Chapter 26

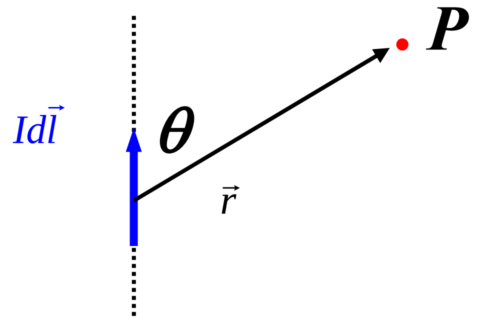
Sources of Magnetic Field

Biot-Savart Law (P614)

Magnetic equivalent to C's law by Biot & Savart

Magnetic field due to an infinitesimal current:

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^3}}$$

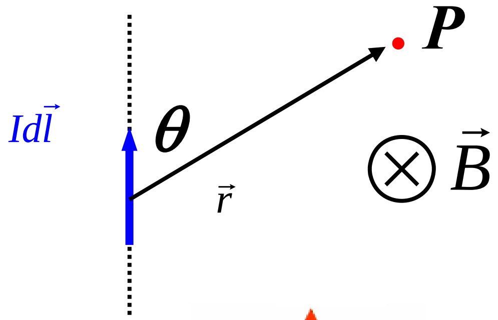


$\mu_0 = 4\pi \times 10^{-7}\, T \cdot m / A$ **Permeability of free space**

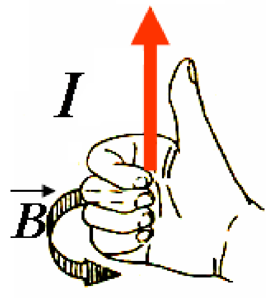
$\vec{r}$: **position vector from** $Id\vec{l}$ **to the field point** ***P***

B due to full current

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \vec{r}}{r^3}}$$



1) Magnitude: $dB = \frac{\mu_0}{4\pi} \frac{Idl \sin\theta}{r^2}$



2) Direction: right-hand rule

3) Total magnetic field due to a full current

$$\boxed{\vec{B} = \int \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \vec{r}}{r^3}}$$

Notice: directions

A straight current

Example1: Magnetic field of a straight current.

Solution: Infinitesimal current

$$r = a/\sin\theta$$

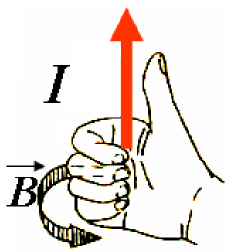
$$x = -a\cot\theta$$

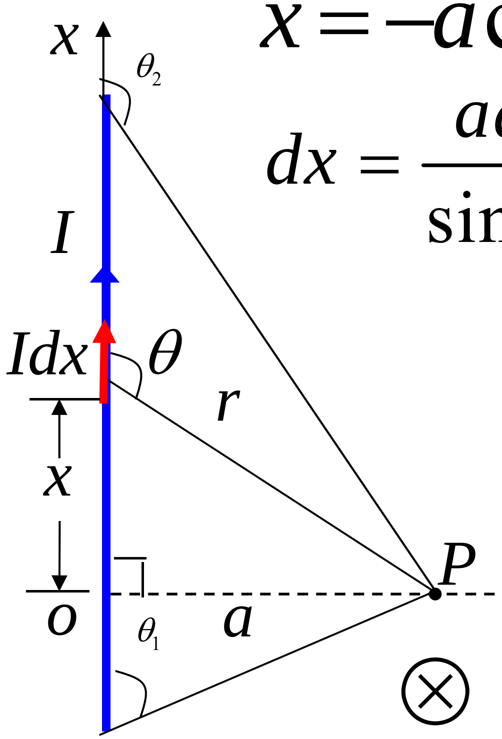
$$dx = \frac{a\,d\theta}{\sin^2\theta}$$

$$dB = \frac{\mu_0}{4\pi}\frac{Idx\sin\theta}{r^2} = \frac{\mu_0 I}{4\pi a}\sin\theta d\theta$$

