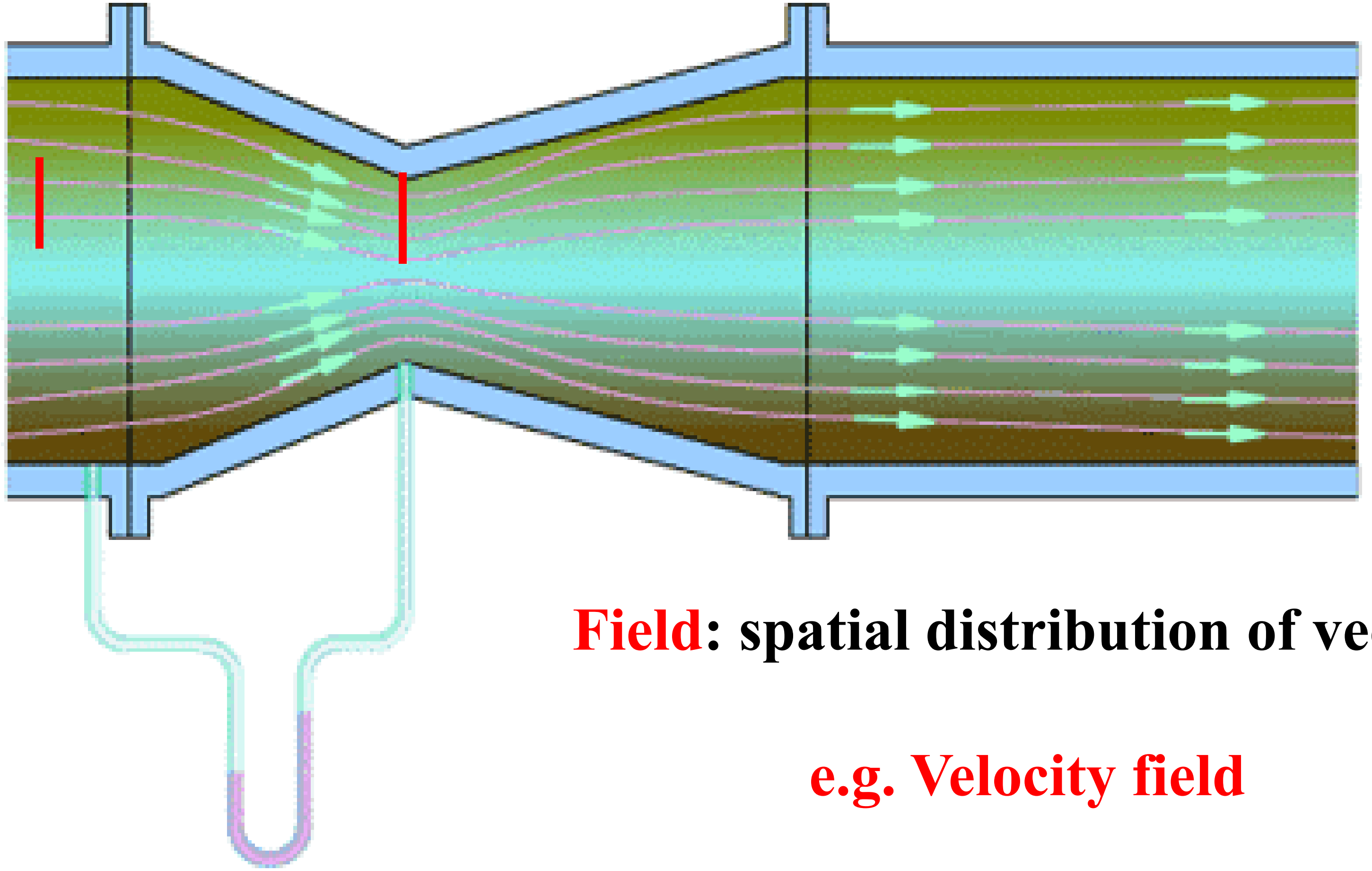
Chapter 20

Gauss's Law

Field lines

Direction?
Magnitude?

Visualize the electric field → electric field lines



Field: spatial distribution of vector

e.g. Velocity field

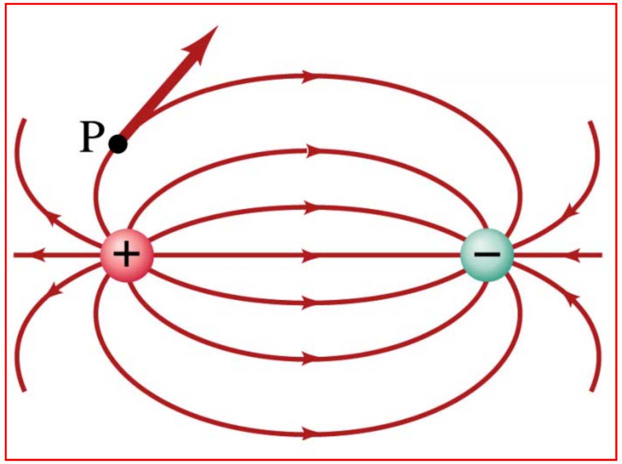
Direction:
tangential direction of the line

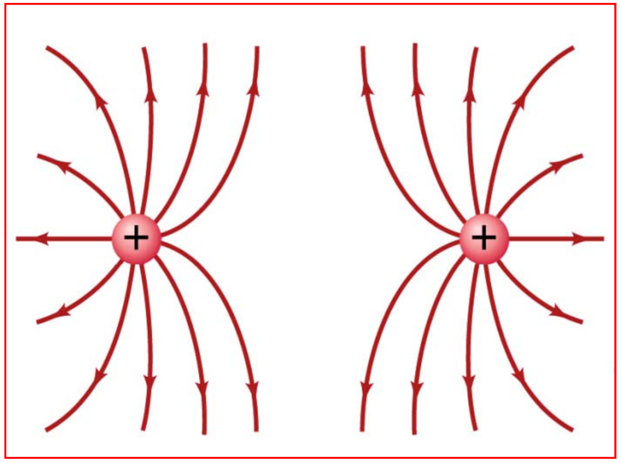
Magnitude:
density of the line

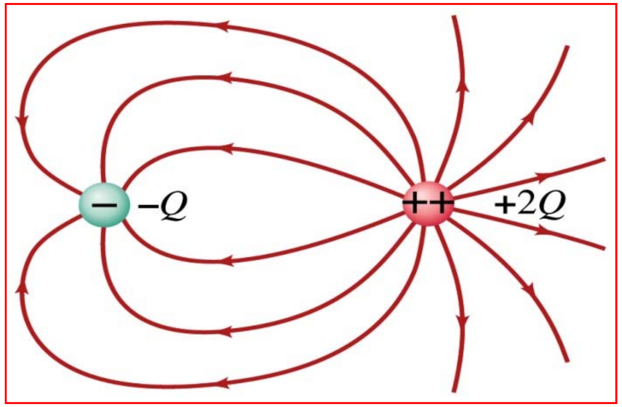
Field lines

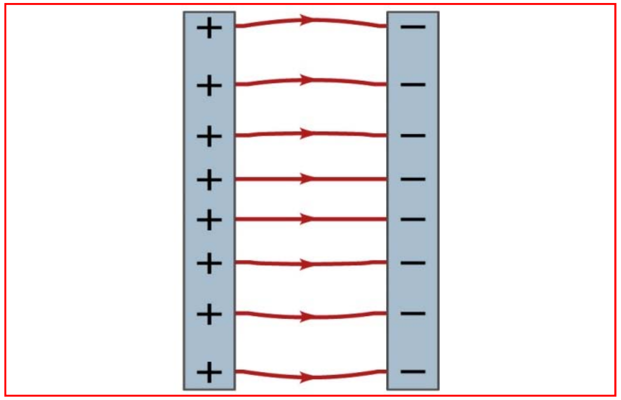
Direction?
Magnitude?

