1. Show that if p is a prime integer then $\mathbb{Z}_p$ is a field.

   Suppose $d \in \mathbb{N}$ and $p \in \mathbb{Z}_p$ then $\exists!\, q, r \in \mathbb{Z}$ and $0 \leq r \leq d-1$ such that $a = p \times q + r$.

2. Show that $\mathbb{Q}\left[\sqrt{2}\right]$ is a field assuming that $\mathbb{Q}$ and $\mathbb{R}$ are fields.

$$\mathbb{Q}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} \,\middle|\, a, b \in \mathbb{Q}\right\} \tag{1}$$

(F 1) $\forall\, a, b \in \mathbb{F}$, $a + b = b + a$ and $a \times b = b \times a$

For the addition statement:

$$\left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) \tag{2}$$

$$(a + c) + (b + d)\sqrt{2} \tag{3}$$

For the commuted addition:

$$\left(c + d\sqrt{2}\right) + \left(a + b\sqrt{2}\right) \tag{4}$$

$$(c + a) + (d + b)\sqrt{2} \tag{5}$$

Because $a, c, d, b \in \mathbb{Q}$, which is a field, expression 5 can be rewritten as:

$$(a + c) + (b + d)\sqrt{2} \tag{6}$$

which is the same as expression 3. Therefore $\mathbb{Q}\left[\sqrt{2}\right]$ satisfies the additive part of axiom (F 1) because $(a + c), (b + d) \in \mathbb{Q}$

For the multiplication statement from the axiom:

$$\left(a + b\sqrt{2}\right) \times \left(c + d\sqrt{2}\right) \tag{7}$$

$$(ac + 2bd) + (ad + bc)\sqrt{2} \tag{8}$$

For the commuted version:

$$\left(c + d\sqrt{2}\right) \times \left(a + b\sqrt{2}\right) \tag{9}$$

$$(ca + 2db) + (cb + da)\sqrt{2} \tag{10}$$

Because $c, a, d, b \in \mathbb{Q}$, which is a field, expression 10 can be rewritten as:

$$\left(a + b\sqrt{2}\right) \times \left(c + d\sqrt{2}\right) \tag{11}$$

which is the same as expression 7. Therefore $\mathbb{Q}\left[\sqrt{2}\right]$ satisfies axiom (F 1) because $(ac + 2bd), (ad + bc) \in \mathbb{Q}$

(F 2) $\forall\, a, b, c \in \mathbb{F}$, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$

For addition:

$$\left(a + b\sqrt{2}\right) + \left[\left(c + d\sqrt{2}\right) + \left(e + f\sqrt{2}\right)\right] \tag{12}$$

$$\left(a + b\sqrt{2}\right) + \left((c + e) + (d + f)\sqrt{2}\right) \tag{13}$$

$$(a + c + e) + (b + d + f)\sqrt{2} \tag{14}$$

$$\left[\left(ab\sqrt{2}\right) + \left(c + d\sqrt{2}\right)\right] + \left(e + f\sqrt{2}\right) \tag{15}$$

$$\left((a + c) + (b + d)\sqrt{2}\right) + \left(e + f\sqrt{2}\right) \tag{16}$$

$$(a + c + e) + (b + d + f)\sqrt{2} \tag{17}$$

The equality of expressions 14 and 17 prove the associativity of addition for the set $\mathbb{Q}\left[\sqrt{2}\right]$ as $(a + c + e), (b + d + f) \in \mathbb{Q}$

For multiplication:

$$\left(a + b\sqrt{2}\right)\left[\left(c + d\sqrt{2}\right)\left(e + \sqrt{2}\right)\right] \tag{18}$$

$$\left(a + b\sqrt{2}\right)\left(ce + cf\sqrt{2} + ed\sqrt{2} + 2df\right) \tag{19}$$

$$(ace + 2(adf + bcf + bed)) + (acf + bce + aed)\sqrt{2} \tag{20}$$

$$\left[\left(a + b\sqrt{2}\right)\left(c + d\sqrt{2}\right)\right]\left(e + f\sqrt{2}\right) \tag{21}$$

$$\left(ac + ad\sqrt{2} + bc\sqrt{2} + 2bd\right)\left(e + f\sqrt{2}\right) \tag{22}$$

$$(ace + 2\,(bde + adf + bcf)) + (acf + ade + bce)\,\sqrt{2} \tag{23}$$

Because $a, b, c, d, e, f \in \mathbb{Q}$ by the law of associativity (for $\mathbb{Q}$) expression 23 can be rewritten as:

$$(ace + 2\,(adf + bcf + bed)) + (acf + bce + aed)\,\sqrt{2} \tag{24}$$

which is the same as expression 20 therefore proving the associativity axiom for multiplication for $\mathbb{Q}\left[\sqrt{2}\right]$ as $(ace + 2\,(adf + bcf + bed))\,, (acf + bce + aed) \in \mathbb{Q}$

