Assignment

3.1: 5, 12; 3.2: 5(beg), 6(adf), 14, 20; 3.3: 2(ad), 3(ad), 7(bd), 9, 10

Work

3.1

5. Prove that E is an elementary matrix if and only if E^t is.
Claim: $E \rightsquigarrow E^t$

$$I_n = \begin{bmatrix} e_1 & e_2 & \cdots & e_i & \cdots & e_j & \cdots & e_n \end{bmatrix} \tag{1}$$

   (a) Claim: The interchange of any two rows i and j is equivalent to interchanging any two columns i and j
   By applying the interchange to E it follows that:

$$E = \begin{bmatrix} e_1 & e_2 & \cdots & e_j & \cdots & e_i & \cdots & e_n \end{bmatrix} \tag{2}$$

$$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & e_j & \cdots & e_i & \cdots & e_n \end{bmatrix} = E \tag{3}$$

   (b) Claim: Multiplying any row i with nonzero scalar c is equivalent to multiplying any column j with the same scalar c.
   By applying the scaling to E is follows that:

$$E = \begin{bmatrix} e_1 & e_2 & \cdots & ce_j & \cdots & e_n \end{bmatrix} \tag{4}$$

$$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & ce_i & \cdots & e_n \end{bmatrix} = E \tag{5}$$

   (c) Claim: Adding any scalar multiple of row i to row j is equivalent to adding any scalar multiple of column i it column j
   By applying the replacement to E it follows that:

$$E = \begin{bmatrix} e_1 & e_2 & \cdots & ce_i + e_j & \cdots & e_n \end{bmatrix} \tag{6}$$

$$\Rightarrow E^t = \begin{bmatrix} e_1 & e_2 & \cdots & ce_j + e_i & \cdots & e_n \end{bmatrix} \tag{7}$$

$$\therefore E^t \text{ is elementary} \tag{8}$$

12. Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

   (a) For $m = 2$

      i. If $a_{11} = 0$ and $a_{21} \neq 0$ interchanging rows 1 and 2 creates an upper triangular matrix.

      ii. If $a_{11} \neq 0$ adding the row 1 scaled by a_{21}/a_{11} and subtracted from row 2 creates an upper triangular matrix.

(b) For $m = k$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \tag{9}$$

i. If $m > n$

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \\ & \text{\huge 0} & & a_{n+1,n} \\ & & & \vdots \\ & & & a_{mn} \end{pmatrix} \tag{10}$$

ii. If $m < n$

$$A \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} \cdots & a_{1n} \\ & a_{22} & & \vdots & & \vdots \\ & \text{\huge 0} & \ddots & \vdots & & \vdots \\ & & & a_{mm} & a_{m,m+1} \cdots & a_{mn} \end{pmatrix} \tag{11}$$

(c) For $m = k + 1$

i. If $m > n$, assume the $m = k$ case holds

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & \text{\huge 0} & & a_{nn} \\ & & & \vdots \\ & & & a_{mn} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,n} \end{pmatrix} \tag{12}$$

Using row operations of type 3 on row $m + 1$ from row 1 to row n in order and make $a_{m+1,1} = 0$ in each row with row operations of type 3 on row $m + 1$ for i from 1 to n.

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \\ & \text{\huge 0} & & \vdots \\ & & & a_{mn} \\ & & & a_{m+1,n} \end{pmatrix} \tag{13}$$

ii. If $m < n$, assume the $m = k$ case holds

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ & a_{22} & & \vdots & & & \vdots \\ 0 & & \ddots & \vdots & & & \vdots \\ & & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,n} & a_{m+1,m+1} & \cdots & a_{m+1,n} \end{pmatrix} \tag{14}$$

Using row operations of type 3 on row $m+1$ from row 1 to row m in order and make $a_{m+1,j} = 0$ in each row i apply a row operation of type 3 on row $m+1$ for i from 1 to m

$$A \rightsquigarrow \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ & a_{22} & & \vdots & & & \vdots \\ & & \ddots & \vdots & & & \vdots \\ & 0 & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \\ & & & & a_{m+1,m+1} & \cdots & a_{m+1,n} \end{pmatrix} \tag{15}$$

3.2

5. For each of the following matrices, compute the rank and the inverse if it exists.

(b) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{16}$$

The rank of the matrix is 1, and it is not invertible.

