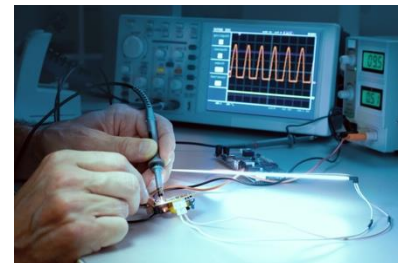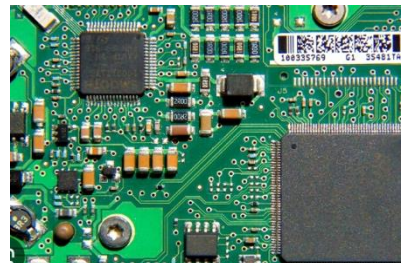# Circuit Analysis and Design

Academic Year 2025/2026 – Semester 1
Lecture 14 – RLC Circuits

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Agenda

❑ **Solution of the second-order differential equations**

❑ **Step response of a series RLC circuit**

❑ **Step response of a parallel RLC circuit**

❑ **General second-order circuits**

Introduction

❑ **We analyze circuits with two energy storage elements:**

▪ **an inductor and a capacitor, or two capacitors, or two inductors, by solving second-order differential equations without or with the input signals.**

❑ **If there is no input signal,**

▪ **The response of the circuit is called the zero input response (source free response, natural response, and transient response). The initial conditions on the energy storage elements cause the zero input response. Once the initial energy is exhausted, the response becomes zero.**

❑ **If there are inputs,**

▪ **the solution of the differential equation is the sum of the complementary solution and the particular solution. The complementary solution is the solution to the homogeneous equation, which is the differential equation without the input. The particular solution is the solution to the entire differential equation including the input. The form of the particular solution is similar to the input.**

Solution of the Second-Order Differential Equations to Constant Input

❑ **The second-order differential equations with constant coefficients with constant input can be written as**

$$\frac{d^2v(t)}{dt^2} + a_1\frac{dv(t)}{dt} + a_0v(t) = b_0 \quad (1)$$

❑ **The solution to Equation (1) is the sum of the** **complementary solution** $v_c(t)$ **and the particular solution** $v_p(t)$**.**

❑ **The complementary solution is the solution to homogeneous differential equation given by**

$$\frac{d^2v(t)}{dt^2} + a_1\frac{dv(t)}{dt} + a_0v(t) = 0 \quad (2)$$

❑ **Depending on the coefficients, there are three cases**

▪ **(1) overdamped, (2) critically damped, and (3) underdamped as discussed earlier.**

❑ **The coefficients of the complementary solution are found by applying the initial conditions to sum** $v_c(t) + v_p(t)$ **after finding the particular solution.**

Solution of the Second-Order Differential Equations to Constant Input (Continued)

❑ **The particular solution** is the solution to the original differential equation given by Equation (1), including the input. **The form of the particular solution is similar to the input signal. For the constant input, the particular solution will be a constant.** Let the particular solution be

$$v_p(k) = K$$

❑ Substituting this proposed solution to Equation (1), we obtain

$$\frac{d^2 K}{dt^2} + a_1 \frac{dK}{dt} + a_0 K = b_0 \quad (3)$$

❑ Since K is a constant, $dK/dt = 0$ and $d^2K/dt^2 = 0$, and Equation (3) becomes

$$a_0 K = b_0$$

❑ Thus, for constant input signal, the particular solution to Equation (1) is given by

$$v_p(t) = K = \frac{b_0}{a_0} \quad (4)$$

Solution of the Second-Order Differential Equations to Constant Input (Continued)

❑ **The characteristic equation to the homogeneous equation given by Equation (2) is**

$$s^2 + a_1 s + a_0 = 0 \qquad (5)$$

❑ **The roots of the characteristic equation are given by**

$$s_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}, \quad s_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0} \quad (6)$$

❑ **Let**

$$\alpha = \frac{a_1}{2}, \quad \omega_0 = \sqrt{a_0} \quad (7)$$

❑ **Then, Equation (6) becomes**

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad (8)$$

Solution of the Second-Order Differential Equations to Constant Input (Continued)

❑ **Case 1: Overdamped ($\alpha > \omega_0$, $\zeta > 1$, $a_1 > 2\sqrt{a_0}$)**

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

▪ *When the two roots are real and distinct, the complementary solution is :*

$$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

▪ **Complete solution:**

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- **To find A1 and A2:**

$$v(0) = \frac{b_0}{a_0} + A_1 + A_2 = V_0 \qquad \frac{dv(0)}{dt} = A_1 s_1 + A_2 s_2 = Dv_0$$

Solution of the Second-Order Differential Equations to Constant Input (Continued)

❑ **Case 2: Critically damped ($\alpha = \omega_0$, $\zeta = 1$, $a_1 = 2\sqrt{a_0}$)**

$$s_1 = -\alpha, \quad s_2 = -\alpha$$

▪ *When the two roots are real and equal, the complementary solution is :*

$$v_c(t) = A_1 e^{s_1 t} + A_2 t e^{s_2 t}$$

▪ **Complete solution:**

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 t e^{s_2 t}$$

• **To find A1 and A2:**

$$v(0) = \frac{b_0}{a_0} + A_1 = V_0 \qquad \frac{dv(0)}{dt} = A_1 s_1 + A_2 = D v_0$$

Solution of the Second-Order Differential Equations to Constant Input (Continued)

❑ **Case 3: Underdamped ($\alpha < \omega_0$, $\zeta < 1$, $a_1 < 2\sqrt{a_0}$)**

$$\beta = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1-\zeta^2}$$

$$s_1 = -\alpha + j\beta, \quad s_2 = -\alpha - j\beta$$

▪ *When the two roots are complex conjugates, the complementary solution is :*

$$v_c(t) = e^{-\alpha t}\left[B_1\cos(\beta t) + B_2\sin(\beta t)\right]$$

▪ **Complete solution:**

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + e^{-\alpha t}\left[B_1\cos(\beta t) + B_2\sin(\beta t)\right]$$

▪ **To find B1 and B2:**

$$v(0) = \frac{b_0}{a_0} + B_1 = V_0 \qquad \frac{dv(0)}{dt} = -\alpha B_1 + \beta B_2 = Dv_0$$

Step Response of a Series RLC Circuit

❑ **The switch in the circuit shown in Figure 8.11 is closed at time $t = 0$. Let the initial voltage across the capacitor at $t = 0$ be V_0, and the initial current through the inductor at $t = 0$ be I_0.**

❑ **Sum the voltage drops around the mesh:**

$$-V_s + Ri(t) + L\frac{di(t)}{dt} + v(t) = 0 \quad (1)$$

❑ **The current through capacitor:** $\quad i(t) = C\frac{dv(t)}{dt} \quad (2)$

❑ **Substitute (2) into (1):**

$$-V_s + RC\frac{dv(t)}{dt} + LC\frac{d^2v(t)}{dt^2} + v(t) = 0 \quad (3)$$

❑ **Rearrange (3):** $\dfrac{d^2v(t)}{dt^2} + \dfrac{R}{L}\dfrac{dv(t)}{dt} + \dfrac{1}{LC}v(t) = \dfrac{1}{LC}V_s \quad (4)$

FIGURE 8.11

❑ *$a_2 = 1$, $a_1 = R/L$, $a_0 = 1/(LC)$, $b_0 = V_s/(LC)$*

❑ *$\alpha = R/(2L)$, $\omega_0 = \sqrt{a_0} = 1/\sqrt{LC}$*

❑ **Particular solution:** $v_p(t) = b_0/a_0 = V_s$

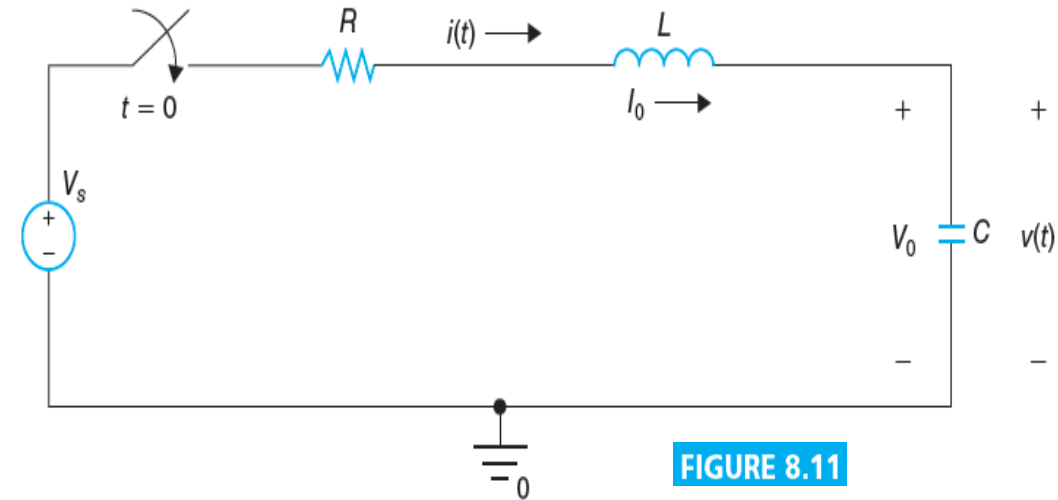$$Dv_0 = \frac{dv(0)}{dt} = \frac{i(0)}{C} = \frac{I_0}{C}$$

