Assignment

Section 6.3: 1, 2, 3, 9; Section 6.4: 1, 2, 9, 11; Section 6.5: 1, 2, 5, 10, 21

Work

6.3

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every linear operator has an adjoint.

   (b) Every linear operator on V has the form $x \to \langle x, y \rangle$ for some $y \in \mathsf{V}$.

   (c) For every linear operator T on V and every ordered basis β for V, we have $[\mathsf{T}^*]_\beta = ([\mathsf{T}]_\beta)^*$.

   (d) The adjoint of a linear operator is unique.

   (e) For any linear operators T and U and scalars a and b,

   $$(a\mathsf{T} + b\mathsf{U})^* = a\mathsf{T}^* + b\mathsf{U}^*$$

   (f) For any $n \times n$ matrix A, we have $(\mathsf{L}_A)^* = \mathsf{L}_A$

   (g) For any linear operator T, we have $(\mathsf{T}^*)^* = \mathsf{T}$

3.

9.

6.4

1. Label the following statements as true or false. Assume the underlying inner product spaces are finite-dimensional.

   (a) Every self-adjoint operator is normal.
   True

   (b) Operators and their adjoints have the same eigenvectors.
   False

   (c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.
   False

   (d) A real or complex matrix A is normal if and only if L_A is normal.
   True

(e) The eigenvalues of a self-adjoint operator must be real.
True

(f) The identity and zero operators are self-adjoint.
True

(g) Every normal operator is diagonalizable.
False

(h) Every self-adjoint operator is diagonalizable.
True

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(a) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a,b) = (2a - 2b, -2a + 5b)$
Suppose β is the standard ordered basis for R^2

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{1}$$

$$\Rightarrow ([\mathsf{T}]_\beta)^* = ([\mathsf{T}^*]) = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{2}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{3}$$

$$\det \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = 0 \tag{4}$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \tag{5}$$

$$\Rightarrow \lambda_1 = 6 \tag{6}$$

$$\lambda_2 = 1 \tag{7}$$

- For $\lambda_1 = 6$

$$[\mathsf{T}]_\beta - 6I_2 = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

$$\Rightarrow x_1 = -\frac{1}{2}x_2 \tag{11}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{12}$$

- For $\lambda_2 = 1$

$$[\mathsf{T}]_\beta - I_2 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \tag{13}$$
$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$
$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$
$$\Rightarrow x_1 = 2x_2 \tag{16}$$
$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{17}$$

Suppose

$$v_1' = (-\frac{1}{2}, 1) \qquad\qquad v_2' = (2, 1) \tag{18}$$

Let

$$v_1 = v_1' \tag{19}$$
$$v_2 = v_2' - \frac{\langle v_2', v_1 \rangle}{\|v_1\|^2} v_1 \tag{20}$$

$$\langle v_2', v_1 \rangle = 0 \tag{21}$$
$$\Rightarrow v_2 = v_2' \tag{22}$$
$$\|v_1\|^2 = \frac{5}{4} \tag{23}$$
$$\Rightarrow \|v_1\| = \frac{\sqrt{5}}{2} \tag{24}$$
$$\Rightarrow o_1 = \frac{1}{\sqrt{5}}(-1, 2) \tag{25}$$
$$\|v_2\|^2 = 5 \tag{26}$$
$$\Rightarrow \|v_2\| = 5 \tag{27}$$
$$\Rightarrow o_2 = \frac{1}{\sqrt{5}}(2, 1) \tag{28}$$

An orthonormal basis is

$$\gamma = \left\{ \frac{1}{\sqrt{5}}(-1, 2), \frac{1}{\sqrt{3}}(2, 1) \right\} \tag{29}$$

The eigenvector $\frac{1}{\sqrt{5}}(-1, 2)$ corresponds to the eigenvalue 6, and the eigenvector $\frac{1}{\sqrt{3}}(2, 1)$ corresponds to the eigenvalue 1.

