

## Assignment

1.5: 2(bdfg), 11, 15; 1.6: 20, 24, 31; 2.1: 6, 12, 14; 2.2: 2(bcg), 8, 11

## Work

### 1.5

2. Determine whether the following sets are linearly dependent or linearly independent.

(b)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

$$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (1)$$

$$a_1 = -a_2 \quad (2)$$

$$-a_1 = a_2 \quad (3)$$

$$\implies a_1 = a_2 = 0 \quad (4)$$

Only the trivial solution exists. The set is linearly independent.

(d)

$$\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\} \text{ in } P_3(\mathbb{R})$$

$$a_1 (x^3 - x) + a_2 (2x^2 + 4) + a_3 (-2x^3 + 3x^2 + 2x + 6) = 0 \quad (5)$$

$$a_1 = 2a_3 \quad (6)$$

$$2a_2 = -3a_3 \quad (7)$$

$$a_1 = t \quad (8)$$

$$a_2 = -\left(\frac{3}{4}\right)t \quad (9)$$

$$a_3 = +\left(\frac{1}{2}\right)t \quad (10)$$

The set is linearly dependent.

(f)

 $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$  in  $\mathbb{R}^3$ 

$$a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 0 \quad (11)$$

$$a + 2b - c = 0 \quad (12)$$

$$-a + 2c = 0 \quad (13)$$

$$2a + b - c = 0 \quad (14)$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 0 \end{pmatrix} \begin{array}{l} \boxed{1} \\ \leftarrow + \\ \boxed{2} \\ \leftarrow + \end{array}^{-2} \mid \cdot \frac{1}{2} \begin{array}{l} \boxed{3} \\ \leftarrow + \\ \boxed{5} \end{array} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Only the trivial solution exists. This set is linearly independent.

(g)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\} \text{ in } \mathbf{M}_{2 \times 2}(\mathbb{R})$$

$$a \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = 0 \quad (15)$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 4 & 0 \end{pmatrix} \begin{array}{l} \boxed{2} \\ \leftarrow + \\ \boxed{4} \\ \leftarrow + \end{array}^{-1} \begin{array}{l} \boxed{1} \\ \leftarrow + \\ \boxed{2} \\ \leftarrow + \end{array}^{-1} \begin{array}{l} \boxed{-2} \\ \leftarrow + \end{array} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a = -3t \quad (16)$$

$$b = -t \quad (17)$$

$$c = -t \quad (18)$$

$$d = t \quad (19)$$

There exists a non-trivial solution, therefore the set is linearly dependent.

11. Let  $S = \{u_1, u_2, \dots, u_n\}$  be a linearly independent subset of a vector space  $\mathbf{V}$  over the field  $\mathbb{Z}_2$ . How many vectors are there in  $\text{span}(S)$ ? Justify your answer.

$$\mathbb{Z}_2 = \{0, 1\} \quad (20)$$

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n \neq 0, \forall c_1, \dots, c_n \in \{0, 1\} \quad (21)$$

unless all  $c_i = 0$ .

$$\implies \text{card}(\text{span}(S)) = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n} \quad (22)$$

$$= \sum_{i=1}^n \binom{n}{i} \quad (23)$$

$$= 2^n \quad (24)$$

15. Let  $S = \{u_1, u_2, \dots, u_n\}$  be a finite set of vectors. Prove that  $S$  is linearly dependent if and only if  $u_1 = 0$  or  $u_{k+1} \in \text{span}(\{u_1, u_2, \dots, u_k\})$  for some  $k$  ( $1 \leq k < n$ ).

**Forward Direction:**

Claim:  $S$  is linearly dependent

Suppose  $u_1 = 0$

Assume  $S$  is linearly independent.

Take the linear combination:

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0 \text{ such that } c_1 \neq 0, c_2, \dots, c_n = 0 \quad (25)$$

$$\implies c_1 u_1 = 0 \not\leq \text{Contradiction!} \quad (26)$$

There exists a non-trivial representation of the zero vector therefore  $S$  is linearly dependent.

Suppose  $u_{k+1} \in \text{span}(\{u_1, \dots, u_k\})$

Assume  $S$  is linearly independent.

