

Assignment

Appendix E: Prove theorems E.3, E.5, E.6, E.7; Section 5.2: 2(bde), 3(ade), 7, 9, 11

Work

Appendix E

3. Let $f(x)$ be a polynomial with coefficients from a field F , and let T be a linear operator on a vector space V over F . Then the following statements are true

- (a) $f(T)$ is a linear operator on V .
- (b) If β is a finite ordered basis for V and $A = [T]_\beta$, then $[f(T)]_\beta = f(A)$.

Suppose $f(x)$ is a polynomial with coefficients in F

Suppose $T \in \mathcal{L}(V)$ and V is a vector space over F .

- (a) Claim: $f(T)$ is a linear operator on V .

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (1)$$

$$f(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I_V \quad (2)$$

Lemma: $T^n \in \mathcal{L}(V) \forall n \in \mathbb{Z}^+$.

Suppose $y, z \in V$.

Suppose $n = 1$

$$T(az + y) = aT(z) + T(y) \quad (3)$$

Suppose true for $1 \leq n \leq k$.

Suppose $n = k + 1$

$$T^{k+1}(az + y) = T(T^k(az + y)) \quad (4)$$

$$= T(aT^k(z) + T^k(y)) \quad (5)$$

$$= aT^{k+1}(z) + T^{k+1}(y) \quad (6)$$

□

$$\begin{aligned} \therefore f(T(az + y)) &= a_n T^n(az + y) + a_{n-1} T^{n-1}(az + y) + \cdots + \\ &\quad + a_1 T(az + y) + a_0(az + y) \end{aligned} \quad (7)$$

$$\begin{aligned} f(T(az + y)) &= a_n(aT^n(z) + T^n(y)) + a_{n-1}(aT^{n-1}(z) + T^{n-1}(y)) + \cdots + \\ &\quad + a_1(aT(z) + T(y)) + a_0(az + y) \end{aligned} \quad (8)$$

$$f(\mathsf{T}(az + y)) = a(a_n \mathsf{T}^n(z) + a_{n-1} \mathsf{T}^{n-1}(z) + \cdots + a_1 \mathsf{T}(z) + a_0 z) + (a_n \mathsf{T}^n(y) + a_{n-1} \mathsf{T}^{n-1}(y) + \cdots + a_1 \mathsf{T}(y) + a_0 y) \quad (9)$$

$$f(\mathsf{T}(az + y)) = af(\mathsf{T})(z) + f(\mathsf{T})(y) \in \mathsf{V} \quad (10)$$

$$\Rightarrow f(\mathsf{T}) \in \mathcal{L}(\mathsf{V}) \quad (11)$$

(b) Claim: If β is a finite ordered basis for V and $A = [\mathsf{T}]_\beta$, then $[f(\mathsf{T})]_\beta = f(A)$.

$$f(\mathsf{T}) = a_n \mathsf{T}^n + a_{n-1} \mathsf{T}^{n-1} + \cdots + a_1 \mathsf{T} + a_0 \mathsf{I}_\mathsf{V} \quad (12)$$

$$\Rightarrow [f(\mathsf{T})]_\beta = [a_n \mathsf{T}^n + a_{n-1} \mathsf{T}^{n-1} + \cdots + a_1 \mathsf{T} + a_0 \mathsf{I}_\mathsf{V}]_\beta \quad (13)$$

$$= a_n [\mathsf{T}^n]_\beta a_{n-1} [\mathsf{T}^{n-1}]_\beta + \cdots + a_1 [\mathsf{T}]_\beta + a_0 [\mathsf{I}_\mathsf{V}]_\beta \quad (14)$$

$$= a_n \left([\mathsf{T}]_\beta\right)^n + a_{n-1} \left([\mathsf{T}]_\beta\right)^{n-1} + \cdots + a_1 [\mathsf{T}]_\beta + a_0 \mathsf{I}_\mathsf{V} \quad (15)$$

$$= a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 \quad (16)$$

$$= f(A) \quad (17)$$

5. Let T be a linear operator on a vector space V over a field F , and let A be an $n \times n$ matrix with entries from F . If $f_1(x)$ and $f_2(x)$ are relatively prime polynomials with entries from F , then there exist polynomials $q_1(x)$ and $q_2(x)$ with entries from F such that

$$(a) \quad q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$$

$$(b) \quad q_1(A)f_1(A) + q_2(A)f_2(A) = I.$$

Suppose $\mathsf{T} \in \mathcal{L}(\mathsf{V})$, such that V is a vector space over F , and $A \in \mathsf{M}_{n \times n}(F)$

