\forall x \in V$

$$\langle T(x), x \rangle = \langle x, T^*(x) \rangle = \langle x, T(x) \rangle \quad (129)$$

$$\overline{\langle T(x), x \rangle} = \langle x, T(x) \rangle \quad (130)$$

$$\Rightarrow \langle T(x), x \rangle = \overline{\langle T(x), x \rangle} \quad (131)$$

$$\Rightarrow \langle T(x), x \rangle \text{ is real}$$

(b) Suppose T satisfies $\langle T(x), x \rangle = 0 \forall x \in V$

Claim: $T = T_0$

Let $z = x + y$

$$\langle T(z), z \rangle = \langle T(x + y), x + y \rangle \quad (132)$$

$$= \langle T(x + y), x \rangle + \langle T(x + y), y \rangle \quad (133)$$

$$= \langle T(x) + T(y), x \rangle + \langle T(x) + T(y), y \rangle \quad (134)$$

$$= \langle T(x), x \rangle + \langle T(y), x \rangle + \langle T(x), y \rangle + \langle T(y), y \rangle \quad (135)$$

$$= \langle T(y), x \rangle + \langle T(x), y \rangle \quad (136)$$

$$= 0 \quad (137)$$

$$\Rightarrow \langle T(y), x \rangle = -\langle T(x), y \rangle \quad (138)$$

Let $x = x + iy$

$$\langle T(z), z \rangle = \langle T(x + iy), x + iy \rangle \quad (139)$$

$$= \langle T(x + iy), x \rangle + \langle T(x + iy), iy \rangle \quad (140)$$

$$= \langle T(x) + T(iy), x \rangle + \langle T(x) + T(iy), iy \rangle \quad (141)$$

$$= \langle T(x), x \rangle + \langle T(iy), x \rangle + \langle T(x), iy \rangle + \langle T(iy), iy \rangle \quad (142)$$

$$= \langle T(iy), x \rangle + \langle T(x), iy \rangle \quad (143)$$

$$= i\langle T(y), x \rangle + -i\langle T(x), y \rangle \quad (144)$$

$$= 0 \quad (145)$$

$$\Rightarrow \langle T(y), x \rangle = \langle T(x), y \rangle \quad (146)$$

$$\Rightarrow \langle T(y), x \rangle = \langle T(x), y \rangle = -\langle T(x), y \rangle \quad (147)$$

$$\Rightarrow \langle T(x), x \rangle = \langle T(x), y \rangle = 0 \quad \forall x, y \in V \quad (148)$$

Suppose x, y are nonzero.

$$\langle T(y), x \rangle = \langle 0, x \rangle = 0 \quad \forall x \in V \quad (149)$$

$$\Rightarrow T(y) = 0 \quad \forall y \in V \quad (150)$$

$$\Rightarrow T = T_0 \quad (151)$$

(c) Suppose $\langle \mathbb{T}(x), x \rangle$ is real for all $x \in \mathbb{V}$

Claim: $\mathbb{T} = \mathbb{T}^*$

$$\langle \mathbb{T}(x), x \rangle = \overline{\langle \mathbb{T}(x), x \rangle} \quad \forall x \in \mathbb{V} \quad (152)$$

$$\overline{\langle \mathbb{T}(x), x \rangle} = \langle x, \mathbb{T}(x) \rangle \quad (153)$$

$$\langle \mathbb{T}(x), x \rangle = \langle x, \mathbb{T}^*(x) \rangle \quad (154)$$

$$\Rightarrow \langle x, \mathbb{T}(x) \rangle = \langle x, \mathbb{T}^*(x) \rangle \quad \forall x \in \mathbb{T}(x) \quad (155)$$

$$\Rightarrow \mathbb{T}(x) = \mathbb{T}^*(x) \quad \forall x \in \mathbb{V} \quad (156)$$

$$\Rightarrow \mathbb{T} = \mathbb{T}^* \quad (157)$$

6.5

- 1.
- 2.
- 5.
- 10.
- 21.