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \dots + a_k u_k \quad (27)$$

Take the linearly combination:

$$c_1 u_1 + c_2 u_2 + \dots + c_k u_k + c_{k+1} + \dots + c_n u_n = 0 \quad (28)$$

Choose  $c_i = (-a_i)$  for  $i(1 \leq i \leq k)$  and  $c_{k+1} = 1$

$$\implies u_{k+1} = 0 \not\leq \text{Contradiction!} \quad (29)$$

There exists a non-trivial representation of the zero vector therefore  $S$  is linearly dependent.

**Reverse Direction:**

Suppose  $u_{k+1} \in \text{span}(\{u_1, \dots, u_k\})$

Claim:  $S$  is linearly dependent

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \dots + a_k u_k \quad (30)$$

$$-u_{k+1} = (-1)(a_1 u_1 + a_2 u_2 + \dots + a_k u_k) \quad (31)$$

$$= (-a_1) u_1 + (-a_2) u_2 + \dots + (-a_k) u_k \quad (32)$$

Take the linear combination of all  $u_1, \dots, u_n$

$$((-a_1) u_1 + (-a_2) u_2 + \dots + (-a_k) u_k) + (1u_{k+1} + 0u_{k+2} + \dots + 0u_n) = 0 \quad (33)$$

## 1.6

20. Let  $V$  be a vector space having dimension  $n$ , and let  $S$  be a subset of  $V$  that generates  $V$ .

- (a) Prove that there is subset of  $S$  that is a basis for  $V$ . (Be careful not to assume that  $S$  is finite.)
- (b) Prove that  $S$  contains at least  $n$  vectors.

- (a) Claim: There exists a subset of  $S$  that is a basis for  $V$ . Suppose that  $\beta$  is a basis for  $V$ . This implies  $\beta \subseteq \text{span}(S)$ . Furthermore all vectors in  $\beta$  can be expressed as linear combinations of vectors in  $S$ . Collect these vectors into a set  $S^*$ . This set is finite because linear combinations comprise of a finite number of vectors.

**Lemma:**  $\text{span}(S^*) = V$

**Forward Containment:** Suppose  $x \in \text{span}(S^*)$

$$S^* \subseteq S \subseteq V \quad (34)$$

$$\implies x \in V \text{ (by theorem 1.5)} \quad (35)$$

**Reverse Containment:** Suppose  $x \in V$

By definition of  $S^*$ ,  $\beta \subseteq \text{span}(S^*)$

This implies vectors from  $\beta$  can be represented as linear combinations and thus linear combinations of the basis vectors can be formed. Every vector in  $V$  is a linear combination of vectors in  $\beta$

$$\implies x \in \text{span}(S^*) \quad (36)$$

Therefore there exists a subset of  $S^*$  (and hence a subset of  $S$ ) that is basis for  $V$  (by theorem 1.9)

(b)

$$\dim(V) = n \quad (37)$$

$$\implies \text{card}(\beta) = n \quad (38)$$

$$\beta \subseteq S \quad (39)$$

$$\implies \text{card}(\beta) \leq \text{card}(S) \quad (40)$$

$$\therefore n \leq \text{card}(S) \quad (41)$$

24. Let  $f(x)$  be a polynomial of degree  $n$  in  $P_n(\mathbb{R})$ . Prove that for any  $g(x) \in P_n(\mathbb{R})$  there exist scalars  $c_0, c_1, \dots, c_n$  such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \dots + c_n f^{(n)}(x)$$

where  $f^{(n)}(x)$  denotes  $n$ th derivative of  $f(x)$ .

Given vector space  $P_n(\mathbb{R})$  where  $\dim(P_n(\mathbb{R})) = n + 1$

Suppose  $S$  is a subset of  $P_n(\mathbb{R})$  and  $S = \{f(x), f'(x), f''(x), \dots, f^{(n)}(x)\}$

$$\implies \text{card}(S) = n + 1 \quad (42)$$

If  $S$  is linearly independent, then  $\text{span}(S) = P(\mathbb{R})$  where  $g(x) \in P_n(\mathbb{R})$