Suppose $f_1(x), f_2(x) \in \mathsf{P}(F)$ such that they are relatively prime.

$$(a) \quad \text{Claim: } \exists q_1(x) \text{ and } q_2(x) \text{ such that } q_1(\mathsf{T})f_1(\mathsf{T}) + q_2(\mathsf{T})f_2(\mathsf{T}) = \mathsf{I}$$

Because $f_1(x)$ and $f_2(x)$ are relatively prime there exists polynomials $q_1(x)$ and $q_2(x)$ such that

$$f_1(x)q_1(x) + f_2(x)q_2(x) = 1 \quad (18)$$

It follows that

$$f_1(\mathsf{T})q_1(\mathsf{T}) + f_2(\mathsf{T})q_2(\mathsf{T}) = \mathsf{I}_\mathsf{V} \quad (19)$$

$$(b) \quad \text{Claim: } \exists q_1(x) \text{ and } q_2(x) \text{ such that } q_1(A)f_1(A) + q_2(A)f_2(A) = I_n$$

$$f_1(x)q_1(x) + f_2(x)q_2(x) = 1 \quad (20)$$

$$\Rightarrow f_1(A)q_1(A) + f_2(A)q_2(A) = I_n \quad (21)$$

6. Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

Claim: Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible, and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

Because $\phi(x)$ is irreducible, $f(x)$ does not divide $\phi(x)$. Since $\phi(x)$ does not divide $f(x)$ it follows that $\phi(x)$ and $f(x)$ are relatively prime.

7. Any two distinct irreducible monic polynomials are relatively prime.

Lemma: All factors of an irreducible monic polynomial $\phi(x)$ are either of the form $c \neq 0, c \in F$ or $d\phi(x), d \neq 0, d \in F$

Suppose $f(x), \phi(x) \in P(F)$ and $\phi(x)$ is an irreducible polynomial.

Suppose $f(x)$ divides $\phi(x)$

$$\Rightarrow \phi(x) = f(x)g(x) \quad \text{for some } g(x) \in P(F) \quad (22)$$

Case 1 $\deg(f(x)) \notin \mathbb{Z}^+$

$$f(x) \neq 0 \because \phi(x) \neq 0 \quad (23)$$

$$\Rightarrow \deg(f(x)) = 0 \quad (24)$$

$$\Rightarrow f(x) = c \quad \text{for some } c \in F \quad (25)$$

Case 2 $\deg(f(x)) \in \mathbb{Z}^+$

$$\phi(x) = f(x)q(x) \quad (26)$$

Because $\phi(x)$ is irreducible, it cannot be expressed as a product of polynomials both possessing positive degree.

$$\Rightarrow \deg(q(x)) \leq 0 \quad (27)$$

$$\Rightarrow q(x) = \frac{1}{d} \quad \text{for some nonzero } d \in F \quad (28)$$

$$\Rightarrow \phi(x) = \frac{f(x)}{d} \quad (29)$$

$$\Rightarrow d\phi(x) = f(x) \quad (30)$$

□

By lemma, suppose the factors of ϕ_1 are c and $d\phi_1$ where $c, d \in F$ and $c, d \neq 0$. suppose the factors of ϕ_2 are e and $g\phi_2$ where $e, g \in F, e, g \neq 0$

Claim: $d\phi_1 \neq g\phi_2$

Suppose $g\phi_2 | \phi_1$

$$\Rightarrow g\phi_2 = d\phi_1 \quad (31)$$

$$\Rightarrow \phi_2 = \frac{d}{g}\phi_1 \Rightarrow \phi_2 = \frac{d}{g}(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) \quad (32)$$

$$\Rightarrow \phi_2 = \frac{d}{g}x^n + \frac{d}{g}a_{n-1}x^{n-1} + \cdots + \frac{d}{g}a_1x + a_0 \quad (33)$$

$$\Rightarrow \frac{d}{g} = 1 \text{ because } \phi_2 \text{ is monic.} \quad (34)$$

$$\Rightarrow \phi_2 = \phi_1 \not\vdash \text{Contradiction!} \quad (35)$$

Theorem states ϕ_1 and ϕ_2 are distinct polynomials.

