

1 Assignment

Section 1.2: 7, 20, 21; Section 1.3: 5, 10, 20, 23, 25; Section 1.4: 2, 4, 11, 13

2 Work

1.2

7. Let $S = \{0, 1\}$ and $F = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

Claim: $\forall t \in S, f = g$

- For $t = 0$

$$f(0) = (2 \times 0) + 1 = g(0) = 1 + (4 \times 0) - (2 \times 0^2) \quad (1)$$

$$f(0) = g(0) = 0 \quad (2)$$

- For $t = 1$

$$f(1) = (2 \times 1) + 1 = g(1) = 1 + (4 \times 1) - (2 \times 1^2) \quad (3)$$

$$f(1) = g(1) = 3 \quad (4)$$

Claim: $\forall t \in S, f + g = h$

- For $t = 0$

$$f(0) + g(0) = h(0) = 5^0 + 1 \quad (5)$$

$$f(0) + g(0) = h(0) = 2 \quad (6)$$

- For $t = 1$

$$f(1) + g(1) = h(1) = 5^1 + 1 \quad (7)$$

$$f(1) + g(1) = h(1) = 6 \quad (8)$$

20. Let \mathbf{V} be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in \mathbf{V}$ and real number t define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

$$t \{a_n\} = \{ta_n\}$$

Prove that, with these operations, \mathbf{V} is a vector space over \mathbb{R} .

Definition: $\{a_n\}$ is any sequence where $\sigma: \mathbb{Z}^+ \rightarrow \mathbb{R}$ given by $\sigma(n) = a_n$

Claim: $\{a_n + b_n\} \in \mathbf{V}$

Since the sequences $\{a_n\}$ and $\{b_n\}$ are defined as sequences whose elements exist in \mathbb{R} and addition is a closed binary operation on \mathbb{R} , the sum of any two elements in $\{a_n\}$ and $\{b_n\}$ must also exist in \mathbb{R} .

Claim: $\{ta_n\} \in \mathbf{V}$

Since the sequence $\{a_n\}$ is defined as a sequence whose elements exist in \mathbb{R} and multiplication is a closed binary operation on \mathbb{R} , the product of an element of $\{a_n\}$ and t an element of \mathbb{R} must also exist in \mathbb{R} .

(VS 1) $\forall x, y \in \mathbf{V}, x + y = y + x$

Claim: $\{a_n\} + \{b_n\} = \{b_n\} + \{a_n\}$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} = \{b_n +_{\mathbb{R}} a_n\} = \{b_n\} + \{a_n\} \quad (9)$$

(VS 2) $\forall x, y, z \in \mathbf{V}, (x + y) + z = x + (y + z)$

Claim: $(\{a_n\} + \{b_n\}) + \{c_n\} = \{a_n\} + (\{b_n\} + \{c_n\})$

$$(\{a_n\} + \{b_n\}) + \{c_n\} = \{a_n +_{\mathbb{R}} b_n\} + \{c_n\} \quad (10)$$

$$= \{a_n +_{\mathbb{R}} b_n +_{\mathbb{R}} c_n\} \quad (11)$$

$$= \{a_n\} + \{b_n +_{\mathbb{R}} c_n\} \quad (12)$$

$$= \{a_n\} + (\{b_n\} + \{c_n\}) \quad (13)$$

(VS 3) $\exists 0 \in \mathbf{V}$ such that $\forall x \in \mathbf{V}, x + 0 = x$

Suppose $\{b_n\} \in \mathbf{V}$ such that $\forall n \in \mathbb{Z}^+, \sigma(n) = b_n = 0$

Claim: $\{a_n\} + 0_{\mathbf{V}} = \{a_n\}$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} = \{a_n +_{\mathbb{R}} 0\} = \{a_n\} \quad (14)$$

$$\implies \{b_n\} = 0_{\mathbf{V}} \quad (15)$$

(VS 4) $\forall x \in \mathbf{V} \exists y \in \mathbf{V}$ such that $x + y = 0$

Suppose $\{b_n\} \in \mathbf{V}$ such that $\sigma(n) = b_n = -a_n$

Claim: $\{a_n\} + \{b_n\} = 0$

$$\{a_n\} + \{b_n\} = \{a_n +_{\mathbb{R}} b_n\} \quad (16)$$

$$\{a_n +_{\mathbb{R}} (-a_n)\} = 0_{\mathbf{V}} \quad (17)$$

(VS 5) $\forall x \in \mathbf{V}, 1 \times x = x$

Claim: $1 \times \{a_n\} = \{a_n\}$

$$1 \times \{a_n\} = \{1 \times_{\mathbb{R}} a_n\} = \{a_n\} \quad (18)$$

(VS 6) $\forall a, b \in F, \forall x \in \mathbf{V}, (ab)x = a(bx)$

Suppose $s, t \in \mathbb{R}$

Claim: $(s \times_{\mathbb{R}} t) \{a_n\} = s(t \{a_n\})$

$$(s \times_{\mathbb{R}} t) \{a_n\} = \{(s \times_{\mathbb{R}} t) \times_{\mathbb{R}} a_n\} \quad (19)$$