(e) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & {}^{1}/_{6} & {}^{-1}/_{3} & {}^{1}/_{2} \\ 0 & 1 & 0 & {}^{1}/_{2} & 0 & {}^{-1}/_{2} \\ 0 & 0 & 1 & {}^{-1}/_{6} & {}^{1}/_{3} & {}^{1}/_{2} \end{array}\right) \tag{17}$$

It follows that the rank is 3 and the inverse is

$$\begin{pmatrix} {}^{1}/_{6} & {}^{-1}/_{3} & {}^{1}/_{2} \\ {}^{1}/_{2} & 0 & {}^{-1}/_{2} \\ {}^{-1}/_{6} & {}^{1}/_{3} & {}^{1}/_{2} \end{pmatrix} \tag{18}$$

(g) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \\ \\ | \cdot -1 \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right) \tag{19}$$

It follows that the rank is 4 and the inverse is

$$\begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix} \tag{20}$$

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

   (a) $\mathsf{T}\colon \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $\mathsf{T}(f(x)) = f'' + 2f'(x) - f(x)$

$$\mathsf{T}(1) = -1 \qquad \mathsf{T}(x) = 2 - x \qquad \mathsf{T}(x^2) = 2a + 4x - x^2 \tag{21}$$

$$\Rightarrow [\mathsf{T}]_\alpha^\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \tag{22}$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \tag{23}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\Rightarrow [\mathsf{T}^{-1}]_\beta^\alpha = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix} \tag{24}$$

$$\mathsf{T}^{-1}(c + bx + ax^2) = -ax^2 - (4a + b) - (a + 2b + c) \tag{25}$$

(d) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{P}_2(\mathbb{R})$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2$$

$$\mathsf{T}(1,0,0) = 1 + x + x^2 \qquad \mathsf{T}(0,1,0) = 1 - x \qquad \mathsf{T}(0,0,1) = 1 + x \tag{26}$$

$$[\mathsf{T}]_\alpha^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{27}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{28}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & {}^1/_2 & {}^{-1}/_2 & 0 \\ 0 & 0 & 1 & {}^1/_2 & {}^1/_2 & -1 \end{array}\right)$$

$$\Rightarrow [\mathsf{T}^{-1}]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 1 \\ {}^1/_2 & {}^{-1}/_2 & 0 \\ {}^1/_2 & {}^1/_2 & -1 \end{pmatrix} \tag{29}$$

$$\mathsf{T}^{-1}(ax^2 + bx + c) = \left(a, \left(\frac{1}{2}\right)c - \left(\frac{1}{2}\right)b, \left(\frac{1}{2}\right)c + \left(\frac{1}{2}\right)b - a\right) \tag{30}$$

(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{R}^4$ defined by

$$\mathsf{T}(A) = (\operatorname{tr}(A), \operatorname{tr}(A^t), \operatorname{tr}(EA), \operatorname{tr}(AE)),$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{31}$$

$$\mathsf{T}(A) = (a+d, a+d, c+b, c+b) \tag{32}$$

$$\mathsf{T}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) = (1,1,0,0) \tag{33}$$
$$\mathsf{T}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) = (0,0,1,1) \tag{34}$$
$$\mathsf{T}\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = (0,0,1,1) \tag{35}$$
$$\mathsf{T}\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) = (1,1,0,0) \tag{36}$$

$$[\mathsf{T}]_\alpha^\beta = \begin{pmatrix}1&0&0&1\\1&0&0&1\\0&1&1&0\\0&1&1&0\end{pmatrix} \tag{37}$$

$$\begin{pmatrix}1&0&0&1\\1&0&0&1\\0&1&1&0\\0&1&1&0\end{pmatrix}\begin{array}{l} \\ \\ {}^{-1} \\ {}_{+}\end{array} \rightsquigarrow \begin{pmatrix}1&0&0&1\\1&0&0&1\\0&1&1&0\\0&0&0&0\end{pmatrix} \tag{38}$$

T is not invertible.

