Assignment

Section 5.4: 4, 15, 17, 19, 41; Section 6.1: 3, 8, 12, 17; Section 6.2: 2(a,c,g,j), 6, 7, 15

Work

5.4

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(\mathsf{T})$-invariant for any polynomial $g(t)$.

   Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$

   Let W be a T-invariant subspace of V

   Lemma: $\mathsf{T}^k(\mathsf{W})$ is T-invariant for all $k \in \mathbb{Z}^+$

   Proof by induction.

   Base case: Suppose $k = 1$

$$\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W} \tag{1}$$
$$\Rightarrow \mathsf{T}^2(\mathsf{W}) = \mathsf{T}(\mathsf{T}(\mathsf{W})) \subseteq \mathsf{W} \tag{2}$$

   Suppose $\mathsf{T}^k(\mathsf{W})$ is T-invariant for $1 \le k \le n$.

   Suppose $k = n + 1$

$$\mathsf{T}^{n+1}(\mathsf{W}) \subseteq \mathsf{T}(\mathsf{T}^n(\mathsf{W})) \tag{3}$$
$$\mathsf{T}^n(\mathsf{W}) \subseteq \mathsf{W} \tag{4}$$
$$\Rightarrow \mathsf{T}^{n+1}(\mathsf{W}) = \mathsf{T}(\mathsf{T}^n(\mathsf{W})) \subseteq \mathsf{W} \tag{5}$$

$$\therefore \mathsf{T}^k(\mathsf{W}) \text{ is } \mathsf{T}\text{-invariant for all } k \in \mathbb{Z}^+ \qquad \square$$

   Suppose $w \in \mathsf{W}$ and $g(t) \in \mathsf{P}(F)$ such that

$$g(t) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{6}$$

$$\Rightarrow g(\mathsf{T}) = a_n\mathsf{T}^n + a_{n-1}\mathsf{T}^{n-1} + \cdots + a_1\mathsf{T} + a_0\mathsf{I_V} \tag{7}$$
$$\Rightarrow g(\mathsf{T})(\mathsf{W}) = a_n\mathsf{T}^n(w) + a_{n-1}\mathsf{T}^{n-1}(w) + \cdots + a_1\mathsf{T}(w) + a_0\mathsf{I_V} \tag{8}$$
$$\Rightarrow g(\mathsf{T})(w) \in \mathsf{W}\ \forall w \in \mathsf{W} \tag{9}$$

$$\therefore \mathsf{W} \text{ is } g(\mathsf{T})\text{-invariant}$$

15. Use Cayley-Hamilton theorem to prove its corollary for matrices.

Corollary to Cayley-Hamilton theorem for matrices Let $A \in \mathsf{M}_{n\times n}(F)$ and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = O$, the $n \times n$ zero matrix.

Suppose $A \in \mathsf{M}_{n\times n}(F)$ and let $f(t)$ be the characteristic polynomial of A.

Suppose β is the standard ordered basis of F^n

$$\Rightarrow A = [\mathsf{L}_A]_\beta \tag{10}$$

A and L_A have the same characteristic polynomial by the definition of characteristic polynomial for functions.

$$f(A) = [f(\mathsf{L}_A)]_\beta \text{ by theorem E.3} \tag{11}$$
$$f(\mathsf{L}_A) = 0 \in \mathsf{F}^n \text{ by Cayley Hamilton} \tag{12}$$
$$\Rightarrow f(A) = [0]_\beta = O \in \mathsf{M}_{n\times n}(F) \tag{13}$$

17. Let A be an $n \times n$ matrix. Prove that

$$\dim\left(\mathrm{span}(\{I_n, A, A^2, \dots\})\right) \leq n$$

The characteristic polynomial of A is

$$f(t) = (-1)t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \tag{14}$$

$$\Rightarrow f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0 \tag{15}$$
$$\Rightarrow A^n = (-1)^{n+1}a_{n-1}A^{n-1} + \cdots + (-1)^{n+1}A + (-1)^{n+1}a_0 I_n \tag{16}$$
$$\Rightarrow A^n \in \mathrm{span}(\{I_n, A, A^2, \dots, A^{n-1}\}) \tag{17}$$

Claim: $A^k \in \mathrm{span}(\{I_n A, A^2, \dots, A^{n-1}\})\ \forall k \in \mathbb{Z}, k \geq n$

Proof by induction.