$$= \{s \times_{\mathbb{R}} (t \times_{\mathbb{R}} a_n)\} \quad (20)$$

$$= s \{t \times_{\mathbb{R}} a_n\} \quad (21)$$

$$= s(t \{a_n\}) \quad (22)$$

(VS 7) $\forall a \in F, \forall x, y \in V, a(x + y) = ax + ay$

Suppose $t \in \mathbb{R}$

Claim: $t(\{a_n\} + \{b_n\}) = t\{a_n\} + t\{b_n\}$

$$t(\{a_n\} + \{b_n\}) = t\{a_n +_{\mathbb{R}} b_n\} \quad (23)$$

$$= \{t \times_{\mathbb{R}} (a_n +_{\mathbb{R}} b_n)\} \quad (24)$$

$$= \{(t \times_{\mathbb{R}} a_n) +_{\mathbb{R}} (t \times_{\mathbb{R}} b_n)\} \quad (25)$$

$$= \{t \times_{\mathbb{R}} a_n\} + \{t \times_{\mathbb{R}} b_n\} \quad (26)$$

$$= t\{a_n\} + t\{b_n\} \quad (27)$$

(VS 8) $\forall a, b \in F, \forall x \in V, (a + b)x = ax + bx$

Suppose $s, t \in \mathbb{R}$

Claim: $(s +_{\mathbb{R}} t)\{a_n\} = s\{a_n\} + t\{a_n\}$

$$(s +_{\mathbb{R}} t)\{a_n\} = \{(s +_{\mathbb{R}} t)a_n\} \quad (28)$$

$$= \{s \times_{\mathbb{R}} a_n + t \times_{\mathbb{R}} a_n\} \quad (29)$$

$$= \{s \times_{\mathbb{R}} a_n\} + \{t \times_{\mathbb{R}} a_n\} \quad (30)$$

$$= s\{a_n\} + t\{a_n\} \quad (31)$$

21. Let V and W be vector spaces over a field F . Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1)$$

Claim: $(v_1 + v_2, w_1 + w_2) \in Z$

$$v_1, v_2 \in V \qquad w_1, w_2 \in W \qquad (32)$$

$$\implies (v_1 + v_2) \in V \qquad \implies (w_1 + w_2) \in W \qquad (33)$$

$$\implies (v_1 + v_2, w_1 + w_2) \in Z \qquad (34)$$

$$\therefore + : Z \times Z \rightarrow Z \qquad (35)$$

Claim: $(cv_1, cw_1) \in Z$

$$v_1 \in V \qquad w_1 \in W \qquad (36)$$

$$\implies (c \times_V v_1) \in V \qquad \implies (c \times_W w_1) \in W \qquad (37)$$

$$\implies (c \times_V v_1, c \times_W w_1) \in Z \qquad (38)$$

$$(cv_1, cw_1) \in Z \qquad (39)$$

$$\therefore \times_Z : F \times Z \rightarrow Z \qquad (40)$$

(VS 1) $\forall x, y \in \mathbf{V}, x + y = y + x$

Suppose $z_1, z_2 \in \mathbf{Z}$ where $z_1 = (v_1, w_1)$ and $z_2 = (v_2, w_2)$

Claim: $z_1 + z_2 = z_2 + z_1$

$$z_1 +_{\mathbf{Z}} z_2 = (v_1, w_1) +_{\mathbf{Z}} (v_2, w_2) \quad (41)$$

$$= (v_1 +_{\mathbf{V}} v_2, w_1 +_{\mathbf{W}} w_2) \quad (42)$$

$$= (v_2 +_{\mathbf{V}} v_1, w_2 +_{\mathbf{W}} w_1) \quad (43)$$

$$= (v_2, w_2) +_{\mathbf{Z}} (v_1, w_1) \quad (44)$$

$$= z_2 +_{\mathbf{Z}} z_1 \quad (45)$$

(VS 2) $\forall x, y, z \in \mathbf{V}, (x + y) + z = x + (y + z)$

Suppose $z_1, z_2, z_3 \in \mathbf{Z}$ where $z_1 = (v_1, w_1)$, $z_2 = (v_2, w_2)$ and $z_3 = (v_3, w_3)$

Claim: $(z_1 +_{\mathbf{Z}} z_2) +_{\mathbf{Z}} z_3 = z_1 +_{\mathbf{Z}} (z_2 +_{\mathbf{Z}} z_3)$

$$(z_1 +_{\mathbf{Z}} z_2) +_{\mathbf{Z}} z_3 = (v_1 +_{\mathbf{V}} v_2, w_1 +_{\mathbf{W}} w_2) +_{\mathbf{V}} (v_3, w_3) \quad (46)$$

$$= (v_1 +_{\mathbf{V}} v_2 +_{\mathbf{V}} v_3, w_1 +_{\mathbf{W}} w_2 +_{\mathbf{W}} w_3) \quad (47)$$

$$= (v_1, w_1) +_{\mathbf{Z}} (v_2 +_{\mathbf{V}} v_3, w_2 +_{\mathbf{W}} w_3) \quad (48)$$

$$= z_1 +_{\mathbf{Z}} (z_2 +_{\mathbf{Z}} z_3) \quad (49)$$

(VS 3) $\exists 0 \in \mathbf{V}$ such that $\forall x \in \mathbf{V}, x + 0 = x$