Claim:  $S$  is linearly independent

$$d_0 f(x) + d_1 f'(x) + d_2 f''(x) + d_3 f'''(x) + \dots + d_n f^{(n)}(x) = 0 \quad (43)$$

$$d_0 = 0 \because x^n \text{ term only exists in } f(x) \quad (44)$$

$$\implies d_1 = 0 \because x^{n-1} \text{ term only exists in } f'(x) \quad (45)$$

$$\implies d_2 = 0 \because x^{n-2} \text{ term only exists in } f''(x) \quad (46)$$

$$\vdots \quad (47)$$

$$\implies d_n = 0 \because x^0 \text{ term only exists in } f^{(n)}(x) \quad (48)$$

31. Let  $W_1$  and  $W_2$  be subspaces of  $V$  having dimensions  $m$  and  $n$ , respectively, where  $m \geq n$ .

- (a) Prove that  $\dim(W_1 \cap W_2) \leq n$   
 $W_1 \cap W_2$  is a vector space. (by theorem 1.4)

$$W_1 \cap W_2 \subseteq W_2 \quad (49)$$

$$\implies \dim(W_1 \cap W_2) \leq \dim(W_2) \text{ (by theorem 1.11)} \quad (50)$$

$$\dim(W_1 \cap W_2) \leq n \quad (51)$$

(b)

Suppose  $\beta_1$  is a basis for  $W_1$  such that  $\beta_1 = \{v_1, v_2, \dots, v_m\}$   
 Suppose  $\beta_2$  is a basis for  $W_2$  such that  $\beta_2 = \{u_1, u_2, \dots, u_n\}$

**Lemma:**  $\text{span}(\beta_1 \cup \beta_2) = \text{span}(\beta_1) + \text{span}(\beta_2)$   
 Suppose  $x \in \text{span}(\beta_1 \cup \beta_2)$  such that for  $c_i \in F$

$$x = c_1v_1 + c_2v_2 + \dots + c_mv_m + c_{m+1}u_1 + \dots + c_{m+n}u_n$$

Let  $c_1v_1 + c_2v_2 + \dots + c_mv_m = v \implies v \in \text{span}(\beta_1)$

Let  $c_{m+1}u_1 + c_{m+2}u_2 + \dots + c_{m+n}u_n = u \implies u \in \text{span}(\beta_2)$

$$\implies x \in \text{span}(\beta_1) + \text{span}(\beta_2) \quad (52)$$

Suppose  $y \in \text{span}(\beta_1) + \text{span}(\beta_2)$  such that for  $a_i, b_i \in F$

$$y = a_1v_1 + a_2v_2 + \dots + a_mv_m + b_1u_1 + b_2u_2 + \dots + b_nu_n$$

$$\implies y \in \text{span}(\beta_1 \cup \beta_2) \quad (53)$$

a priori  $W_1 + W_2$  is a subspace of  $V$  (by HW2 Q.1.3.23)

Some subset of  $\beta_1 \cup \beta_2$  is a basis for  $W_1 + W_2$  (by theorem 1.9) Define this subset to be  $\beta$ .

$$\implies \text{card}(\beta) \leq \text{card}(\beta_1 \cup \beta_2) \quad (54)$$

$$\text{card}(\beta_1 \cup \beta_2) = \text{card}(\beta_1) + \text{card}(\beta_2) - \text{card}(\beta_1 \cap \beta_2) \quad (55)$$

$$\implies \text{card}(\beta) \leq m + n \quad (56)$$

## 2.1

6.  $\mathsf{T}: \mathsf{M}_{n \times n}(F) \rightarrow F$  defined by  $\mathsf{T}(A) = \text{tr}(A)$ . Recall (Example 4, Section 1.3) that

$$\text{tr}(A) = \sum_{i=1}^n A_{ii}$$

(a) Suppose  $A, B \in \mathsf{M}_{n \times n}(F)$  and  $c \in F$

$$\mathsf{T}(cA + B) = \sum_{i=1}^n (ca_{ii} + b_{ii}) \quad (57)$$

$$= \sum_{i=1}^n ca_{ii} + \sum_{i=1}^n b_{ii} \quad (58)$$

$$= c \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \quad (59)$$

$$= c\mathsf{T}(A) + \mathsf{T}(B) \quad (60)$$

(b)

$$\dim(\mathsf{M}_{n \times n}(F)) = n^2 \quad (61)$$

$$R(\mathsf{T}) = \{\mathsf{T}(x) : x \in \mathsf{M}_{n \times n}(F)\} \quad (62)$$

