

Assignment

2.3: 13,15,16,17; 2.4: 2(bef), 5, 17, 20; 2.5: 3(cd), 6(bc), 10, 13

Work

2.3

13. Let A and B be $n \times n$ matrices. Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.

Claim: $\text{tr}(AB) = \text{tr}(AB)$

$$\text{tr}(AB) = \sum_{i=1}^n (AB)_{ii} \quad (1)$$

$$= \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki} \quad (2)$$

$$= \sum_{k=1}^n \sum_{i=1}^n A_{ki} B_{ik} \quad (3)$$

$$= \sum_{k=1}^n \sum_{i=1}^n B_{ik} A_{ki} \quad (4)$$

$$= \sum_{i=1}^n (BA)_{ii} \quad (5)$$

$$= \text{tr}(BA) \quad (6)$$

15. Let M and A be matrices for which the product matrix MA is defined. If the j th column of A is a linear combination of a set of columns of A , prove that the j th column MA is linear combination of the corresponding columns of MA with the same corresponding coefficients.

Suppose $M \in \mathbf{M}_{m \times n}(F)$, $A \in \mathbf{M}_{n \times p}$, $MA \in \mathbf{M}_{m \times p}$

Suppose $v_j = j^{\text{th}}$ column of A and $u_j = j^{\text{th}}$ column of MA

$$v_j = \sum_{i=1}^p a_i v_i, \quad a_i \in F \quad (7)$$

Claim: $u_j = \sum_{i=1}^p a_i u_i$

$$Mv_j = M \sum_{i=1}^p a_i v_i = \sum_{i=1}^p a_i Mv_i \quad (8)$$

$$\Rightarrow u_j = u_j = \sum_{i=1}^p a_i u_i \quad (\text{by theorem 2.13}) \quad (9)$$

16. Let V be a finite-dimensional vector space, and let $T: V \rightarrow V$ be linear.

(a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$

(b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k

(a) Suppose $x \in R(T^2)$

$$\exists a \in V \text{ such that } T(T(a)) = x \quad (10)$$

$$\Rightarrow x \in R(T) \quad (11)$$

$$\Rightarrow R(T^2) \subseteq R(T) \quad (12)$$

$$\text{rank}(T) = \text{rank}(T^2) \quad (13)$$

$$\therefore R(T) = R(T^2) \quad (14)$$

Suppose $x \in N(T)$

$$\Rightarrow T(x) = 0 \quad (15)$$

$$\Rightarrow T(T(x)) = 0 \quad (16)$$

$$\Rightarrow x \in N(T^2) \quad (17)$$

By dimension theory:

$$\dim(R(T)) + \dim(N(T)) = \dim(R(T^2)) + \dim(N(T^2)) \quad (18)$$

$$\dim(N(T)) = \dim(N(T^2)) \quad (\text{by 14}) \quad (19)$$

$$\Rightarrow N(T) = N(T^2) \quad (20)$$

Claim: $R(T) \cap N(T) = \{0\}$

Suppose $u \in R(T) \cap N(T)$

$$\exists y \in V \text{ such that } T(y) = u \quad (21)$$

$$T^2(y) = 0 \quad (22)$$

$$\therefore y \in N(T^2) \quad (23)$$

$$\therefore y \in N(T) \quad (\text{by 20}) \quad (24)$$

$$u = T(y) = 0 \quad (25)$$

Claim: $V = N(T) \oplus R(T)$

Suppose β_N is a basis for $N(T)$ and β_R is a basis for $R(T)$

$$N(T) \cap R(T) = \{0\} \quad (26)$$

$$\Rightarrow \text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0\} \quad (27)$$

$$\therefore \beta_N \cap \beta_R = \{\} \quad (28)$$

$$\beta_N = \{n_1, n_2, \dots, n_k\} \Rightarrow \text{card}(\beta_N) = k \quad (29)$$

$$\beta_R = \{r_1, r_2, \dots, r_m\} \Rightarrow \text{card}(\beta_R) = m \quad (30)$$

For i from 1 to k if $n_i \in \text{span}(\beta_R)$

$$\text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0, n_i\} \not\subseteq \text{Contradiction! with 27} \quad (31)$$