(b) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$
Suppose β is the standard ordered basis of R^3

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{pmatrix} \tag{30}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} -1 & 0 & 4 \\ 1 & 5 & -2 \\ 0 & 0 & 5 \end{pmatrix} \tag{31}$$

$$\Rightarrow \mathsf{T}^* \neq \mathsf{T} \tag{32}$$

$$([\mathsf{T}]_\beta)^*[\mathsf{T}]_\beta \neq ([\mathsf{T}]_\beta)^* \tag{33}$$

T is neither normal nor adjoint.

(c) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(a, b) = (2a + ib, a + 2b)$
Suppose β is the standard ordered basis of C^2

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \tag{34}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \tag{36}$$

$$\Rightarrow \mathsf{T} \text{ is normal.}$$

$$\det \begin{pmatrix} 2 - \lambda & i \\ 1 & 2 - \lambda \end{pmatrix} \tag{37}$$

$$\Rightarrow (2 - \lambda)^2 = i \tag{38}$$

$$\Rightarrow \lambda_1 = \left(2 + \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right) \tag{39}$$

$$\lambda_2 = \left(2 - \frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right) \tag{40}$$

- For $\lambda_1 = \left(2 + \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right)$

$$[\mathsf{T}]_\beta - \lambda_1 I_n = \begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) & i \\ 1 & -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{42}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2}(1+i) & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{43}$$

$$\Rightarrow x_1 = \frac{\sqrt{2}}{2}(1+i)x_2 \tag{44}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{45}$$

- For $\lambda_2 = \left(2 - \frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)$

$$[\mathsf{T}]_\beta - \lambda_2 I_2 = \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \tag{46}$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{47}$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2}(1+i) & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{48}$$

$$\Rightarrow x_1 = -\frac{\sqrt{2}}{2}(1+i)x_2 \tag{49}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -\frac{\sqrt{2}}{2}(1+i) \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{50}$$

Suppose

$$w_1 = \left(\frac{\sqrt{2}}{2}(1+i), 1\right) \qquad w_2 = \left(-\frac{\sqrt{2}}{2}(1+i), 1\right) \tag{51}$$

Let

$$v_1 = w_1 \tag{52}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \tag{53}$$

$$\langle w_2, v_1 \rangle = 0 \tag{54}$$
$$\Rightarrow v_2 = w_2 \tag{55}$$
$$\|v_1\|^2 = 2 \tag{56}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{57}$$
$$\Rightarrow o_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right) \tag{58}$$
$$\|v_2\|^2 = 2 \tag{59}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{60}$$
$$\Rightarrow o_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right) \tag{61}$$

An orthonormal basis is

$$\gamma = \left\{ \left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right) \right\} \tag{62}$$

The eigenvector $\left(\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right)$ corresponds to the eigenvalue of $\left(2 + \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\right)$. The eigenvector $\left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2}\right)$ corresponds to the eigenvalue of $\left(2 - \frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)$.

(d) $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ and T is defined by $\mathsf{T}(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, \mathrm{d}t$$

Suppose β is the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{63}$$
$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{64}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{65}$$

It follows that T is neither self-adjoint nor normal.

(e) $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$ and T is defined by $\mathsf{T}(A) = A^t$.
Suppose β is the standard ordered basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{66}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{67}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{68}$$

$$[\mathsf{T}]_\beta - \lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \tag{69}$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{70}$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) = 0 \tag{71}$$

$$\Rightarrow \lambda_1 = 1 \tag{72}$$

$$\lambda_2 = -1 \tag{73}$$

- For $\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{74}$$

$$\Rightarrow x_2 = x_3 \tag{75}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t, s, r \in \mathbb{R} \right\} \tag{76}$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{77}$$

$$\Rightarrow x_1 = x_4 = 0 \tag{78}$$

$$x_2 = -x_3 \tag{79}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{80}$$

Suppose

$$w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{81}$$

$$w_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{82}$$

Let

$$v_1 = w_1 \tag{83}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{84}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{85}$$

$$v_4 = w_4 - \left(\sum_{i=1}^{3} \frac{\langle w_4, v_i \rangle}{\|v_1\|^2} v_i \right) \tag{86}$$

