Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

Work

1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

(b)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

$$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

$$a_1 - a_2 = 0 \tag{2}$$
$$-2a_1 + a_1 = 0 \tag{3}$$
$$-a_1 + 2a_2 = 0 \tag{4}$$
$$\implies a_1 = a_2 = 0 \tag{5}$$

Only the trivial solution exists. The set is linearly independent.

(d)

$$\left\{ x^3 -, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6 \right\} \text{ in } \mathsf{P}_3(\mathbb{R})$$

$$a_1\left(x^3 - x\right) + a_2\left(2x^2 + 4\right) + a_3\left(-2x^3 + 3x^2 + 2x + 6\right) = 0 \tag{6}$$

$$a_1 = 2a_3 \tag{7}$$
$$2a_2 = -3a_3 \tag{8}$$

$$a_1 = t \tag{9}$$
$$a_2 = -\left({}^3/_4\right) t \tag{10}$$
$$a_3 = +\left({}^1/_2\right) t \tag{11}$$

The set is linearly dependent.

(f)

$$\{(1,-1,2),(1,-2,1),(1,1,4)\} \text{ in } \mathsf{R}^3$$

$$a\begin{bmatrix}1\\-1\\2\end{bmatrix}+b\begin{bmatrix}2\\0\\1\end{bmatrix}+c\begin{bmatrix}-1\\2\\-1\end{bmatrix}=0 \tag{12}$$

$$a+2b-c=0 \tag{13}$$
$$-a+2c=0 \tag{14}$$
$$2a+b-c=0 \tag{15}$$

$$\begin{pmatrix}1&2&-1&0\\-1&0&2&0\\2&1&-1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Only the trivial solution exists. This set is linearly independent.

(g)

$$\left\{\begin{pmatrix}1&0\\-2&0\end{pmatrix},\begin{pmatrix}0&-1\\1&1\end{pmatrix},\begin{pmatrix}-1&2\\1&0\end{pmatrix},\begin{pmatrix}2&1\\-4&4\end{pmatrix}\right\} \text{ in } \mathsf{M}_{2\times 2}(\mathbb{R})$$

$$a\begin{pmatrix}1&0\\-2&1\end{pmatrix}+b\begin{pmatrix}0&-1\\1&1\end{pmatrix}+c\begin{pmatrix}-1&2\\1&0\end{pmatrix}+d\begin{pmatrix}2&1\\-4&4\end{pmatrix}=0 \tag{16}$$

$$\begin{pmatrix}1&0&-1&2&0\\-2&1&1&-4&0\\0&-1&2&1&0\\1&1&0&4&0\end{pmatrix} \to \begin{pmatrix}1&0&-1&2&0\\0&1&-1&0&0\\0&0&1&1&0\\0&0&0&0&0\end{pmatrix}$$

$$a=-3t \tag{17}$$
$$b=-t \tag{18}$$
$$c=-t \tag{19}$$
$$d=t \tag{20}$$

There exists a non-trivial solution, therefore the set is linearly dependent.

11. Let $S=\{u_1,u_2,\dots,u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in span(S)? Justify your answer.

$$\mathbb{Z}_2=\{0,1\} \tag{21}$$
$$c_1u_1+c_2u_2+\cdots+c_nu_n\neq 0,\ \forall c_1,\dots,c_n\in\{0,1\} \tag{22}$$

unless all $c_i=0$.

$$\implies \operatorname{card}(\operatorname{span}(S))=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n} \tag{23}$$
$$=\sum_{i=1}^{n}\binom{n}{i} \tag{24}$$
$$=2^n \tag{25}$$

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}\left(\{u_1, u_2, \ldots, u_k\}\right)$ for some $k\,(1 \leq k < n)$.