$$B = \int_{\theta_1}^{\theta_2}\frac{\mu_0 I}{4\pi a}\sin\theta d\theta$$

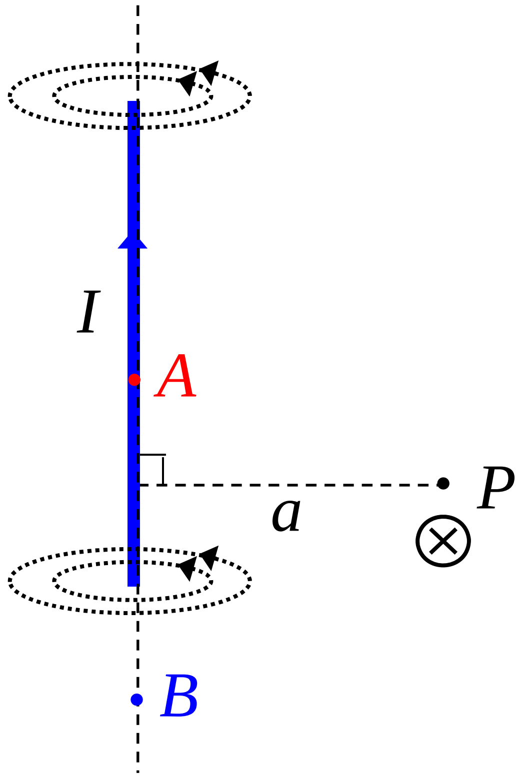
$$= \frac{\mu_0 I}{4\pi a}\left|\cos\theta_1 - \cos\theta_2\right|$$





$$B=\frac{\mu_0 I}{4\pi a}\left|\cos\theta_1-\cos\theta_2\right|$$

Discussion:



1) Infinite straight current

$$\boxed{B=\frac{\mu_0 I}{2\pi a}} \quad □ \quad E=\frac{\lambda}{2\pi\varepsilon_0 a}$$

2) Magnetic field at point *A* or *B*: **No field!**

$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{r}}{r^3}=0 \quad \Rightarrow \vec{B}=\int d\vec{B}=0$$

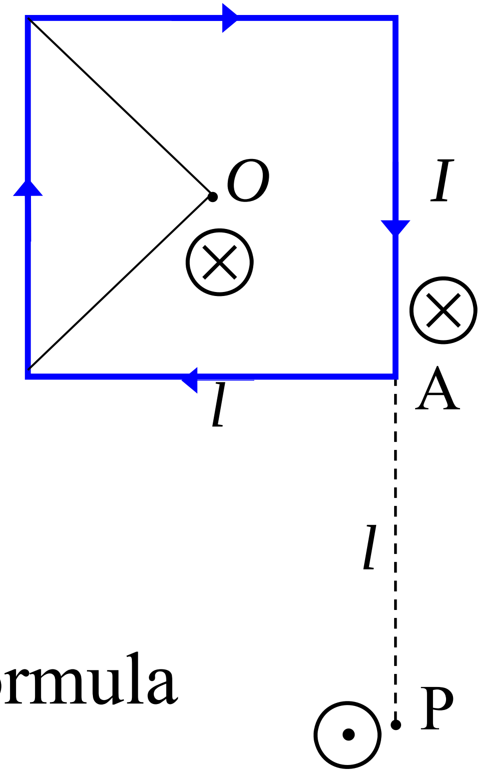
Square loop

Example2: $\vec{B}$ **of a square loop at O, A and P.**

Solution: $$B = \frac{\mu_0 I}{4\pi a}\left|\cos\theta_1 - \cos\theta_2\right|$$

$$B_O = \frac{\mu_0 I}{4\pi \cdot l/2}\left|\cos 45^{o} - \cos 135^{o}\right| \quad \times 4$$

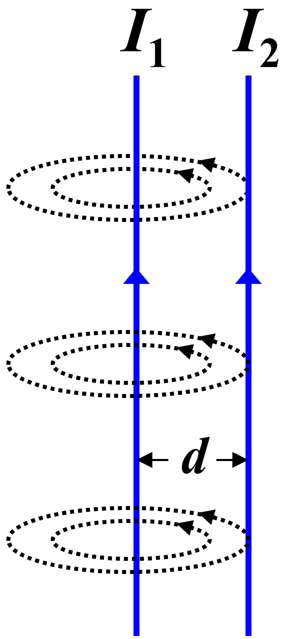
$$B_A = \frac{\mu_0 I}{4\pi l}\left|\cos 45^{o} - \cos 90^{o}\right| \ \times 2$$



B_P can also be obtained by the formula

*Force between parallel wires (P605)