$$\langle w_2, v_1 \rangle = 0 \tag{87}$$
$$\Rightarrow v_2 = w_2 \tag{88}$$
$$\langle w_3, v_1 \rangle = 0 \tag{89}$$
$$\langle w_3, v_2 \rangle = 0 \tag{90}$$
$$\Rightarrow v_3 = w_3 \tag{91}$$
$$\langle w_4, v_1 \rangle = 0 \tag{92}$$
$$\langle w_4, v_2 \rangle = 0 \tag{93}$$
$$\langle w_4, v_3 \rangle = 0 \tag{94}$$
$$\Rightarrow v_4 = w_4 \tag{95}$$
$$\|v_1\|^2 = 2 \tag{96}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{97}$$
$$\Rightarrow o_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{98}$$
$$\|v_2\|^2 = 1 \tag{99}$$
$$\Rightarrow \|v_2\| = 1 \tag{100}$$
$$\Rightarrow o_2 = v_2 \tag{101}$$
$$\|v_3\|^2 = 1 \tag{102}$$
$$\Rightarrow \|v_3\| = 1 \tag{103}$$
$$\Rightarrow o_3 = v_3 \tag{104}$$
$$\|v_4\|^2 = 2 \tag{105}$$
$$\Rightarrow \|v_4\| = \sqrt{2} \tag{106}$$
$$\Rightarrow o_4 = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{107}$$

An orthonormal basis is

$$\gamma = \left\{ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right\} \tag{108}$$

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$ and T is defined by $\mathsf{T}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$

Suppose β is the standard ordered basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$

$$\Rightarrow [\mathsf{T}]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{109}$$

$$([\mathsf{T}]_\beta)^* = [\mathsf{T}^*]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{110}$$

$$\Rightarrow \mathsf{T} \text{ is self adjoint.}$$

$$[\mathsf{T}]_\beta - \lambda I_4 = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix} \tag{111}$$

$$\det \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix} = 0 \tag{112}$$

$$(\lambda - 1)^2(\lambda + 1) \tag{113}$$

$$\Rightarrow \lambda_1 = 1 \tag{114}$$

$$\lambda_2 = -1 \tag{115}$$

- For $\lambda_1 = 1$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{116}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{117}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R} \right\} \tag{118}$$

$$\tag{119}$$

- For $\lambda_2 = -1$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{120}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{121}$$

$$\Rightarrow x_1 = -x_3 \tag{122}$$

$$x_2 = -x_4 \tag{123}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\} \tag{124}$$

Suppose

$$w_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{125}$$

$$w_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \tag{126}$$

Let

$$v_1 = w_1 \tag{127}$$

$$v_2 = w_2 - \frac{\langle w_2, v_2 \rangle}{\|v_1\|^2} v_1 \tag{128}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 \right) \tag{129}$$

$$v_4 = w_4 - \left(\sum_{i=1}^{3} \frac{\langle w_4, v_i \rangle}{\|v_1\|^2} v_i \right) \tag{130}$$

$$\langle w_2, v_a \rangle = 0 \tag{131}$$

$$\Rightarrow v_2 = w_2 \tag{132}$$

$$\langle w_3, v_2 \rangle = 0 \tag{133}$$

$$\langle w_3, v_1 \rangle = 0 \tag{134}$$

$$\Rightarrow v_3 = w_3 \tag{135}$$

$$\langle w_4, v_1 \rangle = 0 \tag{136}$$

$$\langle w_4, v_2 \rangle = 0 \tag{137}$$

$$\langle w_4, v_3 \rangle = 0 \tag{138}$$

$$\Rightarrow v_4 = w_4 \tag{139}$$

$$\|v_1\|^2 = 2 \tag{140}$$
$$\Rightarrow \|v_1\| = \sqrt{2} \tag{141}$$
$$\|v_2\|^2 = 2 \tag{142}$$
$$\Rightarrow \|v_2\| = \sqrt{2} \tag{143}$$
$$\|v_3\|^2 = 2 \tag{144}$$
$$\Rightarrow \|v_3\| = \sqrt{2} \tag{145}$$
$$\|v_4\|^2 = 2 \tag{146}$$
$$\Rightarrow \|v_4\| = \sqrt{2} \tag{147}$$

An orthonormal basis is

$$\gamma = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\} \tag{148}$$

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(\mathsf{T}) = N(\mathsf{T}^*)$ and $R(\mathsf{T}) = R(\mathsf{T}^*)$.

