Assignment

Section 6.6: 5, 7, 8; Section 6.8: 17, 18, 19, 20

Work

6.6

5. Let T be a linear operator on a finite-dimensional inner product space V.

   (a) If T is an orthogonal projection, prove that $\|\mathsf{T}(x)\| \leq \|x\|$ for all $x \in \mathsf{V}$. Give an example of a projection for which this inequality does not hold. What can be concluded about a projection for which the inequality is actually an equality for all $x \in \mathsf{V}$.

   (b) Suppose that T is a projection such that $\|\mathsf{T}(x)\| \leq \|x\|$ for $x \in \mathsf{V}$. Prove that T is a orthogonal projection.

   (a) Suppose T is an orthogonal projection of V on a subspace W.
   Suppose $x \in \mathsf{V}$ can be written in the form $x = x_1 + x_2$ such that $x_1 \in \mathsf{W}$ and $x_2 \in \mathsf{W}^\perp$

$$\Rightarrow \mathsf{T}(x) = x_1 \tag{1}$$
$$\Rightarrow \|\mathsf{T}(x)\| = \|x_1\| \tag{2}$$
$$\|x\| = \|x_1 + x_2\| \tag{3}$$
$$\Rightarrow \|x\| = \sqrt{\|x_1\|^2 + \|x_2\|^2} \quad \because x_1 \perp x_2 \tag{4}$$
$$\sqrt{\|x_1\|^2 + \|x_2\|^2} \geq \|x_1\| \tag{5}$$
$$\Rightarrow \|x\| \geq \|\mathsf{T}(x)\| \tag{6}$$

   This inequality does not hold for

$$\mathsf{T}(x) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x, \quad x \in \mathsf{R}^2 \tag{7}$$

   Let $x = (1, 2)$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{8}$$
$$\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \sqrt{5} \tag{9}$$
$$\left\| \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\| = 3 \tag{10}$$
$$\sqrt{5} < 3 \tag{11}$$

   If the inequality is actually an equality then T is an orthogonal linear operator.

(b) Suppose that T is a projection such that $\|\mathsf{T}(x)\| \leq \|x\|$ for $x \in \mathsf{V}$. Prove that T is an orthogonal projection.

Case 1 Suppose $\|\mathsf{T}(x)\| = \|x\|$

$$\mathsf{T}(x) = \lambda_1 \mathsf{T}_1(x) + \lambda_2 \mathsf{T}_2(x) + \cdots + \lambda_k \mathsf{T}_k \tag{12}$$
$$x = \mathsf{T}_1(x) + \mathsf{T}_2(x) + \cdots + \mathsf{T}_k(x) \tag{13}$$
$$\|x\| = \sqrt{\|\mathsf{T}_1(x)\|^2 + \|\mathsf{T}_2(x)\|^2 + \cdots + \|\mathsf{T}_k(x)\|^2} \tag{14}$$
$$\|\mathsf{T}(x)\| = \sqrt{\lambda_1^2 \|\mathsf{T}_1(x)\|^2 + \lambda_1^2 \|\mathsf{T}_2(x)\|^2 + \cdots + \lambda_1^2 \|\mathsf{T}_k(x)\|^2} \tag{15}$$
$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_k = 1 \tag{16}$$
$$\Rightarrow \mathsf{T}^* = \mathsf{T}_1^* + \mathsf{T}_2^* + \cdots + \mathsf{T}_k^* = \mathsf{T}_1 + \mathsf{T}_2 + \cdots + \mathsf{T}_k = \mathsf{T} \tag{17}$$
$$\Rightarrow \mathsf{T}^2 = \mathsf{T}_1^2 + \mathsf{T}_2^2 + \cdots + \mathsf{T}_k^2 = \mathsf{T}_1 + \mathsf{T}_2 + \cdots + \mathsf{T}_k = \mathsf{T} \tag{18}$$

It follows from Theorem 6.24 that T is an orthogonal projection.

Case 2 Suppose $\|\mathsf{T}(x)\| < \|x\|$

7. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k$ of T to prove the following results.