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EXAMPLE 8.11

❑ **Let $R = 1.5\ k\Omega$, $L = 50\ mH$, $C = 0.1\ \mu F$, $V_0 = 5\ V$, $I_0 = 6\ mA$, and $V_s = 10\ V$ in the circuit shown in Figure 8.11. Find the voltage $v(t)$ across the capacitor, current $i(t)$ through the capacitor, voltage $v_R(t)$ across the resistor, and the voltage $v_L(t)$ across the inductor. Plot $v(t)$, $i(t)$, $v_R(t)$, and $v_L(t)$.**

❑ $a_2 = 1$, $a_1 = R/L = 30000$, $a_0 = 1/(LC) = 2\times10^8$, $b_0 = V_s/(LC) = 2\times10^9$

❑ $\alpha = R/(2L) = 15000$, $\omega_0 = \sqrt{a_0} = 14142.1356$, $\alpha > \omega_0$, **Case 1 (Overdamped)**

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -10000, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -20000$$

❑ **Particular solution** $v_p(t) = b_0/a_0 = V_s = 10\ V$

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 10 + A_1 e^{-10000t} + A_2 e^{-20000t}$$

FIGURE 8.11

❑ $v(0) = 10 + A_1 + A_2 = V_0 = 5$ **(1)**

❑ $dv(0)/dt = A_1 s_1 + A_2 s_2 = I_0/C$

➔ $-10000A_1 - 20000A_2 = 60000$ **(2)**

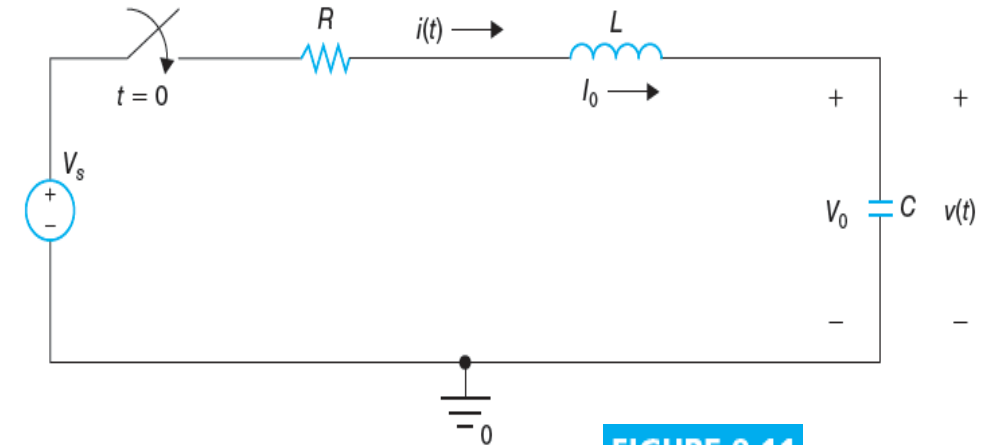❑ $A_1 = -4\ V$, $A_2 = -1\ V$

EXAMPLE 8.11 (Continued)

$$v(t) = \left(10 - 4e^{-10000t} - e^{-20000t}\right)u(t) \text{ V}$$

❑ **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = \left(4e^{-10000t} + 2e^{-20000t}\right)u(t) \text{ mA}$$

❑ **The voltage across the resistor is given by**

$$v_R(t) = Ri(t) = \left(6e^{-10000t} + 3e^{-20000t}\right)u(t) \text{ V}$$

❑ **The voltage across the inductor is given by**

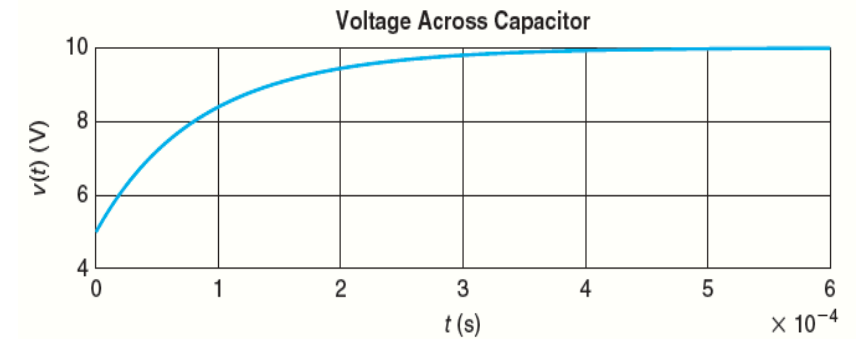$$v_L(t) = L\frac{di(t)}{dt} = \left(-2e^{-10000t} - 2e^{-20000t}\right)u(t) \text{ V}$$

Voltage Across Capacitor

Current Through the Capacitor

Voltage Across the Resistor

Voltage Across the Inductor

FIGURE 8.12

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EXAMPLE 8.12

❑ **Let** $R = 400\ \Omega$, $L = 100\ mH$, $C = 2.5\ \mu F$, $V_0 = 6\ V$, $I_0 = 5\ mA$, **and** $V_s = 16\ V$ **in the circuit shown in Figure 8.11. Find the voltage** $v(t)$ **across the capacitor, current** $i(t)$ **through the capacitor, voltage** $v_R(t)$ **across the resistor, and voltage** $v_L(t)$ **across the inductor. Plot** $v(t)$, $i(t)$, $v_R(t)$, **and** $v_L(t)$.

❑ $a_2 = 1$, $a_1 = R/L = 4000$, $a_0 = 1/(LC) = 4 \times 10^6$, $b_0 = V_s/(LC) = 64 \times 10^6$

❑ $\alpha = R/(2L) = 2000$, $\omega_0 = \sqrt{a_0} = 2000$, $\alpha = \omega_0$, **Case 2 (Critically damped)**

$$s_1 = -\alpha = -2000, \quad s_2 = -\alpha = -2000$$

❑ **Particular solution** $v_p(t) = b_0/a_0 = V_s = 16\ V$

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 t e^{s_2 t} = 16 + A_1 e^{-2000t} + A_2 t e^{-2000t}$$

❑ $v(0) = 16 + A_1 = V_0 = 6 \Rightarrow A_1 = -10\ V$
❑ $dv(0)/dt = A_1 s_1 + A_2 = I_0/C$
❑ $-2000A_1 + A_2 = 2000$ (2)
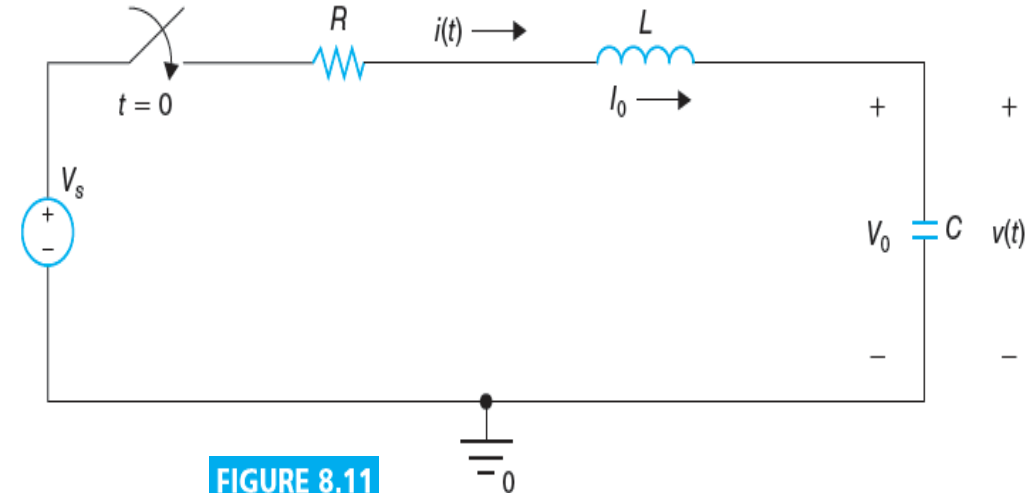❑ $A_1 = -10\ V$, $A_2 = -18000\ V$