True for $k = n$

Suppose true for $n \leq k \leq n + i - 1$ for some $i \in \mathbb{Z}^+, i \geq 2$

Suppose $k = n + i$

$$\Rightarrow A^{k+i} = A^n \cdot A^i \tag{18}$$
$$A^k \in \mathrm{span}(\{I_n A, A^2, \dots, A^{n-1}\}) \tag{19}$$
$$i < n + i \tag{20}$$
$$\Rightarrow A^i \in \mathrm{span}(\{I_n A, A^2, \dots, A^{n-1}\}) \tag{21}$$
$$\Rightarrow A^{i+n} = A^i \cdot A^n \in \mathrm{span}(\{I_n A, A^2, \dots, A^{n-1}\}) \tag{22}$$

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0, a_1, \dots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

Proof by induction on k.

Suppose $k = 1$

$$\Rightarrow A = -a_0 \tag{23}$$
$$\det(A - tI_1) = \det(-a_0 - t) \tag{24}$$
$$= -a_0 - t \tag{25}$$
$$= (-1)^1(a_0 + t^1) \tag{26}$$

Suppose true for $2 \leq k \leq n-1$

Suppose $k = n$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \tag{27}$$

$$\Rightarrow A - tI_n = \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} - t \end{pmatrix} \tag{28}$$

$$\Rightarrow \det(A - tI_n) = (-t)(-1)^2 \det \tilde{A}_{11} + (a_0)(-1)^{n+1} \det \tilde{A}_{1n} \tag{29}$$

$$\det \tilde{A}_{11} = \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} - t \end{pmatrix} \tag{30}$$

$$= (-a)^{n-1}(a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) \tag{31}$$

$$\det \tilde{A}_{1n} = \det \begin{pmatrix} 1 & -t & 0 & \cdots & 0 \\ 0 & 1 & -t & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -t \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{32}$$

$$= 1 \because (\tilde{A}_{1n})_{ii} = 1 \forall i (1 \leq i \leq n-1) \tag{33}$$

$$\Rightarrow \det(A - tI_n) = (-1)(t)(-1)^{n-1}(a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) + (a_0)(-1)^n \tag{34}$$

$$= (-1)^n (a_0 + a_1 + a_2 t + \cdots + a_{n-1} t^{n-2} + t^{n-1}) \tag{35}$$

41. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}$$

Find the characteristic polynomial of A. *Hint:* First prove that A has rank 2 and that $\text{span}(\{(1,1,\ldots,1),(1,2,\ldots,n)\})$ is L_A-invariant.

Hint #2 Show that $\text{span}(\{(1,1,\ldots,1),(1,2,\ldots,n)\})$ is L_A-invariant.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} \tag{36}$$

Let

$$b = \frac{(n)(n+1)}{2} \tag{37}$$

$$Av_1 = \begin{pmatrix} b \\ n^2+b \\ 2n^2+b \\ \cdots \\ (n-1)(n^2)+b \end{pmatrix} + \begin{pmatrix} 0 \\ n^2 \\ 2n^2 \\ \cdots \\ (n-1)n^2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix} \tag{38}$$

$$= n^2(v_2 - v_1) + b(v_1) \tag{39}$$

$$= (b - n^2)v_1 + n^2(v_2) \tag{40}$$

$$Av_2 = \begin{pmatrix} \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} (n+i)i \\ \sum_{i=1}^{n} (2n+i)i \\ \vdots \\ \sum_{i=1}^{n} \left((n)(n-1)+i\right)i \end{pmatrix} \tag{41}$$

$$= v_1\left(\sum_{i=1}^{n} i^2\right) + \begin{pmatrix} 0 \\ \sum_{i=1}^{n} ni \\ \sum_{i=1}^{n} 2ni \\ \vdots \\ \sum_{i=1}^{n} (n-1)(n)i \end{pmatrix} \tag{42}$$

$$Av_2 = \sum_{i=1}^{n} (i^2) v_1 + nb(v_2 - v_1) \tag{43}$$

$$\sum_{i=1}^{n} (i^2) = \frac{2n^3 + 2n^2 + n}{6} = a \tag{44}$$

$$\Rightarrow Av_2 = av_1 + nb(v_2 - v_1) \tag{45}$$

$$= (a - nb)v_1 + nb(v_2) \tag{46}$$

Hint #2 Show A has rank 2

Suppose x is a column of A

$$\Rightarrow x = \begin{pmatrix} 0 + k \\ n + k \\ \vdots \\ (n-1)(n) + k \end{pmatrix} \tag{47}$$

$$= kv_1 + n \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \end{pmatrix} \tag{48}$$

$$= kv_1 + n(v_2 - v_1) \tag{49}$$

$$= (k - n)v_1 + nv_2 \tag{50}$$

Every column of A is in $\text{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\})$

$$\Rightarrow \text{Col}(A) \subseteq \text{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\}) \tag{51}$$

$$\Rightarrow \text{rank}(A) \leq \dim\left(\text{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\})\right) = 2 \tag{52}$$