For i from 1 to m , if $r_i \in \text{span}(\beta_N)$

$$\text{span}(\beta_N) \cap \text{span}(\beta_R) = \{0, r_i\} \not\subseteq \text{Contradiction! with 27} \quad (32)$$

$$\therefore \beta_N \cup \beta_R \text{ is linearly independent} \quad (33)$$

$$\text{card}(\beta_N \cup \beta_R) = \text{card}(\beta_N) + \text{card}(\beta_R) - \text{card}(\beta_N \cap \beta_R) \quad (34)$$

$$= \text{card}(\beta_N) + \text{card}(\beta_R) \quad (35)$$

$$= k + m = n \quad (36)$$

From 33 and 36 it follows that

$$\mathbf{V} = \text{span}(\beta_N \cup \beta_R) \quad (37)$$

Claim: $\text{span}(\beta_N \cup \beta_R) = \text{span}(\beta_N) + \text{span}(\beta_R) = N(\mathbf{T}) + R(\mathbf{T})$

(\supseteq) since $\text{span}(\beta_N)$ and $\text{span}(\beta_R)$ are both contained in \mathbf{V} it follows that

$$\text{span}(\beta_N) + \text{span}(\beta_R) \subseteq \mathbf{V} \quad (38)$$

(\subseteq) Suppose $x \in \text{span}(\beta_N \cup \beta_R)$

$$x = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} \cdots + a_{k+m} v_{k+m} \quad (39)$$

$$a_1 v_1 + \cdots + a_k v_k \in \text{span}(\beta_R) \quad (40)$$

$$a_{k+1} v_{k+1} + \cdots + a_{k+m} v_{k+m} \in \text{span}(\beta_N) \quad (41)$$

$$\Rightarrow x \in \text{span}(\beta_N) + \text{span}(\beta_R) \quad (42)$$

$$\Rightarrow N(\mathbf{T}) \oplus R(\mathbf{T}) = \mathbf{V} \quad (43)$$

(b) Claim: $\text{rank}(\mathbf{T}^k) = \text{rank}(\mathbf{T}^{k+1})$ for some integer k

$$0 \leq \cdots \leq \dim(R(\mathbf{T}^3)) \leq \dim(R(\mathbf{T}^2)) \leq \dim(R(\mathbf{T})) \leq n \quad (44)$$

Suppose $x \in R(\mathbf{T}^{k+1})$

$$\exists a \in \mathbf{V} \text{ such that } \mathbf{T}(\mathbf{T}(a)) = x \quad (45)$$

$$\Rightarrow x \in R(\mathbf{T}^k) \quad (46)$$

$$\Rightarrow R(\mathbf{T}^k) \leq R(\mathbf{T}^{k+1}) \quad (47)$$

Since there is a lower bound 0 and an upper bound n in 44 it follows that for some integer k

$$\dim(R(\mathbf{T}^{k+1})) = \dim(R(\mathbf{T}^k)) \quad (48)$$

$$\Rightarrow R(\mathbf{T}^{k+1}) = R(\mathbf{T}^k) \text{ (by 47 and 48)} \quad (49)$$

$$\Rightarrow \text{rank}(\mathbf{T}^k) = \text{rank}(\mathbf{T}^{k+1}) \text{ for some integer } k \quad (50)$$

Claim: $R(T^{k+m}) = R(T^k) \forall m \geq 1$ and for some integer k

For $m = 1$ by 49

For $m = j \rightarrow R(T^{k+j}) = R(T^k)$

For $m = j + 1$

$$R(T^{k+j+1}) = T(R(T^{k+j})) = T(R(T^k)) = R(T^{k+1}) = R(T^k) \quad (51)$$

$$\Rightarrow R(T^{k+j+1}) = R(T^k) \quad (52)$$

$$\Rightarrow \text{rank}(T^{k+m}) = \text{rank}(T^k) \text{ for some integer } k \text{ and for all integer } m \quad (53)$$

$$\Rightarrow \text{rank}(T^{2k}) = \text{rank}(T^k) \text{ for some } k \quad (54)$$

Apply the same method as in part (a) and it follows that

$$V = R(T^k) \oplus N(T^k) \text{ for some } k \quad (55)$$

17. Let V be a vector space. Determine all linear transformations $T: V \rightarrow V$ such that $T = T^2$.

Suppose $W = \{y: T(y) = y\}$

Claim: $\forall T \in \mathcal{L}(V), T = T^2 \Leftrightarrow T$ is a projection of W along $N(T)$

Claim: $W \cap N(T) = \{0\}$

Suppose $x \in W \cap N(T)$

$$\Rightarrow T(x) = x \quad (56)$$

$$\Rightarrow T(x) = 0 \quad (57)$$

$$\Rightarrow x = 0 \quad (58)$$

$$\Rightarrow W \cap N(T) = \{0\} \quad (59)$$

$N(T)$ is a subspace of V (by theorem 2.1)

