

## Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

## Work

### 3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$\begin{aligned}x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\x_2 + 2x_4 &= 3\end{aligned}$$

$$\left(\begin{array}{cccc|c}1 & 2 & -1 & 3 & 2 \\2 & 4 & -1 & 6 & 5 \\0 & 1 & 0 & 2 & 3\end{array}\right) \begin{array}{c} \boxed{-2} \leftarrow + \quad \boxed{+} \leftarrow \\ \boxed{+} \leftarrow \quad \boxed{1} \leftarrow \\ \boxed{-2} \leftarrow \end{array} \rightsquigarrow \left(\begin{array}{cccc|c}1 & 0 & 9 & -4 & -3 \\0 & 1 & 0 & 2 & 3 \\0 & 0 & 1 & 0 & 1\end{array}\right) \quad (1)$$

$$x_1 = -3 + 4x_4 \quad (2)$$

$$x_2 = 3 - 2x_4 \quad (3)$$

$$x_3 = 1 \quad (4)$$

$$x_4 = x_4 \quad (5)$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \quad (6)$$

(j)

$$2x_1 + 3x_3 - 4x_5 = 5 \quad (7)$$

$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8 \quad (8)$$

$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2 \quad (9)$$

$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8 \quad (10)$$

$$\left(\begin{array}{ccccc|c}2 & 0 & 3 & 0 & -4 & 5 \\3 & -4 & 8 & 3 & 0 & 8 \\1 & -1 & 2 & 1 & -1 & 2 \\-2 & 5 & -9 & -3 & -5 & -8\end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c}1 & 0 & 0 & 0 & -2 & 1 \\0 & 1 & 0 & 0 & -3 & 0 \\0 & 0 & 1 & 0 & 0 & -1 \\0 & 0 & 0 & 1 & -2 & -1\end{array}\right) \quad (11)$$

$$x_1 = 2x_5 + 1 \quad (12)$$

$$x_2 = 3x_5 \quad (13)$$

$$x_3 = -1 \quad (14)$$

$$x_4 = 2x_5 - 1 \quad (15)$$

$$x_5 = x_5 \quad (16)$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \quad (17)$$

8. Let  $W$  denote the subspace of  $\mathbb{R}^5$  consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate  $W$ . Find a subset  $\{u_1, u_2, \dots, u_8\}$  that is a basis for  $W$ .

$$\mathbb{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \dots, x_5 \in \mathbb{R} \right\} \quad (18)$$

$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \quad (19)$$

$$\rightsquigarrow \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ & & 1 & -2 & 0 & -2 & 0 & 1 \\ & & & & 1 & -4 & 0 & -2 \\ & 0 & & & & & 1 & -1 \\ & & & & & & & 0 \end{pmatrix} \quad (20)$$

It follows that  $\{u_1, u_3, u_5, u_7\}$  is linearly independent by theorem 3.16. Therefore  $\{u_1, u_3, u_5, u_7\}$  is a basis for  $W$ .

11. Let  $V$  be as in Exercise 10.

(a) Show that  $S = \{(1, 2, 1, 0, 0)\}$  is a linearly independent subset of  $V$ .

(b) Extend  $S$  to a basis for  $V$ .

(a) Claim:  $S$  is a linearly independent subset of  $V$ .

For  $x \in S, x = (1, 2, 1, 0, 0)$

$$1 + (-2)(2) + 3(1) + (-1)(0) + (2)(0) = 0 \quad (21)$$

$$\Rightarrow x \in V \quad (22)$$

$$\Rightarrow S \subseteq V \quad (23)$$

Suppose  $cx = 0$

$$c(1, 2, 1, 0, 0) = 0 \quad (24)$$

$$\Rightarrow (c, 2c, c, 0, 0) = 0 \quad (25)$$

$$\Rightarrow c = 0 \quad (26)$$

It follows that  $S$  is linearly independent.