Forward Direction:

Suppose S is linearly dependent.
Claim $u_1 = 0$ or $u_{k+1} \in \text{span}\left(\{u_1, u_2, \ldots, u_k\}\right)$ for some $k\,(1 \leq k < n)$.
Let $a_1u_1 + a_2u_2 + \cdots + a_{k+1}u_{k+1} = 0,\ a_i \in F$

Case 1 $\exists k$ such that $a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0,\ a_{k+1} \neq 0$

$$\implies \frac{-a_1}{a_{k+1}}u_1 + \frac{-a_2}{a_{k+1}} + \cdots + \frac{a_k}{a_{k+1}}u_k = u_{k+1} \tag{26}$$

$$\implies u_{k+1} \in \text{span}(\{u_1, \ldots, u_k\}) \tag{27}$$

Case 2 $\nexists k$ such that $a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0,\ a_{k+1} \neq 0$

$$\implies a_{k+1} = 0\ \forall k(1 \leq k < n) \tag{28}$$

$$\implies a_2 = a_3 = \cdots = a_{k+1} = 0\ \forall k(1 \leq k < n) \tag{29}$$

$$\implies a_1u_1 = 0 \tag{30}$$

Because S is linearly dependent $a_1 \neq 0$

$$\implies u_1 = 0 \tag{31}$$

Reverse Direction:
Suppose $u_{k+1} \in \text{span}(\{u_1, \ldots, u_k\})$
Claim: S is linearly dependent

$$\begin{aligned} u_{k+1} &= a_1u_1 + a_2u_2 + \cdots + a_ku_k && (32)\\ -u_{k+1} &= (-1)\left(a_1u_1 + a_2u_2 + \cdots + a_ku_k\right) && (33)\\ &= \left(-a_1\right)u_1 + \left(-a_2\right)u_2 + \cdots + \left(-a_k\right)u_k && (34) \end{aligned}$$

Take the linear combination of all $u_1, \ldots, u_n$

$$\left(\left(-a_1\right)u_1 + \left(a_2\right)u_2 + \cdots + \left(-a_k\right)u_k\right) + \left(1u_{k+1} + 0u_{k+2} + \cdots + 0u_n\right) = 0 \tag{35}$$

Suppose $u_1 = 0$
Claim: S is linearly dependent
Let:

$$\begin{aligned} a_1u_1 + a_2u_n + \cdots + a_nu_n &= 0,\ \forall a_i \in F && (36)\\ \text{such that } a_1u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n &= 0,\ a_1 \neq 0 && (37)\\ a_1u_1 &= 0 && (38) \end{aligned}$$

1.6

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

    (a) Prove that there is subset of S that is a basis for V. (Be careful not to assume that S is finite.)

    (b) Prove that S contains at least n vectors.

    (a) Claim: There exists a subset of S that is a basis for V. Suppose that β is a basis for V. This implies $\beta \subseteq \text{span}(S)$. Furthermore all vectors in β can be expressed as linear combinations of vectors in S. Collect these vectors into a set S^*. This set is finite because linear combinations comprise a finite number of vectors.

    Lemma: $\text{span}(S^*) = \mathsf{V}$
    Forward Containment: Suppose $x \in \text{span}(S^*)$

$$S^* \subseteq S \subseteq \mathsf{V} \tag{39}$$

$$\implies x \in \mathsf{V} \text{ (by theorem 1.5)} \tag{40}$$

    Reverse Containment: Suppose $x \in \mathsf{V}$
    By definition of S^*, $\beta \subseteq \text{span}(S*)$

    This implies vectors from β can be represented as linear combinations and thus linear combinations of the basis vectors can be formed. Every vector in V is a linear combination of vectors in β

$$\implies x \in \text{span}(S^*) \tag{41}$$

    Therefore there exists a subset of S^* (and hence a subset of S) that is basis for V (by theorem 1.9)

    (b)

$$\dim(\mathsf{V}) = n \tag{42}$$

$$\implies \text{card}(\beta) = n \tag{43}$$

$$\beta \subseteq S \tag{44}$$

$$\implies \text{card}(\beta) \leq \text{card}(S) \tag{45}$$

$$\therefore n \leq \text{card}(S) \tag{46}$$

24. Let $f(x)$ be a polynomial of degree n in $\mathsf{P}_n(\mathbb{R})$. Prove that for any $g(x) \in \mathsf{P}_n(\mathbb{R})$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

where $f^{(n)}(x)$ denotes nth derivative of $f(x)$.