Long parallel straight wires with current I_1 and I_2



Magnetic field due to I_1 : $B_1 = \dfrac{\mu_0 I_1}{2\pi d}$

Ampere force on I_2 : $F = B_1 I_2 L$

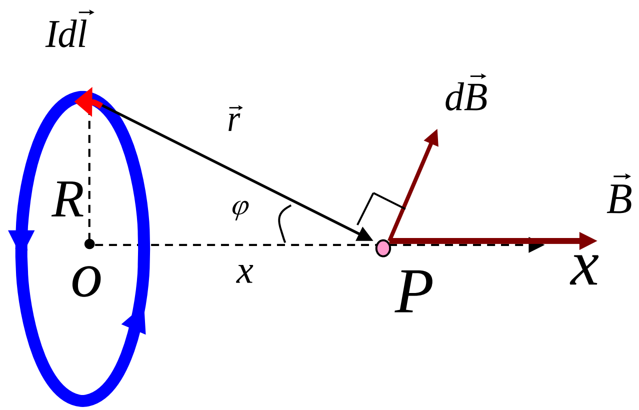
***F* per unit length:** $\dfrac{F}{L} = B_1 I_2 = \dfrac{\mu_0 I_1 I_2}{2\pi d}$

Operational definitions of Ampere & Coulomb

Circular current

Magnetic field of a circular current on the axis.

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}$$

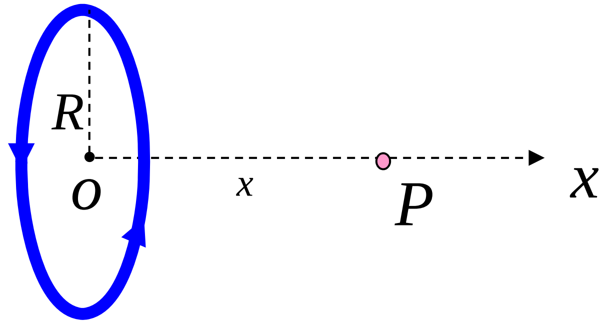


From the symmetry:

$$\therefore\ B = \int \frac{\mu_0 Idl}{4\pi r^2}\sin\varphi$$

$$= \frac{\mu_0 I\sin\varphi}{4\pi r^2}\cdot 2\pi R = \frac{\mu_0 IR^2}{2r^3} = \frac{\mu_0 IR^2}{2\left(R^2+x^2\right)^{3/2}}$$

$$\boxed{B=\frac{\mu_0 I R^2}{2\left(R^2+x^2\right)^{3/2}}}$$



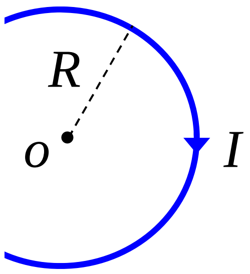
Discussion:

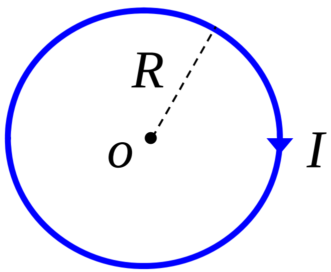
1) Magnetic dipole moment $\mu = I \cdot \pi R^2$

$$B=\frac{\mu_0}{2\pi}\frac{\mu}{\left(R^2+x^2\right)^{3/2}} \approx \frac{\mu_0}{2\pi}\frac{\mu}{x^3} \quad \left(x \gg R\right) \qquad E \approx \frac{p}{2\pi\varepsilon_0 x^3}$$

2) *B* at the center of a circular / arc current:

$$\boxed{B=\frac{\mu_0 I}{2R}\times\frac{l}{2\pi R}}$$

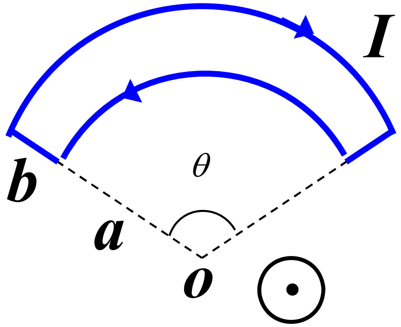




Combined currents

Example3: Magnetic field at point O.

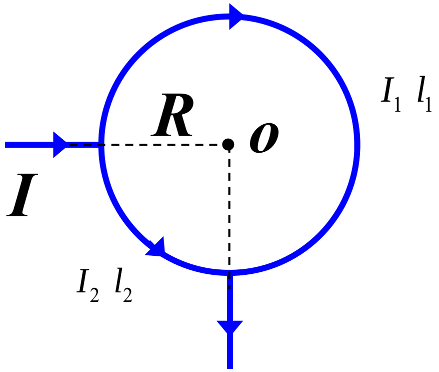
(a)



$$B = \frac{\mu_0 I}{2a} \cdot \frac{\theta}{2\pi} - \frac{\mu_0 I}{2(a+b)} \cdot \frac{\theta}{2\pi}$$

