Assignment

Section 4.4: 1, 5, 6; Section 5.1: 3(bc), 4(ceh), 7, 12, 14, 15, 19, 22

Work

4.4

1. Label the statements as true or false.

(a)	True	(g)	True
(b)	True	(h)	False
(c)	True	(i)	True
(d)	False	(j)	True
(e)	False	(k)	True
(f)	True		

5. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix}$$

where A is a square matrix. Prove that $\det(M) = \det(A)$

Suppose $A \in \mathsf{M}_{k\times k}(F)$

Perform Type 3 operations such that the partition $(A\ B)$ becomes an upper triangular matrix.

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & I \end{pmatrix} \tag{1}$$

M' is an upper triangular matrix so the determinant of M is the product of its diagonal terms.

$$\Rightarrow \det(A) = \prod_{i=1}^{n} M'_{ii} \tag{2}$$

$$M'_{ii} = 1 \quad \text{if } (k+1 \leq i \leq n) \tag{3}$$

$$\Rightarrow \prod_{i=1}^{n} M'_{ii} = \prod_{i=1}^{k} M'_{ii} \tag{4}$$

The first k diagonal terms of M' are the diagonal terms of A. It follows that

$$\prod_{i=1}^{k} M'_{ii} = \prod_{i=1}^{k} A'_{ii} \tag{5}$$

Because A' was obtained from A using Type 3 operations, and M' was obtained from M using Type 3 operations

$$\det(A') = \det(A) \tag{6}$$
$$\det(M') = \det(M) \tag{7}$$
$$\therefore \det(M) = \det(M') = \det(A') = \det(A) \tag{8}$$

6. Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

M can be reduced using strictly Type 3 row operations such that the partitions $(A\ B)$ become an upper triangular matrix $(A'\ B')$. M can also be reduced using strictly Type 3 row operations such that the partition $(O\ C)$ becomes the matrix $(O\ C')$ where C' is an upper triangular matrix in $\mathsf{M}_{(n-k)\times(n-k)}(F)$ where $A \in \mathsf{M}_{n\times n}(F)$.

$$\Rightarrow M' = \begin{pmatrix} A' & B' \\ O & C' \end{pmatrix} \tag{9}$$

M' is upper triangular, so the determinant of M' is a the product of the diagonal terms.

$$\det(M') = \prod_{i=1}^{n} M'_{ii} \tag{10}$$
$$= \left(\prod_{i=1}^{k} M'_{ii}\right)\left(\prod_{i=k+1}^{n} M'_{ii}\right) \tag{11}$$

$$M'_{ii} = A'_{ii} \quad \forall i\ (1 \le i \le k) \tag{12}$$

$$\Rightarrow \prod_{i=1}^{k} M'_{ii} = \prod_{i=1}^{k} A'_{ii} \tag{13}$$
$$= \det(A') \tag{14}$$

$$M'_{k+1,k+1} = C'_{ii} \quad \forall i\ (1 \le i \le n-k) \tag{15}$$

$$\Rightarrow \prod_{i=k+1}^{k} M'_{ii} = \prod_{i=1} n - kC'_{ii} \tag{16}$$
$$= \det(C') \tag{17}$$

Matrices C', A' and M' were obtained respectively from the matrices C, A and M strictly using Type 3 row operations. It follows that

$$\det(M') = \det(M) \tag{18}$$
$$\det(A') = \det(A) \tag{19}$$
$$\det(C') = \det(C) \tag{20}$$

$$\therefore \det(M) = \det(A) \cdot \det(C) \tag{21}$$

5.1

3. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,

   (i) Determine all the eigenvalues of A.

   (ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

   (iii) If possible, find a basis for F^n consisting of eigenvectors of A.

   (iv) If succesful in finding such a basis, determine and invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   (b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$

   (i)

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} = 0 \tag{22}$$

$$\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 0 & 4-2\lambda & 3-\lambda \end{pmatrix} \tag{23}$$

$$\begin{aligned} \det(A - \lambda I) &= (-\lambda)(\lambda-1)(\lambda-3) + 3(4-2\lambda) \\ &\quad - \lambda(4-2\lambda) - 2(3-\lambda) && (24) \\ &= \lambda^3 + 6\lambda^2 - 11\lambda + 6 && (25) \\ &= (\lambda-3)(\lambda-2)(-\lambda_1) = 0 && (26) \end{aligned}$$

$$\Rightarrow \lambda = \{3, 2, 1\} \tag{27}$$

(ii) • For $\lambda = 3$

$$Av = 3v \tag{28}$$
$$(A - 3I)v = 0 \tag{29}$$
$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ -0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right) v = 0 \tag{30}$$
$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} v = 0 \tag{31}$$
$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

$$x_1 = -t \tag{33}$$
$$x_2 = 0 \tag{34}$$
$$x_3 = t \tag{35}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{36}$$