Suppose $z \in \mathbf{Z}$ where $z = (v, w)$ and $0_{\mathbf{Z}} = (0_{\mathbf{V}}, 0_{\mathbf{W}})$

Claim: $z +_{\mathbf{Z}} 0_{\mathbf{Z}} = z$

$$z + 0_{\mathbf{Z}} = (v, w) +_{\mathbf{Z}} (0_{\mathbf{V}}, 0_{\mathbf{W}}) \quad (50)$$

$$= (v +_{\mathbf{V}} 0_{\mathbf{V}}, w +_{\mathbf{W}} 0_{\mathbf{W}}) \quad (51)$$

$$= (v, w) \quad (52)$$

$$= z \quad (53)$$

(VS 4) $\forall x \in \mathbf{V} \exists y \in \mathbf{V}$ such that $x + y = 0$

Suppose $z_1, z_2 \in \mathbf{Z}$ where $z_1 = (v, w)$ and $z_2 = (-v, -w)$

Claim: $z_1 +_{\mathbf{Z}} z_2 = 0_{\mathbf{Z}}$

$$z_1 +_{\mathbf{Z}} z_2 = (v, w) +_{\mathbf{Z}} (-v, -w) \quad (54)$$

$$= (v +_{\mathbf{V}} (-v), w +_{\mathbf{W}} (-w)) \quad (55)$$

$$= (0_{\mathbf{V}}, 0_{\mathbf{W}}) \quad (56)$$

$$= 0_{\mathbf{Z}} \quad (57)$$

(VS 5) $\forall x \in \mathbf{V}, 1 \times x = x$ Suppose $z \in \mathbf{Z}$ where $z = (v, w)$

Claim: $1 \times_{\mathbf{Z}} z = z$

$$1 \times_{\mathbf{Z}} z = 1 \times_{\mathbf{Z}} (v, w) \quad (58)$$

$$= (1 \times_{\mathbf{V}} v, 1 \times_{\mathbf{W}} w) \quad (59)$$

$$= (v, w) \quad (60)$$

$$= z \quad (61)$$

(VS 6) $\forall a, b \in F, \forall x \in \mathbf{V}, (ab)x = a(bx)$

Suppose $z \in \mathbf{Z}$ where $z = (v, w)$ and $a, b \in F$

Claim: $(a \times_F b) \times_{\mathbf{Z}} z = a \times_{\mathbf{Z}} (b \times_{\mathbf{Z}} z)$

$$(a \times_F b) \times_{\mathbf{Z}} z = (ab \times_{\mathbf{V}} v, ab \times_{\mathbf{W}} w) \quad (62)$$

$$= a \times_{\mathbf{Z}} (b \times_{\mathbf{V}} v, b \times_{\mathbf{W}} w) \quad (63)$$

$$= a \times_{\mathbf{Z}} (b \times_{\mathbf{Z}} z) \quad (64)$$

(VS 7) $\forall a \in F, \forall x, y \in V, a(x + y) = ax + ay$

Suppose $z_1, z_2 \in Z$ where $z_1 = (v_1, w_1)$, $z_2 = (v_2, w_2)$ and $a \in F$

Claim: $a \times_Z (z_1 +_Z z_2) = a \times_Z z_1 +_Z a \times_Z z_2$

$$a \times_Z (z_1 +_Z z_2) = a \times_Z (v_1 + v_2, w_1 + w_2) \quad (65)$$

$$= (a \times_V (v_1 + v_2), a \times_W (w_1 + w_2)) \quad (66)$$

$$= (a \times_W v_1 +_W a \times_W v_2, a \times_W w_1 +_W a \times_W w_2) \quad (67)$$

$$= (a \times_V v_1, a \times_W w_1) +_Z (a \times_V v_2, a \times_W w_2) \quad (68)$$