(F 3) $\exists\ 0 \in \mathbb{F}$ such that $0 + a = a + 0 = a,\ \forall a \in \mathbb{F}$ and $\exists\ 1 \in \mathbb{F}$ such that $1 \times a = a \times 1 = a,\ \forall, a \in \mathbb{F}$

For the additive identity:

$$\left(a + b\sqrt{2}\right) + \left(0 + 0\sqrt{2}\right) = a + b\sqrt{2} \tag{25}$$

Therefore $\left(0 + 0\sqrt{2}\,\right) \in \mathbb{Q}\left[\sqrt{2}\right]$ is the additive identity because $0 \in \mathbb{Q}$

For the multiplicative identity:

$$\left(a + b\sqrt{2}\right) + \left(1 + 0\sqrt{2}\right) = a + b\sqrt{2} \tag{26}$$

Therefore $\left(1 + 0\sqrt{2}\,\right) \in \mathbb{Q}\left[\sqrt{2}\right]$ is multiplicative identity because $0, 1 \in \mathbb{Q}$

(F 4) $\forall\ a \in \mathbb{F}\ \exists\ b \in \mathbb{F}$ such that $a + b = b + a = 0$ and $\forall\ a \in \mathbb{F}\ \exists\ b \in \mathbb{F}$ such that $a \times c = c \times a = 0$

For the additive inverse:

$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\,\sqrt{2}\right) = 0 \tag{27}$$

Therefore $\left((-a) + (-b)\,\sqrt{2}\,\right) \in \mathbb{Q}\left[\sqrt{2}\right]$ is the additive inverse of $\left(a + b\sqrt{2}\right)$ because $(-a), (-b) \in \mathbb{Q}$.

For the multiplicative inverse:

$$\left(a+b\sqrt{2}\right)\times\frac{1}{a+b\sqrt{2}} \tag{28}$$

$$\left(a+b\sqrt{2}\right)\times\frac{a-b\sqrt{2}}{a^2+b^2\sqrt{2}} \tag{29}$$

$$\left(a+b\sqrt{2}\right)\times\left[\frac{a}{a^2+2b^2}+\left(\frac{-b}{a^2+2b^2}\right)\sqrt{2}\right]=1 \tag{30}$$

Therefore $\left[\frac{a}{a^2+2b^2}+\left(\frac{-b}{a^2+2b^2}\right)\sqrt{2}\right]\in\mathbb{Q}\left[\sqrt{2}\right]$ is the multiplicative inverse because $\left(\frac{a}{a^2+2b^2}\right),\left(\frac{-b}{a^2+2b^2}\right)\in\mathbb{Q}$ and $\mathbb{Q}\left[\sqrt{2}\right]$ satisfies the axiom.

(F 5) $\forall\, a,b,c\in\mathbb{F},\ a\times(b+c)=a\times b+a\times c$

$$\left(a+b\sqrt{2}\right)\times\left[\left(c+d\sqrt{2}\right)+\left(e+f\sqrt{2}\right)\right] \tag{31}$$

$$ac+ad\sqrt{2}+ae+af\sqrt{2}+bc\sqrt{2}+2bd+eb\sqrt{2}+2bf \tag{32}$$

$$(ac+ae+2bd+2bf)+(ad+af+bc+eb)\sqrt{2} \tag{33}$$

$$\left(a+b\sqrt{2}\right)\left(c+d\sqrt{2}\right)+\left(a+b\sqrt{2}\right)\left(e+f\sqrt{2}\right) \tag{34}$$

$$ac+ad\sqrt{2}+bc\sqrt{2}+2bd+ae+af\sqrt{2}+be\sqrt{2}+2bf \tag{35}$$

$$(ac+ae+2bd+2bf)+(ad+af+bc+eb)\sqrt{2} \tag{36}$$

Therefore $\mathbb{Q}\left[\sqrt{2}\right]$ satisfies the axiom. Furthermore $\mathbb{Q}\left[\sqrt{2}\right]$ is a field.