Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

Work

3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\ x_2 + 2x_4 &= 3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 9 & -4 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \tag{1}$$

$$x_1 = -3 + 4x_4 \tag{2}$$
$$x_2 = 3 - 2x_4 \tag{3}$$
$$x_3 = 1 \tag{4}$$
$$x_4 = x_4 \tag{5}$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{6}$$

(j)

$$2x_1 + 3x_3 - 4x_5 = 5 \tag{7}$$
$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8 \tag{8}$$
$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2 \tag{9}$$
$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8 \tag{10}$$

$$\left(\begin{array}{ccccc|c} 2 & 0 & 3 & 0 & -4 & 5 \\ 3 & -4 & 8 & 3 & 0 & 8 \\ 1 & -1 & 2 & 1 & -1 & 2 \\ -2 & 5 & -9 & -3 & -5 & -8 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \tag{11}$$

$$x_1 = 2x_5 + 1 \tag{12}$$
$$x_2 = 3x_5 \tag{13}$$
$$x_3 = -1 \tag{14}$$
$$x_4 = 2x_5 - 1 \tag{15}$$
$$x_5 = x_5 \tag{16}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{17}$$

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate W. Find a subset $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

$$\mathsf{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \ldots, x_5 \in \mathbb{R} \right\} \tag{18}$$

$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \tag{19}$$

$$\rightsquigarrow \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ & & 1 & -2 & 0 & -2 & 0 & 1 \\ & & & & 1 & -4 & 0 & -2 \\ & & \text{\Huge 0} & & & & 1 & -1 \\ & & & & & & & 0 \end{pmatrix} \tag{20}$$

It follows that $\{u_1, u_3, u_5, u_7\}$ is linearly independent by theorem 3.16. Therefore $\{u_1, u_3, u_5, u_7\}$ is a basis for W.

11. Let V be as in Exercise 10.

(a) Show that $S = \{(1, 2, 1, 0, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

(a) Claim: S is a linearly independent subset of V.
For $x \in S, x = (1, 2, 1, 0, 0)$

$$\begin{aligned} 1 + (-2)(2) + 3(1) + (-1)(0) + (2)(0) &= 0 \qquad (21)\\ \Rightarrow x &\in \mathsf{V} \qquad (22)\\ \Rightarrow S &\subseteq \mathsf{V} \qquad (23)\end{aligned}$$

Suppose $cx - 0$

$$\begin{aligned} c(1, 2, 1, 0, 0) &= 0 \qquad (24)\\ \Rightarrow (c, 2c, c, 0, 0) &= 0 \qquad (25)\\ \Rightarrow c &= 0 \qquad (26)\end{aligned}$$

It follows that S is linearly independent.

(b) Suppose $x_1, x_2, \ldots, x_5 \in F\colon x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 + x_4 - 2x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad (27)$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (28)$$

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad (29)$$

Where β is a basis V.

$$(S\,|B) = \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (30)$$

$$S_\beta{}^1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{31}$$

Claim $(A\,|b)$ in row reduced echelon form ipmlies A in row reduced echelon form

Suppose $(A\,|b)$ is an element of $\mathsf{M}_{m\times(n+1)}(F)$ such that. $(A\,|b)$ is in row reduced echelon form. To complete this proof, it is necessary to show that A satisfies the definition of reduced row echelon form.

14. (a) Claim: any row of A containing a nonzero entry precedes any row in which all the entries are zero (if any)

    Case 1: The last column consists entirely of zeroes. $(A\,|b)$ is in row reduced echelon form, so removing the last column still satisfies this requirement

    Case 2: There is a nonzero value in the last column Because there is a nonzero entry, it is the first nonzero entry of its row. However, since it is in the last row position, all other values in the row are zero. By property C of row reduced echelon form, all subsequent rows (if any) must consist entirely of zeroes. Thus, the partition of row reduced echelon form of $(A\,|b)$ that removes the last row yields a matrix in which any row containing a nonzero entry precedes any row in which all the entries are zero (if any)

    (b) Claim: The first nonzero entry in each row is the only nonzero entry in its column.