FIGURE 8.11

EXAMPLE 8.12 (Continued)

$$v(t) = \left(16 - 10e^{-2000t} - 18000te^{-2000t}\right)u(t) \text{ V}$$

❑ **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = \left(0.005e^{-2000t} + 90te^{-2000t}\right)u(t) \text{ A}$$
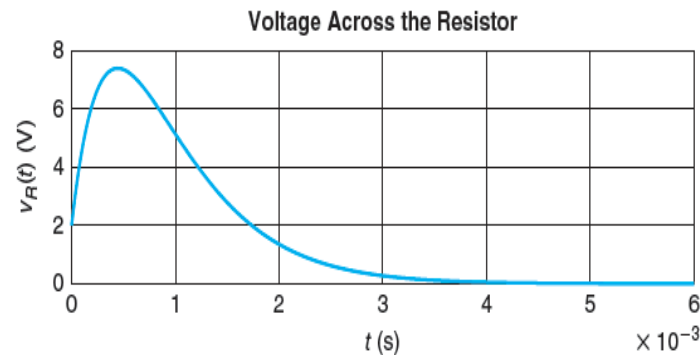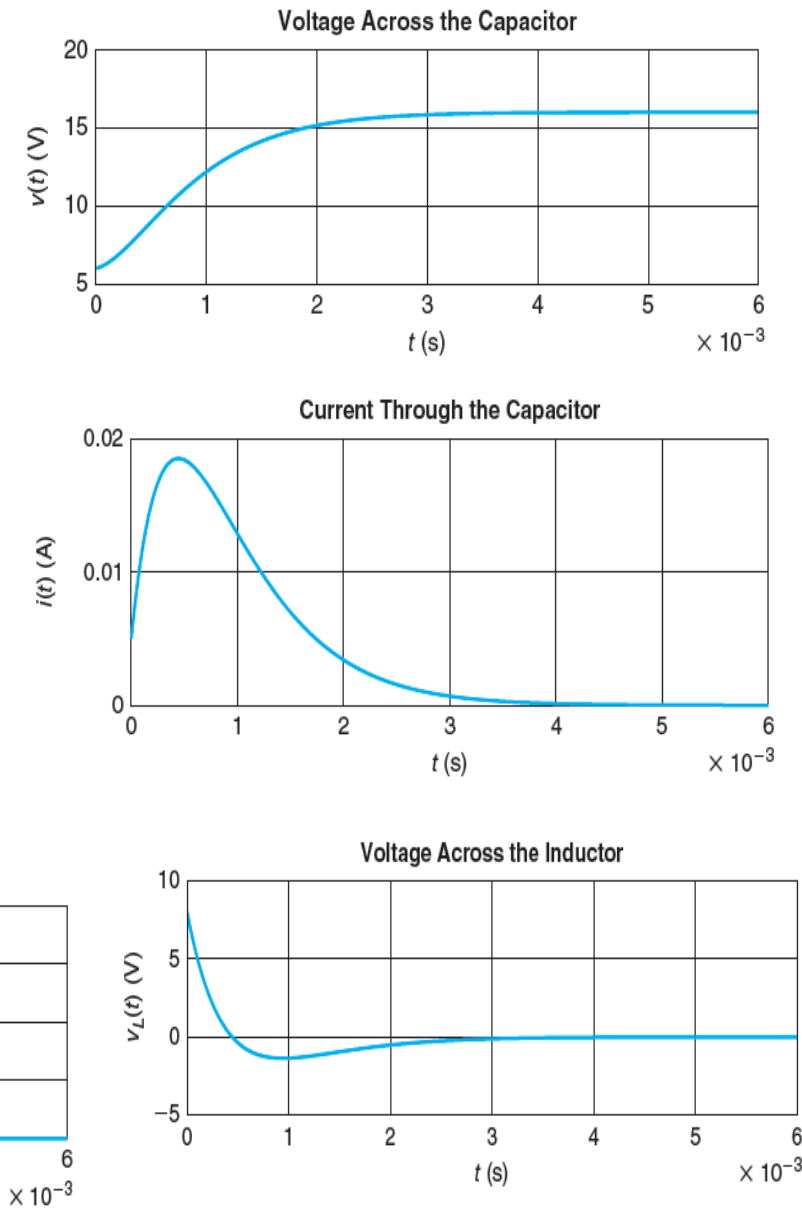
FIGURE 8.13

❑ **The voltage across the resistor is given by**

$$v_R(t) = Ri(t) = \left(2e^{-2000t} + 36000te^{-2000t}\right)u(t) \text{ V}$$

❑ **The voltage across the inductor is given by**

$$v_L(t) = L\frac{di(t)}{dt} = \left(-8e^{-2000t} - 18000te^{-2000t}\right)u(t) \text{ V}$$

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EXAMPLE 8.13

❑ **Let R = 400 Ω, L = 100 mH, C = 0.5 μF, V_0 = 5 V, I_0 = 4 mA, and V_s = 15 V in the circuit shown in Figure 8.11. Find the voltage $v(t)$ across the capacitor, current $i(t)$ through the capacitor, voltage $v_R(t)$ across the resistor, and voltage $v_L(t)$ across the inductor. Plot $v(t)$, $i(t)$, $v_R(t)$, and $v_L(t)$.**

❑ *a_2 = 1, a_1 = R/L = 4000, a_0 = 1/(LC) = 2×10⁷, b_0 = V_s/(LC) = 30×10⁷*

❑ *α = R/(2L) = 2000, ω_0 = $\sqrt{a_0}$ = 4472.135955, α < ω_0,* **Case 3 (Underdamped)**

$$\beta = \sqrt{\omega_0^2 - \alpha^2} = 4000, \quad s_1 = -\alpha + j\beta = -2000 + j4000, \quad s_2 = -\alpha - j\beta = -2000 - j4000$$

❑ **Particular solution** $v_p(t) = b_0/a_0 = V_s = 15$ V

$$v(t) = v_p(t) + v_c(t) = \frac{b_0}{a_0} + e^{-\alpha t}\left[B_1 \cos(\beta t) + B_2 \sin(\beta t)\right] = 15 + e^{-2000t}\left[B_1 \cos(4000t) + B_2 \sin(4000t)\right]$$

FIGURE 8.11

❑ $v(0) = 15 + B_1 = V_0 = 5 \Rightarrow B_1 = -10$ V
❑ $dv(0)/dt = -\alpha B_1 + \beta B_2 = I_0/C$
❑ $-2000B_1 + 4000B_2 = 8000$
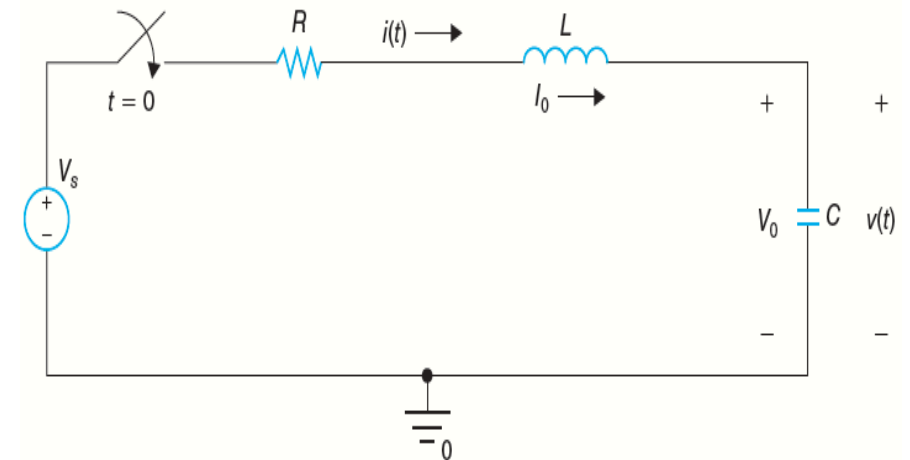❑ $B_1 = -10$ V, $B_2 = -3$ V