$\text{rank}(A) \neq 0$ because $A \neq 0$.

Suppose $\text{rank}(A) = 1$.

It follows that every column of A could be expressed as a scalar multiple of one vector $z \in \mathsf{F}^n$.

Suppose x, y are district columns of A and $c \in F$

$$\Rightarrow x = (k_1 - n)v_1 + nv_1 \qquad y = (k_2 - n)v_1 + nv_2 \tag{53}$$

Suppose $cx = y$.

$$\Rightarrow c(k_1 - n)v_1 + cnv_2 = (k_2 - n)v_1 + nv_2 \tag{54}$$

$$\Rightarrow c(k_1 - n) = (k_2 - n) \text{ and } cn = n \tag{55}$$

$$\Rightarrow n(c - 1) = 0 \tag{56}$$

Case 1 $c = 1$

$$\Rightarrow x = y \lightning \text{ Contradiction! } x, y \text{ are distinct} \tag{57}$$

Case 2 $n = 0$

Impossible.

Therefore $\text{rank}(A) \neq 1$ and it follows that $\text{rank}(A) = 2$.

By the dimension theorem,

$$\text{rank}(\mathsf{L}_A) + \text{Nullity}(\mathsf{L}_A) = \dim(\mathsf{L}_A) \tag{58}$$
$$\dim(\mathsf{L}_A) = n \tag{59}$$
$$\text{rank}(\mathsf{L}_A) = \text{rank}(A) = 2 \tag{60}$$
$$\Rightarrow \text{Nullity}(\mathsf{L}_A) = n - 2 \tag{61}$$
$$\Rightarrow \text{Nullity}(A) = n - 2 \tag{62}$$

Let the eigenspace of eigenvalue zero be E_0

$$E_0 = \text{N}(A = 0 \cdot I_n) = \text{N}(A) \tag{63}$$
$$\Rightarrow \dim E_0 = \text{Nullity}(A) = n - 2 \tag{64}$$

Let m be the algebraic multiplicity of eigenvalue zero.

$$m \geq \dim E_0 = n - 2 \text{ by theorem 5.7} \tag{65}$$

Suppose $\mathsf{W} = \text{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\})$

Suppose $\alpha = \{(1, 1, \ldots, 1), (1, 2, \ldots, n)\}$

W is L_A-invariant. It suffices to calculate the characteristic polynomial of the restriction of L_A to W because its characteristic polynomial divides the characteristic polynomial of L_A by theorem 5.21. L_A and A posses the same characteristic polynomial because $[\mathsf{L}_A]_\gamma = A$ for some ordered basis γ of F^n.

$$Av_1 = (b - n^2)v_1 + (n^2)v_2 \tag{66}$$
$$Av_2 = (a - nb)v_1 + (nb)v_2 \tag{67}$$

$$\Rightarrow [\mathsf{L}_{A_\mathsf{W}}]_\alpha = \begin{pmatrix} b - n^2 & a - nb \\ n^2 & nb \end{pmatrix} \tag{68}$$

$$\Rightarrow \det\left([\mathsf{L}_{A_\mathsf{W}}]_\alpha - tI_2\right) = \det\begin{pmatrix} b - n^2 - t & a - nb \\ n^2 & nb - t \end{pmatrix} \tag{69}$$

$$\Rightarrow t = \frac{1}{2}\left(\frac{n^3 + n}{2} \pm \frac{n}{2\sqrt{2}}\sqrt{3n^4 + 4n^3 + 6n^2 - 4n + 3}\right) \tag{70}$$

It follows that the characteristic polynomial of $\mathsf{L}_{A_\mathsf{W}}$ is

$$f_{\mathsf{L}_{A_\mathsf{W}}}(t) = (t_1 - t)(t_2 - t) \tag{71}$$

$$f_{\mathsf{L}_{A_\mathsf{W}}}(t)|f(t) \tag{72}$$
$$\Rightarrow f(t) = g(t)f_{\mathsf{L}_{A_\mathsf{W}}}(t) = g(t)(t_1 - t)(t_2 - t) \tag{73}$$

