Assignment

Section 6.3: 1, 2, 3, 9; Section 6.4: 1, 2, 9, 11; Section 6.5: 1, 2, 5, 10, 21

Work

6.3

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every linear operator has an adjoint.

   (b) Every linear operator on V has the form $x \to \langle x, y \rangle$ for some $y \in \mathsf{V}$.

   (c) For every linear operator T on V and every ordered basis β for V, we have $[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^*$.

   (d) The adjoint of a linear operator is unique.

   (e) For any linear operators T and U and scalars a and b,

   $$(a\mathsf{T} + b\mathsf{U})^* = a\mathsf{T}^* + b\mathsf{U}^*$$

   (f) For any $n \times n$ matrix A, we have $(\mathsf{L}_A)^* = \mathsf{L}_A$

   (g) For any linear operator T, we have $(\mathsf{T}^*)^* = \mathsf{T}$

3.

9.

6.4

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every self-adjoint operator is normal.
   True

   (b) Operators and their adjoints have the same eigenvectors.
   False

   (c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.
   False

   (d) A real or complex matrix A is normal if and only if L_A is normal.
   True

(e) The eigenvalues of a self-adjoint operator must be real.
True

(f) The identity and zero operators are self-adjoint.
True

(g) Every normal operator is diagonalizable.
False

(h) Every self-adjoint operator is diagonalizable.
True

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(a) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a, b) = (2a - 2b, -2a + 5b)$
Suppose β is the standard ordered basis for R^2

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{1}$$

$$\Rightarrow ([\mathsf{T}]_\beta)^* = ([\mathsf{T}^*]) = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{2}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{3}$$

$$\det \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = 0 \tag{4}$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \tag{5}$$

$$\Rightarrow \lambda_1 = 6 \tag{6}$$

$$\lambda_2 = 1 \tag{7}$$

- For $\lambda_1 = 6$

$$[\mathsf{T}]_\beta - 6I_2 = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

$$\Rightarrow x_1 = -\frac{1}{2} x_2 \tag{11}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{12}$$

- For $\lambda_2 = 1$

$$
\begin{aligned}
[\mathsf{T}]_\beta - I_2 &= \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} && (13)\\
\Rightarrow \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (14)\\
\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (15)\\
\Rightarrow x_1 &= 2x_2 && (16)\\
\Rightarrow E_{\lambda_2} &= \left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} && (17)
\end{aligned}
$$

Suppose

$$
v_1' = (-\frac{1}{2}, 1) \qquad v_2' = (2, 1) \tag{18}
$$

Let

$$
\begin{aligned}
v_1 &= v_1' && (19)\\
v_2 &= v_2' - \frac{\langle v_2', v_1 \rangle}{\|v_1\|^2} v_1 && (20)
\end{aligned}
$$

$$
\begin{aligned}
\langle v_2', v_1 \rangle &= 0 && (21)\\
\Rightarrow v_2 &= v_2' && (22)\\
\|v_1\|^2 &= \frac{5}{4} && (23)\\
\Rightarrow \|v_1\| &= \frac{\sqrt{5}}{2} && (24)\\
\Rightarrow o_1 &= \frac{1}{\sqrt{5}}(-1, 2) && (25)\\
\|v_2\|^2 &= 5 && (26)\\
\Rightarrow \|v_2\| &= 5 && (27)\\
\Rightarrow o_2 &= \frac{1}{\sqrt{5}}(2, 1) && (28)
\end{aligned}
$$

An orthonormal basis is

$$
\gamma = \left\{ \frac{1}{\sqrt{5}}(-1, 2), \frac{1}{\sqrt{3}}(2, 1) \right\} \tag{29}
$$

The eigenvector $\frac{1}{\sqrt{5}}(-1, 2)$ corresponds to the eigenvalue 6, and the eigenvector $\frac{1}{\sqrt{3}}(2, 1)$ corresponds to the eigenvalue 1.