EXAMPLE 8.13 (Continued)

$$v(t) = \left\{ 15 + e^{-2000t} \left[-10\cos(4000t) - 3\sin(4000t) \right] \right\} u(t) \text{ V}$$

❑ **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = e^{-2000t} \left[4\cos(4000t) + 23\sin(4000t) \right] u(t) \text{ mA}$$

❑ **The voltage across the resistor is given by**

$$v_R(t) = Ri(t) = e^{-2000t} \left[1.6\cos(4000t) + 9.2\sin(4000t) \right] u(t) \text{ V}$$

❑ **The voltage across the inductor is given by**

$$v_L(t) = L\frac{di(t)}{dt} = e^{-2000t} \left[8.4\cos(4000t) - 6.2\sin(4000t) \right] u(t) \text{ V}$$

FIGURE 8.14

Voltage Across the Capacitor

Current Through the Capacitor

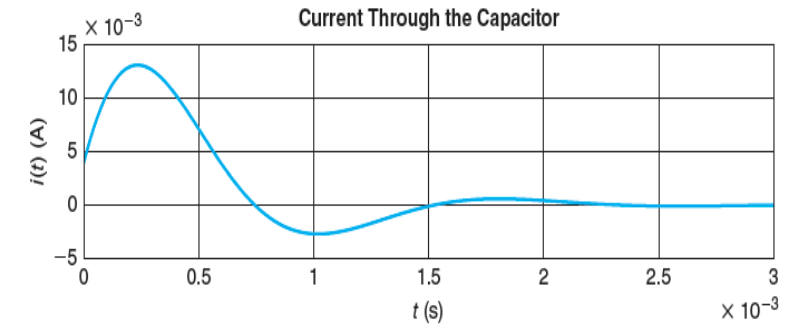
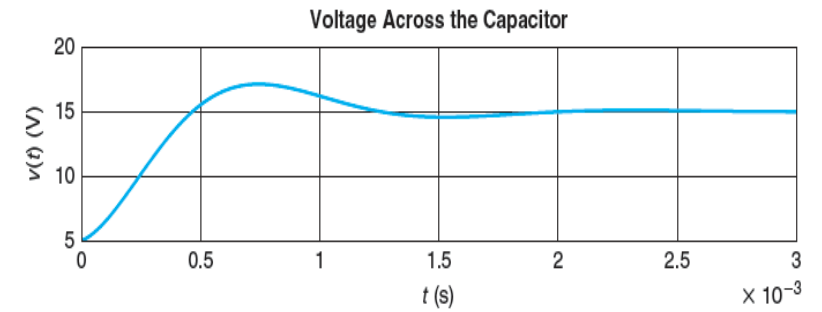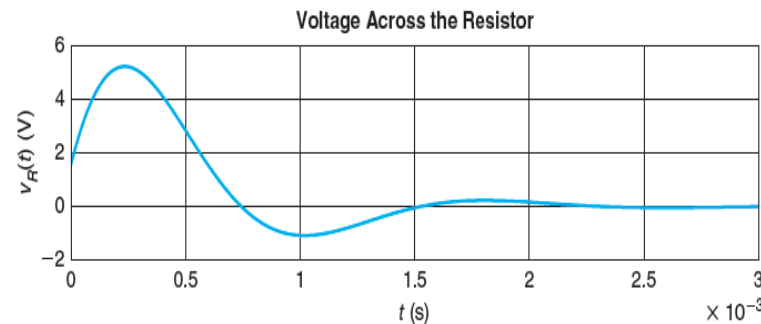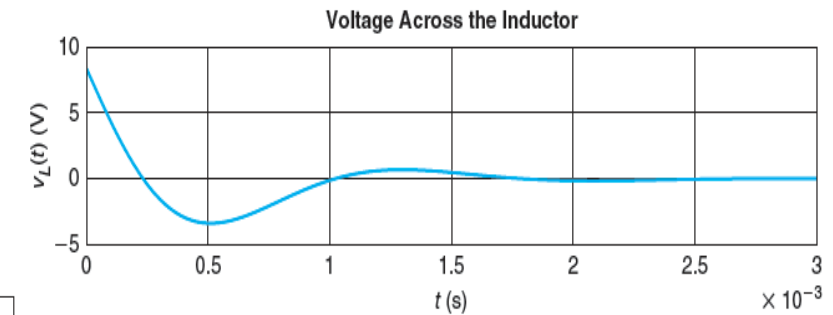
Voltage Across the Inductor

Voltage Across the Resistor

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Step Response of a Parallel RLC Circuit

❑ **Figure 8.18 shows a parallel RLC circuit with a current source I_s. At $t = 0$, the current through the inductor is $i(0) = I_0$ and the voltage across the capacitor is $v(0) = V_0$.**

❑ **Sum the currents away from node *a*:**
$$-I_s + \frac{v(t)}{R} + i(t) + C\frac{dv(t)}{dt} = 0 \quad (1)$$

❑ **The voltage across inductor:**
$$v(t) = L\frac{di(t)}{dt} \quad (2)$$

❑ **Substitute (2) into (1):**
$$-I_s + \frac{L}{R}\frac{di(t)}{dt} + i(t) + LC\frac{d^2i(t)}{dt^2} = 0 \quad (3)$$

❑ **Rearrange (3):** $\dfrac{d^2i(t)}{dt^2} + \dfrac{1}{RC}\dfrac{di(t)}{dt} + \dfrac{1}{LC}i(t) = \dfrac{I_s}{LC} \quad (4)$

❑ $a_2 = 1$, $a_1 = 1/(RC)$, $a_0 = 1/(LC)$, $b_0 = I_s/(LC)$

❑ $\alpha = 1/(2RC)$, $\omega_0 = \sqrt{a_0} = 1/\sqrt{LC}$

❑ **Particular solution:** $i_p(t) = b_0/a_0 = I_s$

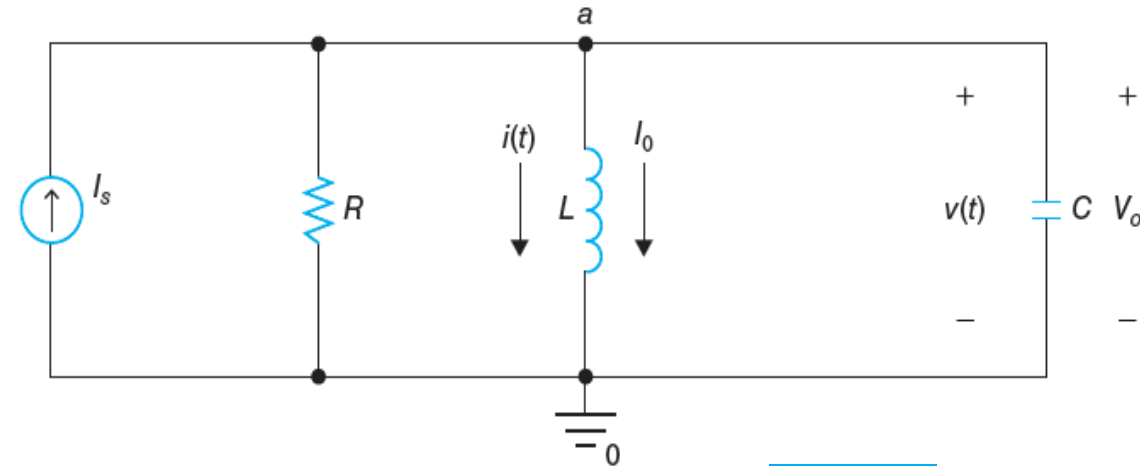$$Di_0 = \frac{di(0)}{dt} = \frac{v(0)}{L} = \frac{V_0}{L}$$

FIGURE 8.18

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EXAMPLE 8.15

- **Let $R = 200\ \Omega$, $L = 20\ mH$, $C = 0.1\ \mu F$, $V_0 = 5\ V$, $I_0 = 4\ mA$, and $I_s = 12\ mA$ in the circuit shown in Figure 8.18. Find the current $i(t)$ through the inductor and voltage $v(t)$ across the inductor and plot $i(t)$ and $v(t)$.**