Claim: W is subspace of V

$$0 \in W \quad (60)$$

Suppose $x, z \in W$

$$T(x + z) = T(x) + T(z) = x + z \quad (61)$$

Suppose $x \in W, c \in F$

$$T(cx) = cT(x) = cx \quad (62)$$

$$W \subseteq V \text{ by definition} \quad (63)$$

Claim $V = W \oplus N(T)$

(\subseteq) Suppose $x \in V$

$$\Rightarrow x = T(x) + (x - T(x)) \quad (64)$$

$$T(x) = T^2(x) \quad (65)$$

$$T(x) \in W \quad (66)$$

$$T(x - T(x)) = T(x) - T^2(x) = 0 \quad (67)$$

$$\Rightarrow x \in W \oplus N(T) \quad (68)$$

$$\Rightarrow V \subseteq W \quad (69)$$

(\supseteq) Suppose $W \subseteq V$

$$N(T) \subseteq V \quad (70)$$

$$\Rightarrow W \oplus N(T) \text{ by closure of } V \quad (71)$$

$$\forall x \in V, x = x_1 + x_2 \text{ for some } x_1 \in W, x_2 \in N(T) \quad (72)$$

$$T(x) = T(x_1) + T(x_2) \quad (73)$$

$$= x_1 + 0 \quad (74)$$

$$= x_1 \quad (75)$$

It follows that T is a projection of W along $N(T)$

Suppose T is a projection of W along $N(T)$

Suppose $x \in W \oplus N(T)$

$$\Rightarrow x = x_1 + x_2 \text{ for some } x_1 \in W, x_2 \in N(T) \quad (76)$$

$$T(x) = T(x_1 + x_2) = T(x_1) + T(x_2) = T(x_1) + 0 = x_1 \quad (77)$$

$$\Rightarrow T(x_1) = 0 \quad (78)$$

$$T^2(x) = T(x_1) \quad (79)$$

$$\Rightarrow T^2(x) = x \quad (80)$$

$$\Rightarrow T^2(x) = T(x) \forall x \in V \quad (81)$$

$$\therefore T = T^2 \quad (82)$$

2.4

2. For each of the following linear transformations T , determine whether T is invertible and justify your answer.

(b) $\mathsf{T}: \mathbb{T}^2 \rightarrow \mathbb{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (3a_1 - 2a_2, a_2, 4a_1)$

Claim: T is 1-1

Suppose $x, y \in \mathbb{R}^2$ such that $x = (a_1, a_2), y = (a_3, a_4)$ and $\mathsf{T}(x) = \mathsf{T}(y)$ for $a_i \in \mathbb{R}$

$$(3a_1 - a_2, a_2, 4a_1) = (3a_3 - a_4, a_4, 4a_3) \quad (83)$$

$$\Rightarrow 3a_1 - a_2 = 3a_3 - a_4 \quad (84)$$

$$a_2 = a_4 \quad (85)$$

$$4a_1 = 4a_3 \quad (86)$$

$$\Rightarrow a_1 = a_3 \quad (87)$$

$$a_2 = a_4 \quad (88)$$

$$\Rightarrow x = y \quad (89)$$

Claim: T is onto

Suppose $x \in \mathbb{R}^3$ such that $x = (b_1, b_2, b_3)$ for $b_i \in \mathbb{R}$

Let $b_2 = a_2, b_3 = 4a_1, b_1 = (\frac{3}{4}b_3 - b_2)$

$$\Rightarrow (b_1, b_2, b_3) = (2a_1 - a_2, a_2, 4a_1) \quad (90)$$

$$\Rightarrow x \in R(\mathsf{T}) \quad (91)$$

$$R(\mathsf{T}) \subseteq M_{n \times n}(\mathbb{R}) \text{ by def of } \mathsf{T} \quad (92)$$

$\therefore \mathsf{T}$ is invertible

(e) $\mathsf{T}: M_{2 \times 2}(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by $\mathsf{T} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$