(b) Uniform conductor ring

$$\left(I_1 \cdot \rho \frac{l_1}{S} = I_2 \cdot \rho \frac{l_2}{S} \right)$$



$$B = \frac{\mu_0 I_1}{2R} \cdot \frac{l_1}{2\pi R} - \frac{\mu_0 I_2}{2R} \cdot \frac{l_2}{2\pi R}$$

$$= \frac{\mu_0}{4\pi R^2}\left(I_1 l_1 - I_2 l_2\right) = 0$$

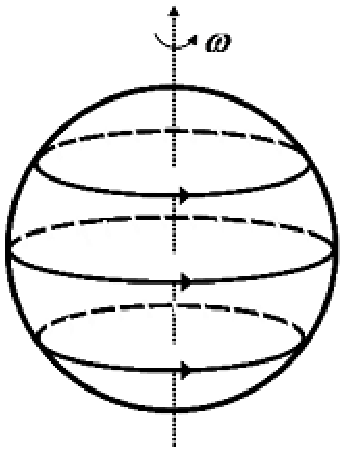
Rotating charged ring

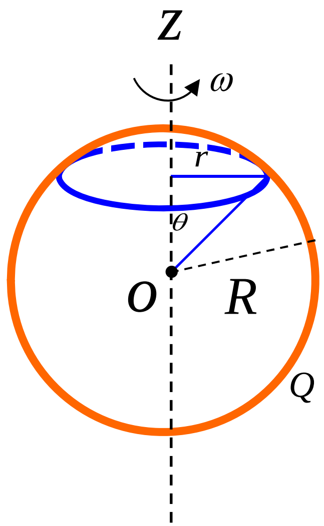
Question: **A uniformly charged ring rotates about *z* axis. Determine the magnetic field at the center.**

$$B = \int_0^{2\pi} \frac{\mu_0}{2} \frac{r^2 \frac{\omega}{2\pi} Q \frac{d\theta}{2\pi}}{R^3}$$

$$= \frac{\mu_0 \omega Q}{8\pi^2 R} \int_0^{2\pi} \sin^2 \theta d\theta$$

$$= \frac{\mu_0 \omega Q}{8\pi R}$$





$$\boxed{B = \frac{\mu_0 I R^2}{2\left(R^2 + x^2\right)^{3/2}}}$$

Magnetic flux

Φ_B : Magnetic flux through an area

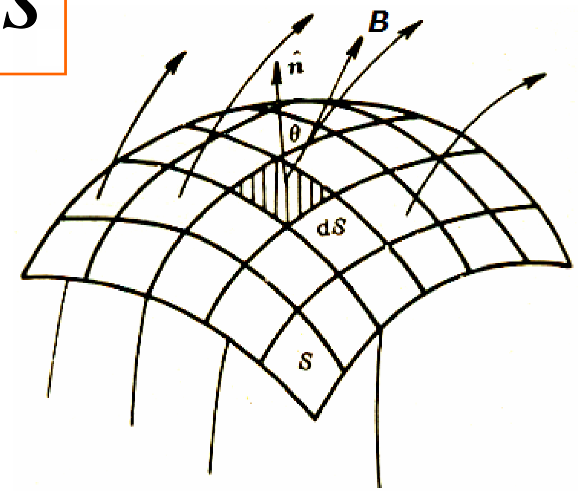
$\propto$ number of field lines passing that area

1) Uniform field: $\Phi_B = \vec{B} \cdot \vec{S}$ **Unit: Wb (Weber)**

2) General case: $\boxed{\Phi_B = \int \vec{B} \cdot d\vec{S}}$

3) Closed surface:

$\oint \vec{B} \cdot d\vec{S} \rightarrow$ **net flux**



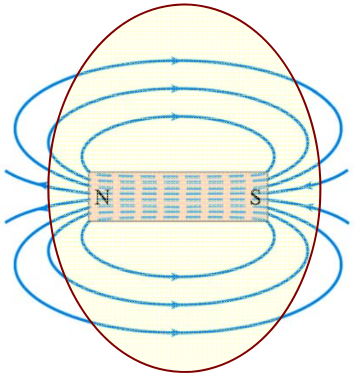
Gauss's law for magnetic field

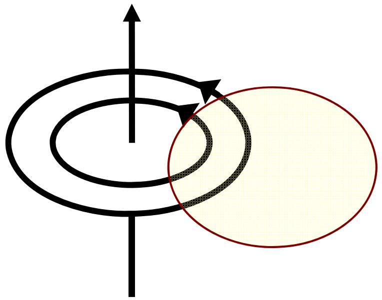
The total magnetic flux through any closed surface is always zero.