(b) Suppose  $x_1, x_2, \dots, x_5 \in F: x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 + x_4 - 2x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (27)$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (28)$$

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (29)$$

Where  $\beta$  is a basis  $V$ .

$$(S|B) = \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (30)$$

$$S_{\beta}^{-1} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (31)$$

## 14. John's email

15. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Suppose  $A \in M_{m \times n}(F)$  such that  $\text{rank}(A) = r \leq \min\{m, n\}$

Proof by induction

Suppose  $n = 1$

### Case 1

$$A = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (32)$$

$A$  is in reduced row echelon form and this the unique representation.

### Case 2

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \text{for some } a_{i1} \neq 0 \quad (33)$$

Execute the following sequence of row operations on  $A$ . Perform type 1 row operation to move  $a_{i1}$  to the first row. Perform type 3 row operations to eliminate all lower terms. Perform type 2 row operation to change the first term in the column to 1.

$$A \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (34)$$

The reduced row echelon form of a column must be of this form. It follows that this matrix is the unique row reduced echelon form.

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<sup>1</sup>We need say what exactly this is

Suppose true for  $1 \leq n \neq k$

Suppose  $n = k + 1$

COME BACK TO THIS

## 4.1

10. The **classical adjoint** of a  $2 \times 2$  matrix  $A \in M_{2 \times 2}(F)$  is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

- (a)  $CA = AC = [\det(A)] I$
- (b)  $\det(C) = \det(A)$
- (c) The classical adjoint of  $A^t$  is  $C^t$
- (d) If  $A$  is invertible, then  $A^{-1} = [\det(A)]^{-1} C$
- (a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{35}$$

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{36}$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{39}$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \tag{41}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \tag{42}$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \tag{43}$$

$$\Rightarrow AC = CA = \det(A)I_2 \tag{44}$$

$$(b) \det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \quad (45)$$

$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \quad (46)$$

It follows that  $D$  is the classical adjoint of  $A^t$

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \quad (47)$$

It follows that  $C^t$  is the classical adjoint of  $A^t$

(d) Suppose  $A$  is invertible

$$\Rightarrow \exists B: AB = BA = I \quad (48)$$

$$\Rightarrow \det(A) \neq 0 \quad (49)$$

$$CA = AC = \det(A)I \quad (50)$$

$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \quad (51)$$

$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \quad (52)$$

## 4.2

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Suppose  $A \in \mathbf{M}_{n \times n}((F))$  and  $A$  is upper triangular.

Proof by induction.

For  $n = 1$ :  $A \in \mathbf{M}_{1 \times 1}(F) \Rightarrow \det(A) = A_{11}$  by definition.

Suppose  $\det(A)$  is the product of its diagonal entries for  $1 \leq n \leq k$

For  $n = k + 1$ :  $A \in \mathbf{M}_{k+1 \times k+1}((F))$

$$\det(A) = \sum_{j=1}^{k+1} (-1)^{k+1+j} A_{k+1,j} \det(\tilde{A}_{k+1,j}) \quad (53)$$

$$A_{n+1,j} = 0 \quad \text{for } 1 \leq j \leq n \quad (54)$$

$$\Rightarrow \det A = (-1)^{(k+1)+(k+1)} A_{k+1,k+1} \det(\tilde{A}_{k+1,k+1}) \quad (55)$$

$$\tilde{A}_{k+1,k+1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k+1} \\ & a_{22} & & a_{2,k+1} \\ & & \ddots & \vdots \\ 0 & & & a_{k+1,k+1} \end{pmatrix} \quad (56)$$

$\tilde{A}_{k+1,k+1} \in \mathbf{M}_{n \times n}(F) \Rightarrow \det(\tilde{A}_{n+1,n+1})$  is a product of diagonal entries by induction hypothesis.

$$(-1)^{(n+1)+(n+1)} = (-1)^{2n+2} = (-1)^2(-1)^{2n} = (-1)^{2n} \quad (57)$$

$$n \in \mathbb{Z}^+ \Rightarrow 2n \text{ is even} \quad (58)$$

$$\Rightarrow (-1)^{2n} = 1 \quad (59)$$

$$\Rightarrow \det(A)A_{n+1,n+1} \prod_{j=1}^n A_{jj} = \prod_{j=1}^{n+1} A_{jj} \quad (60)$$

