Assignment

Section 3.4: 2(fj), 8, 11, 14, 15; Section 4.1: 10; Section 4.2: 23, 29, 30; Section 4.3: 10, 11, 12, 15

Work

3.4

2. Use Gaussian elimination to solve the following systems of linear equations.

(f)

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\ x_2 + 2x_4 &= 3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 9 & -4 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \tag{1}$$

$$\begin{aligned} x_1 &= -3 + 4x_4 && (2) \\ x_2 &= 3 - 2x_4 && (3) \\ x_3 &= 1 && (4) \\ x_4 &= x_4 && (5) \end{aligned}$$

$$S = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \tag{6}$$

(j)

$$\begin{aligned} 2x_1 + 3x_3 - 4x_5 &= 5 && (7) \\ 3x_1 - 4x_2 + 8x_3 + 3x_4 &= 8 && (8) \\ x_1 - x_2 + 2x_3 + x_4 - x_5 &= 2 && (9) \\ -2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 &= -8 && (10) \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 2 & 0 & 3 & 0 & -4 & 5 \\ 3 & -4 & 8 & 3 & 0 & 8 \\ 1 & -1 & 2 & 1 & -1 & 2 \\ -2 & 5 & -9 & -3 & -5 & -8 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right) \tag{11}$$

$$
\begin{align}
x_1 &= 2x_5 + 1 \tag{12}\\
x_2 &= 3x_5 \tag{13}\\
x_3 &= -1 \tag{14}\\
x_4 &= 2x_5 - 1 \tag{15}\\
x_5 &= x_5 \tag{16}
\end{align}
$$

$$
S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{17}
$$

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$
\begin{aligned}
u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6),\\
u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6),\\
u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12),\\
u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7)
\end{aligned}
$$

generate W. Find a subset $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

$$
\mathsf{R}^5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_1, \ldots, x_5 \in \mathbb{R} \right\} \tag{18}
$$

$$
\begin{pmatrix}
2 & -6 & 3 & 2 & -1 & 0 & 1 & 2\\
-3 & 9 & -2 & -8 & 1 & -3 & 0 & -1\\
4 & -12 & 7 & 2 & 2 & -18 & -2 & 1\\
-5 & 15 & -9 & -2 & 1 & 9 & 3 & -9\\
2 & -6 & 1 & 6 & -3 & 12 & -2 & 7
\end{pmatrix} \tag{19}
$$

$$
\rightsquigarrow \begin{pmatrix}
1 & -3 & 0 & 4 & 0 & 1 & 0 & -1\\
 & & 1 & -2 & 0 & -2 & 0 & 1\\
 & & & & 1 & -4 & 0 & -2\\
 & & \text{\huge 0} & & & & 1 & -1\\
 & & & & & & & 0
\end{pmatrix} \tag{20}
$$

It follows that $\{u_1, u_3, u_5, u_7\}$ is linearly independent by theorem 3.16. Therefore $\{u_1, u_3, u_5, u_7\}$ is a basis for W.

11. Let V be as in Exercise 10.

   (a) Show that $S = \{(1, 2, 1, 0, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

(a) Claim: S is a linearly independent subset of V.
For $x \in S, x = (1, 2, 1, 0, 0)$

$$1 + (-2)(2) + 3(1) + (-1)(0) + (2)(0) = 0 \tag{21}$$
$$\Rightarrow x \in \mathsf{V} \tag{22}$$
$$\Rightarrow S \subseteq \mathsf{V} \tag{23}$$

Suppose $cx - 0$

$$c(1, 2, 1, 0, 0) = 0 \tag{24}$$
$$\Rightarrow (c, 2c, c, 0, 0) = 0 \tag{25}$$
$$\Rightarrow c = 0 \tag{26}$$

It follows that S is linearly independent.

(b) Suppose $x_1, x_2, \ldots, x_5 \in F\colon x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 + x_4 - 2x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{27}$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{28}$$

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{29}$$

Where β is a basis V.

$$(S\,|B) = \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

$$S_\beta{}^1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{31}$$

14. John's email

15. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Suppose $A \in \mathsf{M}_{m\times n}(F)$ such that $\operatorname{rank}(A) = r \le \min\{m, n\}$
Proof by induction
Suppose $n = 1$

Case 1

$$A = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{32}$$

A is in reduced row echelon form and this the unique representation.

Case 2

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ A_{m1} \end{pmatrix} \quad \text{for some } a_{i1} \neq 0 \tag{33}$$

Execute the following sequence of row operations on A. Perform type 1 row operation to move a_{i1} to the first row. Perform type 3 row operations to eliminate all lower terms. Perform type 2 row operation to the change the first term in the column to 1.

$$A \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{34}$$

The reduced row echelon form of a column must be of this form. It follows that this matrix is the unique row reduced echelon form.