Given vector space $\mathsf{P}_n(\mathbb{R})$ where $\dim(\mathsf{P}_n(\mathbb{R})) = n + 1$
Suppose S is a subset of $\mathsf{P}_n(\mathbb{R})$ and $S = \{f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\}$

$$\implies \text{card}(S) = n + 1 \tag{47}$$

If S is linearly independent, then $\text{span}(S) = \mathsf{P}(\mathbb{R})$ where $g(x) \in \mathsf{P}_n(\mathbb{R})$
Claim: S is linearly independent

$$d_0 f(x) + d_1 f'(x) + d_2 f''(x) + d_3 f'''(x) + \cdots + d_n f^{(n)}(x) = 0 \tag{48}$$

$$\begin{aligned} d_0 &= 0 \because x^n \text{ term only exists in } f(x) && (49)\\ \implies d_1 &= 0 \because x^{n-1} \text{ term only exists in } f'(x) && (50)\\ \implies d_2 &= 0 \because x^{n-2} \text{ term only exists in } f''(x) && (51)\\ &\vdots && (52)\\ \implies d_n &= 0 \because x^0 \text{ term only exists in } f^{(n)}(x) && (53) \end{aligned}$$

31. Let W_1 and W_2 be subspaces of V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n$
$\mathsf{W}_1 \cap \mathsf{W}_2$ is a vector space. (by theorem 1.4)

$$\mathsf{W}_1 \cap \mathsf{W}_2 \subseteq \mathsf{W}_2 \tag{54}$$

$$\implies \dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq \dim(\mathsf{W}_2)\text{(by theorem 1.11)} \tag{55}$$
$$\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n \tag{56}$$

(b)

Suppose β_1 is a basis for W_1 such that $\beta_1 = \{v_1, v_2, \ldots, v_m\}$
Suppose β_2 is a basis for W_2 such that $\beta_2 = \{u_1, u_2, \ldots, u_n\}$

Lemma: $\operatorname{span}(\beta_1 \cup \beta_2) = \operatorname{span}(\beta_1) + \operatorname{span}(\beta_2)$
Suppose $x \in \operatorname{span}(\beta_1 \cup \beta_2)$ such that for $c_i \in F$

$$x = c_1v_1 + c_2v_2 + \cdots + c_mv_m + c_{m+1} + \cdots + c_{m+n}u_n$$

Let $c_1v_1 + c_2v_2 + \cdots + c_mv_m = v \implies v \in \operatorname{span}(\beta_1)$
Let $c_{m+1}u_1 + c_{m+2}u_2 + \cdots + c_{m+m}u_m = u \implies u \in \operatorname{span}(\beta_2)$

$$\implies x \in \operatorname{span}(\beta_1) + \operatorname{span}(\beta_2) \tag{57}$$

Suppose $y \in \operatorname{span}(\beta_1) + \operatorname{span}(\beta_2)$ such that for $a_i, b_i \in F$

$$y = a_1v_1 + a_2v_2 + \cdots + a_mv_m + b_1u_1 + b_2u_2 + \cdots b_nu_n$$

$$\implies y \in \operatorname{span}(\beta_1 \cup \beta_2) \tag{58}$$

a priori: $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V (by HW2 Q.1.3.23)
Some subset of $\beta_1 \cup \beta_2$ is a basis for $\mathsf{W}_1 + \mathsf{W}_2$ (by theorem 1.9) Define this subset to be β.

$$\implies \operatorname{card}(\beta) \leq \operatorname{card}(\beta_1 \cup \beta_2) \tag{59}$$

$$\operatorname{card}(\beta_1 \cup \beta_2) = \operatorname{card}(\beta_1) + \operatorname{card}(\beta_2) - \operatorname{card}(\beta_1 \cap \beta_2) \tag{60}$$

$$\implies \operatorname{card}(\beta) \leq m + n \tag{61}$$

2.1

6. $\mathsf{T}\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \mathrm{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

   (a) Suppose $A, B \in \mathsf{M}_{n\times n}(F)$ and $c \in F$

$$\begin{aligned}
\mathsf{T}(cA + B) &= \sum_{i=1}^{n}(ca_{ii} + b_{ii}) && (62)\\
&= \sum_{i=1}^{n} ca_{ii} + \sum_{i=1}^{n} b_{ii} && (63)\\
&= c\sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} && (64)\\
&= c\mathsf{T}(A) + \mathsf{T}(B) && (65)
\end{aligned}$$