• For $\lambda = 2$

$$Av = 2v \tag{37}$$
$$(A - 2I)v = 0 \tag{38}$$
$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right) v = 0 \tag{39}$$
$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} v = 0 \tag{40}$$
$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

$$x_1 = -t \tag{42}$$
$$x_2 = t \tag{43}$$
$$x_3 = 0 \tag{44}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{45}$$

- For $\lambda = 1$

$$Av = v \tag{46}$$

$$(A - I)V = 0 \tag{47}$$

$$\left(\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & \\ -1 & & \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \tag{48}$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} v = 0 \tag{49}$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{50}$$

$$x_1 = -t \tag{51}$$

$$x_2 = -t \tag{52}$$

$$x_3 = t \tag{53}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{54}$$

(iii)

$$\beta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \tag{55}$$

(iv)

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \tag{56}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 & -1 \end{array}\right) \tag{57}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \tag{58}$$

$$D = QAQ^{-1} \tag{59}$$
$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -2 & -1 \end{pmatrix} \tag{60}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{61}$$

(c) $\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbb{C}$

(i)

$$\det(A - \lambda I) = \det \begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = 0 \tag{62}$$
$$= -(i-\lambda)(i+\lambda) - 2 = 0 \tag{63}$$
$$= i^2 - \lambda^2 + 2 = 0 \tag{64}$$

$$\lambda^2 = 1 \tag{65}$$
$$\lambda = \pm 1 \tag{66}$$

(ii) • For $\lambda = 1$

$$Av = \lambda v \tag{67}$$
$$(A - I)v = 0 \tag{68}$$
$$\begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix} v = 0 \tag{69}$$
$$\begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i-1 & 1 \\ 0 & 0 \end{pmatrix} \tag{70}$$

$$x_1 = -\frac{t}{i-1} \tag{71}$$
$$x_2 = t \tag{72}$$

$$v = \left\{ t \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{73}$$

• For $\lambda = -1$

$$Av = \lambda v \tag{74}$$
$$(A + I)v = 0 \tag{75}$$
$$\begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix} v = 0 \tag{76}$$
$$\begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} i+1 & 1 \\ 0 & 0 \end{pmatrix} \tag{77}$$

$$x_1 = -\frac{t}{i+1} \tag{78}$$
$$x_2 = t \tag{79}$$

$$v = \left\{ t \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\} \tag{80}$$

(iii)

$$\beta = \left\{ \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{i-1}{2} \\ 1 \end{pmatrix} \right\} \tag{81}$$

(iv)

$$Q = \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \tag{82}$$

$$\left(\begin{array}{cc|cc} \frac{i+1}{2} & \frac{i-1}{2} & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & -1 & \frac{1+i}{2} \end{array}\right) \tag{83}$$
$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \tag{84}$$

$$D = Q^{-1}AQ \tag{85}$$
$$= \begin{pmatrix} 1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \begin{pmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{pmatrix} \tag{86}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{87}$$

4. For each linear operator T on V, find the egienvalues of T and an ordered bases β of V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

   (c) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a,b,c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{88}$$
$$[\mathsf{T}]_\alpha = \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & 6 & 11 \end{pmatrix} \tag{89}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I_3\right) = \det\begin{pmatrix} -4-\lambda & 3 & -6 \\ 6 & -7-\lambda & 12 \\ 6 & -6 & 11-\lambda \end{pmatrix} \tag{90}$$

$$= \det\begin{pmatrix} -4-\lambda & 4 & -6 \\ 6 & -7-\lambda & 12 \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \tag{91}$$

$$= (\lambda+4)(\lambda+7)(\lambda+1) \tag{92}$$

$$- 2(\lambda+1)(\lambda+1)(\lambda+4) - 18(\lambda+1) \tag{93}$$

$$= -\lambda^3 + 3\lambda + 2 = 0 \tag{94}$$

$$= -(\lambda+1)(\lambda-2)(\lambda+1) = 0 \tag{95}$$

$$\Rightarrow \lambda = \{2, -1\} \tag{96}$$

- For $\lambda = 2$

$$[\mathsf{T}]_\alpha v = 0 \tag{97}$$

$$([\mathsf{T}]_\alpha - 2I)v = 0 \tag{98}$$

$$\left(\begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v = 0 \tag{99}$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} v = 0 \tag{100}$$