14. Let $\mathsf{T}, \mathsf{U} \colon \mathsf{V} \to \mathsf{W}$ be linear transformations

    (a) Prove that $\mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$

    (b) Prove that W is finite-dimensional, then $\operatorname{rank}(\mathsf{T}+\mathsf{U}) \leq \operatorname{rank}(\mathsf{T}) + \operatorname{rank}(\mathsf{U})$

    (c) Deduce from (b) that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$ for any $m \times n$ matrices A and B

    (a) Claim: $\mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$

$$\forall x \in \mathsf{V}, (\mathsf{T}+\mathsf{U})(x) = \mathsf{T}(x) + \mathsf{U}(x) \quad \text{where } \mathsf{T}(x) \in \mathsf{R}(\mathsf{T}), \mathsf{U}(x) \in \mathsf{R}(\mathsf{U}) \tag{39}$$
$$\Rightarrow \mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U}) \tag{40}$$

    (b) From (a) it follows that

$$\dim(\mathsf{R}(\mathsf{T}+\mathsf{U})) \leq \dim(\mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})) \tag{41}$$

    From 1.6 exercise 31 (b) it follows that

$$\dim(\mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})) \leq \dim(\mathsf{R}(\mathsf{T})) + \dim(\mathsf{R}(\mathsf{U})) \tag{42}$$
$$\Rightarrow \dim(\mathsf{R}(\mathsf{T}+\mathsf{U})) \leq \dim(\mathsf{R}(\mathsf{T})) + \dim(\mathsf{R}(\mathsf{U})) \tag{43}$$
$$\Rightarrow \operatorname{rank}(\mathsf{T}+\mathsf{U}) \leq \operatorname{rank}(\mathsf{T}) + \operatorname{rank}(\mathsf{U}) \tag{44}$$

    (c) From theorem 3.3 it follows that

$$\operatorname{rank}(A+B) = \operatorname{rank}(\mathsf{L}_{A+B}) \tag{45}$$
$$(A+B)x = Ax + Bx \quad \forall x \in \mathsf{V} \tag{46}$$
$$\Rightarrow \mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B \tag{47}$$
$$\Rightarrow [\mathsf{T}_{A+B}]_\alpha^\beta = [\mathsf{T}_A]_\alpha^\beta + [\mathsf{T}_B]_\alpha^\beta \tag{48}$$
$$\operatorname{rank}(\mathsf{L}_A + \mathsf{L}_B) \leq \operatorname{rank}(\mathsf{L}_A) + \operatorname{rank}(\mathsf{L}_B) \quad \text{by 1.6 ex. 31} \tag{49}$$
$$\Rightarrow \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B) \quad \text{by theorem 3.3} \tag{50}$$

20. Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}$$

(a) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

(b) Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $\text{rank}(B) \leq 2$

(a)

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{51}$$

$$x_1 = s + 3t \qquad x_2 = -2s + t \qquad x_3 = s \qquad x_4 = -2t \qquad x_5 = t \tag{52}$$

$$\Rightarrow x = \left\{ t \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\} \tag{53}$$

It follows that a 5×5 matrix with rank 2 can be made by taking $t = s = 1$ and appended columns of zeros.

$$M = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

(b) From part (a) it follows that $\dim(K_H) = 2$ for $Ax = 0$
Claim: $\forall B \in \mathsf{M}_{5\times 5}(\mathbb{R})$ such that $AB = 0, \operatorname{rank}(B) \leq 2$
Suppose AB=0

$$\Rightarrow B_n \in K_H \forall j \tag{55}$$

$$\{B_j \colon j = 1, 2, \ldots, n\} \subseteq K_H \tag{56}$$

$$\operatorname{span}(B_j) \subseteq K_H \forall j \tag{57}$$

$$\Rightarrow \operatorname{col}(B_j) \subseteq K_K \tag{58}$$

$$\Rightarrow \operatorname{rank}(\operatorname{col}(B_j)) \leq \dim(K_H) = 2 \tag{59}$$

$$\Rightarrow \operatorname{rank}(B) \leq 2 \tag{60}$$

3.3

2. For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution set.

   (a)

$$x_1 + 3x_2 = 0 \tag{61}$$

$$2x_2 + 6x_2 = 0 \tag{62}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \tag{63}$$

$$\Rightarrow x_2 = t \qquad\qquad x_1 = -3t \tag{64}$$

$$\Rightarrow x = \left\{ t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{65}$$

$$\Rightarrow \dim(x) = 1 \tag{66}$$

   Take $t = 1$ it follows that a basis is $\{\left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)\}$