(b) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$
Suppose β is the standard ordered basis of R^3

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{pmatrix} \tag{30}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 5 & -2 \\ 0 & 0 & 5 \end{pmatrix} \tag{31}$$

$$\Rightarrow \mathsf{T}^* \neq \mathsf{T} \tag{32}$$

$$([\mathsf{T}]_\beta)^*[\mathsf{T}]_\beta \neq ([\mathsf{T}]_\beta)^* \tag{33}$$

T is neither normal nor adjoint.

(c)

(d) $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, \mathrm{d}t$$

Suppose β is the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{36}$$

It follows that T is neither self-adjoint nor normal.

(e) $\mathsf{V} = \mathsf{M}_{2\times n}(\mathbb{R})$ and T is defined by $\mathsf{T}(A) = A^t$.
Suppose β is the standard ordered basis of $\mathsf{M}_{n\times n}(\mathbb{R})$

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(\mathsf{T}) = N(\mathsf{T}^*)$ and $R(\mathsf{T}) = R(\mathsf{T}^*)$.

Claim: $N(\mathsf{T}) = N(\mathsf{T}^*)$

($\subseteq$) Suppose $x \in N(\mathsf{T})$

$$\Rightarrow \mathsf{T}(x) = 0 \cdot x \tag{37}$$

$$\Rightarrow \mathsf{T}^*(x) = \bar{0} \cdot x = 0 \tag{38}$$

$$\Rightarrow x \in N(\mathsf{T}^*) \tag{39}$$

($\supseteq$) Suppose $x \in N(\mathsf{T}^*)$

$$\Rightarrow \mathsf{T}^*(x) = 0 \cdot x \tag{40}$$
$$\Rightarrow (\mathsf{T}^*)^*(x) = \bar{0} \cdot x = x \tag{41}$$
$$(\mathsf{T}^*)^*(x) = \mathsf{T} \tag{42}$$
$$\Rightarrow \mathsf{T}(x) = 0 \tag{43}$$
$$\Rightarrow x \in N(\mathsf{T}) \tag{44}$$

Claim: $R(\mathsf{T}) = R(\mathsf{T}^*)$

$$N(\mathsf{T}) = N(\mathsf{T}^*) \tag{45}$$
$$N(\mathsf{T}) = R(\mathsf{T}^*)^\perp \quad \text{(Problem 6.3.12)} \tag{46}$$
$$\Rightarrow R(\mathsf{T}^*)^\perp = R(\mathsf{T})^\perp \tag{47}$$
$$\mathsf{V} = R(\mathsf{T}^*)^\perp \oplus R(\mathsf{T}^*) = R(\mathsf{T})^\perp \oplus R(\mathsf{T}) \tag{48}$$

($\subseteq$) Suppose $x \in R(\mathsf{T})$

$$\Rightarrow x \in R(\mathsf{T})^\perp \oplus R(\mathsf{T}) \tag{49}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^\perp \oplus R(\mathsf{T}^*) \tag{50}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^\perp \text{ or } x \in R(\mathsf{T}^*) \tag{51}$$
$$R(\mathsf{T}^*) = N(\mathsf{T}) \text{ and } x \notin N(\mathsf{T}) \tag{52}$$
$$\Rightarrow x \in R(\mathsf{T}^*) \tag{53}$$

($\supseteq$) Suppose $x \in R(\mathsf{T}^*)$

$$\Rightarrow x \in R(\mathsf{T}^*)^\perp \oplus R(\mathsf{T}^*) \tag{54}$$
$$\Rightarrow x \in (\mathsf{T})^\perp \oplus R(\mathsf{T}) \tag{55}$$
$$\Rightarrow x \in R(\mathsf{T})^\perp \text{ or } x \in R(\mathsf{T}) \tag{56}$$
$$R(\mathsf{T})^\perp = N(\mathsf{T}^*) \text{ and } x \notin N(\mathsf{T}^*) \tag{57}$$
$$\Rightarrow x \in R(\mathsf{T}) \tag{58}$$

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

   (a) If T is self-adjoint, then $\langle \mathsf{T}(x), x \rangle$ is real for all $x \in \mathsf{V}$.