$$\begin{pmatrix} -6 & 3 & -6 \\ 6 & -9 & 12 \\ 6 & -6 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & {}^{1}/_{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{101}$$

$$x_1 = -\frac{1}{2}t \tag{102}$$

$$x_2 = t \tag{103}$$

$$x_3 = t \tag{104}$$

$$v = \left\{ t\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{105}$$

- For $\lambda = -1$

$$[\mathsf{T}_\alpha]v = -v \tag{106}$$

$$[\mathsf{T}]_\alpha]v = 0 \tag{107}$$

$$\left(\begin{pmatrix}-4 & 3 & -6\\ 6 & -7 & 12\\ 6 & -6 & 11\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) v = 0 \tag{108}$$

$$\begin{pmatrix}-3 & 3 & -6\\ 6 & -6 & 12\\ 6 & -6 & 12\end{pmatrix} v = 0 \tag{109}$$

$$\begin{pmatrix}-3 & 3 & -6\\ 6 & -6 & 12\\ 6 & -6 & 12\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & -1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{110}$$

$$x_1 = s - 2t \tag{111}$$

$$x_2 = s \tag{112}$$

$$x_3 = t \tag{113}$$

$$v = \left\{ s\begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\end{pmatrix} : s, t \in \mathbb{R}^3 \right\} \tag{114}$$

$$\beta = \left\{ \begin{pmatrix}-\frac{1}{2}\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\} \tag{115}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{116}$$

$$Q = \begin{pmatrix}{}^{-1}/_2 & 1 & -2\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \tag{117}$$

$$\left(\begin{array}{ccc|ccc}{}^{-1}/_2 & 1 & -2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & {}^{2}/_5 & {}^{-2}/_5 & {}^{4}/_5\\ 0 & 1 & 0 & {}^{2}/_5 & {}^{3}/_5 & {}^{4}/_5\\ 0 & 0 & 1 & {}^{-2}/_5 & {}^{2}/_5 & {}^{1}/_5\end{array}\right) \tag{118}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix}{}^{2}/_5 & {}^{-2}/_5 & {}^{4}/_5\\ {}^{2}/_5 & {}^{3}/_5 & {}^{4}/_5\\ {}^{-2}/_5 & {}^{2}/_5 & {}^{1}/_5\end{pmatrix} \tag{119}$$

$$[\mathsf{T}]_\beta = \begin{pmatrix}{}^{2}/_5 & {}^{-2}/_5 & {}^{4}/_5\\ {}^{2}/_5 & {}^{3}/_5 & {}^{4}/_5\\ {}^{-2}/_5 & {}^{2}/_5 & {}^{1}/_5\end{pmatrix} \begin{pmatrix}-4 & 3 & -6\\ 6 & -7 & 12\\ 6 & -6 & 11\end{pmatrix} \begin{pmatrix}{}^{-1}/_2 & 1 & -2\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \tag{120}$$

$$= \begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{121}$$

(e) $\mathsf{V} = \mathsf{P}_3(\mathbb{R})$ and $\mathsf{T}(f(x)) = xf'(x) + f(2)x + f(3)$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{122}$$

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{123}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I\right) = 0 \tag{124}$$

$$= \det \begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} \tag{125}$$

$$= (1-\lambda)(3-\lambda)(2-\lambda) - 6(2-\lambda) \tag{126}$$

$$= \lambda(2-\lambda)(\lambda-4) \tag{127}$$

$$\lambda = \{0, 2, 4\} \tag{128}$$

- For $\lambda = 0$

$$Av = 0v \tag{129}$$

$$Av = 0 \tag{130}$$

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{131}$$

$$x_1 = -3t \tag{132}$$

$$x_2 = t \tag{133}$$

$$x_3 = 0 \tag{134}$$

$$v = \left\{ t \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{135}$$

- For $\lambda = 2$

$$Av = 2v \tag{136}$$

$$(A - 2I) = v \tag{137}$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} v = 0 \tag{138}$$

$$\begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 3 & 4 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{pmatrix} \tag{139}$$

$$x_1 = -\frac{39}{4}t + 9t \tag{140}$$

$$x_2 = -\frac{12}{4}t \tag{141}$$

$$x_3 = t \tag{142}$$

$$v = \left\{ t \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{143}$$

- For $\lambda = 4$

$$Av = 4v \tag{144}$$

$$(A - 4I) = v \tag{145}$$

$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} v = 0 \tag{146}$$

$$\begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 21 \\ 1 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{147}$$