(a) If g is a polynomial, then

$$g(\mathsf{T}) = \sum_{i=1}^{k} g(\lambda_i) \mathsf{T}_i$$
$$g(\mathsf{T}) = a_n \mathsf{T}^n + a_{n-1} \mathsf{T}^{n-1} + \cdots + a_1 \mathsf{T} \tag{19}$$

$$\begin{aligned} & g(\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k) = \\ & = a_n (\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k)^n + \\ & \qquad + a_{n+1} (\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k)^{n-1} + \cdots + \\ & \qquad\qquad + a_1 (\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k) \end{aligned} \tag{20}$$

It follows that the cross terms cancel because $\mathsf{T}_i \mathsf{T}_j = 0$ for $i \neq j$

$$\begin{aligned} & \Rightarrow g(\mathsf{T}) = a_n \left(\lambda_1^n \mathsf{T}^n + \lambda_2^n \mathsf{T}_2^n + \cdots + \lambda_k^n \mathsf{T}_k^n\right) + \\ & \qquad + a_{n-1} \left(\lambda_1^{n-1} \mathsf{T}^{n-1} + \lambda_2^{n-1} \mathsf{T}^{n-1} + \cdots + \lambda_k^{n-1} \mathsf{T}_k^{n-1}\right) + \cdots + \\ & \qquad\qquad + a_1 \left(\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k\right) \end{aligned} \tag{21}$$

It follows that $\mathsf{T}^n = \mathsf{T}$ because T_i is a projection.

$$\begin{aligned} & \Rightarrow g(\mathsf{T}) = a_n \left(\lambda_1^n \mathsf{T} + \lambda_2^n \mathsf{T}_2 + \cdots + \lambda_k^n \mathsf{T}_k\right) + \\ & \qquad + a_{n-1} \left(\lambda_1^{n-1} \mathsf{T} + \lambda_2^{n-1} \mathsf{T} + \cdots + \lambda_k^{n-1} \mathsf{T}_k\right) + \cdots + \\ & \qquad\qquad + a_1 \left(\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k\right) \end{aligned} \tag{22}$$

Regroup terms.

$$\begin{aligned}\Rightarrow g(\mathsf{T}) = \big(a_n\lambda_1^n\mathsf{T}_1 + a_{n-1}\lambda_1^{n-1}\mathsf{T}_1 + \cdots + \lambda_1\mathsf{T}_1\big) + \\ + \big(a_n\lambda_2^n\mathsf{T}_2 + a_{n-1}\lambda_2^{n-1}\mathsf{T}_2 + \cdots + a_1\lambda_2\mathsf{T}_2\big) + \cdots + \\ + \big(a_n\lambda_k^n\mathsf{T}_k + a_{n-1}\lambda_k^{n-1}\mathsf{T}_k + \cdots + a_1\lambda_k\mathsf{T}_k\big) \quad (23)\end{aligned}$$

Factoring out each T_i yields

$$g(\mathsf{T}) = \mathsf{T}_1 g(\lambda_1) + \mathsf{T}_2 g(\lambda_2) + \cdots + \mathsf{T}_k g(\lambda_k) \quad (24)$$

(b) If $\mathsf{T}^n = \mathsf{T}_0$ for some n, then $\mathsf{T} = \mathsf{T}_0$

$$\begin{aligned} \mathsf{T} &= \lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k && (25)\\ \mathsf{T}_0 &= \mathsf{T}^n && (26)\\ &= (\lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k)^n && (27)\\ &= \lambda_1^n\mathsf{T}_1^n + \lambda_2^n\mathsf{T}_2^n + \cdots + \lambda_k^n\mathsf{T}_k^n && (28)\\ &= \lambda_1^n\mathsf{T}_1 + \lambda_2^n\mathsf{T}_2 + \cdots + \lambda_k^n\mathsf{T}_k && (29)\end{aligned}$$

Let λ_i^n be the corresponding eigenvalue of T_0. All eigenvalues of T_0 must be zero, therefore $\lambda_i^n = \lambda_i$

$$\begin{aligned} \mathsf{T}_0 &= \lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k && (30)\\ &= \mathsf{T} && (31)\end{aligned}$$

(c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

($\Rightarrow$) Suppose $\mathsf{UT} = \mathsf{TU}$.