   (b) If T satisfies $\langle \mathsf{T}(x), x \rangle = 0$ for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}_0$.

   (c) If $\langle \mathsf{T}(x), x \rangle$ is real for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}^*$.

(a) Claim: If T is self-adjoint then $\langle \mathsf{T}(x), x\rangle$ is real $\forall x \in \mathsf{V}$

$$\langle \mathsf{T}(x), x\rangle = \langle x, \mathsf{T}^*(x)\rangle = \langle x, \mathsf{T}(x)\rangle \tag{59}$$
$$\overline{\langle \mathsf{T}(x), x\rangle} = \langle x, \mathsf{T}(x)\rangle \tag{60}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle = \overline{\langle \mathsf{T}(x), x\rangle} \tag{61}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle \text{ is real}$$

(b) Suppose T satisfies $\langle \mathsf{T}(x), x\rangle = 0\ \forall x \in \mathsf{V}$
Claim: $\mathsf{T} = \mathsf{T}_0$
Let $z = x + y$

$$\langle \mathsf{T}(z), z\rangle = \langle \mathsf{T}(x+y), x+y\rangle \tag{62}$$
$$= \langle \mathsf{T}(x+y), x\rangle + \langle \mathsf{T}(x+y), x\rangle \tag{63}$$
$$= \langle \mathsf{T}(x)+\mathsf{T}(y), x\rangle + \langle \mathsf{T}(x)+\mathsf{T}(y), y\rangle \tag{64}$$
$$= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle + \langle \mathsf{T}(y), y\rangle \tag{65}$$
$$= \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle \tag{66}$$
$$= 0 \tag{67}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = -\langle \mathsf{T}(x), y\rangle \tag{68}$$

Let $x = x + iy$

$$\langle \mathsf{T}(z), z\rangle = \langle \mathsf{T}(x+iy), x+iy\rangle \tag{69}$$
$$= \langle \mathsf{T}(x+iy), x\rangle + \langle \mathsf{T}(x+iy), x\rangle \tag{70}$$
$$= \langle \mathsf{T}(x)+\mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x)+\mathsf{T}(iy), iy\rangle \tag{71}$$
$$= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle + \langle \mathsf{T}(iy), iy\rangle \tag{72}$$
$$= \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle \tag{73}$$
$$= i\langle \mathsf{T}(y), x\rangle + -i\langle \mathsf{T}(x), y\rangle \tag{74}$$
$$= 0 \tag{75}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle \tag{76}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle = -\langle \mathsf{T}(x), y\rangle \tag{77}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle = \langle \mathsf{T}(x), y\rangle = 0 \quad \forall x, y \in \mathsf{V} \tag{78}$$

Suppose x, y are nonzero.

$$\langle \mathsf{T}(y), x\rangle = \langle 0, x\rangle = 0 \quad \forall x \in \mathsf{V} \tag{79}$$
$$\Rightarrow \mathsf{T}(y) = 0 \quad \forall y \in \mathsf{V} \tag{80}$$
$$\Rightarrow \mathsf{T} = \mathsf{T}_0 \tag{81}$$

(c) Suppose $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$
Claim: $\mathsf{T} = \mathsf{T}^*$

$$\langle \mathsf{T}(x), x\rangle = \overline{\langle \mathsf{T}(x), x\rangle} \quad \forall x \in \mathsf{V} \tag{82}$$
$$\overline{\langle \mathsf{T}(x), x\rangle} = \langle x, \mathsf{T}(x)\rangle \tag{83}$$
$$\langle \mathsf{T}(x), x\rangle = \langle x, \mathsf{T}^*(x)\rangle \tag{84}$$
$$\Rightarrow \langle x, \mathsf{T}(x)\rangle = \langle x\mathsf{T}^*(x)\rangle \quad \forall x \in \mathsf{T}(x) \tag{85}$$
$$\Rightarrow \mathsf{T}(x) = \mathsf{T}^*(x) \quad \forall x \in \mathsf{V} \tag{86}$$
$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{87}$$

6.5

1.

2.

5.

10.

21.