29. Prove that if  $E$  is an elementary matrix, then  $\det(E^t) = \det(E)$ .

(a) **Types 1 & 2**

$$E^t = E \quad (\text{by HW.3.1.5}) \quad (61)$$

$$\Rightarrow \det(E^t) = \det(E) \quad (62)$$

(b) **Type 3**

$E^t$  is an type 3 elementary matrix (by HW.3.1.5)  $\det(E) = \det(I) = 1$  for any type elementary operation on  $I_n$

$$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \quad (63)$$

$$\Rightarrow \det(E) = \det(E^t) = 1 \quad (64)$$

30. Let the rows of  $A \in \mathbf{M}_{n \times n}(F)$  be  $a_1, a_2, \dots, a_n$  and let  $B$  be the matrix in which the rows are  $a_n, a_{n-1}, \dots, a_1$ . Calculate  $\det(B)$  in terms of  $\det(A)$ .

(a)  **$n$  is even**

In  $A$ , swap

$$a_{n-1} \text{ with } a_1 \quad (65)$$

$$a_{n-2} \text{ with } a_2 \quad (66)$$

$\vdots$

$$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \quad (67)$$

From the fact that  $\frac{n}{2}$  swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n}{2}} \det(A) \quad (68)$$

(b) **n is odd** In  $A$ , swap

$$a_{n-1} \text{ with } a_1 \quad (69)$$

$$a_{n-2} \text{ with } a_2 \quad (70)$$

$\vdots$

$$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \quad (71)$$

From the fact that  $n - \frac{n+1}{2}$  swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n-1}{2}} \det(A) \quad (72)$$

### 4.3

10. The **classical adjoint** of a  $2 \times 2$  matrix  $A \in \mathbf{M}_{2 \times 2}(F)$  is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a)  $CA = AC = [\det(A)] I$

(b)  $\det(C) = \det(A)$

(c) The classical adjoint of  $A^t$  is  $C^t$

(d) If  $A$  is invertible, then  $A^{-1} = [\det(A)]^{-1} C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (73)$$

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \quad (74)$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \quad (75)$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \quad (76)$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \quad (77)$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (78)$$

$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \quad (79)$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \quad (80)$$



$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \quad (81)$$

$$\Rightarrow AC = CA = \det(A)I_2 \quad (82)$$

$$(b) \det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \quad (83)$$

$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \quad (84)$$

It follows that  $D$  is the classical adjoint of  $A^t$

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \quad (85)$$

It follows that  $C^t$  is the classical adjoint of  $A^t$

(d) Suppose  $A$  is invertible

$$\Rightarrow \exists B: AB = BA = I \quad (86)$$

$$\Rightarrow \det(A) \neq 0 \quad (87)$$

$$CA = AC = \det(A)I \quad (88)$$

$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \quad (89)$$

$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \quad (90)$$

12. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Suppose  $A \in \mathbf{M}_{m \times n}(F)$  such that  $\text{rank}(A) = r \leq \min\{m, n\}$

Proof by induction

Suppose  $n = 1$

**Case 1**

$$A = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (91)$$

$A$  is in reduced row echelon form and this the unique representation.

**Case 2**

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ A_{m1} \end{pmatrix} \quad \text{for some } a_{i1} \neq 0 \quad (92)$$

Execute the following sequence of row operations on  $A$ . Perform type 1 row operation to move  $a_{i1}$  to the first row. Perform type 3 row operations to eliminate all lower terms. Perform type 2 row operation to change the first term in the column to 1.

$$A \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (93)$$

The reduced row echelon form of a column must be of this form. It follows that this matrix is the unique row reduced echelon form.

Suppose true for  $1 \leq n \neq k$

Suppose  $n = k + 1$

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