Claim  $R(\mathsf{T}) = F$

$R(\mathsf{T}) \subseteq F$  by definition of  $\mathsf{T}$

Suppose  $c \in F$ , and  $x \in \mathsf{M}_{n \times n}(F)$  such that

$$x = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$\text{tr}(cx) = c \cdot \text{tr}(x) \quad (63)$$

$$= c \cdot 1 \quad (64)$$

$$= c \quad (65)$$

$$\therefore c \in R(\mathsf{T}) \quad (66)$$

Claim:  $\{1\}$  is a basis for  $F$

$$c \cdot 1 = 0 \quad (67)$$

$$\implies c = 0 \text{ (by cancellation law)} \quad (68)$$

$$\therefore \{1\} \text{ is linearly independent} \quad (69)$$

$$c \cdot 1 = c \text{ for } c \in F \quad (70)$$

$$\text{span}(\{1\}) = F \quad (71)$$

$$\therefore \{1\} \text{ is a basis for } R(\mathbf{T}) \quad (72)$$

$$\implies \text{rank}(\mathbf{T}) = 1 \quad (73)$$

$$\implies \text{nullity}(\mathbf{T}) = n^2 - 1 \text{ (by the dimension theorem)} \quad (74)$$

Claim: Basis  $\beta_n$  for  $N(\mathbf{T})$  is a modification of a standard basis for  $\mathbf{M}_{n \times n}(F)$  in which each matrix containing a 1 in a diagonal entry is replaced with a matrix containing 1 in the same entry and  $-1$  in entry  $(n, n)$  and the matrix where all entries but  $(n, n) = 1$  are zero are removed from the set.

Claim:  $\beta_n$  is linearly independent

Suppose  $x \in \text{span}(\beta_n)$  such that  $x = a_1 u_1 + a_2 u_1 + \cdots + a_{n^2-1} u_{n^2-1}$

$$\begin{pmatrix} a_{1,1} & \cdots & \cdots & \cdots & a_{1,n} \\ \vdots & a_{2,2} & & & \\ \vdots & & \ddots & & \\ \vdots & & & a_{n-1,n-1} & \\ a_n & \cdots & \cdots & \cdots & A \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n \times n} \quad (75)$$

Where  $A = (-a_{1,1}) + (-a_{2,2}) + \cdots + (-a_{n-1,n-1})$

Therefore all the entries of the matrix are zero. Furthermore there only exists the trivial representation. As such by corollary 2 of theorem 1.10  $\beta_n$  is a basis for  $N(\mathbf{T})$ .

Suppose:

$$x = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}$$

$$\text{tr} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} = 0 \quad (76)$$

$$\implies \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} \in N(\mathbf{T}) \quad (77)$$

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad (78)$$

$$\implies N(\mathbf{T}) \neq \{0\} \quad (79)$$

Therefore  $\mathbf{T}$  is not one-to-one by theorem 2.4.

$\mathbf{T}$  was shown to be onto by equation 72



12. Is there a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that  $T(1, 0, 3) = (1, 1)$  and  $T(-2, 0, -6) = (2, 1)$ ?

$$T(-2(1, 0, 3)) = T(-2, 0, -6) = (2, 1) \quad (80)$$

$$-2T((1, 0, 3)) = -2 \times (1, 1) = (-2, -2) \quad (81)$$

$$-2T((1, 0, 3)) \neq T((-2, 0, -6)) \quad (82)$$

$$\implies T(cx) \neq cT(x) \quad \forall c \in F \quad (83)$$

Therefore  $T$  is not a linear transformation.

14. Let  $V$  and  $W$  be vector spaces and  $T: V \rightarrow W$  be linear.

- (a) Prove that  $T$  is one-to-one if and only if  $T$  carries linearly independent subsets of  $V$  onto linearly independent subsets of  $W$ .
- (b) Suppose that  $T$  is one-to-one and that  $S$  is a subset of  $V$ . Prove that  $S$  is linearly independent if and only if  $T(S)$  is linearly independent.
- (c) Suppose  $\beta = \{v_1, v_2, \dots, v_n\}$  is a basis for  $V$  and  $T$  is one-to-one and onto. Prove that  $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$  is a basis for  $W$ .