Visualize the electric field → electric field lines



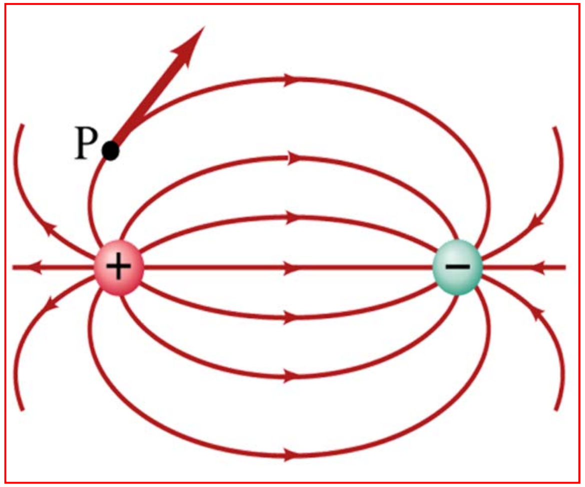






Properties of field lines

- **Direction of electric field $\vec{E}$:**
  tangent to the field line at any point
- **Magnitude of electric field $\vec{E}$:**
  $\propto$ number of lines crossing unit area $\perp$ them
- **Field lines start from + charges, end on – charges**
- **Field lines never cross each other; and there are no closed field lines.**

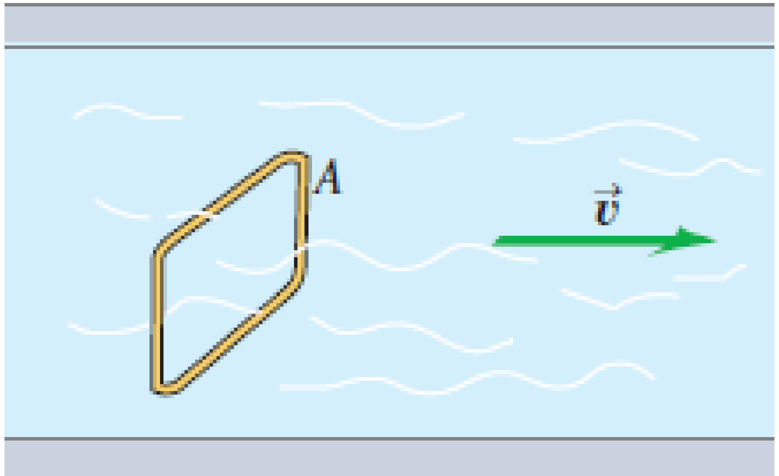


Electric flux

Flux: rate of flow of energy or particles across a given surface

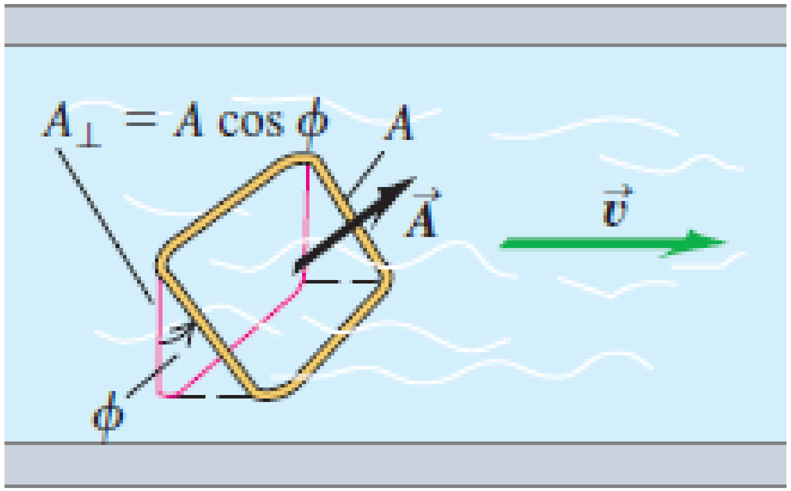
Flux of fluid: volume of fluid across a given surface

$\vec{v} \perp A$



$$\frac{dV}{dt} = \frac{A \cdot dx}{dt} = A \cdot v$$

$\vec{v}$ not $\perp A$



$$\frac{dV}{dt} = \frac{\vec{A} \cdot \vec{i}dx}{dt} = \vec{A} \cdot \vec{v}$$

Electric flux

Φ_E : Electric flux through an area

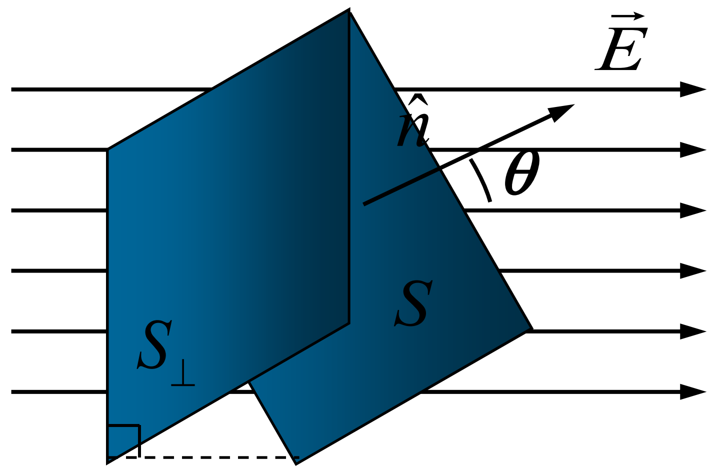
$\propto$ number of field lines passing that area

1) Uniform field:

$$\Phi_E = ES_\perp$$

$$= ES\cos\theta$$

$$\boxed{\Phi_E = \vec{E}\cdot\vec{S}}$$



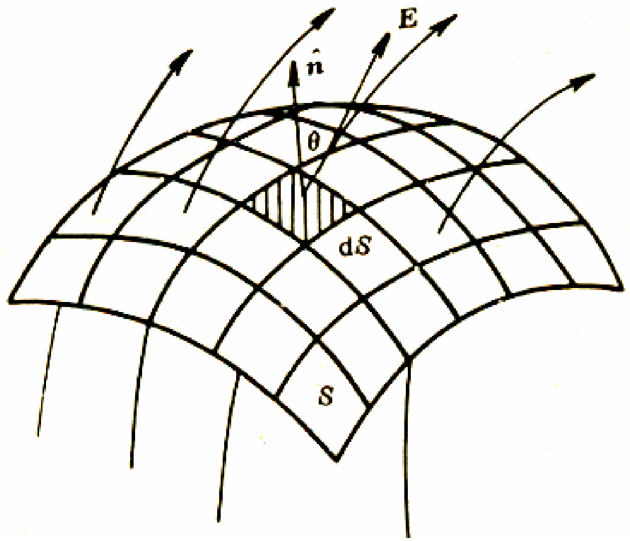
Q: $\vec{E} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ **, pass 5m^2 on *xoy* plane?**