Claim: T is not 1-1

Suppose $x, y \in M_{2 \times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} \quad (93)$$

$$\Rightarrow x \neq y \quad (94)$$

$$\mathsf{T}(x) = (0 \cdot 1) + (2 \cdot 0)x + (1 + 4)x^2 = 5x^2 \quad (95)$$

$$\mathsf{T}(y) = (0 \cdot 1) + (2 \cdot 0)x + (3 + 2)x^2 = 5x^2 \quad (96)$$

$\therefore \mathsf{T}$ is not invertible

(f) $T: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Claim: T is 1-1

Suppose $x, y \in M_{2 \times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad y = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad (97)$$

Suppose $T(x) = T(y)$ for $a, b, c, \dots, h \in \mathbb{R}$

$$\Rightarrow \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} \quad (98)$$

$$\Rightarrow a+b = e+f \quad (99)$$

$$a = e \quad (100)$$

$$c = g \quad (101)$$

$$c+d = g+h \quad (102)$$

$$\Rightarrow a = e \quad (103)$$

$$b = f \quad (104)$$

$$c = g \quad (105)$$

$$d = h \quad (106)$$

$$\Rightarrow x = y \quad (107)$$

Claim: T is onto

Suppose $x \in M_{2 \times 2}(\mathbb{R})$ such that

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for } a, b, c, d \in \mathbb{R} \quad (108)$$

Let $e, f, g, h \in \mathbb{R}$ such that

$$e = b \quad f = e + a \quad (109)$$

$$g = c \quad h = -g + d \quad (110)$$

$$x = \begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} \quad (111)$$

$$\Rightarrow x \in R(T) \quad (112)$$

$$R(T) \subseteq M_{2 \times 2}(\mathbb{R}) \text{ by definition of } T \quad (113)$$

$\therefore T$ is invertible

5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Claim: A^t is invertible

$$\Rightarrow (AB)^t = (BA)^t = I^t = I \quad (114)$$

Lemma: $(AB)^t = B^t A^t$

$$(AB)^t_{ij} = (AB)_{ji} \quad (115)$$

$$= \sum_{k=1}^n A_{jk} B_{kj} \quad (116)$$

$$(B^t A^t)_{ij} = \sum_{k=1}^n B^t_{ik} A^t_{kj} \quad (117)$$

$$= \sum_{k=1}^n B_{kj} A_{jk} \quad (118)$$

$$= \sum_{k=1}^n A_{jk} B_{kj} \quad (119)$$

$$\Rightarrow B^t A^t = A^t B^t = I \quad (120)$$

Claim: $(A^{-1})^t = (A^t)^{-1}$

$$B = A^{-1} \quad (121)$$

$$(A^t)^{-1} = B^t = (A^{-1})^t \quad (122)$$

17. Let V and W be finite-dimensional vector spaces and $T: V \rightarrow W$ be an isomorphism.

Let V_0 be a subspace of V .

(a) Prove that $T(V_0)$ is a subspace of W .

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Let V_0 be a subspace of V

(a) Claim: $0 \in V_0$

$$0 \in V_0 \quad (123)$$

$$\Rightarrow T(0) = 0 \quad (124)$$

Claim: $\forall x, y \in T(V_0), x + y \in T(V_0)$

Suppose β_{V_0} is basis for V_0

$$\Rightarrow V_0 = \text{span}(\beta_{V_0}) \quad (125)$$

Suppose $x, y \in \mathsf{T}(\mathsf{V}_0)$

$$\Rightarrow x = \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad (126)$$

$$y = \mathsf{T}(b_1u_1 + b_2u_2 + \cdots + b_nu_n) \quad \text{for } a_i, b_i \in F, u_i \in \beta_{\mathsf{V}_0} \quad (127)$$

$$\Rightarrow x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \quad (128)$$

$$y = b_1\mathsf{T}(u_1) + b_2\mathsf{T}(u_2) + \cdots + b_n\mathsf{T}(u_n) \quad (129)$$

$$\Rightarrow x + y = (a_1 + b_1)\mathsf{T}(u_1) + (a_2 + b_2)\mathsf{T}(u_2) + \cdots + (a_n + b_n)\mathsf{T}(u_n) \quad (130)$$

$$= \mathsf{T}((a_1 + b_1)u_1 + (a_2 + b_2)u_2 + \cdots + (a_n + b_n)u_n) \quad (131)$$

$$\Rightarrow x + y \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \quad (132)$$

Claim: $\forall x \in \mathsf{T}(\mathsf{V}_0)$ and $\forall c \in F, cx \in \mathsf{T}(\mathsf{V}_0)$

Suppose $x \in \mathsf{T}(\mathsf{V}_0)$ such that

$$x = \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad \text{for } u_i \in \beta_{\mathsf{V}_0}, a_i \in F \quad (133)$$

$$\Rightarrow x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \quad (134)$$

$$cx = c(a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n)) \quad (135)$$

$$= ca_1\mathsf{T}(u_1) + ca_2\mathsf{T}(u_2) + \cdots + ca_n\mathsf{T}(u_n) \quad (136)$$

$$= \mathsf{T}(ca_1u_1 + ca_2u_2 + \cdots + ca_nu_n) \quad (137)$$

$$\Rightarrow cx \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \quad (138)$$

$\mathsf{T}(\mathsf{V}_0) \subseteq \mathsf{W}$ by the definition of T . It follows that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W .