- *$a_2 = 1$, $a_1 = 1/(RC) = 50000$, $a_0 = 1/(LC) = 5\times10^8$, $b_0 = I_s/(LC) = 48\times10^5$, $\alpha = 1/(2RC) = 25000$, $\omega_0 = \sqrt{a_0} = 22360.6798$, $\alpha > \omega_0$,* **Case 1 (Overdamped)**
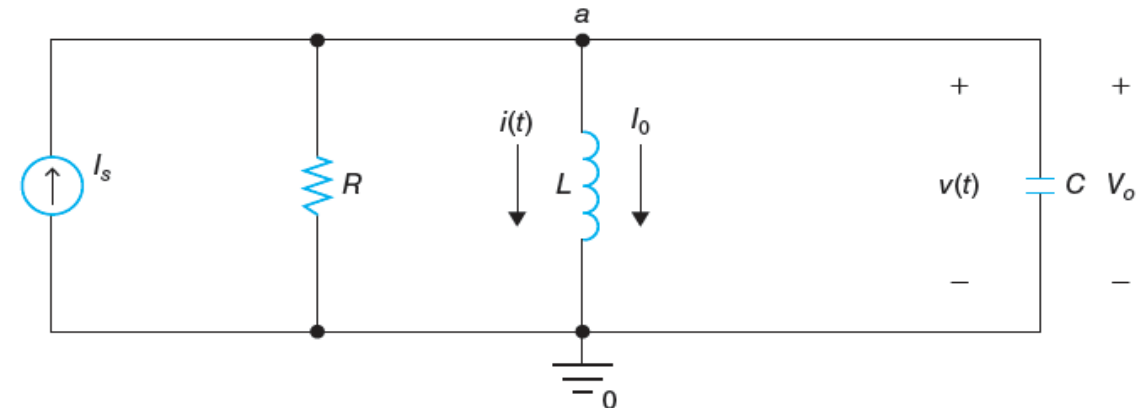
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -13819.6601, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -36180.3399$$

- **Particular solution $i_p(t) = b_0/a_0 = I_s = 12\ mA$**

$$i(t) = i_p(t) + i_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 0.012 + A_1 e^{-13819.6601t} + A_2 e^{-36180.3399t}$$

FIGURE 8.18

- $i(0) = 0.012 + A_1 + A_2 = I_0 = 0.004$ (1),
  - $di(0)/dt = A_1 s_1 + A_2 s_2 = V_0/L,$
- $-13819.6601 A_1 - 36180.3399 A_2 = 250$ (2)
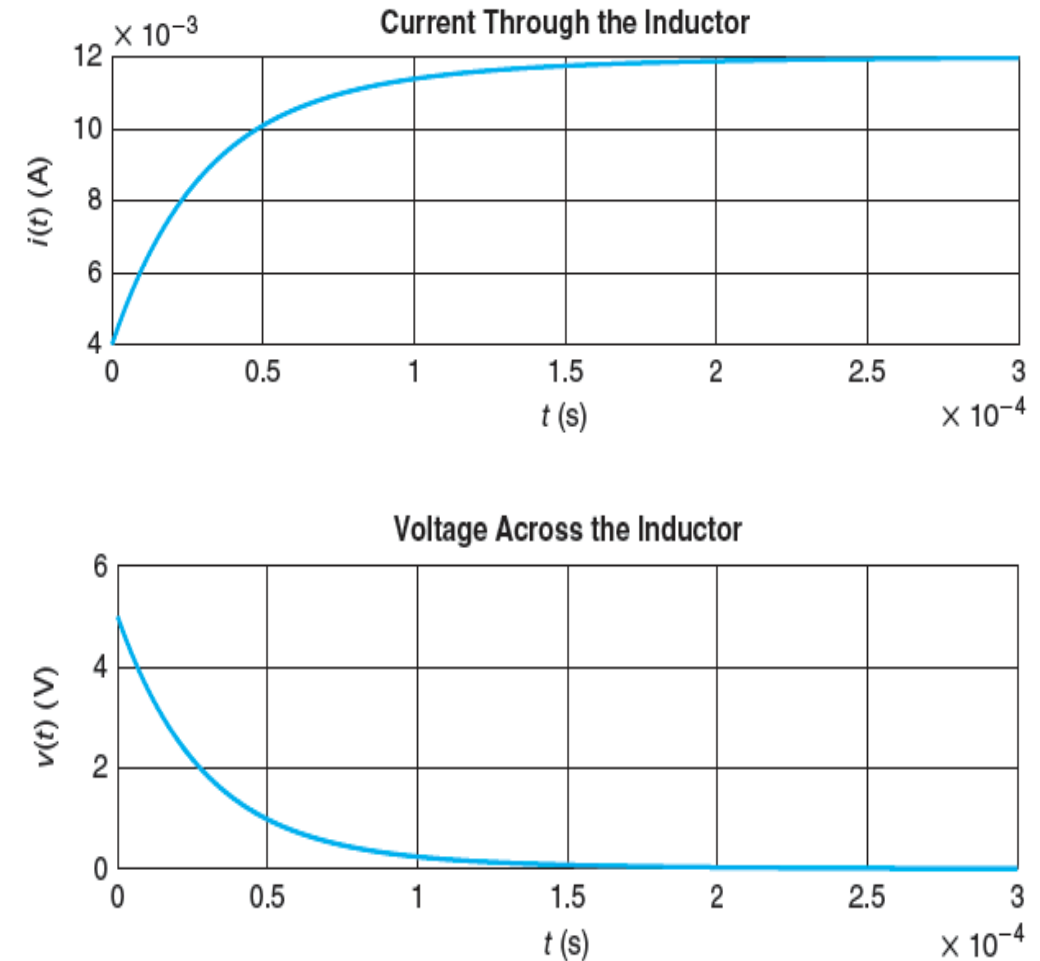- $A_1 = -1.7639\ mA$, $A_2 = -6.2361\ mA$

EXAMPLE 8.15 (Continued)

$$i(t) = \left(12 - 1.7639e^{-13819.6601t} - 6.2361e^{-36180.3399t}\right)u(t) \text{ mA}$$

❑ **The voltage *v(t)* across the inductor is given by**

$$v(t) = L\frac{di(t)}{dt} = \left(0.48754e^{-13819.6601t} + 4.5125e^{-36180.3399t}\right)u(t) \text{ V}$$

FIGURE 8.19

Current Through the Inductor

Voltage Across the Inductor

EXAMPLE 8.16

❑ **Let** $R = 2.5\ k\Omega$, $L = 40\ mH$, $C = 1.6\ nF$, $V_0 = 5\ V$, $I_0 = 2\ mA$, **and** $I_s = 8\ mA$ **in the circuit shown in Figure 8.18. Find the current** $i(t)$ **through the inductor and voltage** $v(t)$ **across the inductor and plot** $i(t)$ **and** $v(t)$.

❑ $a_2 = 1$, $a_1 = 1/(RC) = 250000$, $a_0 = 1/(LC) = 1.5625 \times 10^{10}$, $b_0 = I_s/(LC) = 12.5 \times 10^7$, $\alpha = 1/(2RC) = 125000$, $\omega_0 = \sqrt{a_0} = 125000$, $\alpha = \omega_0$, **Case 2 (Critically damped)**

$$s_1 = -\alpha = -125000, \quad s_2 = -\alpha = -125000$$

❑ **Particular solution** $i_p(t) = b_0/a_0 = I_s = 8\ mA$

$$i(t) = i_p(t) + i_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 t e^{s_2 t} = 0.008 + A_1 e^{-125000t} + A_2 t e^{-125000t}$$