Electric flux

2) General case:

$$d\Phi_E = \vec{E} \cdot d\vec{S}$$

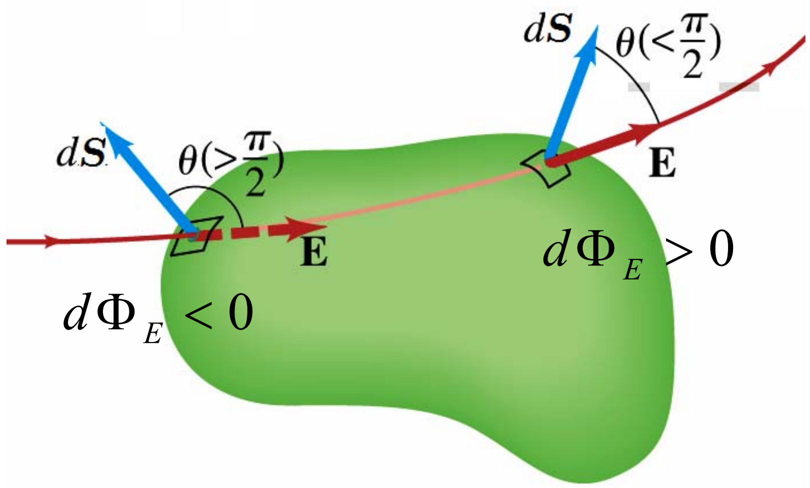
$$\boxed{\Phi_E = \int \vec{E} \cdot d\vec{S}}$$



3) Closed surface:

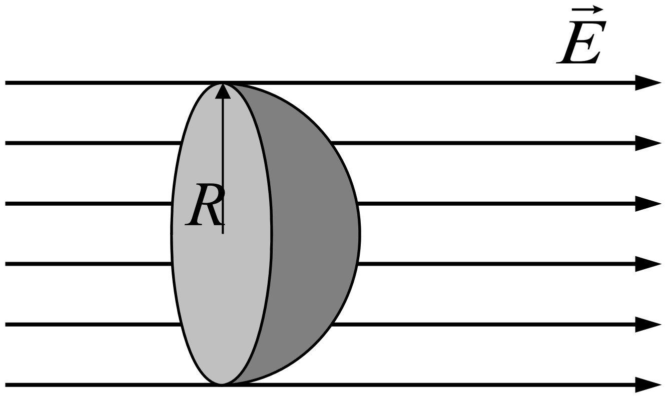
outward → positive

$\oint \vec{E} \cdot d\vec{S}$ → **net flux**



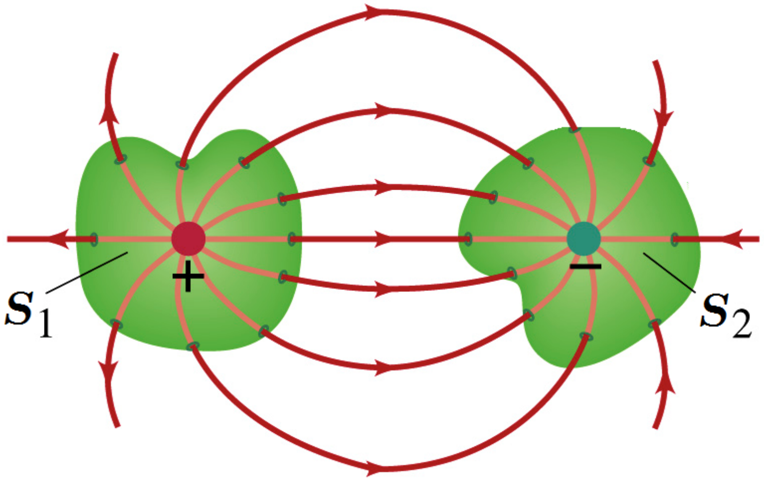
Examples of electric flux

a) Hemispherical surface in uniform field



$$\Phi_E = E \cdot \pi R^2$$

b) Closed surfaces



$$\Phi_E(S_1) > 0$$

$$\Phi_E(S_2) < 0$$

Gauss's law

Electric flux through a closed surface is given by the net charge Q_{in} enclosed within that surface.

$$\boxed{\oint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q_{in}}{\varepsilon_0}} \rightarrow \text{Gauss's law}$$

where ε_0 is the permittivity of free space

$\oint$ is over the value of $\vec{E}$ on a closed surface

Q_{in} is the net charge enclosed by that surface

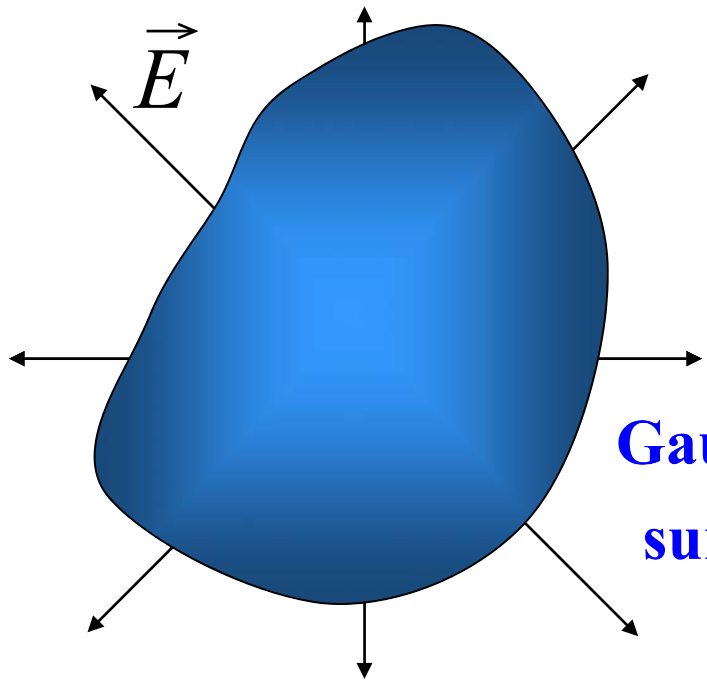
Validity of Gauss's law

1) Point charge Q,

spherical surface:

$$\oint \vec{E} \cdot d\vec{S} = \oint E \cdot dS$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2} \oint dS = \frac{Q}{\varepsilon_0}$$



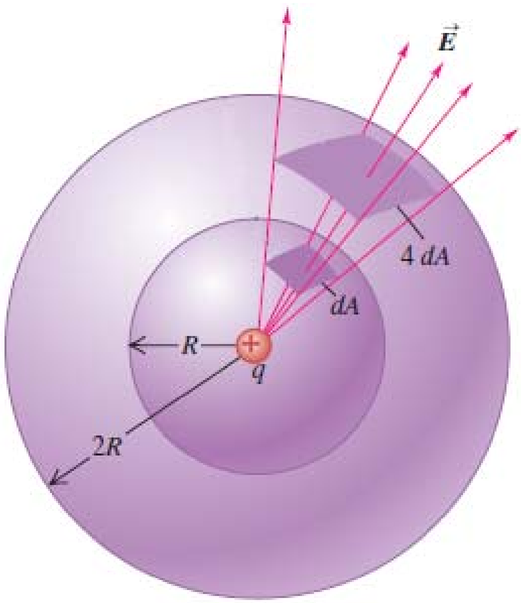
2) Point charge Q inside any closed surface

Same field lines pass through!