$$\begin{aligned} \mathsf{T} &= \lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k && (32)\\ \mathsf{UT} &= \lambda_1\mathsf{UT}_1 + \lambda_2\mathsf{UT}_2 + \cdots + \lambda_k\mathsf{UT}_k && (33)\\ \mathsf{TU} &= \lambda_1\mathsf{T}_1\mathsf{U} + \lambda_2\mathsf{T}_2\mathsf{U} + \cdots + \lambda_k\mathsf{T}_k\mathsf{U} && (34)\end{aligned}$$

$$\Rightarrow \lambda_1\mathsf{T}_1\mathsf{U} + \lambda_2\mathsf{T}_2\mathsf{U} + \cdots + \lambda_k\mathsf{T}_k\mathsf{U} = \lambda_1\mathsf{UT}_1 + \lambda_2\mathsf{UT}_2 + \cdots + \lambda_k\mathsf{UT}_k \quad (35)$$

$$\Rightarrow \mathsf{UT}_i = \mathsf{T}_i\mathsf{U} \quad (36)$$

($\Leftarrow$) Suppose $\mathsf{UT}_i = \mathsf{T}_i\mathsf{U}$

$$\begin{aligned} \mathsf{UT} &= \lambda_1\mathsf{UT}_i + \lambda_2\mathsf{UT}_2 + \cdots + \lambda_k\mathsf{UT}_k && (37)\\ &= \lambda_1\mathsf{T}_1\mathsf{U} + \lambda_2\mathsf{T}_2\mathsf{U} + \cdots + \lambda_k\mathsf{T}_k\mathsf{U} && (38)\\ &= \mathsf{TU} && (39)\end{aligned}$$

(d) There exists a nomral operator U on V such that $\mathsf{U}^2 = \mathsf{T}$.

$$\begin{aligned} \mathsf{T} &= \lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k && (40)\\ &= \lambda_1\mathsf{T}_1^2 + \lambda_2\mathsf{T}_2^2 + \cdots + \lambda_k\mathsf{T}_k^2 && (41)\\ &= \mathsf{U}^2 && (42)\end{aligned}$$

$$\Rightarrow \mathsf{U} = \sqrt{\lambda_1}\mathsf{T}_1 + \sqrt{\lambda_2}\mathsf{T}_2 + \cdots + \sqrt{\lambda_k}\mathsf{T}_k \quad (43)$$

(e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$

($\Rightarrow$) Suppose T is invertible

$$\mathsf{T} = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{44}$$

Let $\mathsf{T}^{-1} = \mathsf{U}$

$$\mathsf{U} = \frac{1}{\lambda_1}\mathsf{U}_1 + \frac{1}{\lambda_2}\mathsf{U}_2 + \cdots + \frac{1}{\lambda_k}\mathsf{U}_k \tag{45}$$

$$\Rightarrow \lambda_i \neq 0 \tag{46}$$

($\Leftarrow$) Suppose $\lambda_i \neq 0$

$$\mathsf{T} = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{47}$$

$\Rightarrow \exists \mathsf{U}$ with eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k}$

$$\mathsf{U} = \frac{1}{\lambda_1}\mathsf{U}_1 + \frac{1}{\lambda_2}\mathsf{U}_2 + \cdots + \frac{1}{\lambda_k}\mathsf{U}_k \tag{48}$$

$$\mathsf{UT} = \frac{\lambda_1}{\lambda_1}\mathsf{U}_1\mathsf{T}_1 + \frac{\lambda_2}{\lambda_2}\mathsf{U}_2\mathsf{T}_2 + \cdots + \frac{\lambda_k}{\lambda_k}\mathsf{U}_k\mathsf{T}_k \tag{49}$$

$$\mathsf{UT} = \mathsf{U}_1\mathsf{T}_1 + \mathsf{U}_2\mathsf{T}_2 + \cdots + \mathsf{U}_k\mathsf{T}_k \tag{50}$$

Because each $R(\mathsf{T}_i) \subseteq \mathsf{V}$ and each U_i is an orthogonal projection each $\mathsf{U}_i\mathsf{T}_i$ is an orthogonal projection.

Let $\mathsf{S} = \mathsf{UT}$ with $\mathsf{S}_i = \mathsf{U}_i\mathsf{T}_i$

$$\mathsf{S} = \mathsf{S}_1 + \mathsf{S}_2 + \cdots + \mathsf{S}_k \tag{51}$$

$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_k = 1 \tag{52}$$

$$\Rightarrow \mathsf{S} = \mathsf{I}_\mathsf{V} \tag{53}$$

$$\Rightarrow \mathsf{U} = \mathsf{T}^{-1} \tag{54}$$

(f) T is a projection if and only if every eigenvalue of T is 1 or 0.