   (b)

$$\begin{aligned}
\dim(\mathsf{M}_{n\times n}(F)) &= n^2 && (66)\\
R(\mathsf{T}) &= \{\mathsf{T}(x) \colon x \in \mathsf{M}_{n\times n}(F)\} && (67)
\end{aligned}$$

   Claim $R(\mathsf{T}) = F$
   $R(\mathsf{T}) \subseteq F$ by definition of T
   Suppose $c \in F$, and $x \in \mathsf{M}_{n\times n}(F)$ such that

$$x = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$\begin{aligned}
\mathrm{tr}(cx) &= c \cdot \mathrm{tr}(x) && (68)\\
&= c \cdot 1 && (69)\\
&= c && (70)
\end{aligned}$$

$$\therefore c \in R(\mathsf{T}) \qquad (71)$$

$$\implies \mathsf{T} \text{ is onto} \qquad (72)$$

   Claim: $\{1\}$ is a basis for F

$$c \cdot 1 = 0 \qquad (73)$$

$$\implies c = 0 \ \text{ (by cancellation law)} \qquad (74)$$

$$\therefore \{1\} \text{ is linearly independent} \qquad (75)$$

$$c \cdot 1 = c \text{ for } c \in F \tag{76}$$

$$\operatorname{span}(\{1\}) = F \tag{77}$$

$$\therefore \{1\} \text{ is a basis for } R(\mathsf{T}) \tag{78}$$

$$\implies \operatorname{rank}(\mathsf{T}) = 1 \tag{79}$$

$$\implies \operatorname{nullity}(\mathsf{T}) = n^2 - 1 \text{ (by the dimension theorem)} \tag{80}$$

Claim: Basis β_n for $N(\mathsf{T})$ is a modification of a standard basis for $\mathsf{M}_{n\times n}(F)$ in which each matrix containing a 1 in a diagonal entry is replaced with a matrix containing 1 in the same entry and -1 in entry (n, n) and the matrix where all entries but $(n, n) = 1$ are zero are removed from the set.
Claim: β_n is linearly independent
Suppose $x \in \operatorname{span}(\beta_n)$ such that $x = a_1u_1 + a_2u_1 + \cdots + a_{n^2-1}u_{n^2-1}$

$$\begin{pmatrix} a_{1,1} & \cdots & \cdots & \cdots & a_{1,n} \\ \vdots & a_{2,2} & & & \\ \vdots & & \ddots & & \\ \vdots & & & a_{n-1,n-1} & \\ a_n & \cdots & \cdots & \cdots & A \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n} \tag{81}$$

Where $A = (-a_{1,1}) + (-a_{2,2}) + \cdots + (-a_{n-1,n-1})$
Therefore all the entries of the matrix are zero. Furthermore there only exists the trivial representation. As such by corollary 2 of theorem 1.10 β_n is a basis for $N(\mathsf{T})$.
Suppose:

$$x = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}$$

$$\operatorname{tr}\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} = 0 \tag{82}$$

$$\implies \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} \in N(\mathsf{T}) \tag{83}$$

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \tag{84}$$

$$\implies N(\mathsf{T}) \neq \{0\} \tag{85}$$

Therefore T is not one-to-one by theorem 2.4.

12. Is there a linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1,0,3) = (1,1)$ and $\mathsf{T}(-2,0,-6) = (2,1)$?

$$\mathsf{T}(-2(1,0,3)) = \mathsf{T}(-2,0,-6) = (2,1) \tag{86}$$
$$-2\mathsf{T}((1,0,3)) = -2 \times (1,1) = (-2,-2) \tag{87}$$

$$-2\mathsf{T}((1,0,3)) \neq \mathsf{T}((-2,0,-6)) \tag{88}$$
$$\implies \mathsf{T}(cx) \neq c\mathsf{T}(x)\ \forall c \in F \tag{89}$$

Therefore T is not a linear transformation.