Since the degree of $g(t)f_{\mathsf{L}_{A_\mathsf{W}}}(t)$ is 2, the degree of $g(t)$ is $n-2$ It follows that the algebraic multiplicity $m = n - 2$.

$$\Rightarrow g(t) = (-t)^{n-2} \tag{74}$$

$$\Rightarrow f(t) = (t_1 - t)(t_2 - t)(-t)^{n-2} \tag{75}$$
$$= (-1)^n(t)^{n-2}(t_1 - t)(t_2 - t) \tag{76}$$

6.1

3. In $\mathsf{C}([0,1])$, let $f(t)=t$ and $g(t)=e^t$. Compute $\langle f,g\rangle$ (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy-Schwartz inequality and the triangle inequality.

$$\langle f,g\rangle := \int_0^1 f(t)g(t)\,\mathrm{d}t \tag{77}$$

$$\langle f,g\rangle = \int_0^1 te^t\,\mathrm{d}t = 1 \tag{78}$$

$$\|f\| = \sqrt{\langle f,f\rangle} = \int_0^1 t^2\,\mathrm{d}t = \frac{1}{\sqrt{3}} \tag{79}$$

$$\|g\| = \sqrt{\langle g,g\rangle} = \int_0^1 e^{2t}\,\mathrm{d}t = \sqrt{\frac{e^2-1}{2}} \tag{80}$$

$$\|f+g\| = \sqrt{\langle f+g,f+g\rangle} \tag{81}$$

$$\begin{aligned}\langle f+g,f+g\rangle &= \langle f+g,f\rangle + \langle f+g,g\rangle &(82)\\ &= \langle f,f\rangle + \langle g,f\rangle + \langle f,g\rangle + \langle g,g\rangle &(83)\\ &= \frac{3e^2+11}{6} &(84)\end{aligned}$$

$$\Rightarrow \|f+g\| = \sqrt{\frac{3e^2+11}{6}} \tag{85}$$

Cauchy-Schwartz Inequality

$$\|f\|\,\|g\| \overset{?}{\geq} \langle f,g\rangle \tag{86}$$

$$\sqrt{\frac{3e^2+11}{6}} \overset{?}{\geq} 1 \tag{87}$$

$$\Rightarrow e^2 \geq 7 \checkmark \tag{88}$$

Triangle Inequality

$$\|f\| + \|g\| \overset{?}{\geq} \|f+g\| \tag{89}$$

$$\frac{1}{\sqrt{3}} + \sqrt{\frac{e^2-1}{6}} \overset{?}{\geq} \sqrt{\frac{3e^2+11}{6}} \tag{90}$$

$$\Rightarrow \sqrt{\frac{e^2-1}{6}} \geq 1 \checkmark \tag{91}$$

8. Provide reasons why each of the following is not an inner product on the given vector spaces.

   (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on R^2.
   Suppose $(a,b) = (c,d) = (1,1)$

   $$\langle (1,1),(1,1)\rangle = 1 - 1 = 0. \tag{92}$$

   This is not an inner product because it fails Property D of an inner product.

   (b) $\langle A, B\rangle = \mathrm{tr}(A+B)$ on $\mathsf{M}_{n\times n}(\mathbb{R})$.
   Suppose $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

   $$\langle A, A\rangle = \mathrm{tr}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 0 \tag{93}$$

   This is not an inner product because it fails Property D of an inner product.

   (c) $\langle f(x), g(x)\rangle = \int\limits_0^1 f'(t)g'(t)\ \mathrm{d}t$ on $\mathsf{P}(\mathbb{R})$, where $'$ denotes differentiation.
   Suppose $f(x) \equiv 1$

   $$\langle f(x), f(x)\rangle = \int\limits_0^1 0 \cdot 1\ \mathrm{d}t = 0 \tag{94}$$

   This is not an inner product because it fails Property D of an inner product.

12. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \, \|v_i\|^2$$

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{i=1}^{k} a_i v_i \right\rangle \tag{95}$$

$$= \left\langle a_1 v_1, \sum_{i=1}^{k} a_i v_i \right\rangle + \left\langle a_2 v_2, \sum_{i=1}^{k} a_i v_i \right\rangle + \cdots + \left\langle a_k v_k, \sum_{i=1}^{k} a_i v_i \right\rangle \tag{96}$$

Fix some j $(1 \le j \le k)$

$$\langle a_j v_j, a_m v_m\rangle = 0\ \forall m\ (1 \le m \le k),\ m \ne j \because A \text{ is orthogonal} \tag{97}$$

$$\Rightarrow \left\langle a_j v_j, \sum_{i=1}^{k} a_i v_i \right\rangle = \langle a_j v_j, a_j v_j\rangle \tag{98}$$

$$= \|a_j v_j\|^2\ \forall j\ (1 \le j \le m) \tag{99}$$

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} \|a_i v_i\|^2 \tag{100}$$

$$= \sum_{i=1}^{k} a_i^2 \langle v_i, v_i \rangle \tag{101}$$

$$= \sum_{i=1}^{k} a_i^2 \|v_i\|^2 \tag{102}$$

$$= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2 \tag{103}$$

$$\tag{104}$$

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

    Suppose $x, y \in \mathsf{V}$

    Suppose $\mathsf{T}(x) = \mathsf{T}(y)$

$$\Rightarrow \|\mathsf{T}(x)\| = \|\mathsf{T}(y)\| \tag{105}$$

$$\Rightarrow \|x\| = \|y\| \tag{106}$$

$$\Rightarrow \sqrt{\langle x, x \rangle} = \sqrt{\langle y, y \rangle} \tag{107}$$

$$\Rightarrow \langle x, x \rangle = \langle y, y \rangle \tag{108}$$

$$\Rightarrow \langle x - y, x - y \rangle = 0 \tag{109}$$

$$\Rightarrow x - y = 0 \tag{110}$$

$$\Rightarrow x = y \tag{111}$$

6.2

2. In each part, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for span(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for span(S), and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 6.5 to verify your results.

   (a) $\mathsf{V} = \mathsf{R}^3, S = \{(1,0,1),(0,1,1),(1,3,3)\}$, and $x = (1,1,2)$

$$w_1 = (1,0,1) \qquad w_2 = (0,1,1) \qquad w_3 = (1,3,3) \tag{112}$$

$$v_1 = (1,0,1) \tag{113}$$

$$v_2 = (0,1,1) - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \tag{114}$$

$$v_3 = (1,3,3) - \left(\frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \tag{115}$$

An orthogonal basis is

$$\left\{(1,3,3), \left(-\frac{1}{2}, 1, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$$

The corresponding orthonormal basis is

$$\beta = \left\{\left(\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\} \tag{116}$$

$$\langle x, v_1\rangle = \frac{3}{\sqrt{2}} \qquad \langle x, v_2\rangle = \sqrt{\frac{3}{2}} \qquad \langle x, v_3\rangle = 0 \tag{117}$$

The Fourier coefficients relative to β are

$$\left\{\frac{3}{\sqrt{2}}, \sqrt{\frac{3}{3}}, 0\right\}$$

$$\left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\sqrt{\frac{3}{2}}\right)\left(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right) + (0)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = (1,1,2) \checkmark \tag{118}$$

   (c) $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\ \mathrm{d}t$, $S = \{1, x, x^2\}$, and $h(x) = 1 + x$

$$w_1 = 1 \qquad w_2 = x \qquad w_3 = x^2 \tag{119}$$

$$v_1 = w_1 = 1 \tag{120}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = x - \frac{1}{2} \tag{121}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2\right) = x^2 - x + \frac{1}{6} \tag{122}$$

An orthogonal basis is

$$\left\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\right\}$$

The corresponding orthonormal basis is

$$\beta = \left\{1, \sqrt{3}(2x-1), \sqrt{3}(6x^2 - 6x + 1)\right\} \tag{123}$$

$$\langle h(x), 1\rangle = \frac{3}{2} \tag{124}$$

$$\langle h(x), \sqrt{3}(2x-1)\rangle = \frac{\sqrt{3}}{6} \tag{125}$$

$$\langle h(x), \sqrt{3}(6x^2 - 6x + 1)\rangle = 0 \tag{126}$$

The Fourier coefficients relative to β are

$$\left\{\frac{3}{2}, \frac{\sqrt{3}}{6}, 0\right\}$$

$$\left(\frac{3}{2}\right)(1) + \left(\frac{\sqrt{3}}{6}\right)(\sqrt{3})(2x-1) + \frac{3}{2} + \left(\frac{3}{6}\right)(2x-1) = x + 1 \checkmark \tag{127}$$