($\Rightarrow$) Suppose T is a projection.

$$\mathsf{T}^n = \mathsf{T} \tag{55}$$

$$\Rightarrow \mathsf{T} = (\lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T})^n \tag{56}$$

$$= \lambda_1^n \mathsf{T}_1^n + \lambda_2^n \mathsf{T}_2^n + \cdots + \lambda_k^n \mathsf{T}_k^n \tag{57}$$

$$= \lambda_1^n \mathsf{T}_1 + \lambda_2^n + \cdots + \lambda_k^n \mathsf{T}_k \tag{58}$$

$$\Rightarrow \lambda_1^n \mathsf{T}_1 + \lambda_2^n + \cdots + \lambda_k^n \mathsf{T}_k = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{59}$$

$$\Rightarrow \lambda_i^n = \lambda_i \tag{60}$$

$$\lambda_i^n = \lambda_i \Leftrightarrow \lambda_i = 0 \text{ or } \lambda_i = 1 \tag{61}$$

($\Leftarrow$) Suppose all $\lambda_1 = 0$ or all $\lambda_i = 1$

Case 1: Suppose all $\lambda_i = 1$

$$\mathsf{T} = \mathsf{T}_1 + \mathsf{T}_2 + \cdots + \mathsf{T}_k \tag{62}$$
$$\mathsf{T}^n = (\mathsf{T}_1 + \mathsf{T}_2 + \cdots + \mathsf{T}_k)^n \tag{63}$$
$$= \mathsf{T}_1^n + \mathsf{T}_2^n + \cdots + \mathsf{T}_k^n \tag{64}$$
$$= \mathsf{T}_1 + \mathsf{T}_2 + \cdots + \mathsf{T}_k \tag{65}$$
$$= \mathsf{T} \tag{66}$$

Case 1: Suppose all $\lambda_i = 0$

$$\mathsf{T} = 0 \cdot \mathsf{T}_1 + 0 \cdot \mathsf{T}_2 + \cdots + 0 \cdot \mathsf{T}_k \tag{67}$$
$$= 0 \tag{68}$$
$$= 0^n \tag{69}$$
$$= \mathsf{T}^n \tag{70}$$

(g) $\mathsf{T} = -\mathsf{T}^*$ if and only if every λ_i is an imaginary number.
($\Rightarrow$) Suppose $\mathsf{T} = -\mathsf{T}^*$

$$\mathsf{T} = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{71}$$
$$\mathsf{T}^* = \overline{\lambda}_1 \mathsf{T}_1^* + \overline{\lambda}_2 \mathsf{T}_2^* + \cdots + \overline{\lambda}_k \mathsf{T}_k^* \tag{72}$$
$$= \overline{\lambda}_1 \mathsf{T}_1 + \overline{\lambda}_2 \mathsf{T}_2 + \cdots + \overline{\lambda}_k \mathsf{T}_k \tag{73}$$

$$-\mathsf{T}^* = \mathsf{T} \tag{74}$$
$$\Rightarrow -\overline{\lambda}_1 \mathsf{T}_1 - \overline{\lambda}_2 \mathsf{T}_2 - \cdots - \overline{\lambda}_k \mathsf{T}_k = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{75}$$
$$\Leftrightarrow -\overline{\lambda}_j = \lambda_j \Leftrightarrow \lambda_j = c_j i, \quad c \in \mathbb{R} \tag{76}$$

($\Leftarrow$) Suppose λ_i is imaginary.

$$\mathsf{T} = \lambda_1 \mathsf{T}_1 + \lambda_2 \mathsf{T}_2 + \cdots + \lambda_k \mathsf{T}_k \tag{77}$$
$$\mathsf{T}^* = \overline{\lambda}_1 \mathsf{T}_1^* + \overline{\lambda}_2 \mathsf{T}_2^* + \cdots + \overline{\lambda}_k \mathsf{T}_k^* \tag{78}$$
$$= \overline{\lambda}_1 \mathsf{T}_1 + \overline{\lambda}_2 \mathsf{T}_2 + \cdots + \overline{\lambda}_k \mathsf{T}_k \tag{79}$$

$$\lambda_j = c_j i \Rightarrow \overline{\lambda}_j = -\lambda_j \tag{80}$$

$$\mathsf{T}^* = -\lambda_1 \mathsf{T}_1 - \lambda_2 \mathsf{T}_2 - \cdots - \lambda_k \mathsf{T}_k \tag{81}$$
$$= -\mathsf{T}s \tag{82}$$

