Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

Work

1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

   (b)
$$\left\{\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}, \begin{pmatrix}-2 & 6\\ 4 & -8\end{pmatrix}\right\} \text{in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

   (d)
$$\left\{x^3 -, 2x^2+4, -2x^3+3x^2+2x+6\right\} \text{ in } \mathsf{P}_3\left(\mathbb{R}\right)$$

   (f)
$$\left\{\left(1,-1,2\right), \left(1,-2,1\right), \left(1,1,4\right)\right\} \text{ in } \mathsf{R}^3$$

   (g)
$$\left\{\begin{pmatrix}1 & 0\\ -2 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\ -4 & 4\end{pmatrix}\right\} \text{ in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

11. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in span(S)? Justify your answer.

$$\mathbb{Z}_2 = \{0,1\} \tag{1}$$
$$c_1u_1 + c_2u_2 + \cdots + c_nu_n \neq 0,\ \forall c_1, \ldots, c_n \in \{0,1\} \tag{2}$$

unless all $c_i = 0$.

$$\begin{aligned}\implies \text{card}\left(\text{span}\left(S\right)\right) &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} && (3)\\ &= \sum_{i=1}^{n}\binom{n}{i} && (4)\\ &= 2^n && (5)\end{aligned}$$

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for some $k\,(1 \leq k < n)$.

Forward Direction:
Claim: S is linearly dependent
Suppose $u_1 = 0$
Assume S is linearly independent.
Take the linear combination:

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0 \text{ such that } c_1 \neq 0, c_2, \ldots, c_n = 0 \tag{6}$$

$$\implies c_1u_1 = 0 \lightning \text{ Contradiction!} \tag{7}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \ldots, u_k\})$
Assume S is linearly independent.

$$u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k \tag{8}$$

Take the linearly combination:

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k + c_{k+1} + \cdots + c_nu_n = 0 \tag{9}$$

Choose $c_i = (-a_i)$ for $i(1 \leq i \leq k)$ and $c_{k+1} = 1$

$$\implies u_{k+1} = 0 \lightning \text{ Contradiction!} \tag{10}$$

There exists a non-trivial representation of the zero vector therefore S is linearly dependent.

Reverse Direction:
Suppose $u_{k+1} \in \operatorname{span}(\{u_1, \ldots, u_k\})$
Claim: S is linearly dependent

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k && (11)\\ -u_{k+1} &= (-1)\,(a_1u_1 + a_2u_2 + \cdots + a_ku_k) && (12)\\ &= (-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k && (13) \end{aligned}$$

Take the linear combination of all $u_1, \ldots, u_n$

$$((-a_1)\,u_1 + (a_2)\,u_2 + \cdots + (-a_k)\,u_k) + (1u_{k+1} + 0u_{k+2} + \cdots + 0u_n) = 0 \tag{14}$$

1.6

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

    (a) Prove that there is subset of S that is a basis for V. (Be careful not to assume that S is finite.)

    (b) Prove that S contains at least n vectors.

24. Let $f(x)$ be a polynomial of degree n in $\mathsf{P}_n(\mathbb{R})$. Prove that for any $g(x) \in \mathsf{P}_n(\mathbb{R})$ there exist scalars $c_0, c_1, \dots, c_n$ such that

    $$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

    where $f^{(n)}(x)$ denotes nth derivative of $f(x)$.

    $$\mathsf{P}_n\left(\mathbb{R}\right) = \left\{k \in \mathsf{P}\left(\mathbb{R}\right) : \deg\left(k\right) \le n\right\} \tag{15}$$

    Suppose $g = c_n x_n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_o,\ c_o, \dots, c_n \in \mathbb{R}$[1]

    $$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 && (16)\\ f'(x) &= n a_n x^{n-1} + \left(n-1\right) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1 && (17)\\ f''(x) &= \frac{n!}{(n-1)!}x^{n-2} + \cdots + 4\cdot 3a_3x^2 + 3\cdot 2a_3 x + 2a_2 && (18)\\ &\vdots && (19)\\ f^{(n)}(x) &= \dots && (20) \end{aligned}$$

    $$\begin{pmatrix} a_n & a_{n-1} & \cdots & \cdots & a_1 & a_0 \\ 0 & \frac{n}{(n-1)!}a_n & \cdots & \cdots & \vdots & a_1 \\ \vdots & 0 & \ddots & \vdots & \vdots & 2a_2 \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{n!}{(n-(n-1))!}a_n & (n-1)!a_{n-1} \\ 0 & 0 & 0 & 0 & 0 & n!a_n \end{pmatrix} \begin{pmatrix} x^n \\ x^{n-1} \\ \vdots \\ \vdots \\ x \\ x^0 \end{pmatrix} = \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} \tag{21}$$

    [2]

31. Let W_1 and W_2 are subspaces of V, and find the dimensions of $\mathsf{W}_1, \mathsf{W}_2, \mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

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[1] what the fuck is g here for?
[2] matrix is fucked up

2.1

6. $\mathsf{T}\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \operatorname{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ij}$$

12. Is there a linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1,0,3) = (1,1)$ and $\mathsf{T}(-2,0,-6) = (2,1)$?

14. Let V and W be vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear.

    (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

    (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

    (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

2.2

2. Let β and γ be the standard ordered bases for R^n and R^m respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{R}]_\beta^\gamma$.

    (b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

    (c) $\mathsf{T}\colon \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

    (g) $\mathsf{T}\colon \mathsf{R}^n \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

11. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the exercises of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.