14. Let V and W be vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear.

   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

   (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

   (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

   (a) **Forward Direction:**
   Suppose $S \subseteq \mathsf{V}$ such that S is linearly independent and T is one-to-one.
   Let $\mathsf{T}(S) = \{\mathsf{T}(x)\colon x \in S\}$
   Claim: $\mathsf{T}(S)$ is linearly independent.
   Suppose $x \in \text{span}(\mathsf{T}(S))$ such that $x = c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$ for $c_i \in F$ and $u_i \in \mathsf{T}(S)$

$$\begin{aligned}
\mathsf{T} \text{ is one-to-one} \implies u_i &= \mathsf{T}(v_i) \text{ for some unique } v_i \in S \\
\implies x &= c_1\mathsf{T}(v_1) + c_2\mathsf{T}(v_2) + \cdots + c_n\mathsf{T}(v_n) && (90)\\
&= \mathsf{T}(c_1v_1 + c_2v_2 + \cdots + c_nv_n) && (91)\\
&= 0 && (92)
\end{aligned}$$

$$\implies c_1v_1 + c_2v_2 + \cdots c_nv_n = 0 \quad (93)$$

$$\implies c_1 = c_2 = \cdots = c_n \quad (94)$$

   Therefore there only exists the trivial solution to the linear combination x.
   Reverse Direction: Suppose $S \subseteq \mathsf{T}$ such that S is linearly independent and $\mathsf{T}(S)$ is linearly independent.
   Claim: T is one-to-one
   Suppose $x, y \in \text{span}(S)$ such that $x \neq y$

$$x = a_1u_1 + a_2u_2 + \cdots a_nu_n \quad (95)$$

$$y = b_1u_1 + b_2u_2 + \cdots b_nu_n \quad (96)$$

   Because S is linearly independent each linear combination is unique.

$$\implies a_i \neq b_i \text{ for some } i = 1, 2, \ldots, n \quad (97)$$

$$\begin{aligned}
\text{Let } \mathsf{T}(x) &= \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) && (98)\\
\implies \mathsf{T}(x) &= a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) && (99)\\
\text{Let } \mathsf{T}(y) &= \mathsf{T}(b_1u_1 + b_2u_2 + \cdots + b_nu_n) && (100)\\
\implies \mathsf{T}(y) &= b_1\mathsf{T}(u_1) + b_2\mathsf{T}(u_2) + \cdots + b_n\mathsf{T}(u_n) && (101)\\
\mathsf{T}(u_i) &\in \mathsf{T}(S) \forall i = 1, 2, \ldots, n && (102)\\
\implies \mathsf{T}(x), \mathsf{T}(y) &\in \text{span}(\mathsf{T}(S)) && (103)
\end{aligned}$$

   Because $\mathsf{T}(S)$ is linearly independent each linear combination is unique.

$$\implies \mathsf{T}(x) \neq \mathsf{T}(y) \quad (104)$$

(b) Suppose T is one-to-one, $S \subseteq \mathsf{V}$
Forward Direction:
Suppose S is linearly independent.
Claim $\mathsf{T}(S)$ is linearly independent.
$\mathsf{T}(S)$ is linearly independent by part (a)
Reverse Direction:
Suppose $S = \{v_1, v_2, \ldots, v_m\}$

$$\implies \mathsf{T}(S) = \{\mathsf{T}(v_1) + \mathsf{T}(v_2) + \cdots + \mathsf{T}(v_m)\} \tag{105}$$

Suppose $\mathsf{T}(S)$ is linearly independent.
Claim: S is linearly independent.
Suppose $x \in \operatorname{span}(S)$ such that $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0,\ c_i \in F$

$$\begin{aligned} \implies \mathsf{T}(x) &= \mathsf{T}(c_1v_1 + c_2v_n + \cdots + c_mv_m) && (106)\\ &= \mathsf{T}(0) && (107)\\ &= 0 \ \text{ (by theorem 2.4)} && (108) \end{aligned}$$

$$\mathsf{T}(x) = c_1\mathsf{T}(v_1) + c_2\mathsf{T}(v_2) + \cdots + c_m\mathsf{T}(v_m) = 0 \tag{109}$$

$$\mathsf{T}(v_i) \in \mathsf{T}(S)\ \forall i = 1, 2, \ldots, m \tag{110}$$

Therefore $\mathsf{T}(x) \in \operatorname{span}(\mathsf{T}(S))$

$$\mathsf{T}(S) \ \text{ is linearly independent } \implies c_1 = c_2 = \cdots = c_m = 0 \tag{111}$$