    Removing the last column yields a matrix in which the first nonzero entry in each row is the only nonzero entry in its column.

    (c) Claim: The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

    Case 1: The last column consists entirely of zeroes. Thus, removing the last column from the row reduced echelon form of $(A\,|b)$ yields a matrix in which the first nonzero entry in each row is 1 nd it occurs to the righ tof the first nonzero entry in the preceding row.

    Case 2: There is a non zero value in the last column If there is a nonzero value in the last column, than that row consists entirely of zeroes otherwise and all subsequent rows (if any) consist entire of zeroes. Thus, removing this column from the row reduced echelon form of $(A\,|b)$ yields a matrix that satisfies this property.

[1] We need say what exaclty thsi is

15. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Suppose $A \in \mathsf{M}_{m\times n}(F)$ such that $\text{rank}(A) = r \le \min\{m, n\}$
Proof by induction
Suppose $n = 1$

Case 1

$$A = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{32}$$

A is in reduced row echelon form and this the unique representation.

Case 2

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ A_{m1} \end{pmatrix} \quad \text{for some } a_{i1} \neq 0 \tag{33}$$

Execute the following sequence of row operations on A. Perform type 1 row operation to move a_{i1} to the first row. Perform type 3 row operations to eliminate all lower terms. Perform type 2 row operation to the change the first term in the column to 1.

$$A \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{34}$$

The reduced row echelon form of a column must be of this form. It follows that this matrix is the unique row reduced echelon form.

Suppose true for $1 \le n \neq k$
Suppose $n = k + 1$
COME BACK TO THIS

4.1

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]\, I$

(b) $\det(C) = \det(A)$

(c) The classical adjoint of A^t is C^t

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}\, C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{35}$$
$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{36}$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{37}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{38}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{39}$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{40}$$
$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \tag{41}$$
$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \tag{42}$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \tag{43}$$
$$\Rightarrow AC = CA = \det(A) I_2 \tag{44}$$

(b) $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{45}$$
$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \tag{46}$$

It follows that D is the classical adjoint of A^t

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \tag{47}$$

It follows that C^t is the classical adjoint of A^t

(d) Suppose A is invertible

$$\Rightarrow \exists B \colon AB = BA = I \tag{48}$$
$$\Rightarrow \det(A) \neq 0 \tag{49}$$

$$CA = AC = \det(A)I \tag{50}$$
$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \tag{51}$$
$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \tag{52}$$

4.2

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Suppose $A \in \mathsf{M}_{n\times n}(()F)$ and A is upper triangular.

Proof by induction.

For $n = 1$: $A \in \mathsf{M}_{1\times 1}(F) \Rightarrow \det(A) = A_{11}$ by definition.

Suppose $\det(A)$ is the product of its diagonal entries for $1 \leq n \leq k$

For $n = k+1$: $A \in \mathsf{M}_{k+1\times k+1}(()F)$

$$\det(A) = \sum_{j=1}^{k+1}(-1)^{k+1+j}A_{k+1,j}\det(\tilde{A}_{k+1,j}) \tag{53}$$
$$A_{n+1,j} = 0 \quad \text{for } 1 \leq j \leq n \tag{54}$$
$$\Rightarrow \det A = (-1)^{(k+1)+(k+1)}A_{k+1,k+1}\det(\tilde{A}_{k+1,k+1}) \tag{55}$$
$$\tilde{A}_{k+1,k+1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k+1} \\ & a_{22} & & a_{2,k+1} \\ & \text{\huge 0} & \ddots & \vdots \\ & & & a_{k+1,k+1} \end{pmatrix} \tag{56}$$

$\tilde{A}_{k+1,k+1} \in \mathsf{M}_{n\times n}(F) \Rightarrow \det(\tilde{A}_{n+1,n+1})$ is a product of diagonal entries by induction hypothesis.