→ **Gauss's law for magnetic field**

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \cdot \vec{B} = 0$$





Closed field lines without beginning or end

Ampere's law (1)

The linear integral of $\vec{B}$ around any closed path is equal to μ_0 times the current passing through the area enclosed by the chosen path.

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{in}}$$

Ampere's law

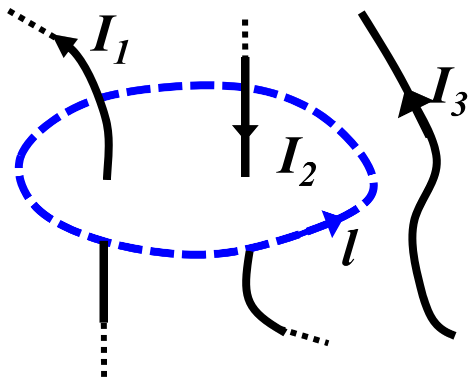
1) **Magnetic field is produced by all currents**

2) **The closed integral only depends on I_{in}**

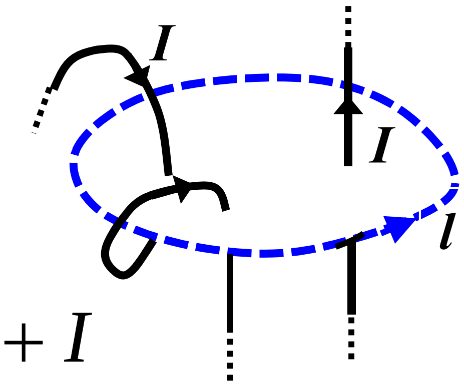
3) **How to count the enclosed current?**

Ampere’s law (2)

The sign of enclosed current: right-hand rule

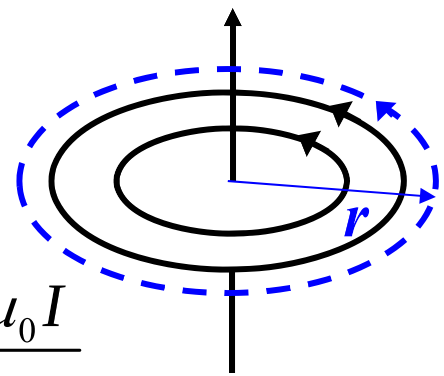


$$I_{in} = I_1 - I_2$$



$$I_{in} = -2I + I$$

1) Circular path around infinite straight current

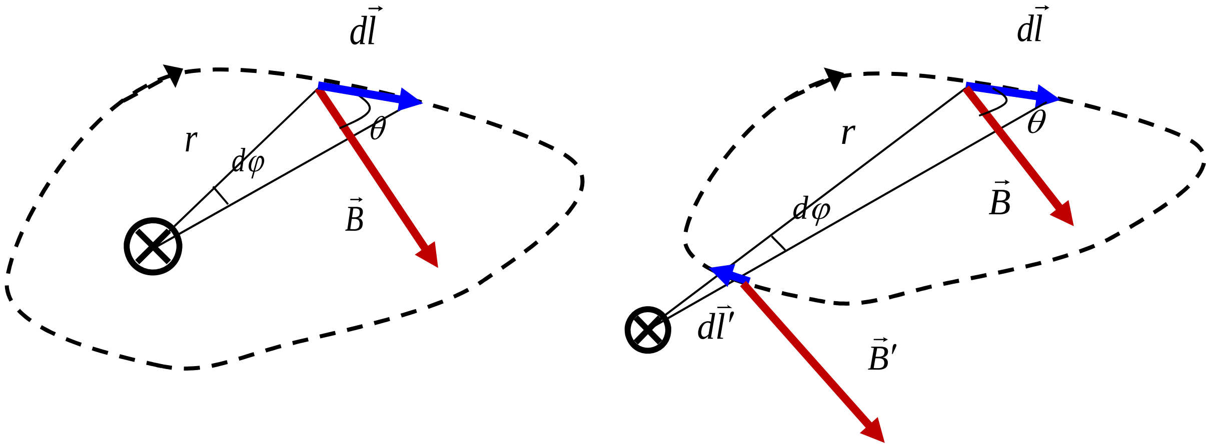


$$\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 I \qquad B = \frac{\mu_0 I}{2\pi r}$$