8. Use Corollary 1 of the spectral theorem to show that if T is a normal operator on a complex finite-dimensional inner product space and U is a linear operator that commutes with T, then U commutes with T^*.

$$
\begin{align}
\mathsf{T} &= \lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k \tag{83}\\
\mathsf{T}^* &= g(\mathsf{T}) \text{ for some polynomial } g \because \mathsf{T} \text{ is normal} \tag{84}
\end{align}
$$

$$
\begin{align}
\mathsf{T}^* &= g(\lambda_1)\mathsf{T}_1^* + g(\lambda_2)\mathsf{T}^* + \cdots + g(\lambda_k)\mathsf{T}_k^* \tag{85}\\
&= g(\lambda_1)\mathsf{T}_1 + g(\lambda_2)\mathsf{T} + \cdots + g(\lambda_k)\mathsf{T}_k \tag{86}
\end{align}
$$

Because $\mathsf{UT} = \mathsf{TU}$ for each T_i, $\mathsf{UT}_i = \mathsf{T}_i\mathsf{U}$

$$
\begin{align}
\Rightarrow \mathsf{T}^*\mathsf{U} &= g(\lambda_1)\mathsf{T}_1\mathsf{U} + g(\lambda_2)\mathsf{TU} + \cdots + g(\lambda_k)\mathsf{T}_k\mathsf{U} \tag{87}\\
&= g(\lambda_1)\mathsf{UT}_1 + g(\lambda_2)\mathsf{UT} + \cdots + g(\lambda_k)\mathsf{UT}_k \tag{88}\\
&= \mathsf{U}g(\lambda_1)\mathsf{T}_1 + \mathsf{U}g(\lambda_2)\mathsf{T} + \cdots + \mathsf{U}g(\lambda_k)\mathsf{T}_k \tag{89}\\
&= \mathsf{UT}^* \tag{90}
\end{align}
$$

6.8

17. For each of the give quadratic forms K on a real inner product space V, find a bilinear form H such that $K(x) = H(x, x)$ for all $x \in \mathsf{V}$. Then find an orthonormal basis β for V such that $\psi_\beta(H)$ is a diagonal matrix.

(a) $K\colon \mathsf{R}^2 \to \mathbb{R}$ defined by $K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -2t_1^2 + 4t_1t_2 + t_2^2$

Let $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} -2 & -t & 2 \\ 2 & 1-t & \end{pmatrix} = (t+3)(t-2) \tag{91}$$

$$\Rightarrow \lambda_1 = -3 \tag{92}$$
$$\lambda_2 = 2 \tag{93}$$

- For $\lambda_1 = -3$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{94}$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{95}$$
$$\Rightarrow x_1 + 2x_2 = 0 \tag{96}$$
$$\Rightarrow E_{\lambda_1} = \left\{ t\begin{pmatrix} 1 \\ {}^{-1}/_{2} \end{pmatrix} : t \in \mathbb{R} \right\} \tag{97}$$

- For $\lambda_2 = 2$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{98}$$
$$\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{99}$$
$$\Rightarrow 2x_1 - x_2 = 0 \tag{100}$$
$$\Rightarrow E_{\lambda_2} = \left\{ t\begin{pmatrix} {}^{1}/_{2} \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{101}$$

$$\left\langle \begin{pmatrix} 1 \\ {}^{-1}/_{2} \end{pmatrix}, \begin{pmatrix} {}^{1}/_{2} \\ 1 \end{pmatrix} \right\rangle = 0 \tag{102}$$
$$\left\| \begin{pmatrix} 1 \\ {}^{-1}/_{2} \end{pmatrix} \right\| = \left\| \begin{pmatrix} {}^{1}/_{2} \\ 1 \end{pmatrix} \right\| = \frac{\sqrt{5}}{2} \tag{103}$$

$$\Rightarrow v_1 = \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ {}^{-1}/_{2} \end{pmatrix} \tag{104}$$

$$v_2 = \frac{2}{\sqrt{5}} \begin{pmatrix} {}^{1}/_{2} \\ 1 \end{pmatrix} \tag{105}$$

$$\Rightarrow \beta = \left\{ \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ {}^{-1}/_{2} \end{pmatrix}, \frac{2}{\sqrt{5}} \begin{pmatrix} {}^{1}/_{2} \\ 1 \end{pmatrix} \right\} \tag{106}$$

$$\Rightarrow Q = \frac{2}{\sqrt{5}} = \begin{pmatrix} 1 & {}^{1}/_{2} \\ {}^{-1}/_{2} & 1 \end{pmatrix} \tag{107}$$

$$Q^t AQ = \psi_\beta(H) = \frac{4}{5} \begin{pmatrix} {}^{-15}/_{4} & 0 \\ 0 & {}^{5}/_{2} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \tag{108}$$