❑ $i(0) = 0.008 + A_1 = I_0 = 0.002 \Rightarrow A_1 = -0.006\ A$,

❑ $di(0)/dt = A_1 s_1 + A_2 = V_0/L$,

❑ $-125000\ A_1 + A_2 = 125 \Rightarrow A_2 = -625\ A$

FIGURE 8.18
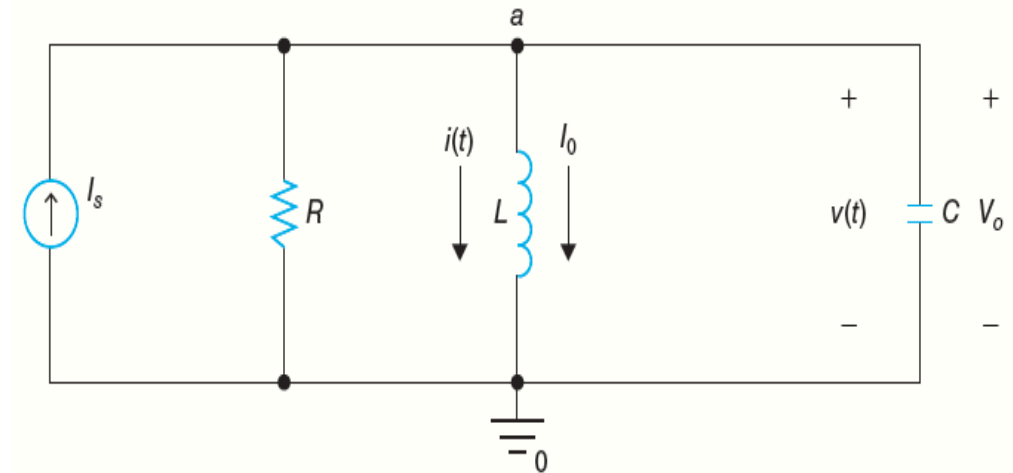
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EXAMPLE 8.16 (Continued)

$$i(t) = \left(0.008 - 0.006e^{-125000t} - 625te^{-125000t}\right)u(t) \text{ A}$$

❑ **The voltage $v(t)$ across the inductor is given by**

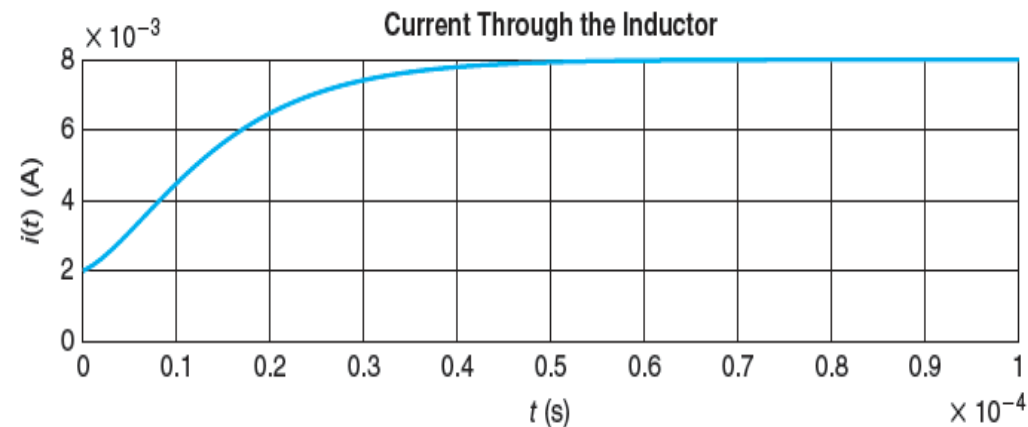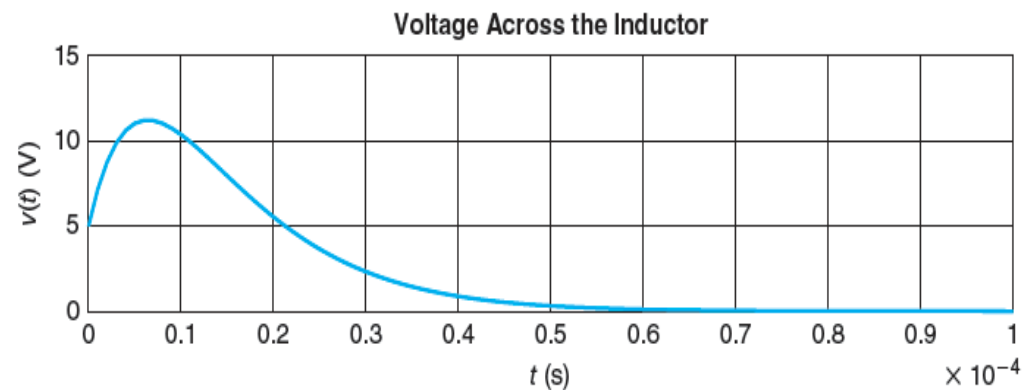$$v(t) = L\frac{di(t)}{dt} = \left(5e^{-125000t} + 3,125,000te^{-125000t}\right)u(t) \text{ V}$$

FIGURE 8.20

EXAMPLE 8.17

❑ **Let R = 5 kΩ, L = 20 mH, C = 5 nF, V_0 = 12 V, I_0 = 1 mA, and I_s = 10 mA in the circuit shown in Figure 8.18. Find the current $i(t)$ through the inductor and voltage $v(t)$ across the inductor and plot $i(t)$ and $v(t)$.**

❑ *a_2 = 1, a_1 = 1/(RC) = 40000, a_0 = 1/(LC) = 1×10^{10}, b_0 = I_s/(LC) = 1×10^8, α = 1/(2RC) = 20000, ω_0 = $\sqrt{a_0}$ = 100000, $\alpha < \omega_0$,* **Case 3 (Underdamped)**

$$\beta = \sqrt{\omega_0^2 - \alpha^2} = 97979.59, \ s_1 = -\alpha + j\beta = -20000 + j97979.59, \quad s_2 = -\alpha - j\beta = -20000 - j97979.59$$

❑ **Particular solution $i_p(t)$ = b_0/a_0 = I_s = 10 mA**

$$i(t) = i_p(t) + i_c(t) = \frac{b_0}{a_0} + A_1 e^{s_1 t} + A_2 t e^{s_2 t} = 0.01 + e^{-20000t} \left[B_1 \cos(97979.59t) + B_2 \sin(97979.59t) \right]$$

❑ *$i(0)$ = 0.01 + B_1 = I_0 = 0.001 $\Rightarrow$*

B_1 = – 9 mA, $di(0)/dt$ = – αB_1 + βB_2 = V_0/L,

❑ *– 20000 B_1 + 97979.59B_2 = 600 $\Rightarrow B_2$ = 4.2866 mA*

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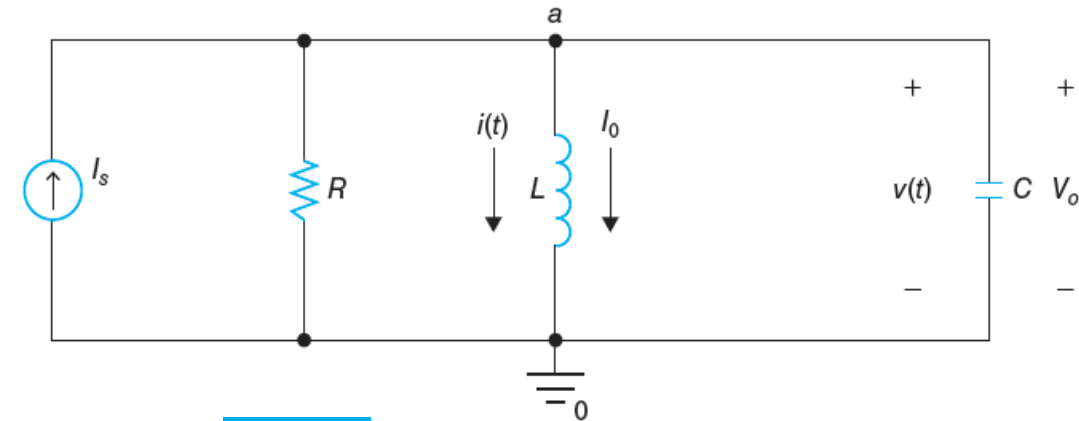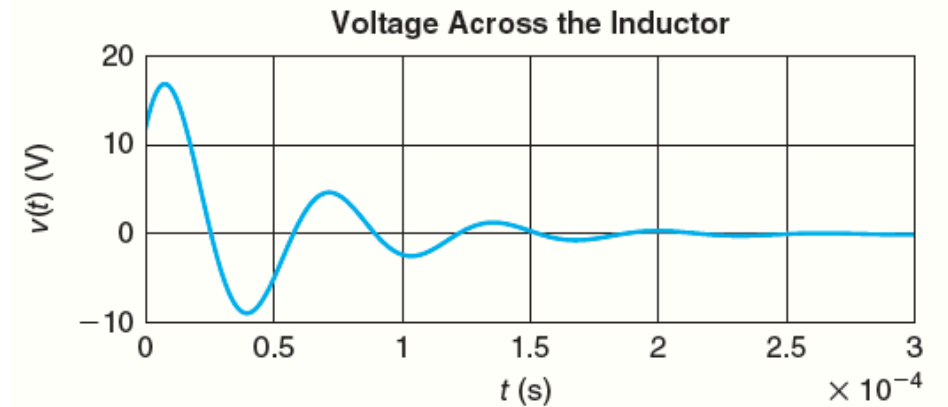
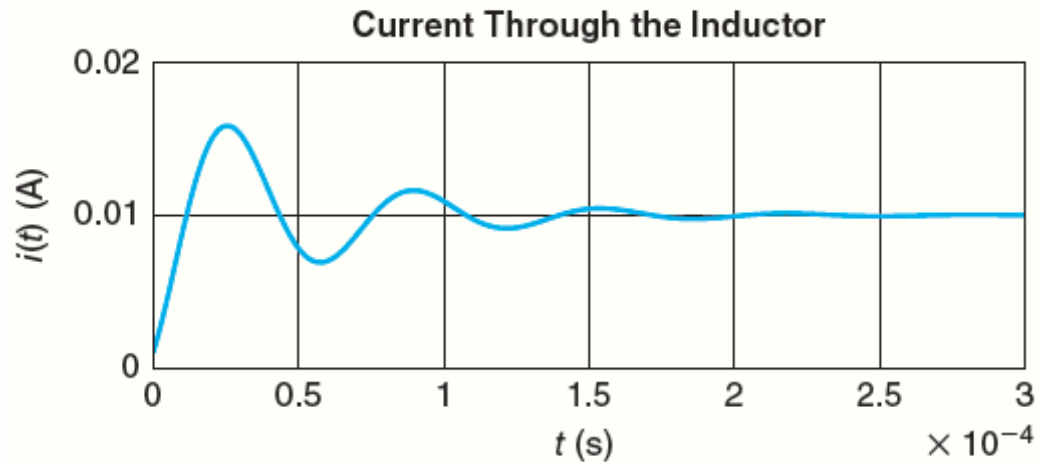# EXAMPLE 8.17 (Continued)