$$= a \times_Z z_1 +_Z a \times_Z z_2 \quad (69)$$

(VS 8) $\forall a, b \in F, \forall x \in V, (a + b)x = ax + bx$

Suppose $z \in Z$ where $z = (v, w)$ and $a, b \in F$

Claim: $(a +_F b) \times_Z z = a \times_Z z +_Z b \times_Z z$

$$(a +_F b) \times_Z z = (a +_F b) \times_Z (v, w) \quad (70)$$

$$= ((a +_F b) \times_V v, (a +_F b) \times_W w) \quad (71)$$

$$= (a \times_V v +_V b \times_V v, a \times_W w +_W b \times_W w) \quad (72)$$

$$= (a \times_V v, a \times_W w) +_Z (b \times_V v, b \times_W w) \quad (73)$$

$$= a \times_Z z +_Z b \times_Z z \quad (74)$$

1.3

5. Prove that $A + A^t$ is symmetric for any square matrix A .

Claim: $A + A^t = (A + A^t)^t$

$$A_{n,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

$$A_{n,n}^t = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{pmatrix}$$

$$A_{n,n} + A_{n,n}^t = \begin{pmatrix} a_{1,1} + a_{1,1} & a_{1,2} + a_{2,1} & \cdots & a_{1,n} + a_{n,1} \\ a_{2,1} + a_{1,2} & a_{2,2} + a_{2,2} & \cdots & a_{2,n} + a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + a_{1,n} & a_{n,2} + a_{2,n} & \cdots & a_{n,n} + a_{n,n} \end{pmatrix}$$

$$(A_{n,n} + A_{n,n}^t)^t = \begin{pmatrix} a_{1,1} + a_{1,1} & a_{2,1} + a_{1,2} & \cdots & a_{n,1} + a_{1,n} \\ a_{1,2} + a_{2,1} & a_{2,2} + a_{2,2} & \cdots & a_{n,2} + a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} + a_{n,1} & a_{2,n} + a_{n,2} & \cdots & a_{n,n} + a_{n,n} \end{pmatrix}$$

10. Prove that $W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 0\}$ is a subspace of F^n , but $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 1\}$ is not.

(a) $0_V \in W$

Suppose $(a_1, a_2, \dots, a_n) \in W_1$

Suppose $(b_1, b_2, \dots, b_n) \in W_1$ such that $b_i = 0$ for integer $i \in [1, n]$

Claim: $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1, a_2, \dots, a_n)$

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \quad (75)$$

$$= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \quad (76)$$

$$= (a_1, a_2, \dots, a_n) \quad (77)$$

(b) $\forall x, y \in W, x + y \in W$

Suppose $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in W_1$

Claim: $(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in W_1$

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \quad (78)$$

$$\sum_{k=1}^n (a_k) + \sum_{k=1}^n (b_k) = \sum_{k=1}^n (a_k + b_k) = 0 \quad (79)$$

$$\therefore (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in W_1 \quad (80)$$

(c) $\forall a \in F, x \in W, ax \in W$

Suppose $c \in F, (a_1, a_2, \dots, a_n) \in W_1$

Claim: $(ca_1, ca_2, \dots, ca_n) \in W_2$

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n) \quad (81)$$

$$\sum_{k=1}^n (ca_k) = c \sum_{k=1}^n (a_k) = 0 \quad (82)$$

$$\therefore (ca_1, ca_2, \dots, ca_n) \in W_1 \quad (83)$$

Suppose $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in W_2$

Claim: $(a_1 + b_1, a_1 + b_2, \dots, a_n + b_n) \notin W_2$

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \quad (84)$$

$$\sum_{k=1}^n (a_k) + \sum_{k=1}^n (b_k) = \sum_{k=1}^n (a_k + b_k) = 2 \quad (85)$$

$$\therefore (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \notin W_2 \quad (86)$$

20. Prove that if W is a subspace of a vector space V and w_1, w_2, \dots, w_n are in W , then

$a_1 w_1 + a_2 w_2 + \dots + a_n w_n \in W$ for any scalars a_1, a_2, \dots, a_n .

Suppose $a_1, a_2, \dots, a_n \in F, w_1, w_2, \dots, w_n \in W$

Claim: $\forall a_1, a_2, \dots, a_n \in F$ and $\forall w_1, w_2, \dots, w_n \in W, a_1 w_1 + a_2 w_2 + \dots + a_n w_n \in W$.

For every integer $i \in [1, n], a_i w_i \in W$ by theorem 1.3.c

$$\implies a_1 w_1 + a_2 w_2 + \dots + a_n w_n \text{ (by theorem 1.2.c)} \quad (87)$$

23. Let W_1 and W_2 be subspaces of a vector space V .

- (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2 .
- (b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$