$$(-1)^{(n+1)+(n+1)} = (-1)^{2n+2} = (-1)^2(-1)^{2n} = (-1)^{2n} \tag{57}$$

$$n \in \mathbb{Z}^+ \Rightarrow 2n \text{ is even} \tag{58}$$
$$\Rightarrow (-1)^{2n} = 1 \tag{59}$$
$$\Rightarrow \det(A)A_{n+1,n+1}\prod_{j=1}^{n} A_{jj} = \prod_{j=1}^{n+1} A_{jj} \tag{60}$$

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

    (a) **Types 1 & 2**

    $$E^t = E \quad \text{(by HW.3.1.5)} \tag{61}$$
    $$\Rightarrow \det(E^t) = \det(E) \tag{62}$$

    (b) **Type 3**

    E^t is an type 3 elementary matrix (by HW.3.1.5) $\det(E) = \det(I) = 1$ for any type elementary operation on I_n

    $$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \tag{63}$$
    $$\Rightarrow \det(E) = \det(E^t) = 1 \tag{64}$$

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$. be $a_1, a_2, \ldots, a_n$ and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

    (a) **n is even**

    In A, swap

    $$a_{n-1} \text{ with } a_1 \tag{65}$$
    $$a_{n-2} \text{ with } a_2 \tag{66}$$
    $$\vdots$$
    $$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \tag{67}$$

    From the fact that ${}^{n}/_{2}$ swaps were performed it follows from Theorem 4.6 that

    $$\det(B) = (-1)^{\frac{n}{2}}\det(A) \tag{68}$$

    (b) **n is odd** In A, swap

    $$a_{n-1} \text{ with } a_1 \tag{69}$$
    $$a_{n-2} \text{ with } a_2 \tag{70}$$
    $$\vdots$$
    $$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \tag{71}$$

    From the fact that $n - \frac{n+1}{2}$ swaps were performed it follows from Theorem 4.6 that

    $$\det(B) = (-1)^{\frac{n-1}{2}}\det(A) \tag{72}$$

4.3

10. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **nilpotent** is for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that M if nilpotent, then $\det(M) = 0$.

$$M^k = 0 \quad \text{for some } k \in \mathbb{Z}^+ \tag{73}$$

$$\Rightarrow \det(M^k) = \det(0) \tag{74}$$
$$\Rightarrow (\det(M))^k = \det(0) \tag{75}$$
$$\Rightarrow (\det(M))^k = 0 \tag{76}$$
$$\Rightarrow \det(M) = 0 \tag{77}$$

11.

12. A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called **orthogonal** if $QQ^t = I$ Prove that Q is orthogonal, then $\det(Q) = \pm 1$.

    Suppose $QQ^t = I$

$$\Rightarrow \det(QQ^t) = \det(I) \tag{78}$$
$$\Rightarrow \det(Q)\det(Q^t) = 1 \tag{79}$$
$$\Rightarrow (\det(Q))^2 = 1 \tag{80}$$
$$\Rightarrow (\det(Q))^2 - 1 = 0 \tag{81}$$
$$\Rightarrow (\det(Q) + 1)(\det(Q) - 1) = 0 \tag{82}$$
$$\Rightarrow \det(Q) = 1 \text{ or } \det(Q) = -1 \tag{83}$$

15. Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

    Suppose $A, B \in \mathsf{M}_{n\times n}(F)$ such that A and B are similar.

$$\Rightarrow A = Q^{-1}BQ \text{ for some } Q^{-1} \text{ invertible} \tag{84}$$

$$\Rightarrow \det(A) = \det(Q^{-1}BQ) \tag{85}$$
$$= \det(Q^{-1})\det(BQ) \tag{86}$$
$$= \det(Q^{-1})\det(B)\det(Q) \tag{87}$$
$$= \det(Q)^{-1}\det(B)\det(Q) \tag{88}$$
$$= \det(B)\det(Q)^{-1}\det(Q) \tag{89}$$
$$= \det(B) \tag{90}$$