   (d)

$$x_1 + x_2 - x_3 = 0 \qquad x_1 - x_2 + x_3 = 0 \qquad x_1 + 2x_2 - 2x_3 = 0 \tag{67}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix} \quad \Big| \cdot \tfrac{1}{3} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{68}$$

$$x_3 = t \qquad\qquad x_2 = t \qquad\qquad x_1 = 0 \tag{69}$$

$$\Rightarrow x = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{70}$$

$$\Rightarrow \dim x = 1 \tag{71}$$

   Take $t = 1$ it follows that a basis is $\left\{\left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)\right\}$

3. Using the results of Exercise 2, find all solutions to the following systems.

   (a)

$$x_1 + 3x_2 = 5 \qquad\qquad 2x_1 + 6x_2 = 10 \tag{72}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 10 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right) \tag{73}$$

$$\Rightarrow x_2 = t \qquad\qquad x_1 = 5 - 3t \tag{74}$$

$$x = \left\{ \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{75}$$

(d)

$$2x_1 + x_2 - x_3 = 5 \qquad x_1 - x_2 + x_3 = 1 \qquad x_1 + 2x_2 - 2x_3 = 4 \tag{76}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & -2 & 4 \end{array}\right) \quad {\scriptstyle -2 \quad -1 \quad -1} \;\Big|\; \cdot \tfrac{1}{3} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{77}$$

$$\Rightarrow x_3 = t \qquad x_2 1 + t \qquad x_1 = 2 \tag{78}$$

$$\Rightarrow x = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{79}$$

7. Determine which of the following systems of linear equations has a solution.

(b)

$$x_1 + x_2 - x_2 = 1 \qquad 2x_1 + x_2 + 3x_2 = 2 \tag{80}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 3 & 2 \end{array}\right) \quad {\scriptstyle -2} \quad \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 5 & 0 \end{array}\right) \tag{81}$$

$$\Rightarrow x_3 = t \tag{82}$$

$$x_2 = 5t \tag{83}$$

$$x_1 = 1 - 4t \tag{84}$$

$$\Rightarrow x = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{85}$$

(d)

$$x_1 + x_2 + 3x_3 - x_4 = 0 \tag{86}$$

$$x_1 + x_2 + x_3 + x_3 = 1 \tag{87}$$

$$x_1 - 2x_2 + x_3 - x_4 = 1 \tag{88}$$

$$4x_1 + x_2 + 8x_3 - x_4 = 0 \tag{89}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & 1 \\ 4 & 1 & 8 & -1 & 0 \end{array}\right) \quad {\scriptstyle -1 \quad -1 \quad -4 \quad -1 \quad -1} \quad \begin{array}{l} \big|\; \cdot -\frac{1}{2} \\ \big|\; \cdot -\frac{1}{2} \end{array} \tag{90}$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & {}^2/_3 & 0 & {}^{-1}/_3 \\ 0 & 0 & 1 & 1 & {}^1/_2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\Rightarrow x_1 = -\frac{5}{2} \qquad x_2 = -2 \qquad x_3 = \frac{5}{2} \qquad x_4 = -2 \tag{91}$$

$$x = \left\{ \begin{pmatrix} -5/2 \\ 2 \\ 5/2 \\ -2 \end{pmatrix} \right\} \tag{92}$$

9. Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in \mathsf{R}(\mathsf{L}_A)$.
   ($\Rightarrow$)
   Suppose $Ax = b$ has a solution

$$\Rightarrow \exists x\colon \mathsf{L}_A(x) = b \tag{93}$$
$$\Rightarrow b \in \mathsf{R}(\mathsf{L}_A) \tag{94}$$

   ($\Leftarrow$)
   Suppose $b \in \mathsf{R}(\mathsf{L}_A)$

$$\Rightarrow \exists x\colon Ax = b \tag{95}$$

10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equation in n unknowns has rank m, then the system has a solution.
    Suppose $A \in \mathsf{M}_{m \times n}(F)$ and $\text{rank}(A) = m$
    Since $\text{rank}(A) = m$ it follows that

$$\text{rank}(A|b) = m \quad \text{for } b \in \mathsf{M}_{m\times 1} \tag{96}$$

    It follows that $Ax = b$ is consistent since $\text{rank}(A|b) = \text{rank}(A)$ by theorem 3.11.