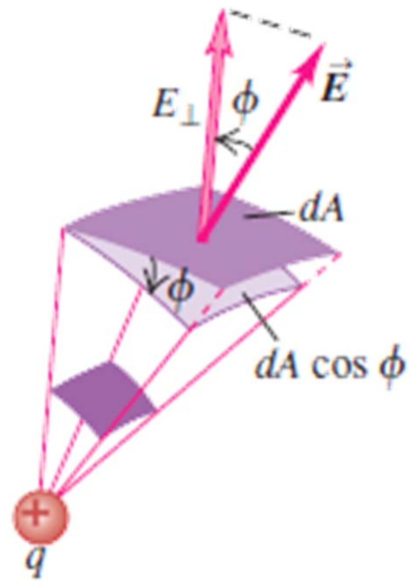
$$\boxed{\oint \vec{E} \cdot d\vec{S} = \frac{Q_{in}}{\varepsilon_0}}$$

Validity of Gauss's law

Qualitative interpretation: Same field lines pass through!



$$
\begin{aligned}
\Phi_E &= \oint_S E \cdot \mathrm{d}S \cdot \cos\phi \\
&= \oint_S \frac{q}{4\pi\varepsilon_0 r^2} \cdot \mathrm{d}S_\perp \\
&= \frac{q}{4\pi\varepsilon_0} \cdot \oint_S \frac{\mathrm{d}S_\perp}{r^2} \\
&= \frac{q}{4\pi\varepsilon_0} \cdot \oint_S \mathrm{d}\Omega \\
&= \frac{q}{4\pi\varepsilon_0} \cdot 4\pi = \frac{q}{\varepsilon_0}
\end{aligned}
$$

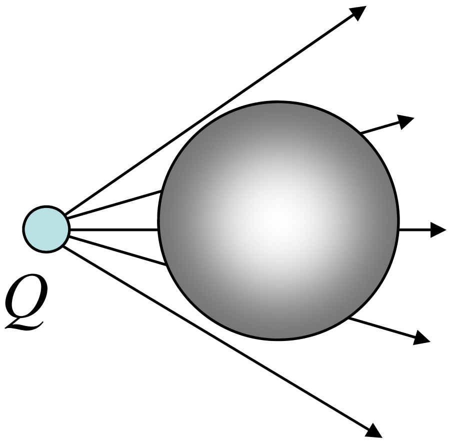


Quantitative interpretation: the same shadow, the same solid angle

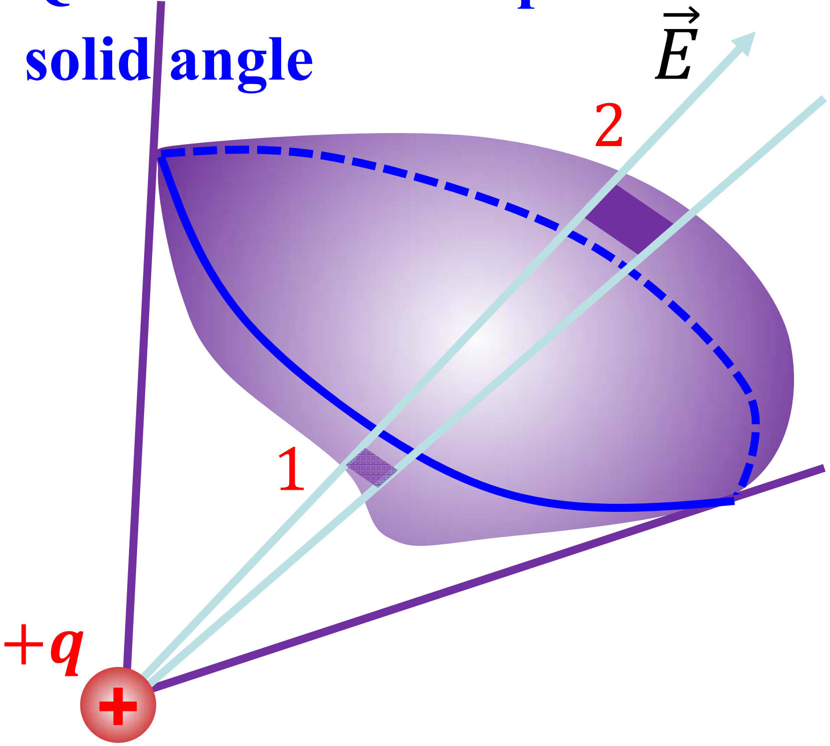
Validity of Gauss's law

3) Q outside the closed surface:

$$\oint \vec{E} \cdot \mathrm{d}\vec{S} = 0$$



Quantitative interpretation: the same shadow, the same solid angle



$$\mathrm{d}\Phi_E = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\mathrm{d}S_\perp}{r^2} = \pm \frac{q}{4\pi\varepsilon_0} \cdot \mathrm{d}\Omega$$

$$\mathrm{d}\Phi_{E1} = -\frac{q}{4\pi\varepsilon_0} \cdot \mathrm{d}\Omega$$

$$\mathrm{d}\Phi_{E2} = +\frac{q}{4\pi\varepsilon_0} \cdot \mathrm{d}\Omega$$

$$\mathrm{d}\Phi_{E1} + \mathrm{d}\Phi_{E2} = 0$$

$$\Phi_E = \int_S \vec{E} \cdot \mathrm{d}\vec{S} = 0$$

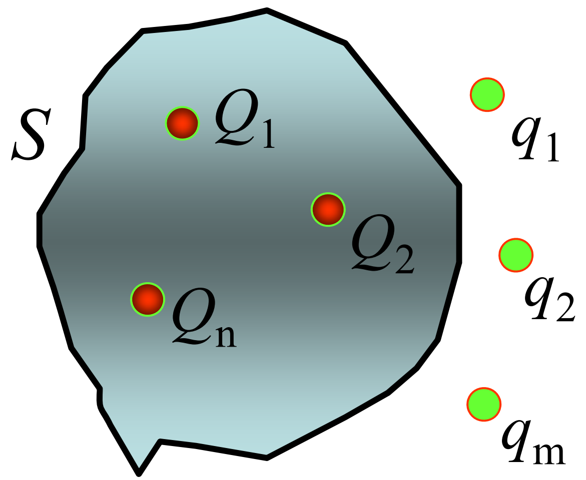
Validity of Gauss's law

4) General case, several charges:

$$\oint \vec{E}\cdot \mathrm{d}\vec{S} = \oint \left(\sum_i \vec{E}_i\right)\cdot \mathrm{d}\vec{S}$$

$$= \sum_i \left(\oint \vec{E}_i \cdot \mathrm{d}\vec{S}\right) = \sum_i \frac{Q_i}{\varepsilon_0}$$

$$= \frac{Q_{in}}{\varepsilon_0}$$



Explanations of Gauss's law

$$\boxed{\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q_{in}}{\varepsilon_0}}$$

1) $\vec{E}$ is produced by **all the charges** within or out of the closed surface.

2) Φ_E depends only on the **net charge inside**.