(b) Claim: $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$

Suppose β_{V_0} is a basis of V_0

Lemma: $\mathsf{T}(\beta_{\mathsf{V}_0})$ is a basis of $\mathsf{T}(\mathsf{V}_0)$

Claim: $\text{span}(\mathsf{T}(\beta_{\mathsf{V}_0})) = \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0}))$

(\subseteq) Suppose $x \in \text{span}(\mathsf{T}(\beta_{\mathsf{V}_0}))$

$$x = a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \quad (139)$$

$$= \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad \text{for } u_i \in \beta_{\mathsf{V}_0}, a_i \in F \quad (140)$$

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n \in \text{span}(\beta_{\mathsf{V}_0}) \quad (141)$$

$$\Rightarrow x \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0})) \quad (142)$$

(\supseteq) Suppose $x \in \mathsf{T}(\text{span}(\beta_{\mathsf{V}_0}))$ such that

$$x = \mathsf{T}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) \quad \text{for } a_i \in F, u_i \in \beta_{\mathsf{V}_0} \quad (143)$$

$$= a_1\mathsf{T}(u_1) + a_2\mathsf{T}(u_2) + \cdots + a_n\mathsf{T}(u_n) \quad (144)$$

$$\Rightarrow x \in \text{span}(\mathsf{T}(\beta_{\mathsf{V}_0})) \quad (145)$$

Claim: $\mathsf{T}(\beta_{\mathsf{V}_0})$ is linearly independent

$$\beta_{\mathsf{V}_0} \text{ is linearly independent and } \mathsf{T} \text{ is 1-1} \quad (146)$$

$$\Rightarrow \mathsf{T}(\beta_{\mathsf{V}_0}) \text{ is linearly independent by HW.2.1.14.a} \quad (147)$$

Claim: $\text{card}(\beta_{V_0}) = \text{card}(T(\beta_{V_0}))$

T is an isomorphism. Equivalently it also invertible and 1-1. Therefore it follows that for every $x \in T(\beta_{V_0})$ there exists a unique vector $y \in \beta_{V_0}$ such that $T(y) = x$

$$\therefore \dim(\beta_{V_0}) = \dim(T(\beta_{V_0})) \quad (148)$$

20. Let $T: V \rightarrow W$ be a linear transformation from an n -dimensional vector space V to an m -dimensional vector space W . Let β and γ be ordered bases for V and W respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_{\beta}^{\gamma}$.

Claim: $\text{nullity}(T) = \text{nullity}(L_A)$

$N(T)$ is a subspace of V . This implies that $\phi_{\beta}(N(T))$ is a subspace of F^n and $\dim(N(T)) = \dim(\phi_{\beta}(N(T)))$ by HW.2.4.17

Claim: $\phi_{\beta}(N(T)) = N(L_A)$

(\subseteq) Suppose $y \in \phi_{\beta}(N(T))$

$$\Rightarrow y = \phi_{\beta}(x) \quad (149)$$

$$\Rightarrow L_A = L_A \phi_{\beta}(x) = \phi_{\gamma} T(x) = \phi_{\gamma}(0) = 0 \quad (150)$$

$$y \in N(L_A) \quad (151)$$

(\supseteq) Suppose $y \in N(L_A)$

$$N(L_A) \subseteq F^n \text{ by definition of } N(L_A) \quad (152)$$

$$\Rightarrow \forall y \in N(L_A), \exists! x \in V \text{ such that } \phi_{\beta}(x) = y \quad (153)$$

