Assignment

2.3: 13,15,16,17; 2.4: 2(bef), 5, 17, 20; 2.5: 3(cd), 6(bc), 10, 13

Work

2.3

13. Let A and B be $n \times n$ matrices. Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.

15. Let M and A be matrices for which the product matrix MA is defined. If the jth column of A is a linear combination of a set of columns of A, prove that the jth column MA is linear combination of the corresponding columns of MA with the same corresponding coefficients.

16. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear.

   (a) If $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$

   (b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k

17. Let V be a vector space. Determine all linear transformations $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T} = \mathsf{T}^2$.

2.4

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

   (b) $\mathsf{T}\colon \mathsf{T}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (3a_1 - 2a_2, a_2, 4a_1)$
   Claim: T is 1-1
   Suppose $x, y \in \mathbb{R}^2$ such that $x = (a_1, a_2), y = (a_3, a_4)$ and $\mathsf{T}(x) = \mathsf{T}(y)$ for $a_i \in \mathbb{R}$

$$(3a_1 - a_2, a_2, 4a_1) = (3a_3 - a_4, a_4, 4a_3) \tag{1}$$

$$\begin{aligned} \implies 3a_1 - a_2 &= 3a_3 - a_4 &(2)\\ a_2 &= a_4 &(3)\\ 4a_1 &= 4a_3 &(4) \end{aligned}$$

$$\begin{aligned} \implies a_1 &= a_3 &(5)\\ a_2 &= a_4 &(6)\\ & &(7) \end{aligned}$$

$$\implies x = y \tag{8}$$

Claim: T is onto
Suppose $x \in \mathbb{R}^3$ such that $x = (b_1, b_2, b_3)$ for $b_i \in \mathbb{R}$
Let $b_2 = a_2, b_3 = 4a_1, b_1 = (\frac{3}{4}b_3 - b_2)$

$$\implies (b_1, b_2, b_3) = (2a_1 - a_2, a_2, 4a_1) \tag{9}$$

$$\implies x \in R(\mathsf{T}) \tag{10}$$

$$R(\mathsf{T}) \subseteq \mathsf{M}_{n\times n}(\mathbb{R}) \text{ by def of } \mathsf{T} \tag{11}$$

$$\therefore \mathsf{T} \text{ is invertible} \tag{12}$$

(e) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$

(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

17. Let V and W be finite-dimensional vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

    (a) Prove that $\mathsf{T}(\mathsf{V}_o)$ is a subspace of W.

    (b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

20. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W Let β and γ be ordered bases for V and W respectively. Prove that $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{L}_A)$ and that $\text{nullity}(\mathsf{T} = \text{nullity}(\mathsf{L}_1)$, where $A = [\mathsf{T}]_\beta^\gamma$.

2.5

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change or coordinate matrix that changes β'-coordinates into β-coordinates.

    (c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = 1 \tag{13}$$

$$2a + 3b + c = 0 \tag{14}$$

$$a = 0 \tag{15}$$
$$b = 1 \tag{16}$$
$$c = 0 \tag{17}$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x \tag{18}$$

$$a = -1 \tag{19}$$
$$b = 0 \tag{20}$$
$$c = 0 \tag{21}$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x^2 \tag{22}$$
$$2a + 3b + c = 1 \tag{23}$$

$$a = 0 \tag{24}$$
$$b = 0 \tag{25}$$
$$c = 1 \tag{26}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

$$a(x^2 - x + 1) + b(x + 1)c(x^2 + 1) = x^2 + x + 4 \tag{28}$$

$$a + c = 1 \tag{29}$$
$$-a + b = 1 \tag{30}$$
$$a + b + c = 4 \tag{31}$$

$$a = 2 \tag{32}$$
$$b = 3 \tag{33}$$
$$c = 1 \tag{34}$$

$$a(x^2 - x + 1) + b(x + 1)c(x^2 + 1) = 4x^2 - 3x + 2 \tag{35}$$

$$a + c = 4 \tag{36}$$
$$-a + b = -3 \tag{37}$$
$$a + b + c = 2 \tag{38}$$

$$a = 1 \tag{39}$$
$$b = -2 \tag{40}$$
$$c = 3 \tag{41}$$

$$a(x^2 - x + 1) + b(x + 1)c(x^2 + 1) = 2x^2 + 3 \tag{42}$$

$$a + c = 2 \tag{43}$$
$$-a + b = 0 \tag{44}$$
$$a + b + c = 3 \tag{45}$$

$$a = 1 \tag{46}$$
$$b = 1 \tag{47}$$
$$c = 1 \tag{48}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \tag{49}$$