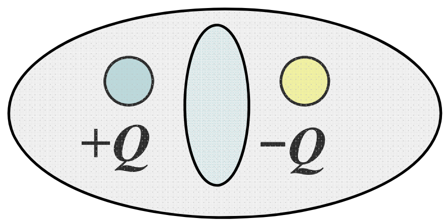
3) More general than Coulomb's law

4) Equivalent differential form: $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$

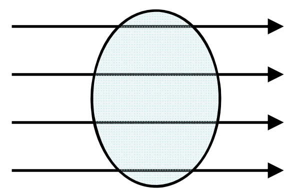
True or false?

Are the following statements true or false?

① $\oint \vec{E} \cdot d\vec{S} = 0 \quad \rightarrow$ **no charge inside the surface**



no net charge



② **No charge inside the surface** $\rightarrow \; E = 0$

③ $\vec{E}$ **is constant inside the surface** $\rightarrow Q_{in} = 0$

Applications of Gauss's law

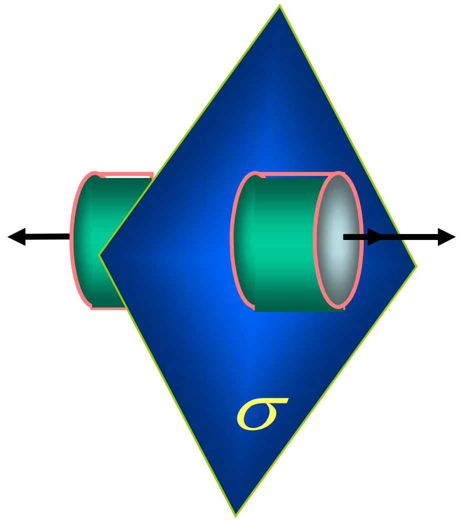
Determine the electric field in a simple way

if the charge distribution is highly symmetric

① **Identify the symmetry of system.**

② **Choose a proper Gaussian surface.**

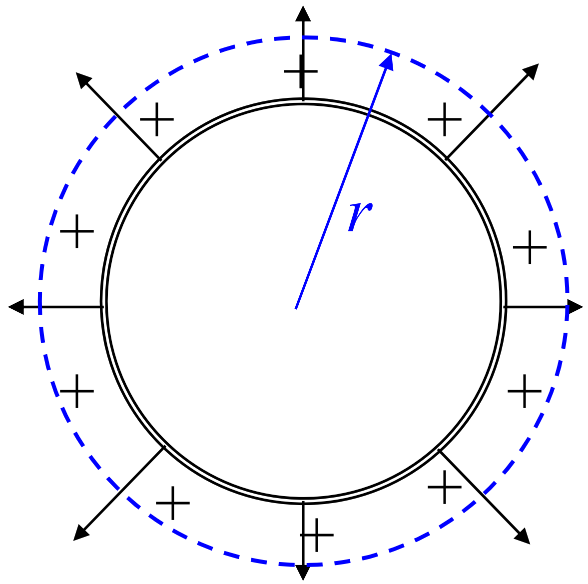
$\perp \vec{E}$ or $// \vec{E}$ **& closed surface**



③ **Apply Gauss's law, write the equation.**

Spherical shell

Example1: Charge *Q* is uniformly distributed on a thin spherical shell of radius *R*. Determine the field (a) outside the shell; (b) inside the shell.



Solution: ***E*** is also symmetric

(a) Choose a Gaussian surface:

$$\oint \vec{E} \cdot d\vec{S} = E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

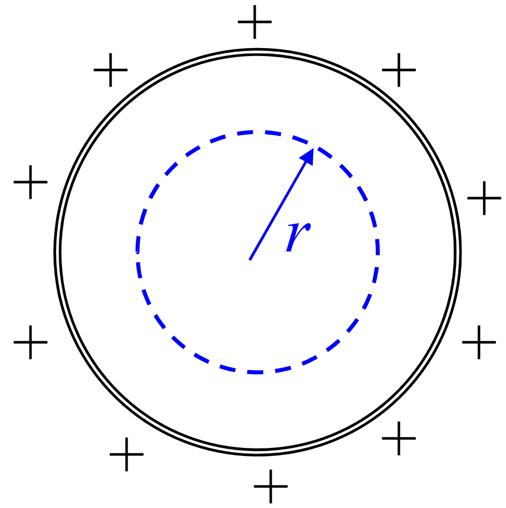
$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\sigma}{\varepsilon_0}$$

Same as point charge!

(b) Gaussian surface inside the shell

$$\oint \vec{E} \cdot d\vec{S} = 0 \quad \Rightarrow \vec{E} = 0$$

No electric field inside !



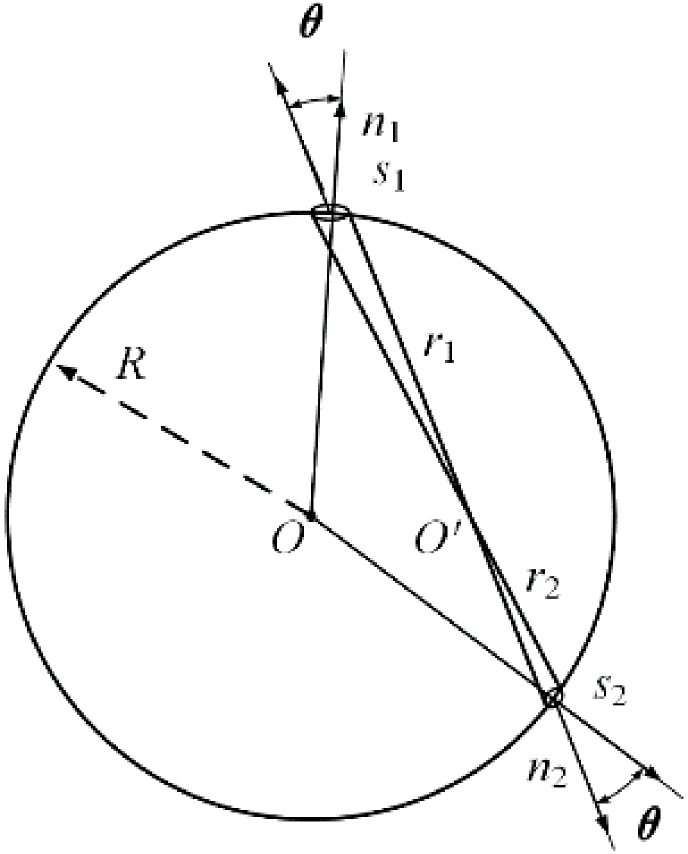
Discussion:

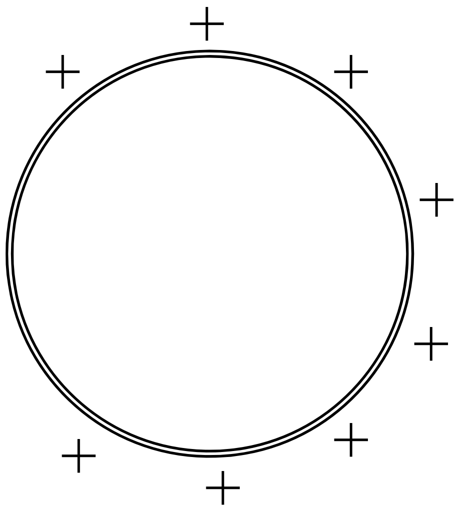
(1) Uniformly charged Conductor sphere

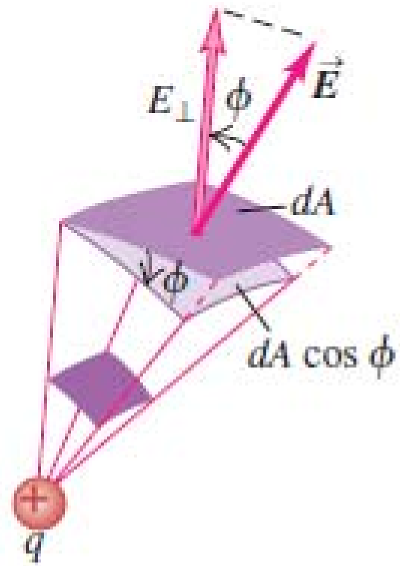
(2) Uniformly charged nonconducting sphere

Outside? Inside? Gravitational field

Thinking: another way to prove $\vec{E} = 0$ in a uniformly charged thin spherical shell?







$$\sum E = \frac{k\sigma}{4\pi\varepsilon_0}\frac{s_1}{r_1^2} - \frac{k\sigma}{4\pi\varepsilon_0}\frac{s_2}{r_2^2}$$

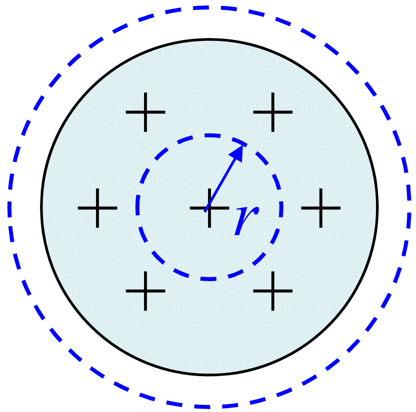
Solid angle: an angle formed by three or more planes intersecting at a common point (the vertex)

$$\Omega = \frac{s_\perp}{r^2}$$

Uniformly charged sphere

Example2: Uniformly charged sphere (*Q*, *R*). Find the field (a) outside and (b) inside the sphere.

Solution: (a) outside the sphere:

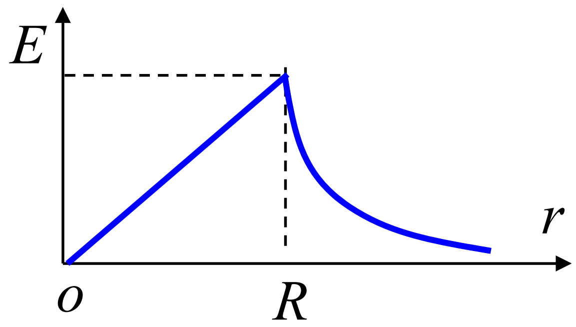


$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

(b) Inside the sphere:

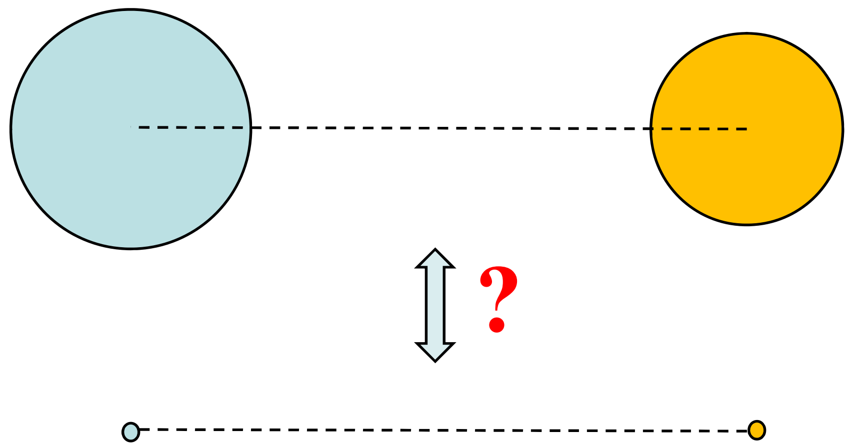
$$\oint \vec{E} \cdot d\vec{S} = E \cdot 4\pi r^2 = \frac{\rho}{\varepsilon_0}\frac{4\pi r^3}{3}$$

$$\Rightarrow E = \frac{\rho r}{3\varepsilon_0} \qquad (\rho = Q / \frac{4\pi R^3}{3})$$



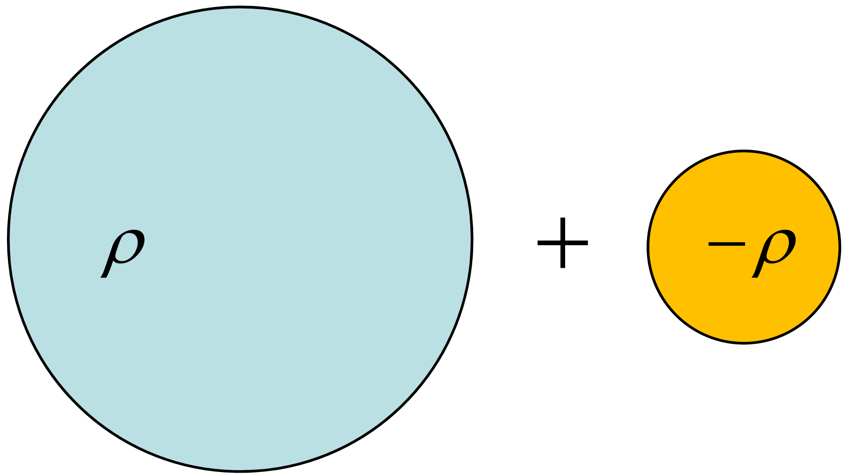
Force between two spheres

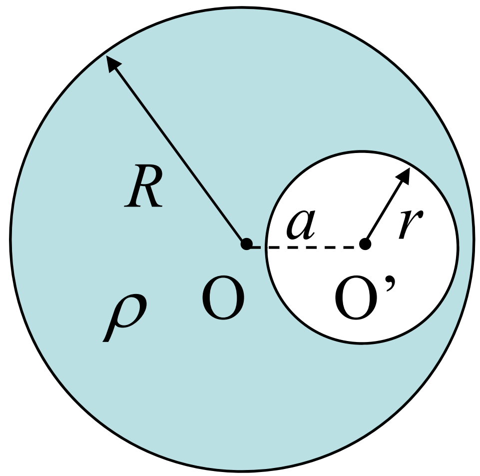
Thinking: **Cavendish used two solid sphere to measure the gravitational force, is the result valid or not? (Compare to the force between two particles)**



Holey sphere

Question: **A uniformly charged sphere (ρ, R) has a spherical hole on it. What is the electric field inside the hole?**