$\therefore \phi_{\beta}$ is an isomorphism

$$L_A(y) = L_A(\phi_{\beta}(x)) = 0 \quad (154)$$

$$L_A \phi_{\beta} = \phi_{\gamma} T \quad (155)$$

$$\Rightarrow \phi_{\gamma}(T(x)) = 0 \Rightarrow T(x) = 0 \quad (156)$$

$\therefore \phi_{\gamma}$ is an isomorphism

$$\therefore y \in \phi_{\beta}(N(T)) \quad (157)$$

$$\phi_{\beta}(N(T)) = N(L_A) \Rightarrow \dim(\phi_{\beta}(N(T))) = \dim(N(L_A)) = \dim(N(T)) \quad (158)$$

$$\therefore \text{nullity}(T) = \text{nullity}(L_A) \quad (159)$$

Claim: $\text{rank}(\mathbf{T}) = \text{rank}(\mathbf{L}_A)$

$R(\mathbf{T})$ is a subspace of \mathbf{W} . This implies that $\phi_\gamma(R(\mathbf{T}))$ is subspace of \mathbf{W} and $\dim(R(\mathbf{T})) = \dim(\phi_\gamma(R(\mathbf{T})))$ by HW.2.4.17

Claim: $\phi_\gamma(R(\mathbf{T})) = R(\mathbf{L}_A)$

(\subseteq) Suppose $x \in \phi_\gamma(R(\mathbf{T}))$

$$\Rightarrow x \in \phi_\gamma(y) \quad \text{for some } y \in R(\mathbf{T}) \quad (160)$$

$$\Rightarrow y \in \mathbf{T}(z) \quad \text{for some } z \in \mathbf{V} \quad (161)$$

$$\phi_\gamma(\mathbf{T}(z)) = (\phi_\gamma \mathbf{T})(x) \quad (162)$$

$$\phi_\gamma \mathbf{T} = \mathbf{L}_A \phi_\beta \quad (163)$$

$$\Rightarrow x = (\mathbf{L}_A \phi_\beta)(z) \quad (164)$$

$$= \mathbf{L}_A(w) \quad \text{for some } w \in \mathbf{F}^n \quad (165)$$

$$\Rightarrow x \in R(\mathbf{L}_A) \quad (166)$$

(\supseteq) Suppose $x \in R(\mathbf{L}_A)$

$$\Rightarrow x = [\mathbf{T}]_\beta^\gamma \phi_\beta(z) \quad \text{for some } z \in \mathbf{V} \quad (167)$$

$$\Rightarrow x = \mathbf{L}_A(\phi_\beta(z)) = (\mathbf{L}_A \phi_\beta)(z) = \phi_\gamma \mathbf{T}(z) \quad (168)$$

$$\mathbf{T}(z) \in R(\mathbf{T}) \quad (169)$$

$$\Rightarrow x \in \phi_\gamma(R(\mathbf{T})) \quad (170)$$

$$\phi_\gamma(R(\mathbf{T})) = R(\mathbf{L}_A) \Rightarrow \dim(\phi_\gamma(R(\mathbf{T}))) = \dim(R(\mathbf{L}_A)) = \dim(R(\mathbf{T})) \quad (171)$$

$$\therefore \text{rank}(\mathbf{T}) = \text{rank}(\mathbf{L}_A) \quad (172)$$

2.5

3. For each of the following pairs of ordered bases β and β' for $\mathbf{P}_2(\mathbb{R})$, find the change or coordinate matrix that changes β' -coordinates into β -coordinates.

(c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = 1 \quad (173)$$

$$2a + 3b + c = 0 \quad (174)$$

$$-a + c = 0 \quad (175)$$

$$b = 1 \quad (176)$$

$$a = 0 \quad (177)$$

$$b = 1 \quad (178)$$

$$c = -3 \quad (179)$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x \quad (180)$$

$$2a + 3b + c = 0 \quad (181)$$

$$a + 1 \quad (182)$$

$$b = 0 \quad (183)$$

$$a = -1 \quad (184)$$

$$b = 0 \quad (185)$$

$$c = 2 \quad (186)$$

$$a(2x^2 - x) + b(3x^2 + 1) + c(x^2) = x^2 \quad (187)$$

$$2a + 3b + c = 1 \quad (188)$$

$$-a = 0b = 0 \quad (189)$$

$$a = 0 \quad (190)$$

$$b = 0 \quad (191)$$

$$c = 1 \quad (192)$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix} \quad (193)$$