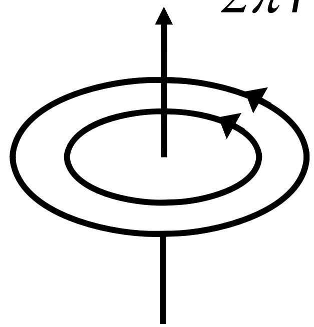
Ampere's law (3)

2) Any path around the current in same plane

$$\vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi r}\cos\theta dl \quad = \frac{\mu_0 I}{2\pi r} r d\varphi \quad = \frac{\mu_0 I}{2\pi} d\varphi$$



$$B = \frac{\mu_0 I}{2\pi r}$$



3) Any path that encloses no current

Equations for *E* & *B* field

Magnetic field is not a conservative field

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{in} \Rightarrow \nabla \times \vec{B} = \mu_0 \vec{j}$$

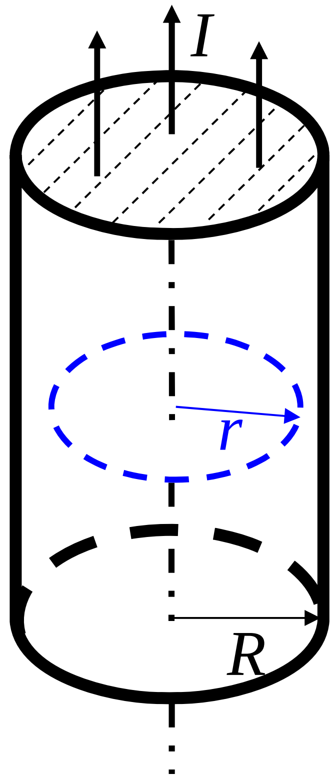
Maxwell equations for steady magnetic field & electrostatic field

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho_E}{\varepsilon_0} \\ \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{j} \end{cases}$$

Applications of Ampere's law → symmetry

Cylindrical current

Example4: The current is uniformly distributed over a long cylindrical conductor. Determine (a) magnetic field; (b) magnetic flux.



Solution: (a) Symmetry of system?

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi r = \mu_0 I_{in}$$

$$\therefore\ r > R:\quad B = \frac{\mu_0 I}{2\pi r} \qquad \left(j = \frac{I}{\pi R^2}\right)$$

$$\therefore\ r < R:\quad B = \frac{\mu_0}{2\pi r} j\cdot \pi r^2 = \frac{\mu_0 j r}{2}$$

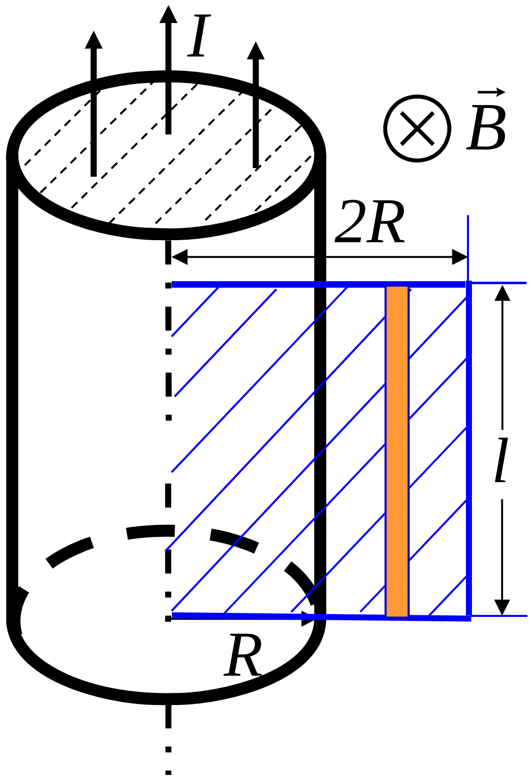
$$r > R: \quad B = \frac{\mu_0 I}{2\pi r}; \qquad r < R: \quad B = \frac{\mu_0 jr}{2} = \frac{\mu_0 Ir}{2\pi R^2}$$