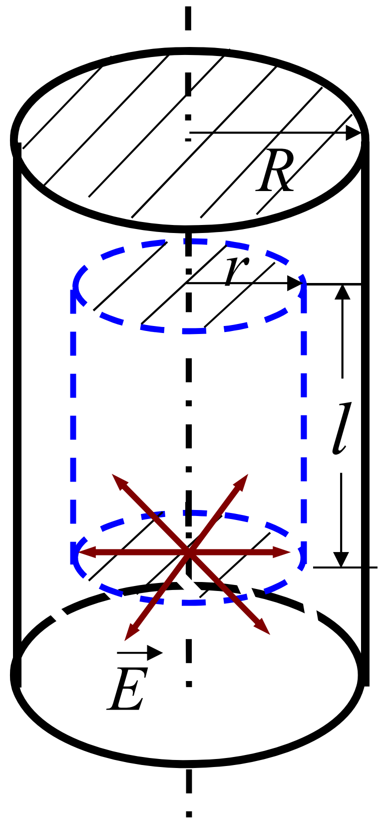


Uniformly charged long cylinder

Example3: **A long cylinder is uniformly charged (** $\boldsymbol{\rho}$**,** $\boldsymbol{R}$ **). Find the field (a) outside and (b) inside.**

Solution: Axial symmetry

Choose Gaussian surface as:



$$\oint \vec{E}\cdot d\vec{S} = \int_{flats} \vec{E}\cdot d\vec{S} + \int_{side} \vec{E}\cdot d\vec{S}$$

$$= E\cdot 2\pi r l = \lambda l / \varepsilon_0$$

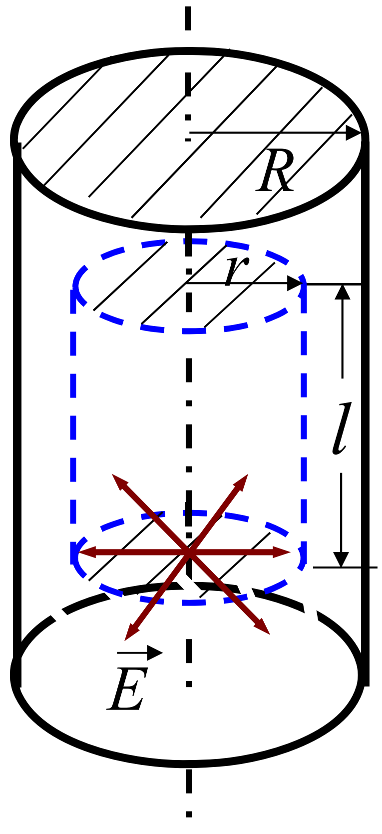
$$\Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad \lambda ?$$

Uniformly charged long cylinder

Example3: **A long cylinder is uniformly charged (ρ, R). Find the field (a) outside and (b) inside.**

Solution: Axial symmetry

Choose Gaussian surface as:



$$\oint \vec{E}\cdot d\vec{S} = \int_{flats} \vec{E}\cdot d\vec{S} + \int_{side} \vec{E}\cdot d\vec{S}$$

$$= E\cdot 2\pi r l = \lambda l / \varepsilon_0$$

$$\Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad \lambda ?$$

(a) outside: $r > R$

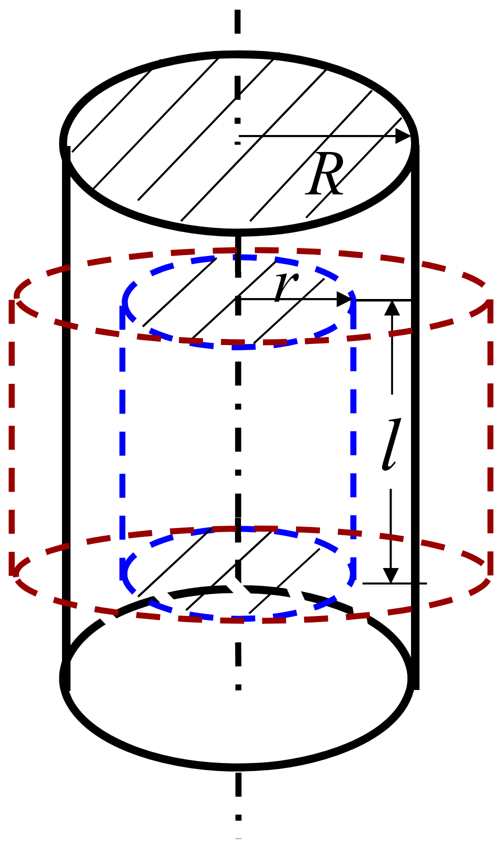
$$E = \frac{\rho \pi R^2 l}{2\pi\varepsilon_0 r l} = \frac{\rho R^2}{2\varepsilon_0 r}$$

(b) inside: $r < R$

$$E = \frac{\rho \pi r^2 l}{2\pi\varepsilon_0 r l} = \frac{\rho r}{2\varepsilon_0}$$

Inside sphere: $E = \dfrac{\rho r}{3\varepsilon_0}$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{Q_{in}}{2\pi\varepsilon_0 r l}$$



What about a cylindrical shell?

Gauss's Law can help us study Conductor in external field

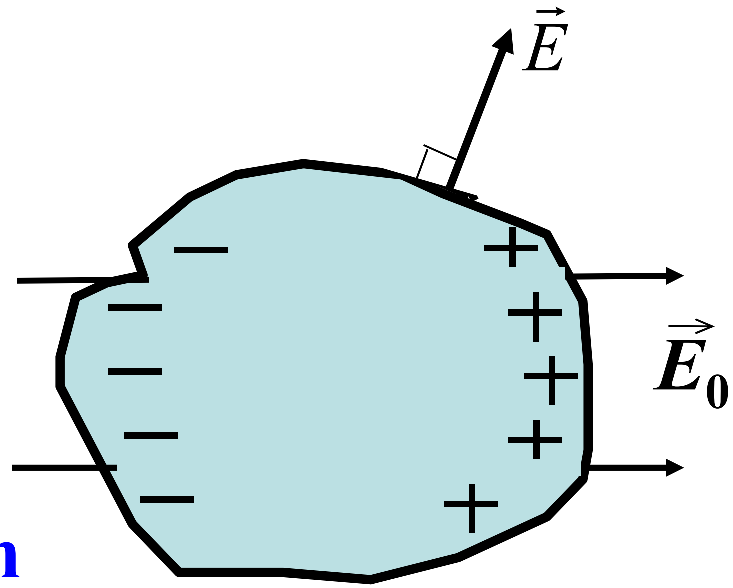
Conductor in external field

⇩

Electrostatic induction

⇩

Electrostatic equilibrium

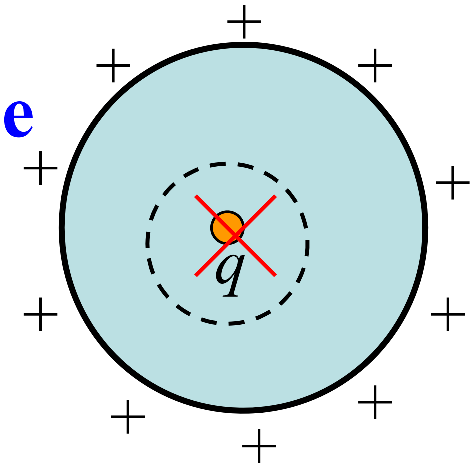


① Electric field inside at any position is 0.