$$x_1 = t \tag{148}$$

$$x_2 = t \tag{149}$$

$$x_3 = 0 \tag{150}$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{151}$$

$$\beta = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -12 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \tag{152}$$

$$\Rightarrow \left\{ (-3 + x), (-3 - 13x + 4x^2), (1 + x) \right\} \tag{153}$$

$$Q = \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \tag{154}$$

$$\left(\begin{array}{ccc|ccc} -3 & -3 & 1 & 1 & 0 & 0 \\ 1 & -13 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & {}^{-1}/_{4} & {}^{1}/_{4} & {}^{5}/_{8} \\ 0 & 1 & 0 & 0 & 0 & {}^{1}/_{4} \\ 0 & 0 & 1 & {}^{1}/_{4} & {}^{3}/_{4} & {}^{21}/_{8} \end{array}\right) \tag{155}$$

$$Q^{-1} = \begin{pmatrix} {}^{-1}/_{4} & {}^{1}/_{4} & {}^{5}/_{8} \\ 0 & 0 & {}^{1}/_{4} \\ {}^{1}/_{4} & {}^{3}/_{4} & {}^{21}/_{8} \end{pmatrix} \tag{156}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{157}$$

$$= \begin{pmatrix} {}^{-1}\!/_4 & {}^{1}\!/_4 & {}^{5}\!/_8 \\ 0 & 0 & {}^{1}\!/_4 \\ {}^{1}\!/_4 & {}^{3}\!/_4 & {}^{21}\!/_8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix} \tag{158}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{159}$$

(h) $\mathsf{V} = \mathsf{M}_{n\times n}(\mathbb{R})$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{160}$$

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{161}$$

$$\det\left([\mathsf{T}]_\alpha - \lambda I\right) = \det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \tag{162}$$

$$\begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} \tag{163}$$

$$\det \begin{pmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda^2 \end{pmatrix} = 0 \tag{164}$$

$$\Rightarrow (1-\lambda)^2(1-\lambda^2) = 0 \tag{165}$$

$$\Rightarrow \lambda = \pm 1 \tag{166}$$

- For $\lambda = 1$

$$[\mathsf{T}]_\alpha v = v \tag{167}$$

$$([[T]_\alpha - I)v = 0 \tag{168}$$

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} v = 0 \tag{169}$$

$$x_1 = t \tag{170}$$
$$x_2 = k \tag{171}$$
$$x_3 = s \tag{172}$$
$$x_4 = t \tag{173}$$

$$v = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t, k, s \in \mathbb{R} \right\} \tag{174}$$

- For $\lambda = -1$

$$[\mathsf{T}]_\alpha v = v \tag{175}$$
$$([\mathsf{T}]_\alpha + I)v = 0 \tag{176}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} v = 0 \tag{177}$$

$$x_1 = -t \tag{178}$$
$$x_2 = 0 \tag{179}$$
$$x_3 = 0 \tag{180}$$
$$x_4 = t \tag{181}$$

$$v = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \tag{182}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{183}$$
$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{184}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{185}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & {}^1/_2 & 0 & 0 & {}^1/_2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & {}^{-1}/_2 & 0 & 0 & {}^1/_2 \end{array}\right) \tag{186}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} {}^1/_2 & 0 & 0 & {}^1/_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ {}^{-1}/_2 & 0 & 0 & {}^1/_2 \end{pmatrix} \tag{187}$$

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q \tag{188}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{189}$$

7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(\mathsf{T})$, as follows: Choose any ordered basis β for V, and define $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta)$.

   (a) Prove that he preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.

   (b) Prove that T is invertible if and only if $\det \mathsf{T} \neq 0$.

   (c) Prove that if T is invertible, then $\det(\mathsf{T}^{-12}) = [\det(\mathsf{T})]^{-1}$.

   (d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T}) \cdot \det(\mathsf{U})$.

   (e) Prove that $\det(\mathsf{T} - \lambda \mathsf{I_V}) = \det[\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

   (a) Suppose Q is the change of coordinates matrix from γ to β.

