Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

Work

3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$x_1 + 2x_2 - x_3 + 3x_4 = 2$$
$$2x_1 + 4x_2 - x_3 + 6x_4 = 5$$
$$x_2 + 2x_4 = 3$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 & \bigm| & 2 \\ 2 & 4 & -1 & 6 & \bigm| & 5 \\ 0 & 1 & 0 & 2 & \bigm| & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 9 & -4 & \bigm| & -3 \\ 0 & 1 & 0 & 2 & \bigm| & 3 \\ 0 & 0 & 1 & 0 & \bigm| & 1 \end{pmatrix} \tag{1}$$

$$x_1 = -3 + 4x_4 \tag{2}$$
$$x_2 = 3 - 2x_4 \tag{3}$$
$$x_3 = 1 \tag{4}$$
$$x_4 = x_4 \tag{5}$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{6}$$

(j)

$$2x_1 + 3x_3 - 4x_5 = 5 \tag{7}$$
$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8 \tag{8}$$
$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2 \tag{9}$$
$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8 \tag{10}$$

$$\begin{pmatrix} 2 & 0 & 3 & 0 & -4 & \bigm| & 5 \\ 3 & -4 & 8 & 3 & 0 & \bigm| & 8 \\ 1 & -1 & 2 & 1 & -1 & \bigm| & 2 \\ -2 & 5 & -9 & -3 & -5 & \bigm| & -8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & \bigm| & 1 \\ 0 & 1 & 0 & 0 & -3 & \bigm| & 0 \\ 0 & 0 & 1 & 0 & 0 & \bigm| & -1 \\ 0 & 0 & 0 & 1 & -2 & \bigm| & -1 \end{pmatrix} \tag{11}$$

$$x_1 = 2x_5 + 1 \tag{12}$$

$$x_2 = 3x_5 \tag{13}$$

$$x_3 = -1 \tag{14}$$

$$x_4 = 2x_5 - 1 \tag{15}$$

$$x_5 = x_5 \tag{16}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{17}$$

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$u_1 = (2, -3, 4, -5, 2), \qquad u_2 = (-6, 9, -12, 15, -6),$$
$$u_3 = (3, -2, 7, -9, 1), \qquad u_4 = (2, -8, 2, -2, 6),$$
$$u_5 = (-1, 1, 2, 1, -3), \qquad u_6 = (0, -3, -18, 9, 12),$$
$$u_7 = (1, 0, -2, 3, -2), \qquad u_8 = (2, -1, 1, -9, 7)$$

generate W. Find a subset $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

$$\mathsf{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \ldots, x_5 \in \mathbb{R} \right\} \tag{18}$$

$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \tag{19}$$

$$\rightsquigarrow \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ & & 1 & -2 & 0 & -2 & 0 & 1 \\ & & & 1 & -4 & 0 & -2 \\ & \mathbf{0} & & & 1 & -1 \\ & & & & & 0 \end{pmatrix} \tag{20}$$

It follows that $\{u_1, u_3, u_5, u_7\}$ is linearly independent by theorem 3.16. Therefore $\{u_1, u_3, u_5, u_7\}$ is a basis for W.

11.

14.

15.

II

4.1

10.

4.2

23.

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

(a) **Types 1 & 2**

$$E^t = E \quad \text{(by HW.3.1.5)} \tag{21}$$
$$\Rightarrow \det(E^t) = \det(E) \tag{22}$$

(b) **Type 3**

E^t is an type 3 elementary matrix (by HW.3.1.5) $\det(E) = \det(I) = 1$ for any type elementary operation on I_n

$$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \tag{23}$$
$$\Rightarrow \det(E) = \det(E^t) = 1 \tag{24}$$

30. Let the rows of $A \in \mathsf{M}_{n \times n}(F)$. be $a_1, a_2, \ldots, a_n$ and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

(a) **n is even**

In A, swap

$$a_{n-1} \text{ with } a_1 \tag{25}$$
$$a_{n-2} \text{ with } a_2 \tag{26}$$
$$\vdots$$
$$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \tag{27}$$

From the fact that $^n/_2$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n}{2}} \det(A) \tag{28}$$

(b) **n is odd** In A, swap

$$a_{n-1} \text{ with } a_1 \tag{29}$$
$$a_{n-2} \text{ with } a_2 \tag{30}$$
$$\vdots$$
$$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \tag{31}$$

From the fact that $n - \frac{n+1}{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n-1}{2}} \det(A) \tag{32}$$

III

4.3

10.

11.

12.