(b) $K\colon \mathsf{R}^2 \to \mathbb{R}$ defined by $K \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} = 7t_1^2 - 8t_1t_2 + t_2^2$

Let $A = \begin{pmatrix} 7 & -4 \\ -4 & 1 \end{pmatrix}$

$$\det \begin{pmatrix} 7-t & -4 \\ -4 & 1-t \end{pmatrix} = (t+1)(t-9) \tag{109}$$

$$\Rightarrow \lambda_1 = -1 \tag{110}$$

$$\lambda_2 = 9 \tag{111}$$

- For $\lambda_1 = -1$

$$\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{112}$$

$$\begin{pmatrix} 0 & 0 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{113}$$

$$\Rightarrow -4x_1 + 2x_2 = 0 \tag{114}$$

$$x_2 = 2x_1 \tag{115}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 1 \\ {}^{1}/_{2} \end{pmatrix} : t \in \mathbb{R} \right\} \tag{116}$$

- For $\lambda_2 = 9$

$$\begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{117}$$

$$\begin{pmatrix} -2 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{118}$$

$$\Rightarrow -2x_1 - 4x_2 = 0 \tag{119}$$

$$-x_1 = 2x_2 \tag{120}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} {}^{-1}/_{2} \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{121}$$

$$\left\langle \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\rangle = 0 \tag{122}$$

$$\left\| \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\| = \frac{\sqrt{5}}{2} \tag{123}$$

$$\Rightarrow v_1 = \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \tag{124}$$

$$v_2 = \frac{2}{\sqrt{5}} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \tag{125}$$

$$\Rightarrow \beta = \left\{ \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}, \frac{2}{\sqrt{5}} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\} \tag{126}$$

$$\Rightarrow Q = \frac{2}{\sqrt{5}} = \begin{pmatrix} 1 & 1/2 \\ -1/2 & 1 \end{pmatrix} \tag{127}$$

$$Q^t A Q = \psi_\beta(H) = \frac{4}{5} \begin{pmatrix} 45/4 & 0 \\ 0 & -5/4 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \tag{128}$$

(c) $K \colon \mathsf{R}^3 \to \mathbb{R}$ defined by$K \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1 t_3$

Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

$$\det \begin{pmatrix} 3-t & 0 & -1 \\ 0 & 3-t & 0 \\ -1 & 0 & 3-t \end{pmatrix} = t^3 + 9t^2 - 26t + 24 \tag{129}$$

$$\Rightarrow \lambda_1 = 2 \tag{130}$$

$$\lambda_2 = 3 \tag{131}$$

$$\lambda_3 = 4 \tag{132}$$

- For $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{133}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{134}$$

$$\Rightarrow x_2 = 0 \tag{135}$$

$$-x_1 + x_3 = 0 \tag{136}$$

$$\Rightarrow x_3 = x_1 \tag{137}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{138}$$

- For $\lambda_2 = 3$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{139}$$

$$\Rightarrow x_1 = 0 \tag{140}$$

$$x_3 = 0 \tag{141}$$

$$x_2 = x_2 \tag{142}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{143}$$

- For $\lambda_3 = 4$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{144}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{145}$$

$$\Rightarrow x_2 = 0 \tag{146}$$

$$-x_1 = x_3 \tag{147}$$

$$\Rightarrow E_{\lambda_3} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{148}$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{149}$$

$$v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{150}$$

$$v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{151}$$

$$\Rightarrow \beta = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\} \tag{152}$$

$$\Rightarrow Q = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & -1\end{pmatrix} \tag{153}$$

$$Q^tAQ = \psi_\beta(H) = \frac{1}{2}\begin{pmatrix}4 & 0 & 0\\0 & 6 & 0\\0 & 0 & 8\end{pmatrix} = \begin{pmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 4\end{pmatrix} \tag{154}$$

18. Let $\mathcal{S}$ be the set of all $(t_1, t_2, t_3) \in \mathsf{R}^3$ for which

$$3t_1^2 + 3t_2^2 + 3t_3^2 - t_1t_3 + 2\sqrt{2}(t_1 + t_3) + 1 = 0$$

Find an orthonormal basis β for R^3 for which teh equation relating the coordinates of points of $\mathcal{S}$ relative to β is simpler. Describe $\mathcal{S}$ geometrically.