(a) **Forward Direction:**

Suppose  $S \subseteq V$  such that  $S$  is linearly independent and  $T$  is one-to-one.

Let  $T(S) = \{T(x) : x \in S\}$

Claim:  $T(S)$  is linearly independent.

Suppose  $x \in \text{span}(T(S))$  such that  $x = c_1u_1 + c_2u_2 + \dots + c_nu_n = 0$  for  $c_i \in F$  and  $u_i \in T(S)$

$$T \text{ is one-to-one} \implies u_i = T(v_i) \text{ for some } v_i \in S$$

$$\implies x = c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n) \quad (84)$$

$$= T(c_1v_1 + c_2v_2 + \dots + c_nv_n) \quad (85)$$

$$= 0 \quad (86)$$

$$\implies c_1v_1 + c_2v_2 + \dots + c_nv_n = 0 \quad (87)$$

$$\implies c_1 = c_2 = \dots = c_n \quad (88)$$

Therefore there only exists the trivial solution to the linear combination.

**Reverse Direction:** Suppose  $S \subseteq T$  such that  $S$  is linearly independent and  $T(S)$  is linearly independent.

Claim:  $T$  is one-to-one

Suppose  $x, y \in \text{span}(S)$  such that  $x \neq y$

$$x = a_1u_1 + a_2u_2 + \dots + a_nu_n \quad (89)$$

$$y = b_1u_1 + b_2u_2 + \dots + b_nu_n \quad (90)$$

Because  $S$  is linearly independent each linear combination is unique.

$$\implies a_i \neq b_i \text{ for some } i = 1, 2, \dots, n \quad (91)$$

$$\text{Let } T(x) = T(a_1u_1 + a_2u_2 + \dots + a_nu_n) \quad (92)$$

$$\implies T(x) = a_1T(u_1) + a_2T(u_2) + \dots + a_nT(u_n) \quad (93)$$

$$\text{Let } T(y) = T(b_1u_1 + b_2u_2 + \dots + b_nu_n) \quad (94)$$

$$\implies T(y) = b_1T(u_1) + b_2T(u_2) + \dots + b_nT(u_n) \quad (95)$$

$$T(u_i) \in T(S) \forall i = 1, 2, \dots, n \quad (96)$$

$$\implies T(x), T(y) \in \text{span}(T(S)) \quad (97)$$

Because  $T(S)$  is linearly independent each linear combination is unique.

$$\implies T(x) \neq T(y) \quad (98)$$

(b) Suppose  $T$  is one-to-one,  $S \subseteq V$

**Forward Direction:**

Suppose  $S$  is linearly independent.

Claim  $T(S)$  is linearly independent.

$T(S)$  is linearly independent by part (a)

**Reverse Direction:**

Suppose  $S = \{v_1, v_2, \dots, v_m\}$

$$\implies T(S) = \{T(v_1) + T(v_2) + \dots + T(v_m)\} \quad (99)$$

Suppose  $x \in \text{span}(S)$  such that  $x = c_1v_1 + c_2v_2 + \dots + c_mv_m$

$$\implies T(x) = T(c_1v_1 + c_2v_2 + \dots + c_mv_m) \quad (100)$$

$$= T(0) \quad (101)$$

$$= 0 \quad (102)$$

Because  $T$  is one-to-one.

$$T(x) = c_1T(v_1) + c_2T(v_2) + \dots + c_mT(v_m) \quad (103)$$

$$T(v_i) \in T(S) \forall i = 1, 2, \dots, m \quad (104)$$

Therefore  $T(x) \in \text{span}(T(S))$  and  $T(S)$  is linearly independent.

$$c_1 = c_2 = \dots = c_m = 0 \quad (105)$$

(c) Suppose  $\beta = \{v_1, v_2, \dots, v_n\}$  is a basis for  $V$  and  $T$  is one-to-one and onto.