(g) $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$, $S = \left\{\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}\right\}$, and $A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$

$$w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \tag{128}$$

$$v_1 = w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \tag{129}$$

$$v_2 = w_2 \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \tag{130}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2\right) = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \tag{131}$$

An orthogonal basis is

$$\left\{\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}\right\}$$

The corresponding orthonormal basis is

$$\beta = \left\{ \begin{pmatrix} {}^{1}/_{2} & {}^{5}/_{6} \\ {}^{-1}/_{6} & {}^{1}/_{6} \end{pmatrix}, \begin{pmatrix} {}^{-\sqrt{2}}/_{3} & {}^{\sqrt{2}}/_{3} \\ {}^{1}/_{\sqrt{2}} & {}^{-1}/_{3\sqrt{2}} \end{pmatrix}, \begin{pmatrix} {}^{1}/_{\sqrt{2}} & {}^{-1}/_{3\sqrt{2}} \\ {}^{\sqrt{2}}/_{3} & {}^{-\sqrt{2}}/_{3} \end{pmatrix} \right\} \tag{132}$$

The Fourier coefficients relative to β are

$$\left\{24, 6\sqrt{2}, -9\sqrt{2}\right\}$$

$$24 \begin{pmatrix} {}^{1}/_{2} & {}^{5}/_{6} \\ {}^{-1}/_{6} & {}^{1}/_{6} \end{pmatrix} + 6\sqrt{2} \begin{pmatrix} {}^{-\sqrt{2}}/_{3} & {}^{\sqrt{2}}/_{3} \\ {}^{1}/_{\sqrt{2}} & {}^{-1}/_{3\sqrt{2}} \end{pmatrix} - 9\sqrt{2} \begin{pmatrix} {}^{1}/_{\sqrt{2}} & {}^{-1}/_{3\sqrt{2}} \\ {}^{\sqrt{2}}/_{3} & {}^{-\sqrt{2}}/_{3} \end{pmatrix} = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix} \checkmark \tag{133}$$

(j) $\mathsf{V} = \mathsf{C}^4$, $S = \{(1, i, 2-i, -1), (2+3i, 2i, 1-i, 2i), (-1+7i, 6+10i, 11-4i, 3+4i)\}$ and $x = (-2+7i, 6+9i, 9-3i, 4+4i)$

$$w_1 = \begin{pmatrix} 1 \\ i \\ 2-i \\ -1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 2+3i \\ 2i \\ 1-i \\ 2i \end{pmatrix} \qquad w_3 = \begin{pmatrix} -2+7i \\ 6+9i \\ 9-3i \\ 4+4i \end{pmatrix} \tag{134}$$

$$v_1 = w_1 = \begin{pmatrix} 1 \\ i \\ 2-i \\ -1 \end{pmatrix} \tag{135}$$

$$v_2 = w_2 \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 1+3i \\ 2i \\ -1 \\ 1+2i \end{pmatrix} \tag{136}$$

$$v_3 = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right) = \begin{pmatrix} -7+i \\ 6+2i \\ 5 \\ 5 \end{pmatrix} \tag{137}$$

An orthogonal basis is

$$\left\{ \begin{pmatrix} 1 \\ i \\ 2-i \\ -1 \end{pmatrix}, \begin{pmatrix} 1+3i \\ 2i \\ -1 \\ 1+2i \end{pmatrix}, \begin{pmatrix} -7+i \\ 6+2i \\ 5 \\ 5 \end{pmatrix} \right\}$$

The corresponding orthonormal basis is

$$\beta = \left\{ \frac{1}{\sqrt{8}} \begin{pmatrix} 1 \\ i \\ 2-i \\ -1 \end{pmatrix}, \frac{1}{2\sqrt{5}} \begin{pmatrix} 1+3i \\ 2i \\ -1 \\ 1+2i \end{pmatrix}, \frac{1}{2\sqrt{35}} \begin{pmatrix} -7+i \\ 6+2i \\ 5 \\ 5 \end{pmatrix} \right\} \tag{138}$$