(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = x^2 + x + 4 \quad (194)$$

$$a + c = 1 \quad (195)$$

$$-a + b = 1 \quad (196)$$

$$a + b + c = 4 \quad (197)$$

$$a = 2 \quad (198)$$

$$b = 3 \quad (199)$$

$$c = -11 \quad (200)$$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = 4x^2 - 3x + 2 \quad (201)$$

$$a + c = 4 \quad (202)$$

$$-a + b = -3 \quad (203)$$

$$a + b + c = 2 \quad (204)$$

$$a = 1 \quad (205)$$

$$b = -2 \quad (206)$$

$$c = 3 \quad (207)$$

$$a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1) = 2x^2 + 3 \quad (208)$$

$$a + c = 2 \quad (209)$$

$$-a + b = 0 \quad (210)$$

$$a + b + c = 3 \quad (211)$$

$$a = 1 \quad (212)$$

$$b = 1 \quad (213)$$

$$c = 1 \quad (214)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix} \quad (215)$$

6. For each matrix A and ordered basis β , find $[\mathbf{L}_A]_\beta$. Also find an invertible matrix Q such that $[\mathbf{L}_A]_\beta = Q^{-1}AQ$.

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (216)$$

$$\mathbf{L}_A(v_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (217)$$

$$\Rightarrow [\mathbf{L}_A(v_1)]_\beta = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (218)$$

$$\mathbf{L}_A(v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (219)$$

$$\Rightarrow [\mathbf{L}_A(v_2)]_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (220)$$

$$\Rightarrow [\mathbf{L}_A] = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \quad (221)$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (222)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array} \right) \xrightarrow[\leftarrow_+]{\leftarrow^{-1}} \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array} \right) \quad (223)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \quad (224)$$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (225)$$

$$\mathbf{L}_A(v_1) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad (226)$$

$$\mathbf{L}_A(v_2) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \quad (227)$$

$$\mathbf{L}_A(v_3) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \quad (228)$$

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (229)$$

$$\begin{array}{rclclcl} a+b+c & = & 1 & a+b+c=1 & b & = -2 \\ \Rightarrow a+c & = & 3 \Rightarrow & a+c=3 & \Rightarrow a & = 2 \\ a+b+2c & = & 2 & c=1 & c & = 1 \end{array}$$

$$\Rightarrow [\mathbf{L}_A(v_1)]_\beta = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad (230)$$

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (231)$$

$$\begin{array}{rclclcl} a+b+c & = & 0 & a+b+c=0 & b & = -3 \\ \Rightarrow a+c & = & 3 \Rightarrow & a=2 & \Rightarrow a & = 2 \\ a+b+2c & = & 1 & c=1 & c & = 1 \end{array}$$

$$\Rightarrow [\mathbf{L}_A(v_2)]_\beta = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (232)$$

$$\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (233)$$

$$\begin{array}{rclclcl} a+b+c & = & 0 & a+b+c=0 & b & = -4 \\ \Rightarrow a+c & = & 4 \Rightarrow & a=2 & \Rightarrow a & = 2 \\ a+b+2c & = & 2 & c=2 & c & = 2 \end{array}$$

$$\Rightarrow [\mathbf{L}_A(v_3)]_\beta = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \quad (234)$$

$$\Rightarrow [\mathbf{L}_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix} \quad (235)$$

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad (236)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{c} \begin{array}{c} \boxed{\begin{array}{c} \leftarrow^{-1} \\ \leftarrow_+ \end{array}}^{-1} \\ \leftarrow_+ \end{array} \begin{array}{c} \boxed{\begin{array}{c} \leftarrow_+ \\ \leftarrow_1 \end{array}}^+ \\ \leftarrow_1 \end{array} \end{array} \mid \cdot -1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \quad (237)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (238)$$

10. Prove that if A and B are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$.

$$\text{tr}(B) = \text{tr}(QAQ^{-1}) \quad (239)$$

$$= \text{tr}((QA)Q^{-1}) \quad (240)$$

$$= \text{tr}(Q^{-1}(QA)) \quad (241)$$

$$= \text{tr}((Q^{-1}Q)A) \quad (242)$$

$$= \text{tr}(A) \text{ (by HW.2.3.13)} \quad (243)$$

13. Let V be a finite-dimensional vector space over a field F , and let $\beta = \{x_1, \dots, x_n\}$ be an ordered basis for V . Let Q be an $n \times n$ invertible matrix with entries from F . Define

$$x'_j = \sum_{i=1}^n Q_{ij}x_i \text{ for } 1 \leq j \leq n$$

and set $\beta' = \{x'_1, x'_2, \dots, x'_n\}$. Prove that β' is a basis for V and hence that Q is a coordinate matrix changing β' -coordinates into β -coordinates.