Claim: $W_1 + W_2 \subseteq V$

Suppose $x \in W_1 + W_2$ such that $z = x + y$ and $x \in W_1, y \in W_2$

$$W_1 \in V \tag{88}$$

$$W_2 \in V \tag{89}$$

$$\implies x \in V \tag{90}$$

$$\implies y \in V \tag{91}$$

$$\implies x + y \in V \tag{92}$$

$$\implies W_1 + W_2 \subseteq V \tag{93}$$

(i) $0 \in W_1$

$$0 \in W_1 \tag{94}$$

$$0 \in W_2 \tag{95}$$

Suppose $z \in W_1 + W_2$ such that $z = x + y$ and $x = y = 0$

$$\implies z = 0 + 0 = 0 \tag{96}$$

(ii) $x + y \in W$ when $x \in W$ and $y \in W$ Suppose $z_1, z_2 \in W$

such that $z_1 = x + y, z_2 = a + b$

Claim: $z_1 + z_2 \in W_1 + W_2$

$$z_1 + z_2 = x + y + a + b \tag{97}$$

$$= (x + a) + (y + b) \tag{98}$$

$$x + a \in W_1 \quad (99)$$

$$y + b \in W_2 \quad (100)$$

$$\implies z_1 + z_2 \in W_2 \quad (101)$$

(iii) $cx \in W$ when $z \in W_1 + W_2$ such that $z = x + y$

Suppose $c \in F$ and $z \in W_1 + W_2$ such that $z = x + y$

$$cz = c(x + y) \quad (102)$$

$$= cx + cy \quad (103)$$

$$cx \in W_1 \quad (104)$$

$$cy \in W_2 \quad (105)$$

$$\implies cz \in W_1 + W_2 \quad (106)$$

Suppose X is a subspace of V and $W_1 \subseteq X$, and $W_2 \subseteq X$

Claim: $W_1 + W_2 \subseteq X$

Suppose $z \in W_1 + W_2$ such that $z = x + y$ for $x \in W_1$ and $y \in W_2$

$$x \in W_1 \implies x \in X \quad (107)$$

$$y \in W_2 \implies y \in X \quad (108)$$

$$\implies x + y \in X \quad (109)$$

25. Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0,$$

we have $b_i = 0$ whenever i is odd. Prove that $P(F) = W_1 \oplus W_2$.

$$W_1 = \{x \in P(F) : c_k = 0 \ \forall \text{ integers } k \in [0, 2] \text{ such that } 2 \mid (k+1)\} \quad (110)$$

$$W_2 = \{x \in P(F) : c_k = 0 \ \forall \text{ integers } k \in [0, 2] \text{ such that } 2 \nmid k\} \quad (111)$$

(a) $0_V \in W$

Suppose $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_1$,

$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in W_1$ such that $b_i = 0$ for every integer $i \in [1, m]$

Claim: $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) =$
 $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0)$

$$(a_n x^n + \cdots + a_2 x^2 + a_0) + (b_m x^m + \cdots + b_3 x^3 + b_1 x) \quad (112)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (0x^m + 0x^{m-2} + \cdots + 0x^3 + 0x) \quad (113)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (0 + 0 + \cdots + 0 + 0) \quad (114)$$

$$(a_n x^n + \cdots + a_0) + 0_{W_1} = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \quad (115)$$

(b) $\forall x, y \in W, x + y \in W$

Suppose $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x),$

$$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in W_1$$

Claim: $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) +$

$$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in W_1$$

Without loss of generality assume $n \leq m \implies \exists k \in [0, \frac{n-1}{2}]$ such that
 $m = n - 2k$

$$\begin{aligned} & (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) + \\ & (b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \end{aligned} \quad (116)$$

$$\begin{aligned} & (a_n x^n + a_{n-2} x^{n-2} + \cdots + \\ & (a_{n-2k} + b_{n-2k}) (x^{n-2k}) + (a_{n-2k-2} + b_{n-2k-2}) (x^{n-2k-2}) + \cdots + \\ & (a_3 + b_3) x^3 + (a_1 + b_1) x \end{aligned} \quad (117)$$

$$\begin{aligned} \implies & (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) + \\ & (b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in W_1 \end{aligned} \quad (118)$$

(c) $\forall a \in F, x \in W, ax \in W$

Suppose $c \in F$ and $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) \in W_1$

Claim: $c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) \in W_1$

$$\begin{aligned} & c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) \\ & = (ca_n x^n + ca_{n-2} x^{n-2} + \cdots + ca_3 x^3 + ca_1 x) \end{aligned} \quad (119)$$

$$ca_n, ca_{n-2}, \dots, ca_1 \in F \quad (120)$$

$$\implies c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) \in W_1 \quad (121)$$

(a) $0_V \in W$

Suppose $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_2$,

$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_2^2 + b_0) \in W_2$ such that $b_i = 0$ for every integer $i \in [1, m]$

Claim: $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (b_m x^m + b_{m-2} x^{m-2} + \cdots + b_2 x^2 + b_0) =$
 $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0)$

$$(a_n x^n + \cdots + a_2 x^2 + a_0) + (b_m x^m + \cdots + b_2 x^2 + b_0) \quad (122)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (0x^m + 0x^{m-2} + \cdots + 0x^2 + 0) \quad (123)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (0 + 0 + \cdots + 0 + 0) \quad (124)$$

$$(a_n x^n + \cdots + a_0) + 0_{W_1} = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \quad (125)$$

(b) $\forall x, y \in W, x + y \in W$

Suppose $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x),$

$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_3 x^3 + b_1 x) \in W_2$

Claim: $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) +$

$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_2 x^2 + b_0) \in W_2$

Without loss of generality assume $n \leq m \implies \exists k \in [0, \frac{n}{2}]$ such that $m = n - 2k$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) +$$

$$(b_m x^m + b_{m-2} x^{m-2} + \cdots + b_2 x^2 + b_0) \quad (126)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots +$$

$$(a_{n-2k} + b_{n-2k}) (x^{n-2k}) + (a_{n-2k-2} + b_{n-2k-2}) (x^{n-2k-2}) + \cdots +$$

$$(a_2 + b_2) x^2 + (a_0 + b_0) \quad (127)$$

$$\begin{aligned} \implies (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + \\ (b_m x^m + b_{m-2} x^{m-2} + \cdots + b_2 x^2 + b_0) \in W_2 \end{aligned} \quad (128)$$