   Claim: $N(\mathsf{T}) = N(\mathsf{T}^*)$

   ($\subseteq$) Suppose $x \in N(\mathsf{T})$

$$\Rightarrow \mathsf{T}(x) = 0 \cdot x \tag{149}$$
$$\Rightarrow \mathsf{T}^*(x) = \bar{0} \cdot x = 0 \tag{150}$$
$$\Rightarrow x \in N(\mathsf{T}^*) \tag{151}$$

   ($\supseteq$) Suppose $x \in N(\mathsf{T}^*)$

$$\Rightarrow \mathsf{T}^*(x) = 0 \cdot x \tag{152}$$
$$\Rightarrow (\mathsf{T}^*)^* (x) = \bar{0} \cdot x = x \tag{153}$$
$$(\mathsf{T}^*)^* (x) = \mathsf{T} \tag{154}$$
$$\Rightarrow \mathsf{T}(x) = 0 \tag{155}$$
$$\Rightarrow x \in N(\mathsf{T}) \tag{156}$$

   Claim: $R(\mathsf{T}) = R(\mathsf{T}^*)$

$$N(\mathsf{T}) = N(\mathsf{T}^*) \tag{157}$$
$$N(\mathsf{T}) = R(\mathsf{T}^*)^{\perp} \quad \text{(Problem 6.3.12)} \tag{158}$$
$$\Rightarrow R(\mathsf{T}^*)^{\perp} = R(\mathsf{T})^{\perp} \tag{159}$$
$$\mathsf{V} = R(\mathsf{T}^*)^{\perp} \oplus R(\mathsf{T}^*) = R(\mathsf{T})^{\perp} \oplus R(\mathsf{T}) \tag{160}$$

($\subseteq$) Suppose $x \in R(\mathsf{T})$

$$\Rightarrow x \in R(\mathsf{T})^{\perp} \oplus R(\mathsf{T}) \tag{161}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \oplus R(\mathsf{T}^*) \tag{162}$$
$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \text{ or } x \in R(\mathsf{T}^*) \tag{163}$$
$$R(\mathsf{T}^*) = N(\mathsf{T}) \text{ and } x \notin N(\mathsf{T}) \tag{164}$$
$$\Rightarrow x \in R(\mathsf{T}^*) \tag{165}$$

($\supseteq$) Suppose $x \in R(\mathsf{T}^*)$

$$\Rightarrow x \in R(\mathsf{T}^*)^{\perp} \oplus R(\mathsf{T}^*) \tag{166}$$
$$\Rightarrow x \in (\mathsf{T})^{\perp} \oplus R(\mathsf{T}) \tag{167}$$
$$\Rightarrow x \in R(\mathsf{T})^{\perp} \text{ or } x \in R(\mathsf{T}) \tag{168}$$
$$R(\mathsf{T})^{\perp} = N(\mathsf{T}^*) \text{ and } x \notin N(\mathsf{T}^*) \tag{169}$$
$$\Rightarrow x \in R(\mathsf{T}) \tag{170}$$

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

    (a) If T is self-adjoint, then $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$.

    (b) If T satisfies $\langle \mathsf{T}(x), x\rangle = 0$ for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}_0$.

    (c) If $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$, then $\mathsf{T} = \mathsf{T}^*$.