Claim $\text{span}(\beta) = \text{span}(\beta')$

Reverse Direction

Suppose $x' \in \text{span}(\beta')$

$$x' = c_1 \left(\sum_{i=1}^n Q_{i1}x_i \right) + c_2 \left(\sum_{i=1}^n Q_{i2}x_i \right) + \dots + c_n \left(\sum_{i=1}^n Q_{in}x_i \right) \quad (244)$$

$$\begin{aligned} x' = & c_1 (Q_{11}x_1 + Q_{21}x_2 + \dots + Q_{n1}x_n) + \\ & + c_2 (Q_{12}x_1 + Q_{22}x_2 + \dots + Q_{n2}x_n) + \\ & + \dots + c_n (Q_{1n}x_1 + Q_{2n}x_2 + \dots + Q_{nn}x_n) \end{aligned} \quad (245)$$

$$\begin{aligned} x' = & (c_1Q_{11} + c_2Q_{12} + \dots + c_nQ_{1n})x_1 + \\ & + (c_1Q_{21} + c_2Q_{22} + \dots + c_nQ_{2n})x_2 + \\ & + \dots + (c_1Q_{n1} + c_2Q_{n2} + \dots + c_nQ_{nn})x_n \end{aligned} \quad (246)$$

$$\Rightarrow x \in \text{span}(\beta') \quad (247)$$

Forward Direction

Suppose $x \in \text{span}(\beta)$

$$x = c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad (248)$$

$$= \sum_{i=1}^n c_i x_i \quad (249)$$

Let $c_i = \sum_{j=1}^n a_j Q_{ij}$

$$x = \sum_{i=1}^n \left(x_i \sum_{j=1}^n a_j Q_{ij} \right) \quad (250)$$

$$x = \sum_{i=1}^n ((a_1 Q_{i1} + a_2 Q_{i2} + \cdots + a_n Q_{in}) x_i) \quad (251)$$

$$\begin{aligned} x = & (a_1 Q_{11} + a_2 Q_{12} + \cdots + a_n Q_{1n})x_1 + \\ & + (a_1 Q_{21} + a_2 Q_{22} + \cdots + a_n Q_{2n})x_2 + \\ & + \cdots + (a_1 Q_{n1} + a_2 Q_{n2} + \cdots + a_n Q_{nn})x_n \end{aligned} \quad (252)$$

$$\begin{aligned} x = & a_1(Q_{11}x_1 + Q_{21}x_2 + \cdots + Q_{n1}x_n) + \\ & + a_2(Q_{12}x_1 + Q_{22}x_2 + \cdots + Q_{n2}x_n) + \\ & + \cdots + a_n(Q_{1n}x_1 + Q_{2n}x_2 + \cdots + Q_{nn}x_n) \end{aligned} \quad (253)$$

$$x = a_1 \sum_{i=1}^n Q_{i1}x_i + a_2 \sum_{i=1}^n Q_{i2}x_i + \cdots + a_n \sum_{i=1}^n Q_{in}x_i \quad (254)$$

$$\Rightarrow x \in \text{span}(\beta') \quad (255)$$

Suppose $x \in \text{span}(\beta')$ such that

$$x = a_1 Qx_1 + a_2 Qx_2 + \cdots + a_n Qx_n = 0 \quad (256)$$

$$\Rightarrow Q^{-1}Q(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 0 \quad (257)$$

$$\Rightarrow I_n(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 0 \quad (258)$$

$$\therefore \beta' \text{ is linearly independent} \quad (259)$$

Claim: $Q = [I_V]_{\beta'}^{\beta}$

$$\mathsf{T}_{[I_V]_{\beta'}^{\beta}}(v_i) = v'_i \text{ by definition of } [I_V]_{\beta'}^{\beta} \quad (260)$$

$$\Rightarrow \mathsf{T}_{[I_V]_{\beta'}^{\beta}}(\beta) = \beta' \quad (261)$$

Define $\mathsf{T}_Q: V \rightarrow V$ such that $\mathsf{T}_Q(x) = Qx, \forall x \in V$

$$\mathsf{T}_Q(\beta) = \beta' \quad (262)$$

$$\Rightarrow \mathsf{T}_{[I_V]_{\beta'}^{\beta}} = \mathsf{T}_Q \text{ (by theorem 2.6)} \quad (263)$$

$$\Rightarrow [I_V]_{\beta'}^{\beta} = Q \quad (264)$$