(c) $\forall a \in F, x \in W, ax \in W$

Suppose $c \in F$ and $(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_2$

Claim: $c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_2$

$$\begin{aligned} c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \\ = (ca_n x^n + ca_{n-2} x^{n-2} + \cdots + ca_2 x^2 + ca_0) \end{aligned} \quad (129)$$

$$ca_n, ca_{n-2}, \dots, ca_0 \in F \quad (130)$$

$$\implies c(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_1 \quad (131)$$

Case I

Suppose $A \in P(F)$, $A \neq 0_P$

Claim: $A \notin W_1 \cap W_2$

Case (i) Suppose $A \in W_2$, $A = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0)$ such that $a_n \neq 0$

By definition of W_2 $2 \nmid (n+1) \implies 2 \mid n$

Claim: $A \notin W_1$

Suppose $A \in W_1$

$$2 \mid n \implies a_n = 0 \not\vdash \text{Contradiction!} \quad (132)$$

$$\implies A \notin W_1 \quad (133)$$

Case (ii) Suppose $A \in W_1, A = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x)$ such that $a_n \neq 0$ By definition of W_1 $2 \nmid n \implies 2 \mid (n+1)$

Claim: $A \notin W_2$

Suppose $A \in W_2$

$$2 \mid (n+) \implies a_n = 0 \not\Leftarrow \text{Contradiction!} \quad (134)$$

$$\implies A \notin W_2 \quad (135)$$

Case II

Suppose $A \in P(F), A = 0$

Claim: $A \in W_1 \cap W_2$

$$0_{W_1} \in W_1 \quad (136)$$

$$0_{W_2} \in W_2 \quad (137)$$

$$0_{W_1} = 0_F = 0_{W_2} \quad (138)$$

$$0_{W_1} = 0_{W_2} \quad (139)$$

$$\implies 0 \in W_1 \cap W_2 \quad (140)$$

$$\implies W_1 \cap W_2 = \{0\} \quad (141)$$

Claim: $P(F) \supseteq W_1 + W_2$

$$W_1 \subseteq P(F) \quad (142)$$

$$W_2 \subseteq P(F) \quad (143)$$

Suppose $x \in W_1, y \in W_2$

$$x + y \in P(F) \quad (144)$$

$$\implies W_1 + W_2 \subseteq P(F) \quad (145)$$

Claim: $P(F) \subseteq W_+ W_2$

Suppose $h \in P(F)$, $h = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$

Case (i) n is odd

$$h = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) + (a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + \cdots + a_2 x^2 + a_0) \quad (146)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_1 x) \in W_2 \quad (147)$$

$$(a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + \cdots + a_2 x^2 + a_0) \in W_1 \quad (148)$$

Case (ii) n is even

$$h = (a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) + (a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + \cdots + a_3 x^3 + a_1 x) \quad (149)$$

$$(a_n x^n + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_0) \in W_2 \quad (150)$$

$$(a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + \cdots + a_3 x^3 + a_1 x) \in W_1 \quad (151)$$

Claim:

$$P(F) = \{ (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + x_0) : c_i \in F \text{ for integer } i \in [1, n] \} = W_1 \oplus W_2$$

$$W_1 \cap W_2 = \{0\} \quad (152)$$

$$W_1 + W_2 = P(F) \quad (153)$$

$$\implies P(F) = W_1 \oplus W_2 \quad (154)$$

1.4

A system of linear equations such as

$$x_1 + 3x_2 + 3x_3 = 4$$

$$x_1 + 4x_2 + x_3 = 5$$

$$3x_1 + x_2 + 5x_3 = 2$$

Can be rewritten as an augmented matrix, with the left most columns representing the coefficients of the variables, and the right hand column representing the right hand side of the linear equations.

$$+ \begin{pmatrix} 1 & 3 & 3 & 4 \\ 1 & 4 & 1 & 5 \\ 3 & 1 & 5 & 2 \end{pmatrix}$$

Solve the following systems of linear equations, if possible.