$$i(t) = \left\{10 + e^{-20000t}\left[-9\cos(97979.59t) + 4.2866\sin(97979.59t)\right]\right\}u(t) \ mA$$

❑ **The voltage *v(t)* across the inductor is given by**

$$v(t) = L\frac{di(t)}{dt} = e^{-20000t}\left[12\cos(97979.59t) + 15.9217\sin(97979.59t)\right]u(t) \ V$$

FIGURE 8.21

General Second-Order Circuits

❑ **We analyze circuits that cannot be reduced to a series RLC circuit or parallel RLC circuit, and circuits that may contain two capacitors or two inductors, rather than one inductor and one capacitor.**

❑ **Different types of inputs → Particular solution is similar to the input**

Input	Particular Solution
A	K
Ae^{-at}	Ke^{-at}
$A\cos(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$
$A\sin(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$
$A_1\cos(\omega t) + A_2\sin(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$
$A_2 t^2 + A_1 t + A_0$	$K_2 t^2 + K_1 t + K_0$

EXAMPLE 8.5

❑ **The switch in the circuit shown in Figure 8.6 has been closed for a long time before it is opened at $t = 0$. Find voltage $v(t)$ across the capacitor and the current $i(t)$ through the capacitor for $t \geq 0$.**

❑ **For $t < 0$, current through R_3-C path is zero. The current through L is $I_0 = 24\ V/1.2\ k\Omega = 20\ mA$. The voltage across the capacitor is $V_0 = R_2 I_0 = 0.8\ k\Omega \times 20\ mA = 16V$.**

❑ **For $t \geq 0$, $R = R_2 + R_3 = 1\ k\Omega$, $L = 0.2\ H$, $C = 0.96 \times 10^{-6}\ F$.**

❑ $a_2 = 1$, $a_1 = R/L = 5000$, $a_0 = 1/(LC) = 5.2083 \times 10^6$, $\alpha = a_1/2 = 2500$, $\omega_0 = \sqrt{a_0} = 2282.1773$, $\alpha > \omega_0$, **overdamped.**
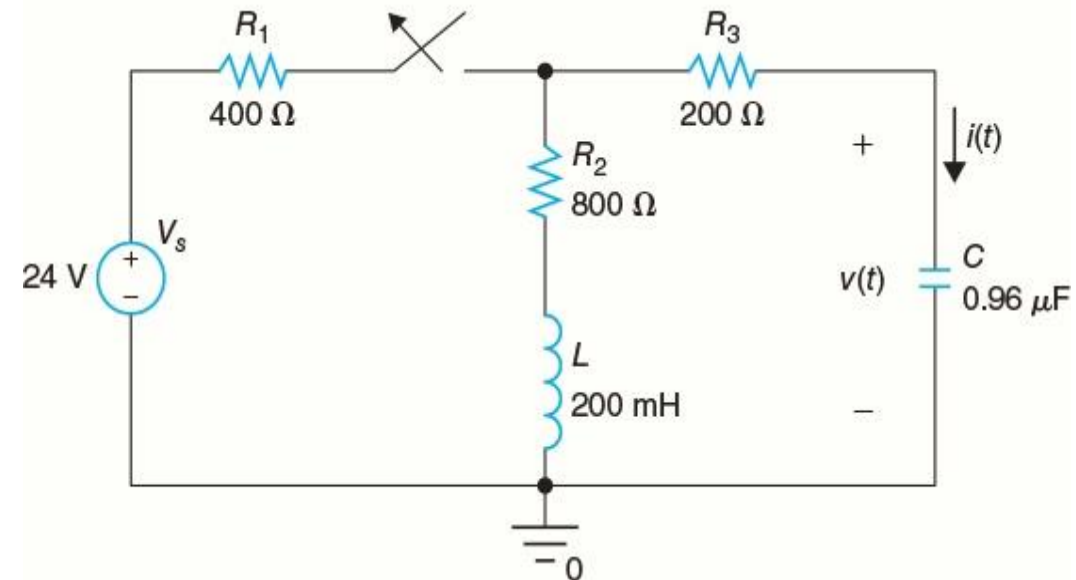
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1479.3793, s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -3520.6207$$

FIGURE 8.6

Circuit for EXAMPLE 8.5.

❑ $A_1 + A_2 = V_0 = 16$ (1),
❑ $dv(0)/dt = A_1 s_1 + A_2 s_2 = -I_0/C$
❑ $-1479.38 A_1 - 3520.62 A_2 = -20833.33$ (2)
❑ **Solving (1) and (2), we get $A_1 = 17.39\ V$, $A_2 = -1.39\ V$**

$$v(t) = \left(17.39 e^{-1479.38t} - 1.39 e^{-3520.62t}\right) u(t)$$

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EXAMPLE 8.10

❑ **The switch in the circuit shown in Figure 8.9 has been closed for a long time before it is opened at $t = 0$. Find current $i(t)$ for $t \geq 0$.**

▪ $v(0) = V_0 = 0$, $i(0) = I_0 = (4V/1\Omega)/2 = 2A$

▪ $R = 1\ \Omega$, $a_2 = 1$, $a_1 = 1/(RC) = 4$, $a_0 = 1/(LC) = 4$, $\alpha = a_1/2 = 2$, $\omega_0 = \sqrt{a_0} = 2$.