Let $A = \begin{pmatrix}3 & 0 & -2\\0 & 3 & 0\\-2 & 0 & 3\end{pmatrix}$

$$\det\begin{pmatrix}3-t & 0 & -2\\0 & 3-t & 0\\-2 & 0 & 3-t\end{pmatrix} = (t-5)(t-3)(t-1) \tag{155}$$

$$\Rightarrow \lambda_1 = 5 \tag{156}$$

$$\lambda_2 = 3 \tag{157}$$

$$\lambda_3 = 1 \tag{158}$$

- For $\lambda_1 = 5$

$$\begin{pmatrix}-2 & 0 & -2\\ 0 & -2 & 0\\ -2 & 0 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{159}$$

$$\begin{pmatrix}0 & 0 & 0\\ 0 & -2 & 0\\ -2 & 0 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{160}$$

$$\Rightarrow x_1 = -x_3 \tag{161}$$

$$\Rightarrow E_{\lambda_1} = \left\{ t\begin{pmatrix}-1\\0\\1\end{pmatrix} : t \in \mathbb{R}\right\} \tag{162}$$

- For $\lambda_2 = 3$

$$\begin{pmatrix}0 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{163}$$

$$\Rightarrow x_2 = x_2 \tag{164}$$

$$\Rightarrow E_{\lambda_2} = \left\{ t\begin{pmatrix}0\\1\\0\end{pmatrix} : t \in \mathbb{R}\right\} \tag{165}$$

- For $\lambda_3 = 1$

$$\begin{pmatrix}2 & 0 & -2\\ 0 & 2 & 0\\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{166}$$

$$\begin{pmatrix}0 & 0 & 0\\ 0 & -2 & 0\\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{167}$$

$$\Rightarrow x_1 = x_3 \tag{168}$$

$$\Rightarrow E_{\lambda_3} = \left\{ t\begin{pmatrix}1\\0\\1\end{pmatrix} : t \in \mathbb{R}\right\} \tag{169}$$

$$Q = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{pmatrix} \tag{170}$$

$$Q^t AQ = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{171}$$

$$\Rightarrow K(x) = 5s_1^2 + 3s_2^2 + s_3^2 \tag{172}$$

$$x = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = Q \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{173}$$

$$\Rightarrow t_1 = -\frac{s_1}{\sqrt{2}} + \frac{s_3}{\sqrt{2}} \tag{174}$$

$$t_2 = s_2 \tag{175}$$

$$t_3 = \frac{s_1}{\sqrt{2}} + \frac{s_3}{\sqrt{2}} \tag{176}$$

$$2\sqrt{2}(t_1 + t_3) + 1 = 2\sqrt{2}\frac{2s_3}{\sqrt{2}} + 1 = 4s_3 + 1 \tag{177}$$

It folows that $x \in \mathcal{S}$ if and only if

$$5s_1^2 + 3s_2^2 + s_3^2 + 4s_3 + 1 = 0 \tag{178}$$

$$5(x')^2 + 3(y')^2 + (z')^2 + 4z' + 2 = 1 \tag{179}$$

$$5(x')^2 + 3(y')^2 + (z' + 2)^2 = 1 \tag{180}$$

It follows that $\mathcal{S}$ is an ellipsoid.

19. Prove the following refinement of Theorem 6.37(d).

(a) If $0 < \text{rank} A < n$ and A has no negative eigenvalues, then f has no local maximum at p.

(b) If $0 < \text{rank} A < n$ and A has no positive eigenvalues, then f has no local minimum at p.