2. (a)

$$\begin{pmatrix} 2 & -2 & -3 & 0 & -2 \\ 3 & -3 & -3 & 5 & 7 \\ 1 & -1 & -2 & -1 & -3 \end{pmatrix} \begin{matrix} \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \end{matrix} \begin{matrix} -2 & -3 & -3 & -3 & -3 \\ -2 & -3 & -3 & -3 & -3 \\ -2 & -3 & -3 & -3 & -3 \end{matrix} \leftarrow + \mid \cdot 1/4 \rightarrow \begin{pmatrix} 1 & -1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 2 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $x_2 = s, x_4 = t$

$$x_1 = -35 + s + 3t \quad (155)$$

$$x_2 = s \quad (156)$$

$$x_3 = 16 - 2t \quad (157)$$

$$x_4 = t \quad (158)$$

(b)

$$\begin{pmatrix} 3 & -7 & 4 & 10 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{pmatrix} \begin{array}{c} \leftarrow \begin{array}{c} \boxed{-3} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{-2} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \\ \leftarrow \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{2} \\ \leftarrow + \end{array} \end{array} \mid \cdot -1 \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$x_1 = -2 \quad (159)$$

$$x_2 = -4 \quad (160)$$

$$x_3 = -3 \quad (161)$$

(c)

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & 1 & 5 \\ 2 & 3 & -1 & 4 & 8 \end{pmatrix} \begin{array}{c} \leftarrow \begin{array}{c} \boxed{-1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{-2} \\ \leftarrow + \end{array} \\ \leftarrow \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \end{array} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 0 & -1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Inconsistent; not solvable.

(d)

$$\begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & -3 \end{pmatrix} \begin{array}{c} \leftarrow \begin{array}{c} \boxed{-1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{-1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \\ \leftarrow \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \\ \leftarrow \begin{array}{c} \boxed{-2} \\ \leftarrow + \end{array} \begin{array}{c} \boxed{1} \\ \leftarrow + \end{array} \end{array} \rightarrow \begin{pmatrix} 1 & 0 & 8 & 0 & -16 \\ 0 & 0 & 0 & -5 & -10 \\ 0 & -1 & 3 & 0 & -4 \end{pmatrix}$$

Let $s = x_3$

$$x_1 = -8s - 16 \quad (162)$$

$$x_2 = 3s + 4 \quad (163)$$

$$x_3 = s \quad (164)$$

$$x_4 = 2 \quad (165)$$

4.

$$(a) \quad x^3 - 3x + 5 \stackrel{?}{=} c_1 (x^3 + 2x^2 - x + 1) + c_2 (x^3 + 3x^2 - 1)$$

$$c_1 + c_2 = 1 \quad (173)$$

$$2c_1 + 3c_2 = 0 \quad (174)$$

$$-c_1 = -3 \quad (175)$$

$$c_1 - c_2 = 5 \quad (176)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & -3 \\ 1 & -1 & 5 \end{pmatrix} \begin{array}{c} \begin{array}{cc} \overline{\quad}^{-2} & \overleftarrow{\quad}^{+} \\ \overleftarrow{\quad}^{+} & \overline{\quad}^{-1} \end{array} \begin{array}{c} \overline{\quad}^1 \\ \overleftarrow{\quad}^{-1} \end{array} \begin{array}{c} \overline{\quad}^{-1} \\ \overleftarrow{\quad}^{+} \end{array} \\ \overleftarrow{\quad}^{+} \\ \overleftarrow{\quad}^{+} \end{array} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

$$x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1) \quad (177)$$

$$(b) \quad 4x^3 + 2x^2 - 6 \stackrel{?}{=} c_1 (x^3 - 2x^3 + 4x + 1) + c_2 (3x^3 - 6x^2 + x + 4)$$

$$c_1 + 3c_2 = 4 \quad (178)$$

$$-2c_1 + -6c_2 = 2 \quad (179)$$

$$4c_1 + c_2 = 0 \quad (180)$$

$$c_1 + 4c_2 = -6 \quad (181)$$

$$\begin{pmatrix} 1 & 3 & 4 \\ -2 & -6 & 2 \\ 4 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix} \begin{array}{c} \overline{\quad}^2 \\ \overleftarrow{\quad}^{+} \end{array} \rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 10 \\ 4 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix}$$

Inconsistent; no linear combinations.

$$(c) \quad -2x^3 - 11x^2 + 3x + 2 \stackrel{?}{=} c_1 (x^3 - ex^2 + 3x - 1) + c_2 (2x^3 + x^2 + 3x - 2)$$

$$c_1 + 2c_2 = -2 \quad (182)$$

$$-2c_1 + c_2 = -11 \quad (183)$$

$$3c_1 + 3c_2 = 3 \quad (184)$$

$$-c_1 = 2 \quad (185)$$

$$\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ | \cdot 1/3 \\ \leftarrow + \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array} \rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & -15 \\ 0 & 1 & 5 \\ -1 & 0 & 2 \end{pmatrix}$$

Inconsistent; no linear combinations.