(b) Magnetic flux through the area:

$$\Phi_B = \int \vec{B} \cdot d\vec{S} \;\; = \int BdS$$

$$= \int_0^R \frac{\mu_0 Ir}{2\pi R^2} ldr + \int_R^{2R} \frac{\mu_0 I}{2\pi r} ldr$$

$$= \frac{\mu_o Il}{4\pi} + \frac{\mu_o Il}{2\pi} \ln 2$$



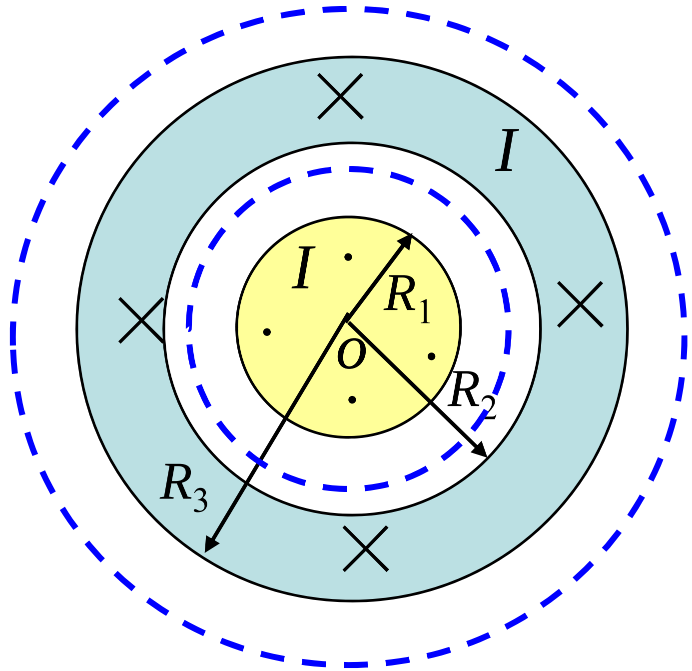
Coaxial cable

Question: **Coaxial cable: a wire surrounded by a cylindrical tube. They carry equal and opposite currents *I* distributed uniformly. What is *B* ?**

$$R_1 < r < R_2: \quad B = \frac{\mu_0 I}{2\pi r}$$

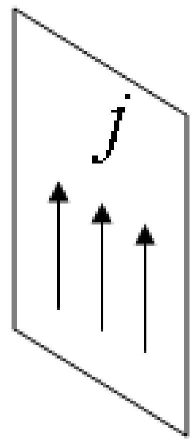
$$r > R_3: \quad B = 0$$

$$r < R_1 \text{ or } \quad R_2 < r < R_3 \ ?$$



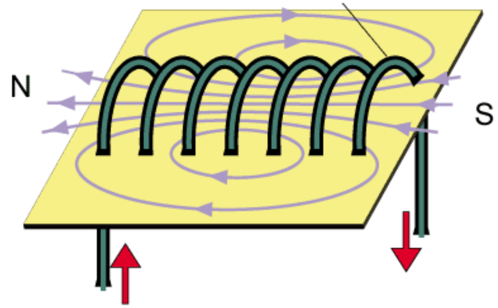
Infinite plane current

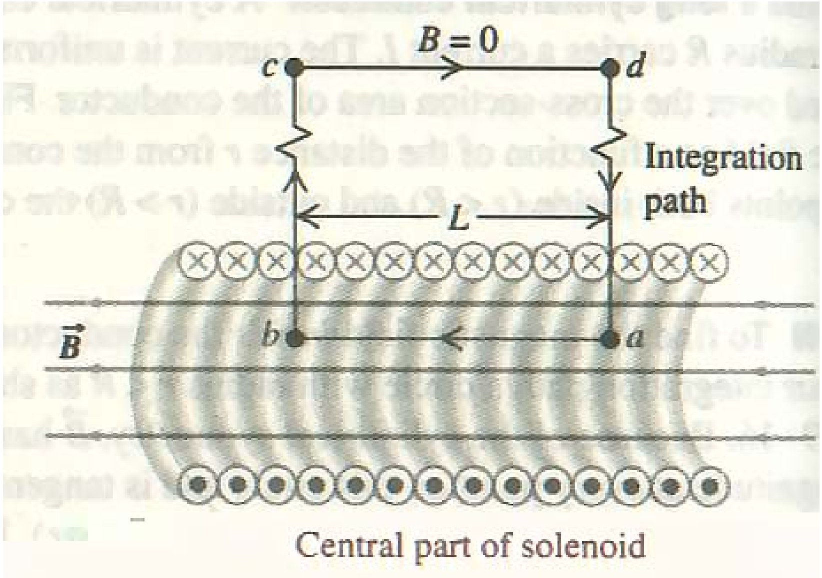
Homework: Determine the magnetic field produced by an infinite plane distributed with uniform current density *j*.