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto.
Claim: $\mathsf{T}(\beta)$ is a basis for W

$$\begin{aligned} R(\mathsf{T}) &= \mathsf{W} && (112)\\ &= \operatorname{span}(\mathsf{T}(\beta))\ \text{(by theorem 2.2)} && (113) \end{aligned}$$

$\mathsf{T}(\beta)$ is linearly independent by part (a)

2.2

2. Let β and γ be the standard ordered bases for R^n and R^m respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{R}]_\beta^\gamma$.

   (b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

$$\mathsf{T}(1,0,0) = (2,1) \tag{114}$$
$$\mathsf{T}(0,1,0) = (3,0) \tag{115}$$
$$\mathsf{T}(0,0,1) = (-1,1) \tag{116}$$

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & -3 & -1 \\ 1 & 0 & 1 \end{pmatrix} \tag{117}$$

   (c) $\mathsf{T}\colon \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

$$\mathsf{T}(1,0,0) = 2 \tag{118}$$
$$\mathsf{T}(0,1,0) = 1 \tag{119}$$
$$\mathsf{T}(0,0,1) = 3 \tag{120}$$

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix} \tag{121}$$

   (g) $\mathsf{T}\colon \mathsf{R}^n \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$

$$\mathsf{T}(1,0,\ldots,0) = 1 \tag{122}$$
$$\mathsf{T}(0,1,\ldots,0) = 0 \tag{123}$$
$$\vdots \tag{124}$$
$$\mathsf{T}(0,1,\ldots,1,0) = 0 \tag{125}$$
$$\mathsf{T}(0,0,\ldots,1) = 1 \tag{126}$$

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{127}$$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.
   Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$
   Suppose $x, y \in \mathsf{V}$

$$x = \sum_{i=1}^{n} a_i v_i \qquad\qquad y = \sum_{i=1}^{n} b_i v_i \tag{128}$$

$$x + y = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n} (a_i v_i + b_i v_i) = \sum_{i=1}^{n} (a_i + b_i) v_i \tag{129}$$

$$\mathsf{T}(x + y) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \mathsf{T}(x) + \mathsf{T}(y) \tag{130}$$

   Suppose $c \in F$ and $x \in \mathsf{V}$

$$x = \sum_{i=1}^{n} a_i v_i \tag{131}$$

$$cx = c \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} c a_i v_i \tag{132}$$

$$\mathsf{T}(cx) = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = c\mathsf{T}(x) \tag{133}$$

11. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the exercises of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.
Suppose β_W is a basis of W

$$\mathsf{T}(\beta_\mathsf{W}) \subseteq \mathsf{W} \tag{134}$$

Suppose $x \in \beta_\mathsf{W}$

$$\implies \mathsf{T}(x) \in \mathsf{W} \tag{135}$$

Therefore $\mathsf{T}(x)$ can be described as a linear combination of vectors in β_W.
Suppose β_W is extended to β (by corollary 2 of theorem 2.2) such that
$\beta = \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ and $u_i \in \beta_\mathsf{W}$ for $i = 1, 2, \ldots, n$
$\implies$ for $u_i, i = 1, 2, \ldots, k;$

$$[\mathsf{T}(u_i)]_\beta = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{136}$$

$$[\mathsf{T}]_\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & & & & \\ a_{21} & a_{22} & \cdots & a_{2k} & & & & \\ \vdots & \vdots & & \vdots & & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & [\mathsf{T}(u_{k+1})]_\beta & [\mathsf{T}(u_{k+2})]_\beta & \cdots & [\mathsf{T}(u_n)]_\beta \\ 0 & 0 & \cdots & 0 & & & & \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & \cdots & 0 & & & & \\ 0 & 0 & \cdots & 0 & & & & \end{pmatrix} \tag{137}$$