$$(d) \quad x^3 + x^2 + 2x + 13 \stackrel{?}{=} c_1 (2x^3 - 2x^2 + 4x + 1) + c_2 (x^3 - x^2 + 2x + 3)$$

$$2c_1 + c_2 = 1 \quad (186)$$

$$-3c_1 - c_2 = 1 \quad (187)$$

$$4c_1 + 2c_2 = 2 \quad (188)$$

$$c_1 + 3c_2 = 13 \quad (189)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ -3 & -1 & 1 \\ 4 & 2 & 2 \\ 1 & 3 & 13 \end{pmatrix} \begin{array}{l} \leftarrow + \quad \leftarrow -3 \quad | \cdot -1 \\ \leftarrow + \quad \leftarrow + \quad | \cdot -1 \\ \leftarrow + \quad \leftarrow + \quad \leftarrow + \\ \leftarrow + \quad \leftarrow + \quad \leftarrow + \end{array} \begin{array}{l} \leftarrow -4 \\ \leftarrow -2 \\ \leftarrow -1 \\ \leftarrow -3 \end{array} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix}$$

$$x^3 + x^2 + 2x + 13 = -2 (2x^3 - 2x^2 + 4x + 1) + 5 (x^3 - x^2 + 2x + 3) \quad (190)$$

(e) $x^3 - 8x^2 + 4x \stackrel{?}{=} c_1(x^3 - 2x^2 + 3x - 1) + c_2(x^3 - 2x + 3)$

$$c_1 + c_2 = 1 \quad (191)$$

$$c_1 = 4 \quad (192)$$

$$3c_1 - 2c_2 = 1 \quad (193)$$

$$-c_1 + 3c_2 = 0 \quad (194)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 3 & -2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{array}{c} \leftarrow^+ \quad \leftarrow^+ \\ \quad \quad \quad \downarrow \\ \quad \quad \quad \text{---}^1 \quad \text{---}^{-1} \\ \quad \quad \quad \downarrow \\ \quad \quad \quad \leftarrow^+ \quad | \cdot 1/3 \quad \text{---}^{-1} \end{array} \rightarrow \begin{pmatrix} 0 & 0 & -16/3 \\ 1 & 0 & 4 \\ 3 & -2 & 1 \\ 0 & 1 & 4/3 \end{pmatrix}$$

Inconsistent; no linear combination.

(f) $6x^3 - 3x^2 + x + 2 \stackrel{?}{=} c_1(x^3 - x^2 + 2x + 3) + c_2(2x^3 - 3x + 1)$

$$c_1 + c_2 = 6 \quad (195)$$

$$c_1 = 3 \tag{196}$$

$$2c_1 - 3c_2 = 1 \quad (197)$$

$$3c_1 + c_2 = 2 \quad (198)$$

$$\begin{pmatrix} 1 & 1 & 6 \\ 1 & 0 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{array}{l} \xleftarrow{+} \quad \xleftarrow{+} \\ | \cdot -1 \quad \downarrow -1 \\ \xleftarrow{+} \quad | \cdot -1/3 \quad \downarrow -1 \end{array} \rightarrow \begin{pmatrix} 0 & 0 & 4/3 \\ 1 & 0 & 3 \\ 0 & 1 & 5/3 \\ 3 & 1 & 2 \end{pmatrix}$$

Inconsistent; no linear combinations.

11. Prove that $\text{span}(\{x\}) = \{ax : a \in F\}$ for any vector x is a vector space. Interpret this result geometrically in \mathbb{R}^3 .

Suppose V is a vector space.

Claim: $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$ Suppose $y \in \text{span}(\{x\})$

$$y = a_1x + a_2x + \cdots + a_nx \text{ for } a_1, \dots, a_n \in F \quad (199)$$

$$= (a_1 + a_2 + \cdots + a_n)x \quad (200)$$

$$(a_1 + a_2 + \cdots + a_n) \in F \quad (201)$$

$$\implies y \in \{ax : a \in F\} \quad (202)$$

Claim: $\text{span}(\{x\}) \supseteq \{ax : a \in F\}$

Suppose $x \in \{ax : a \in F\}$

$$z = bx \text{ for } b \in F \quad (203)$$

bx is a linear combination of 1 term.

$$\implies z \in \text{span}(\{x\}) \quad (204)$$

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

Claim: $\text{span}(S_1) \subseteq \text{span}(S_2)$

Suppose: $y \in \text{span}(S_1)$

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n \text{ for } a_1, a_2, \dots, a_n \in F \text{ and } x_1, x_2, \dots, x_n \in S_1 \quad (205)$$

$$\implies S_1 \subseteq S_2 \forall \text{ integers } i \in [1, n], x_i \in S_2 \quad (206)$$

$$\forall a_1, a_2, \dots, a_n \in F, a_1x_1 + a_2x_2 + \cdots + a_nx_n \in \text{span}(S_2) \quad (207)$$

$$\implies \text{span}(S_1) \subseteq \text{span}(S_2) \quad (208)$$

Claim: $\text{span}(S_2) \subseteq V$

Suppose $y \in \text{span}(S_2)$

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \forall a_1x_1, a_2x_2, \dots, a_n \in F \text{ and } x_1, x_2, \dots, x_n \in S_2 \quad (209)$$

Given $S_2 \subseteq V$

$$x_1, x_2, \dots, x_n \in V \implies a_1x_1 + a_2x_2 + \cdots + a_nx_n \in V \quad (210)$$

$$\implies \text{span}(S_2) \subseteq V \quad (211)$$