Claim:  $T(\beta)$  is a basis for  $W$

$$R(T) = W \quad (106)$$

$$= \text{span}(T(\beta)) \text{ (by theorem 2.2)} \quad (107)$$

$T(\beta)$  is linearly independent by part (a)

## 2.2

2. Let  $\beta$  and  $\gamma$  be the standard ordered bases for  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively. For each linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , compute  $[T]_{\beta}^{\gamma}$ .

(b)  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

$$T(1, 0, 0) = (2, 1) \quad (108)$$

$$T(0, 1, 0) = (3, 0) \quad (109)$$

$$T(0, 0, 1) = (-1, 1) \quad (110)$$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 2 & -3 & -1 \\ 1 & 0 & 1 \end{pmatrix} \quad (111)$$

(c)  $T: \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

$$T(1, 0, 0) = 2 \quad (112)$$

$$T(0, 1, 0) = 1 \quad (113)$$

$$T(0, 0, 1) = -3 \quad (114)$$

$$[T]_{\beta}^{\gamma} = (2 \quad 1 \quad -3) \quad (115)$$

(g)  $T: \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $T(a_1, a_2, \dots, a_n) = a_1 + a_n$

$$T(1, 0, \dots, 0) = 1 \quad (116)$$

$$T(0, 1, \dots, 0) = 0 \quad (117)$$

$$\vdots \quad (118)$$

$$T(0, 1, \dots, 1, 0) = 0 \quad (119)$$

$$T(0, 0, \dots, 1) = 1 \quad (120)$$

$$[T]_{\beta}^{\gamma} = (1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1) \quad (121)$$

8. Let  $V$  be an  $n$ -dimensional vector space with an ordered basis  $\beta$ . Define  $T: V \rightarrow F^n$  by  $T(x) = [x]_\beta$ . Prove that  $T$  is linear.  
 Suppose  $\beta = \{v_1, v_2, \dots, v_n\}$   
 Suppose  $x, y \in V$

$$x = \sum_{i=1}^n a_i v_i \qquad y = \sum_{i=1}^n b_i v_i \qquad (122)$$

$$x + y = \sum_{i=1}^n a_i v_i + \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (a_i v_i + b_i v_i) = \sum_{i=1}^n (a_i + b_i) v_i \qquad (123)$$

$$T(x + y) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = T(x) + T(y) \qquad (124)$$

Suppose  $c \in F$  and  $x \in V$

$$x = \sum_{i=1}^n a_i v_i \qquad (125)$$

$$cx = c \sum_{i=1}^n a_i v_i = \sum_{i=1}^n ca_i v_i \qquad (126)$$

$$T(cx) = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = cT(x) \qquad (127)$$

11. Let  $V$  be an  $n$ -dimensional vector space, and let  $T: V \rightarrow V$  be a linear transformation. Suppose that  $W$  is a  $T$ -invariant subspace of  $V$  (see the exercises of Section 2.1) having dimension  $k$ . Show that there is a basis  $\beta$  for  $V$  such that  $[T]_\beta$  has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where  $A$  is a  $k \times k$  matrix and  $O$  is the  $(n - k) \times k$  zero matrix.

Suppose  $\beta_W$  is a basis of  $W$

$$T(\beta_W) \subseteq W \quad (128)$$

Suppose  $x \in \beta_W$

$$\implies T(x) \in W \quad (129)$$

Therefore  $T(x)$  can be described as a linear combination of vectors in  $\beta_W$ .

Suppose  $\beta_W$  is extended to  $\beta$  (by corollary 2 of theorem 2.2) such that

$\beta = \{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_n\}$  and  $u_i \in \beta_W$  for  $i = 1, 2, \dots, n$

$\implies$  for  $u_i, i = 1, 2, \dots, k$ ;

$$[T(u_i)]_\beta = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (130)$$

$$[T]_\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & [T(u_{k+1})]_\beta & [T(u_{k+2})]_\beta & \cdots & [T(u_n)]_\beta \\ a_{21} & a_{22} & \cdots & a_{2k} & & & & \\ \vdots & \vdots & & \vdots & & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & & & & \\ 0 & 0 & \cdots & 0 & & & & \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & \cdots & 0 & & & & \\ 0 & 0 & \cdots & 0 & & & & \end{pmatrix} \quad (131)$$