[1] We need say what exaclty thsi is

Suppose true for $1 \leq n \neq k$
Suppose $n = k + 1$

COME BACK TO THIS

4.1

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]\, I$

(b) $\det(C) = \det(A)$

(c) The classical adjoint of A^t is C^t

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}\, C$

(a)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{35}$$

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{36}$$

$$AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{12}A_{11} + A_{12}A_{11} \\ A_{22}A_{21} - A_{22}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} \tag{39}$$

$$CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} A_{22}A_{11} - A_{21}A_{12} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{21}A_{11} & -A_{21}A_{11} + A_{11}A_{22} \end{pmatrix} \tag{41}$$

$$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix} = AC \tag{42}$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \tag{43}$$

$$\Rightarrow AC = CA = \det(A)I_2 \tag{44}$$

(b) $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{22}A_{11} - A_{21}A_{12} = \det(A)$

(c)

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{45}$$

$$D = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \tag{46}$$

It follows that D is the classical adjoint of A^t

$$C^t = \begin{pmatrix} A_{22} & -A_{21} \\ A_{12} & A_{11} \end{pmatrix} = D \tag{47}$$

It follows that C^t is the classical adjoint of A^t

(d) Suppose A is invertible

$$\Rightarrow \exists B\colon AB = BA = I \tag{48}$$

$$\Rightarrow \det(A) \neq 0 \tag{49}$$

$$CA = AC = \det(A)I \tag{50}$$

$$\Rightarrow [\det(A)]^{-1}CA = [\det(A)]^{-1} = A[\det(A)]^{-1}C = I \tag{51}$$

$$\Rightarrow A^{-1} = [\det(A)]^{-1}C \tag{52}$$

4.2

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Suppose $A \in \mathsf{M}_{n\times n}(()F)$ and A is upper triangular.

Proof by induction.

For $n = 1$: $A \in \mathsf{M}_{1\times 1}(F) \Rightarrow \det(A) = A_{11}$ by definition.

Suppose $\det(A)$ is the product of its diagonal entries for $1 \leq n \leq k$

For $n = k+1$: $A \in \mathsf{M}_{k+1\times k+1}(()F)$

$$\det(A) = \sum_{j=1}^{k+1} (-1)^{k+1+j} A_{k+1,j} \det(\tilde{A}_{k+1,j}) \tag{53}$$

$$A_{n+1,j} = 0 \quad \text{for } 1 \leq j \leq n \tag{54}$$

$$\Rightarrow \det A = (-1)^{(k+1)+(k+1)} A_{k+1,k+1} \det(\tilde{A}_{k+1,k+1}) \tag{55}$$

$$\tilde{A}_{k+1,k+1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k+1} \\ & a_{22} & & a_{2,k+1} \\ & \text{\huge 0} & \ddots & \vdots \\ & & & a_{k+1,k+1} \end{pmatrix} \tag{56}$$

$\tilde{A}_{k+1,k+1} \in \mathsf{M}_{n\times n}(F) \Rightarrow \det(\tilde{A}_{n+1,n+1})$ is a product of diagonal entries by induction hypothesis.

$$(-1)^{(n+1)+(n+1)} = (-1)^{2n+2} = (-1)^2(-1)^{2n} = (-1)^{2n} \tag{57}$$

$$n \in \mathbb{Z}^+ \Rightarrow 2n \text{ is even} \tag{58}$$
$$\Rightarrow (-1)^{2n} = 1 \tag{59}$$
$$\Rightarrow \det(A)A_{n+1,n+1}\prod_{j=1}^{n} A_{jj} = \prod_{j=1}^{n+1} A_{jj} \tag{60}$$

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

(a) **Types 1 & 2**

$$E^t = E \quad \text{(by HW.3.1.5)} \tag{61}$$
$$\Rightarrow \det(E^t) = \det(E) \tag{62}$$

(b) **Type 3**

E^t is an type 3 elementary matrix (by HW.3.1.5) $\det(E) = \det(I) = 1$ for any type elementary operation on I_n

$$\det(E^t) = \det(I) \text{ because } E^t \text{ is type 3} \tag{63}$$
$$\Rightarrow \det(E) = \det(E^t) = 1 \tag{64}$$

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$. be $a_1, a_2, \ldots, a_n$ and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

(a) **n is even**

In A, swap

$$a_{n-1} \text{ with } a_1 \tag{65}$$
$$a_{n-2} \text{ with } a_2 \tag{66}$$
$$\vdots$$
$$a_{n-\frac{n}{2}+1} \text{ with } a_{n-\frac{n}{2}} \tag{67}$$

From the fact that ${}^n/_2$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n}{2}}\det(A) \tag{68}$$

(b) **n is odd** In A, swap

$$a_{n-1} \text{ with } a_1 \tag{69}$$
$$a_{n-2} \text{ with } a_2 \tag{70}$$
$$\vdots$$
$$a_{n-\frac{n+1}{2}+1} \text{ with } a_{n-\frac{n+1}{2}} \tag{71}$$

From the fact that $n - \frac{n+1}{2}$ swaps were performed it follows from Theorem 4.6 that

$$\det(B) = (-1)^{\frac{n-1}{2}} \det(A) \tag{72}$$

4.3

10. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **nilpotent** is for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that M if nilpotent, then $\det(M) = 0$.

$$M^k = 0 \quad \text{for some } k \in \mathbb{Z}^+ \tag{73}$$

$$\Rightarrow \det(M^k) = \det(0) \tag{74}$$
$$\Rightarrow (\det(M))^k = \det(0) \tag{75}$$
$$\Rightarrow (\det(M))^k = 0 \tag{76}$$
$$\Rightarrow \det(M) = 0 \tag{77}$$

11.

12.

15.