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

   (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

   (c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

10. Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

$$\begin{aligned} \mathrm{tr}(B) &= \mathrm{tr}(QAQ^{-1}) && (50) \\ &= \mathrm{tr}(A(QQ^{-1})) && (51) \\ &= \mathrm{tr}(A) \text{ (by HW.2.3.13)} && (52) \end{aligned}$$

13. Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \text{ for } 1 \leq j \leq n$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is a coordinate matrix changing β'-coordinates into β-coordinates.
Claim $\mathrm{span}(\beta) = \mathrm{span}(\beta')$

Forward Direction
Suppose $x' \in \mathrm{span}(\beta')$

$$x' = c_1 \left(\sum_{i=1}^{n} Q_{i1} x_i \right) + c_2 \left(\sum_{i=1}^{n} Q_{i2} x_i \right) + \cdots + c_n \left(\sum_{i=1}^{n} Q_{in} x_i \right) \tag{53}$$

$$\begin{aligned} x' = c_1 (Q_{11}x_1 + Q_{21}x_2 + \cdots + Q_{n1}x_n) + \\ + c_2 (Q_{12}x_1 + Q_{22}x_2 + \cdots + Q_{n2}x_n) + \\ + \cdots + c_n (Q_{1n}x_1 + Q_{2n}x_2 + \cdots + Q_{nn}x_n) \end{aligned} \tag{54}$$

$$
\begin{aligned}
x' = (c_1Q_{11} + c_2Q_{12} + \cdots + c_nQ_{1n})x_1 + \\
+ (c_1Q_{21} + c_2Q_{22} + \cdots + c_nQ_{2n})x_2 + \\
+ \cdots + (c_1Q_{n1} + c_2Q_{n2} + \cdots + c_nQ_{nn})x_n \quad (55)
\end{aligned}
$$

$$\implies x \in \operatorname{span}(\beta') \qquad (56)$$

Reverse Direction
Suppose $x \in \operatorname{span}(\beta)$

$$x = c_1x_2 + c_2x_2 + \cdots + c_nx_n \qquad (57)$$

$$= \sum_{i=1}^{n} c_ix_i \qquad (58)$$

Let $c_i = \sum_{j=1}^{n} a_jQ_{ij}$

$$x = \sum_{i=1}^{n} \left(x_i \sum_{j=1}^{n} a_jQ_{ij} \right) \qquad (59)$$

$$x = \sum_{i=1}^{n} \left(\left(a_1Q_{i1} + a_2Q_{i2} + \cdots + a_nQ_{in} \right) x_i \right) \qquad (60)$$

$$
\begin{aligned}
x = (a_1Q_{11} + a_2Q_{12} + \cdots + a_nQ_{1n})x_1 + \\
+ (a_1Q_{21} + a_2Q_{22} + \cdots + a_nQ_{2n})x_2 + \\
+ \cdots + (a_1Q_{n1} + a_2Q_{n2} + \cdots a_nQ_{nn})x_n \quad (61)
\end{aligned}
$$

$$
\begin{aligned}
x = a_1(Q_{11}x_1 + Q_{21}x_2 + \cdots + Q_{n1}x_n) + \\
+ a_2(Q_{12}x_1 + Q_{22}x_2 + \cdots + Q_{n2}x_n) + \\
+ \cdots + a_n(Q_{1n}x_1 + Q_{2n}x_2 + \cdots + Q_{nn}x_n) \quad (62)
\end{aligned}
$$

$$x = a_1 \sum_{i=1}^{n} Q_{i1}x_i + a_2 \sum_{i=1}^{n} Q_{i2}x_i + \cdots + a_n \sum_{i=1}^{n} Q_{in}x_i \qquad (63)$$

$$\implies x \in \operatorname{span}(\beta') \qquad (64)$$