▪ $\alpha = \omega_0$, **critically damped**, $s_1 = -\alpha = -2$, $s_2 = -\alpha = -2$
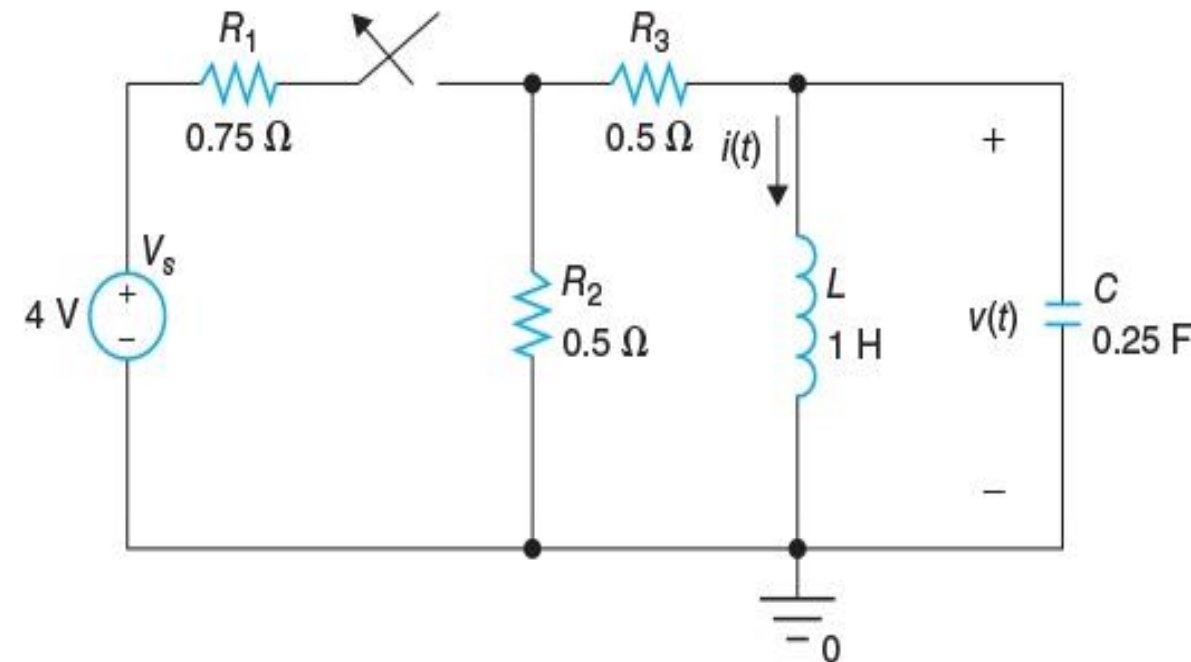
Circuit for EXAMPLE 8.10.

▪ **The current through the inductor is**

$$i(t) = A_1 e^{s_1 t} + A_2 t e^{s_2 t} = A_1 e^{-2t} + A_2 t e^{-2t}$$

▪ $i(0) = A_1 = I_0 = 2\ A$

▪ $di(0)/dt = A_1 s_1 + A_2 = V_0/L$

▪ $-2A_1 + A_2 = 0 \Rightarrow A_2 = 4\ A$

▪ $A_1 = 2\ A$, $A_2 = 4\ A$

$$i(t) = \left(2e^{-2t} + 4t e^{-2t}\right) u(t)\ \text{A}$$

EXAMPLE 8.19

❏ **Find** $v_2(t)$ **for** $t \geq 0$ **for the circuit shown in Figure 8.26. The initial conditions are** $v_1(0) = V_{10} = 3$ V **and** $v_2(0) = V_{20} = 1$ V.

▪ **Sum the currents leaving node** *a*:
$$\frac{v_1(t) - V_s}{R_1} + C_1 \frac{dv_1(t)}{dt} + \frac{v_1(t) - v_2(t)}{R_2} = 0 \quad (1)$$

▪ **Sum the currents leaving node** *b*:
$$\frac{v_2(t) - v_1(t)}{R_2} + C_2 \frac{dv_2(t)}{dt} = 0 \Rightarrow v_1(t) = R_2 C_2 \frac{dv_2(t)}{dt} + v_2(t) \quad (2)$$

▪ **At** $t = 0$, **Equation (2) becomes**
$$\frac{dv_2(0)}{dt} = \frac{v_1(0) - v_2(0)}{R_2 C_2} = 2 \quad (3)$$

▪ **Substitute (2) to (1) and simplify:**
$$\frac{d^2 v_2(t)}{dt^2} + 2.5 \frac{dv_2(t)}{dt} + v_2(t) = 10$$

▪ $v_{2p}(t) = 10$ V

▪ $s^2 + 2.5s + 1 = (s + 0.5)(s + 2) = 0 \Rightarrow s = -0.5, s = -2$

▪ $v_2(t) = 10 + A_1 e^{-0.5t} + A_2 e^{-2t}$, $10 + A_1 + A_2 = 1$,

▪ $dv_2(0)/dt = A_1 s_1 + A_2 s_2 = 2$, $A_1 = -32/3$, $A_2 = 5/3$

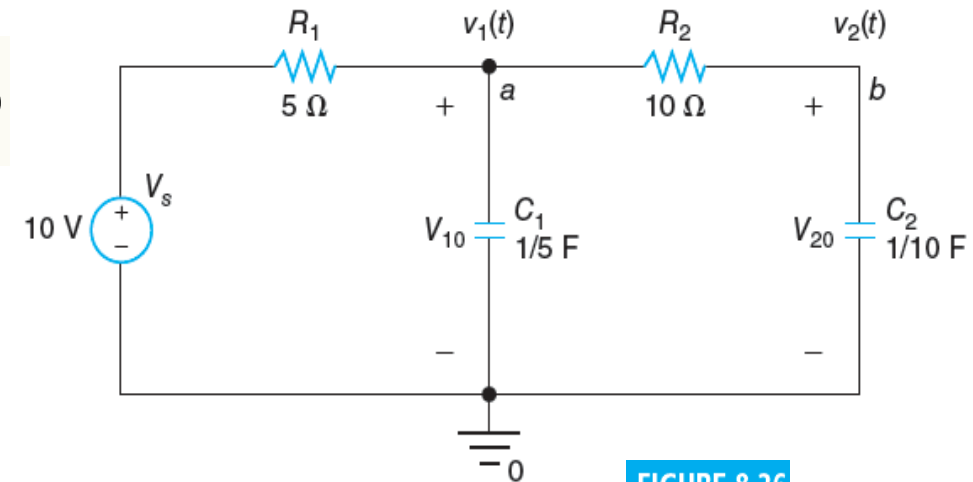▪ $v_2(t) = (10 - 10.6667 e^{-0.5t} + 1.6667 e^{-2t}) u(t)$ V

FIGURE 8.26

Summary

❑ **The second-order differential equations with constant coefficients with constant input can be written as**

$$\frac{d^2 v(t)}{dt^2} + a_1 \frac{dv(t)}{dt} + a_0 v(t) = b_0$$

❑ **The particular solution is $v_p(t) = b_0/a_0$.**

❑ **The complementary solution $v_c(t)$ is the solution to homogeneous differential equation**

$$\frac{d^2 v(t)}{dt^2} + a_1 \frac{dv(t)}{dt} + a_0 v(t) = 0$$

❑ **The complete solution is the sum of the particular solution and the complementary solution. The coefficients are found by applying the initial conditions on the complete solution.**

Summary (Continued)

❑ **The second-order homogeneous differential equation can be written as**

$$\frac{d^2 v(t)}{dt^2} + a_1 \frac{dv(t)}{dt} + a_0 = 0$$

❑ **The characteristic equation is** $s^2 + a_1 s + a_0 = 0$

❑ **Let** $\alpha = a_1/2, \ \omega_0 = \sqrt{a_0}$

❑ **The roots of the characteristic equation are:**

$$s_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0} = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \ \ s_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0} = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

❑ **If** $\alpha > \omega_0$**, case 1 (overdamped), two distinct real roots.**

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \ \ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

❑ $v(0) = A_1 + A_2 = V_0, \ dv(0)/dt = A_1 s_1 + A_2 s_2 = Dv_0$

Summary (Continued)

❑ **If $\alpha = \omega_0$, case 2 (critically damped), two identical real roots**

$$s_1 = -\frac{a_1}{2} = -\alpha, \ \ s_2 = -\frac{a_1}{2} = -\alpha$$

$$v(t) = A_1 e^{s_1 t} + A_2 t e^{s_2 t}$$

❑ $v(0) = A_1 = V_0,\ dv(0)/dt = A_1 s_1 + A_2 = Dv_0$

❑ **If $\alpha < \omega_0$, case 3 (underdamped), complex conjugate roots**

$$\beta = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + j\beta, \ \ s_2 = -\alpha - j\beta$$

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\beta t) + B_2 \sin(\beta t)\right]$$

❑ $v(0) = B_1 = V_0,\ dv(0)/dt = -\alpha B_1 + \beta B_2 = Dv_0$

❑ **Series RLC circuit: $a_1 = R/L$, $a_0 = 1/(LC)$, $\alpha = R/(2L)$, $\omega_0 = \sqrt{a_0}$, $dv(0)/dt = I_0/C$**

❑ **Parallel RLC circuit: $a_1 = 1/(RC)$, $a_0 = 1/(LC)$, $\alpha = 1/(2RC)$, $\omega_0 = \sqrt{a_0}$, $di(0)/dt = V_0/L$**