② Electric field nearly outside ⊥ the surface

Charges on Conductor

1) All the charges are on the surface

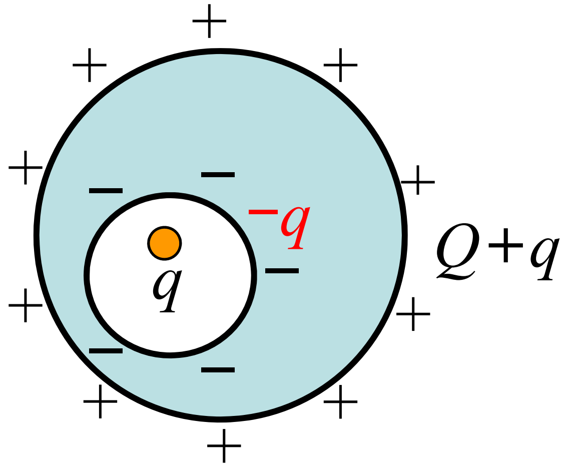


A charge inside? $\oiint \vec{E} \cdot d\vec{S} \neq 0!$

2) For a holey conductor with Q

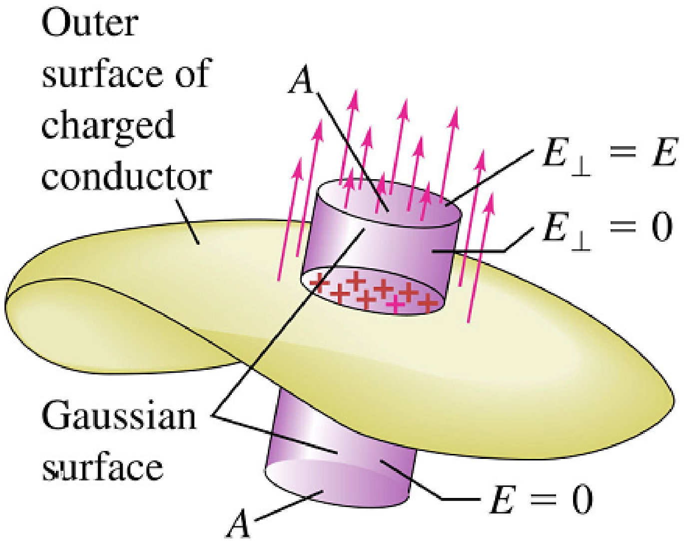
Case A: no other charge inside

Case B: other charges inside



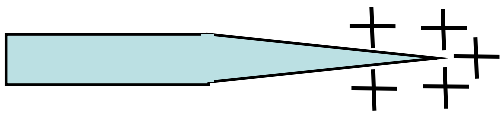
E nearby conductor surface

Electric field just outside the surface of conductor:



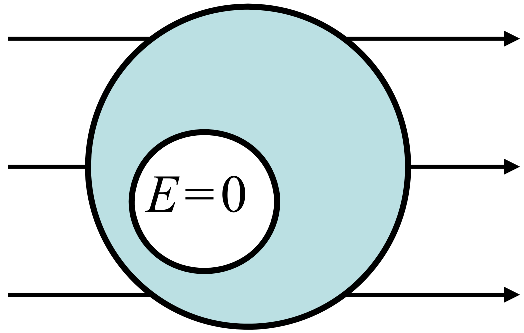
Charge density & curvature of surface

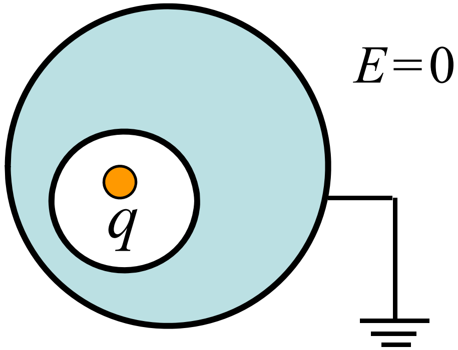
$$\boxed{E = \frac{\sigma}{\varepsilon_0}}$$



Electrostatic shielding

A body placed inside the cavity of conductor will not be affected by the electric field outside.





Faraday cage

Flat metal plates

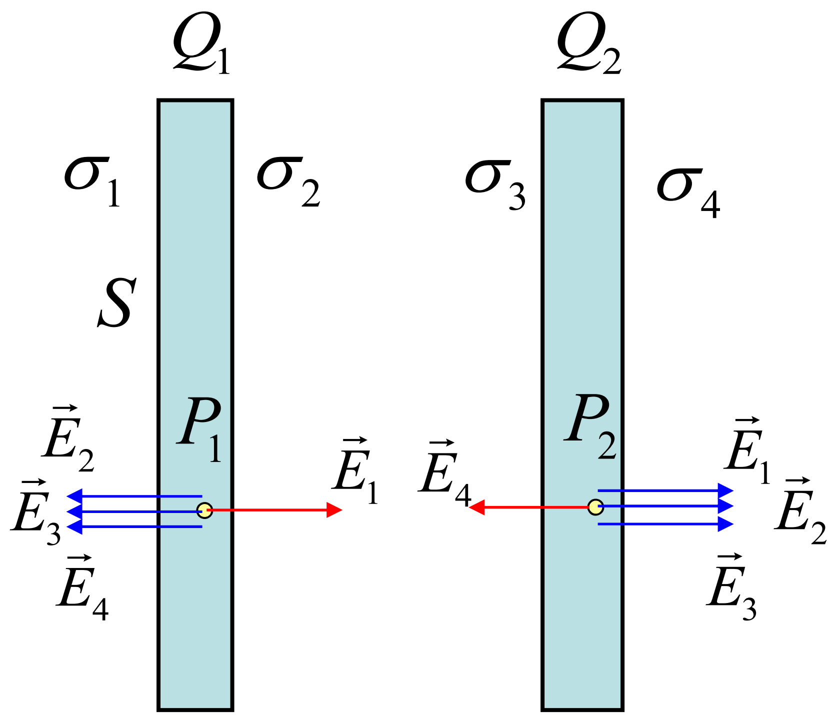
Example4: **Two large flat metal plates with charges Q_1 and Q_2. Determine (a) charges on each surface; (b) electric field between the plates.**

Solution: $(\sigma_1+\sigma_2)S=Q_1$

$$(\sigma_3+\sigma_4)S=Q_2$$

$$\frac{\sigma_1}{2\varepsilon_0}-\frac{\sigma_2}{2\varepsilon_0}-\frac{\sigma_3}{2\varepsilon_0}-\frac{\sigma_4}{2\varepsilon_0}=0$$

$$\frac{\sigma_1}{2\varepsilon_0}+\frac{\sigma_2}{2\varepsilon_0}+\frac{\sigma_3}{2\varepsilon_0}-\frac{\sigma_4}{2\varepsilon_0}=0$$



$$\frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} - \frac{\sigma_3}{2\varepsilon_0} - \frac{\sigma_4}{2\varepsilon_0} = 0 \qquad (\sigma_1 + \sigma_2) S = Q_1$$

$$\frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} + \frac{\sigma_3}{2\varepsilon_0} - \frac{\sigma_4}{2\varepsilon_0} = 0 \qquad (\sigma_3 + \sigma_4) S = Q_2$$

(a)

$$\begin{cases} \sigma_1 = \sigma_4 = \dfrac{Q_1 + Q_2}{2S} \\[2ex] \sigma_2 = \dfrac{Q_1 - Q_2}{2S} = -\sigma_3 \end{cases}$$

(b) E between plates:

$$E = \frac{\sigma_2}{\varepsilon_0} = \frac{Q_1 - Q_2}{2\varepsilon_0 S}$$