    (a) Claim: If T is self-adjoint then $\langle \mathsf{T}(x), x\rangle$ is real $\forall x \in \mathsf{V}$

$$\langle \mathsf{T}(x), x\rangle = \langle x, \mathsf{T}^*(x)\rangle = \langle x, \mathsf{T}(x)\rangle \tag{171}$$
$$\overline{\langle \mathsf{T}(x), x\rangle} = \langle x, \mathsf{T}(x)\rangle \tag{172}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle = \overline{\langle \mathsf{T}(x), x\rangle} \tag{173}$$
$$\Rightarrow \langle \mathsf{T}(x), x\rangle \text{ is real}$$

    (b) Suppose T satisfies $\langle \mathsf{T}(x), x\rangle = 0 \ \forall x \in \mathsf{V}$
    Claim: $\mathsf{T} = \mathsf{T}_0$
    Let $z = x + y$

$$\begin{aligned}
\langle \mathsf{T}(z), z\rangle &= \langle \mathsf{T}(x+y), x+y\rangle && (174)\\
&= \langle \mathsf{T}(x+y), x\rangle + \langle \mathsf{T}(x+y), x\rangle && (175)\\
&= \langle \mathsf{T}(x)+\mathsf{T}(y), x\rangle + \langle \mathsf{T}(x)+\mathsf{T}(y), y\rangle && (176)\\
&= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle + \langle \mathsf{T}(y), y\rangle && (177)\\
&= \langle \mathsf{T}(y), x\rangle + \langle \mathsf{T}(x), y\rangle && (178)\\
&= 0 && (179)
\end{aligned}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = -\langle \mathsf{T}(x), y\rangle \tag{180}$$

Let $x = x + iy$

$$\begin{aligned}
\langle \mathsf{T}(z), z\rangle &= \langle \mathsf{T}(x+iy), x+iy\rangle &(181)\\
&= \langle \mathsf{T}(x+iy), x\rangle + \langle \mathsf{T}(x+iy), x\rangle &(182)\\
&= \langle \mathsf{T}(x)+\mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x)+\mathsf{T}(iy), iy\rangle &(183)\\
&= \langle \mathsf{T}(x), x\rangle + \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle + \langle \mathsf{T}(iy), iy\rangle &(184)\\
&= \langle \mathsf{T}(iy), x\rangle + \langle \mathsf{T}(x), iy\rangle &(185)\\
&= i\langle \mathsf{T}(y), x\rangle + -i\langle \mathsf{T}(x), y\rangle &(186)\\
&= 0 &(187)
\end{aligned}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle \tag{188}$$

$$\Rightarrow \langle \mathsf{T}(y), x\rangle = \langle \mathsf{T}(x), y\rangle = -\langle \mathsf{T}(x), y\rangle \tag{189}$$

$$\Rightarrow \langle \mathsf{T}(x), x\rangle = \langle \mathsf{T}(x), y\rangle = 0 \quad \forall x, y \in \mathsf{V} \tag{190}$$

Suppose x, y are nonzero.

$$\langle \mathsf{T}(y), x\rangle = \langle 0, x\rangle = 0 \quad \forall x \in \mathsf{V} \tag{191}$$

$$\Rightarrow \mathsf{T}(y) = 0 \quad \forall y \in \mathsf{V} \tag{192}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}_0 \tag{193}$$

(c) Suppose $\langle \mathsf{T}(x), x\rangle$ is real for all $x \in \mathsf{V}$
Claim: $\mathsf{T} = \mathsf{T}^*$

$$\langle \mathsf{T}(x), x\rangle = \overline{\langle \mathsf{T}(x), x\rangle} \quad \forall x \in \mathsf{V} \tag{194}$$

$$\overline{\langle \mathsf{T}(x), x\rangle} = \langle x, \mathsf{T}(x)\rangle \tag{195}$$

$$\langle \mathsf{T}(x), x\rangle = \langle x, \mathsf{T}^*(x)\rangle \tag{196}$$

$$\Rightarrow \langle x, \mathsf{T}(x)\rangle = \langle x\mathsf{T}^*(x)\rangle \quad \forall x \in \mathsf{T}(x) \tag{197}$$

$$\Rightarrow \mathsf{T}(x) = \mathsf{T}^*(x) \quad \forall x \in \mathsf{V} \tag{198}$$

$$\Rightarrow \mathsf{T} = \mathsf{T}^* \tag{199}$$

6.5

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