Solenoid

Solenoid: **a long coil of wire with many loops**





$$\oint_{abcd} \vec{B}\cdot d\vec{l} = \mu_0 I_{in}$$

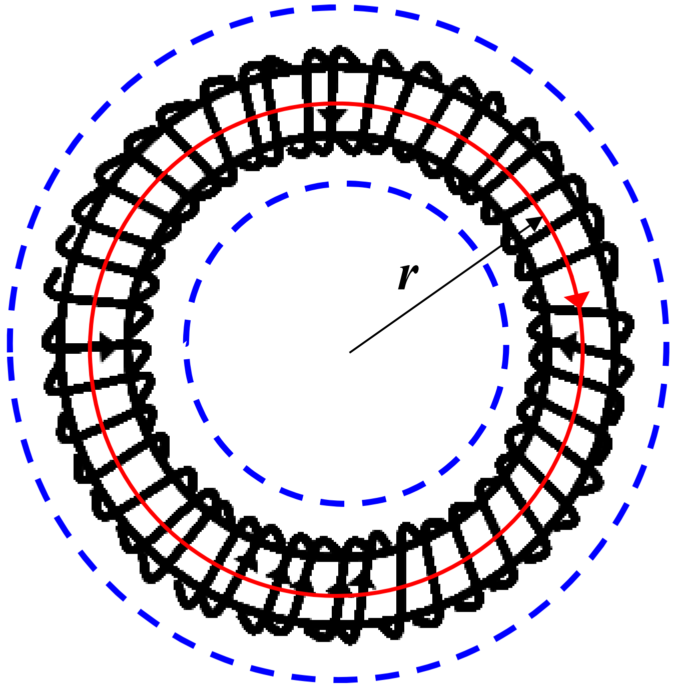
$$\Rightarrow BL = \mu_0 NI \quad \Rightarrow B = \mu_0 \frac{N}{L} I = \mu_0 nI$$

→ Uniform

Toroid

Toroid: **solenoid bent into the shape of a circle**

N loops, current I



Outside: $B = 0$

Inside? $B = \dfrac{\mu_0 N I}{2\pi r}$

Nonuniform field

$r \to \infty$ **becomes solenoid again**

*B produced by moving charge

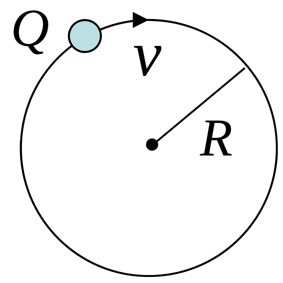
Equations about Ampere force & Lorentz force:

$$d\vec{F} = Id\vec{l} \times \vec{B} \quad \Leftrightarrow \quad \vec{F} = q\vec{v} \times \vec{B}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^3} \quad \Leftrightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \vec{r}}{r^3}$$

Magnetic field produced by a single moving charge

***B* created by rotating charge?**



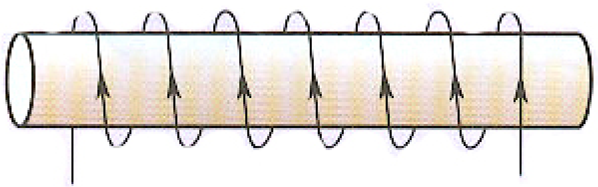
*Ferromagnetic

Increase the magnetic field by ferromagnetics

$$B_0 = \mu_0 nI \qquad \Rightarrow B = K_m B_0 = \mu nI$$

K_m: relative permeability

μ: magnetic permeability

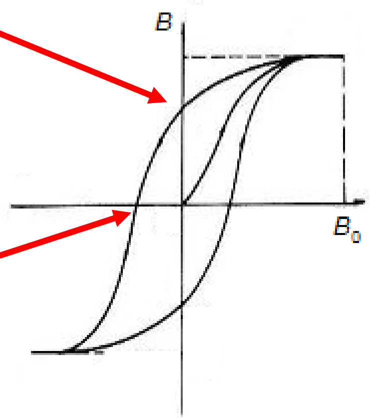


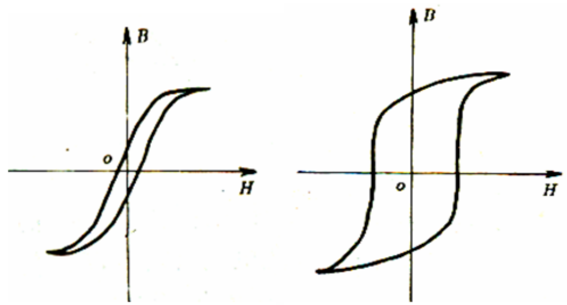
Electromagnet

Hysteresis:

remanence

coercivity





“Hard/soft”