(a) Because the matrix A does not have full rank at least 1 of the eigenvalues of A is equal to zero. Therefore some of the eigenvalues are positive while some are zero. Futhermore because the rank of A is greater than zero at least one non-zero eigenvalue is guarenteed. Without loss of generality assume that $p = 0$. Taking from the proof for Theorem 6.37 in the book

$$f(0) = 0 \leq \sum_{i=1}^{n} \left(\frac{1}{2}\lambda_i - \epsilon \right) s_i^2 < f(x) \tag{181}$$

It follows that because there is at least one non-zero eigenvalue that the sum in the above inequality is greater than zero. Therefore $f(x)$ is greater than $f(0) = 0$ around 0. Therefore there is no local maximum at 0.

(b) Because the matrix A does not have full rank at least 1 of the eigenvalues of A is equal to zero. Therefore some of the eigenvalues are negative while some are zero. Futhermore because the rank of A is greater than zero at least one non-zero eigenvalue is guarenteed. Without loss of generality assume that $p = 0$. Taking from the proof for Theorem 6.37 in the book

$$f(x) < \sum_{i=1}^{n} \left(\frac{1}{2}\lambda_i + \epsilon\right) s_i^2 \geq 0 = f(0) \tag{182}$$

It follows that because there is at least one non-zero eigenvalue that the sum in the above inequality is less than zero. Therefore $f(x)$ is less than $f(0) = 0$ around 0. Therefore there is no local minimum at 0.

20. Prove the following variation of the second-deriviative test for the case of $n = 2$: define

$$D - \left[\frac{\partial^2 f(p)}{\partial t_1^2}\right]\left[\frac{\partial^2 f(p)}{\partial t_2^2}\right] - \left[\frac{\partial^2 f(p)}{\partial t_1 \partial t_2}\right]^2$$

(a) If $D > 0$ and $\frac{\partial^2 f(p)}{\partial t_1^2} > 0$, then f has a local minimum at p.

(b) If $D > 0$ and $\frac{\partial^2 f(p)}{\partial t_1^2} < 0$, then f has a local maximum at p.

(c) If $D < 0$, then f has no local extremum at p.

(d) If $D = 0$, then the test is inconsclusive.

$$\det\begin{pmatrix} \frac{\partial^2 f(p)}{\partial t_1^2} - \lambda & \frac{\partial^2 f(p)}{\partial t_1 \partial t_2} \\ \frac{\partial^2 f(p)}{\partial t_1 \partial t_2} & \frac{\partial^2 f(p)}{\partial t_2^2} - \lambda \end{pmatrix} = 0 \tag{183}$$

$$\Rightarrow \lambda^2 - \lambda\left(\frac{\partial^2 f(p)}{\partial t_1^2} + \frac{\partial^2 f(p)}{\partial t_2^2}\right) + \frac{\partial^2 f(p)}{\partial t_1^2}\frac{\partial^2 f(p)}{\partial t_2^2} - \left(\frac{\partial^2 f(p)}{\partial t_1 \partial t_2}\right)^2 \tag{184}$$

$$\begin{aligned}\Rightarrow 2\lambda = {} & \frac{\partial^2 f(p)}{\partial t_1^2} + \frac{\partial^2 f(p)}{\partial t_2^2} \pm \\ & \pm \sqrt{\left(\frac{\partial^2 f(p)}{\partial t_1^2}\right)^2 - \frac{\partial^2 f(p)}{\partial t_1^2}\frac{\partial^2 f(p)}{\partial t_2^2} + \left(\frac{\partial^2 f(p)}{\partial t_2^2}\right)^2 + 4\left(\frac{\partial^2 f(p)}{\partial t_1 \partial t_2}\right)^2}\end{aligned} \tag{185}$$

(a) Based on the assumptions and equation 185, λ_1 and λ_2 are stricly positive therefore f has a local minimum at p.

(b) Based on the assumptions and equation 185, λ_1 and λ_2 are stricly negative therefore f has a local maximum at p.

(c) Based on the assumptions and equation 185, $\lambda_1 > 0$ and $\lambda_2 < 0$ therefore f has no local extrema at p.

(d)

$$\lambda^2 - \lambda \left(\frac{\partial^2 f(p)}{\partial t_1^2} + \frac{\partial^2 f(p)}{\partial t_2^2} \right) \tag{186}$$

$$\Rightarrow \lambda_1 = 0 \tag{187}$$

$$\lambda_2 = \frac{\partial^2 f(p)}{\partial t_1^2} + \frac{\partial^2 f(p)}{\partial t_2^2} \tag{188}$$

Based on the assumptions and equation 187 the test is inconclusive because at least one of the eigen values is zero.