5.2

2. For each of the following matrices $A \in \mathbf{M}_{n \times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(b)

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \quad (36)$$

$$\det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9 \quad (37)$$

$$= \lambda^2 - 2\lambda - 8 \quad (38)$$

$$= (\lambda - 4)(\lambda + 2) \quad (39)$$

$$\Rightarrow \lambda_1 = 4, \lambda_2 = -2 \quad (40)$$

- For $\lambda_1 = 4$

$$A - 4I_2 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \quad (41)$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad (42)$$

$$\Rightarrow \text{rank}(A - 4I_2) = 1 \quad (43)$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (44)$$

$$\Rightarrow x_1 = x_2 \quad (45)$$

$$S_1 = \left\{ z \begin{pmatrix} 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \quad (46)$$

The eigenvector corresponding to λ_1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- For $\lambda_2 = -2$

$$A + 2I_2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad (47)$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (48)$$

$$\Rightarrow \text{rank}(A + 2I_2) = 1 \quad (49)$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (50)$$

$$\Rightarrow x_1 = -x_2 \quad (51)$$

$$S_2 = \left\{ z \begin{pmatrix} -1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \quad (52)$$

The eigenvector corresponding to λ_2 is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (53)$$

$$(Q|I) = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array} \right) \quad (54)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \quad (55)$$

$$\Rightarrow Q^{-1}AQ = D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \quad (56)$$

(d)

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \quad (57)$$

$$\det(A - \lambda I_3) = \det((A - \lambda I_3)^t) \quad (58)$$

$$\det((A - \lambda I_3)^t) = \det \begin{pmatrix} 7 - \lambda & 8 & 6 \\ -4 & -5 - \lambda & -6 \\ 0 & 0 & 3 - \lambda \end{pmatrix} \quad (59)$$

$$\det(A - \lambda I_3) = (-1)(\lambda - 3)^3(\lambda + 1) \quad (60)$$

$$\lambda_1 = -1, \text{ multiplicity } 1 \quad (61)$$

$$\lambda_2 = 3, \text{ multiplicity } 2 \quad (62)$$

- For $\lambda_1 = -1$

$$A + I_3 = \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \quad (63)$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (64)$$

$$\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (65)$$

$$\Rightarrow x_1 = \frac{2}{3}x_3 \quad (66)$$

$$\Rightarrow x_2 = \frac{4}{3}x_3 \quad (67)$$

$$\Rightarrow S_1 = \left\{ z \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \quad (68)$$

$$\Rightarrow \text{an eigenvector is } \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

- For $\lambda_2 = 3$

$$A - 3I_3 = \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \quad (69)$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (70)$$

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (71)$$

$$\Rightarrow x_1 = x_2 \quad (72)$$

$$S_2 = \left\{ z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : z_1, z_2 \in \mathbb{R} \right\} \quad (73)$$

$$\Rightarrow \text{eigenvectors corresponding to } \lambda_2 = 3 \text{ are } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \quad (74)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 3/2 & -3/2 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 \end{array} \right) \quad (75)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ 3/2 & -3/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix} \quad (76)$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (77)$$

(e)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (78)$$

$$A - \lambda I_3 = \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & 1 - \lambda \end{pmatrix} \quad (79)$$

$$\det(A - \lambda I_3) = (-1) \det \begin{pmatrix} -\lambda & 1 \\ 1 & -1 \end{pmatrix} + (1 - \lambda) \det \begin{pmatrix} -\lambda & 0 \\ 1 & -\lambda \end{pmatrix} \quad (80)$$

$$= (-1)(\lambda - 1) + (1 - \lambda)(\lambda^2) \quad (81)$$

$$= (1 - \lambda)(1 + \lambda)^2 \quad (82)$$

This characteristic polynomial does not split over \mathbb{R} , thus A is not diagonalizable.

3. For each of the following linear operators T on a vector space V , test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(a) $V = P_3(\mathbb{R})$ and T is defined by $T(f(x)) = f'(x) + f''(x)$, respectively.

Let α be the standard ordered basis of $P_3(\mathbb{R})$.

$$T(\alpha) = \{0, 1, 2x + 2, 3x^2 + 6x\} \quad (83)$$

$$\Rightarrow [T]_\alpha = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (84)$$

$$\det([T]_\beta - \lambda I_4) = \det \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^4 \quad (85)$$

$$\Rightarrow \lambda_1 = 0 \text{ with multiplicity } 4 \quad (86)$$

$$\text{rank}([T]_\beta - \lambda_1 I_3) = 3 \quad (87)$$

$$4 - 3 \neq 4 \text{ multiplicity of } \lambda_1 \quad (88)$$

$$\Rightarrow T \text{ is not diagonalizable}$$

9.

11.