The Fourier coefficients are

$$\left\{6\sqrt{2}, 4\sqrt{5}, 2\sqrt{35}\right\}$$

$$\frac{6\sqrt{2}}{\sqrt{8}}\begin{pmatrix}1\\ i\\ 2-i\\ -1\end{pmatrix}+\frac{4\sqrt{5}}{2\sqrt{5}}\begin{pmatrix}1+3i\\ 2i\\ -1\\ 1+2i\end{pmatrix}+\frac{2\sqrt{35}}{2\sqrt{35}}\begin{pmatrix}-7+i\\ 6+2i\\ 5\\ 5\end{pmatrix}=\begin{pmatrix}-2+7i\\ 6+9i\\ 9-3i\\ 4+4i\end{pmatrix}\ \checkmark \tag{139}$$

7. Let β be a basis for a subspace W of an inner product space V, and let $z \in \mathsf{V}$. Prove that $z \in \mathsf{W}^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

   Claim: $x \in \mathsf{W}^{\perp} \Leftrightarrow \langle z, v\rangle = 0 \quad \forall v \in \beta$

   $(\Rightarrow)$

   Suppose $z \in \mathsf{W}^{\perp}$

$$\Rightarrow \langle z, x\rangle = 0 \quad \forall x \in S \tag{140}$$
$$\beta \subseteq S \tag{141}$$
$$\Rightarrow \langle z, y\rangle = 0 \quad \forall y \in \beta \tag{142}$$

   $(\Leftarrow)$

   Suppose $\langle z, v\rangle = 0 \quad \forall v \in \beta$

   Suppose $w \in \mathsf{W}$

$$\operatorname{span}(\beta) = \mathsf{W} \tag{143}$$
$$\Rightarrow w = a_1 v_1 + a_1 v_2 + \cdots + a_n v_n \tag{144}$$

$$\langle z, w\rangle = \left\langle z, \sum_{i=1}^{n} a_i v_i \right\rangle \tag{145}$$
$$= \sum_{i=1}^{n} a_i \langle z, v_i\rangle \tag{146}$$
$$= \sum_{i=1}^{n} a_i \cdot 0 \tag{147}$$
$$= 0 \tag{148}$$

$$\Rightarrow \langle v, w\rangle = 0 \quad \forall w \in \mathsf{W} \tag{149}$$
$$\Rightarrow z \in \mathsf{W}^{\perp} \tag{150}$$

15. Let V be a finite-dimensional inner product space over F.

(a) *Pasrseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in \mathsf{V}$ prov that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$$

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle\cdot,\cdot\rangle$, then for any $x, y \in \mathsf{V}$

$$\langle \phi_\beta(x), \phi_\beta(y)\rangle' = \langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle\,,$$

where $\langle\cdot,\cdot\rangle'$ is the standard inner product on F^n.

(a) Suppose $x, y \in \mathsf{V}$

$$\Rightarrow x = \sum_{i=1}^{n} a_i v_i \qquad\qquad y = \sum_{i=1}^{n} b_i v_i \tag{151}$$

Suppose β is an orthonormal basis.

$$\Rightarrow \langle v_i, v_j\rangle\ \forall v_i, v_j \in \beta \text{ such that } i \neq j\ (1 \leq i, j \leq n) \tag{152}$$

Fix some k $(1 \leq k \leq n)$

$$\langle x, v_k\rangle = \left\langle \sum_{i=1}^{n} a_i v_i, v_k \right\rangle \tag{153}$$

$$= \sum_{i=1}^{n} a_i \langle v_i, v_k\rangle \tag{154}$$

$$= a_k \langle v_k, v_k\rangle \tag{155}$$

$$= a_k \left\|v_k\right\|^2 \tag{156}$$

$$= a_k \tag{157}$$

$$\Rightarrow \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle} = \sum_{i=1}^{n} a_i \overline{b_i} = \langle x, y\rangle \tag{158}$$

(b) Suppose ϕ_β is the coordinate isomorphism from $\mathsf{V} \to \mathsf{F}^n$

$$\Rightarrow \phi_\beta(x), \phi_\beta(y) \in \mathsf{F}^n \tag{159}$$

$$\Rightarrow \phi_\beta(x) = [x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \phi_\beta(y) = [y]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{160}$$

$$\Rightarrow \langle [x]_\beta, [y]_\beta\rangle' = \sum_{i=1}^{n} a_i \overline{b_i} = \langle x, y\rangle \tag{161}$$

$$\therefore \langle \phi_\beta(x), \phi_\beta(y)\rangle' = \langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle \tag{162}$$