$$Q = [\mathsf{I_V}]_\gamma^\beta \Rightarrow Q^{-1} = [\mathsf{I_V}]_\beta^\gamma \tag{190}$$

$$[\mathsf{I_V}]_\gamma^\beta [\mathsf{T}]_\gamma [\mathsf{I_V}]_\beta^\gamma = [\mathsf{T}]_\beta \tag{191}$$

$$\Rightarrow \det([\mathsf{I_V}]_\gamma^\beta [\mathsf{T}]_\gamma [\mathsf{I_V}]_\beta^\gamma) = \det[\mathsf{T}]_\beta \tag{192}$$

$$\Rightarrow \det[\mathsf{I_V}]_\gamma^\beta \det[\mathsf{T}]_\gamma \det[\mathsf{I_V}]_\beta^\gamma \tag{193}$$

$$[\mathsf{I_V}]_\gamma^\beta = ([\mathsf{I_V}]_\beta^\gamma)^{-1} \tag{194}$$

$$\det[\mathsf{I_V}]_\gamma^\beta = \det([\mathsf{I_V}]_\beta^\gamma)^{-1} \tag{195}$$

$$\Rightarrow \det\,[\mathsf{I_V}]_\gamma^\beta \det\,[\mathsf{T}]_\gamma \det\,[\mathsf{I_V}]_\beta^\gamma = \det\,([\mathsf{I_V}]_\beta^\gamma)^{-1} \det\,[\mathsf{T}]_\gamma \det\,[\mathsf{I_V}]_\beta^\gamma \tag{196}$$
$$= \det[\mathsf{T}]_\gamma = \det[\mathsf{T}]_\beta \tag{197}$$

(b) $(\Rightarrow)$

Suppose T is invertible
Suppose β is an ordered basis of V.

$$[\mathsf{I_V}]_\beta = [\mathsf{T}\cdot\mathsf{T}^{-1}]_\beta = [\mathsf{T}]_\beta[\mathsf{T}^{-1}]_\beta \tag{198}$$
$$\Rightarrow \det\,[\mathsf{I_V}]_\beta = \det\,[\mathsf{T}]_\beta \det\,[\mathsf{T}^{-1}]_\beta \tag{199}$$
$$\det\,[\mathsf{I_V}]_\beta = 1 \tag{200}$$
$$\Rightarrow \det\,[\mathsf{T}]_\beta \det\,[\mathsf{T}^{-1}]_\beta = 1 \tag{201}$$
$$\Rightarrow \det\,[\mathsf{T}]_\beta \neq 0 \tag{202}$$
$$\Rightarrow \det\mathsf{T} \neq 0 \tag{203}$$

$(\Leftarrow)$

Suppose $\det\mathsf{T} \neq 0$

$$\Rightarrow \det[\mathsf{T}]_\beta \neq 0 \quad \text{for some ordered basis } \beta \text{ of } \mathsf{V} \tag{204}$$
$$\det\,[\mathsf{T}]_\beta \neq 0 \tag{205}$$
$$\Rightarrow \mathsf{T} \text{ is invertible, by corollary to Th. 2.18} \tag{206}$$

(c) Suppose T is invertible and β is some ordered basis of V.

$$\det\mathsf{I_V} = \det\mathsf{T}\cdot\mathsf{T}^{-1} \tag{207}$$
$$= \det\,[\mathsf{T}\cdot\mathsf{T}^{-1}]_\beta \tag{208}$$
$$= \det\,[\mathsf{T}]_\beta[\mathsf{T}^{-1}]_\beta \tag{209}$$
$$= \det\,[\mathsf{T}]_\beta \det\,[\mathsf{T}^{-1}]_\beta \tag{210}$$
$$= \det\mathsf{T}\det\mathsf{T}^{-1} \tag{211}$$

$$\det\mathsf{I_V} = \det\,[\mathsf{I_V}]_\beta = \det I_n = 1 \tag{212}$$
$$\Rightarrow \det\mathsf{T}\det\mathsf{T}^{-1} = 1 \tag{213}$$
$$\Rightarrow \det\mathsf{T}^{-1} = (\det\mathsf{T})^{-1} \tag{214}$$

(d) Suppose β is an ordered basis of V.

$$\det\mathsf{TU} = \det\,[\mathsf{TU}]_\beta \tag{215}$$
$$= \det\,[\mathsf{T}]_\beta[\mathsf{U}]_\beta \tag{216}$$
$$= \det\,[\mathsf{T}]_\beta \det\,[\mathsf{U}]_\beta \tag{217}$$
$$= \det\mathsf{T}\det\mathsf{U} \tag{218}$$

(e)

$$\begin{aligned}
\det\left([\mathsf{T}]_\beta - \lambda I\right) &= \det\left([\mathsf{T}]_\beta - \lambda[\mathsf{I}_\mathsf{V}]_\beta\right) && (219)\\
&= \det\left([\mathsf{T}]_\beta - [\lambda\mathsf{I}_\mathsf{V}]_\beta\right) && (220)\\
&= \det\left[\mathsf{T}_\beta - \lambda\mathsf{I}_\mathsf{V}\right]_\beta && (221)\\
&= \det\left(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V}\right) && (222)
\end{aligned}$$

12.

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