Circuit Analysis and Design

Academic Year 2025/2026 – Semester 1

Lecture 16 – Analysis of Phasor Transformed Circuits

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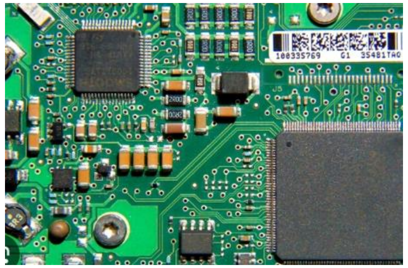
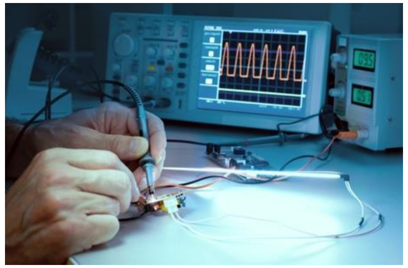




Agenda

- ❑ Sinusoidal signals (ac signals)
- ❑ RMS value
- ❑ Phasors (use for ac circuit analysis)
- ❑ Impedances and admittances (ac equivalent of resistance)
- ❑ Phasor-transformed circuit
- ❑ Series and parallel connection of impedances

Sinusoidal Signals

- Circuits are analyzed in the steady state where the input signal is a sinusoid.

- A cosine wave $v(t) = V_m \cos\left(\frac{2\pi}{T}t + \phi\right)$ has three parameters: amplitude V_m, period T, and phase ϕ.

- As the independent variable t (time) increases, the angle of cosine $2\pi t/T + \phi$ increases. At $t = 0$, the angle of cosine is ϕ. As t is increased from $t = 0$ to $t = T$, the angle increases linearly from ϕ to $2\pi + \phi$.
- Since $\cos(2\pi + \phi) = \cos(2\pi)\cos(\phi) - \sin(2\pi)\sin(\phi) = 1\times\cos(\phi) - 0\times\sin(\phi) = \cos(\phi)$, the value of cosine wave at $t = T$ is identical to the value at $t = 0$.

  - The cosine wave repeats itself every T seconds as shown in Figure 9.1. It is a periodic wave with period T seconds.

  - In 1 second, there are $f = 1/T$ periods (cycles, waves) of cosine wave. The parameter f is called frequency and a unit of $1/s$, called Hertz (Hz).

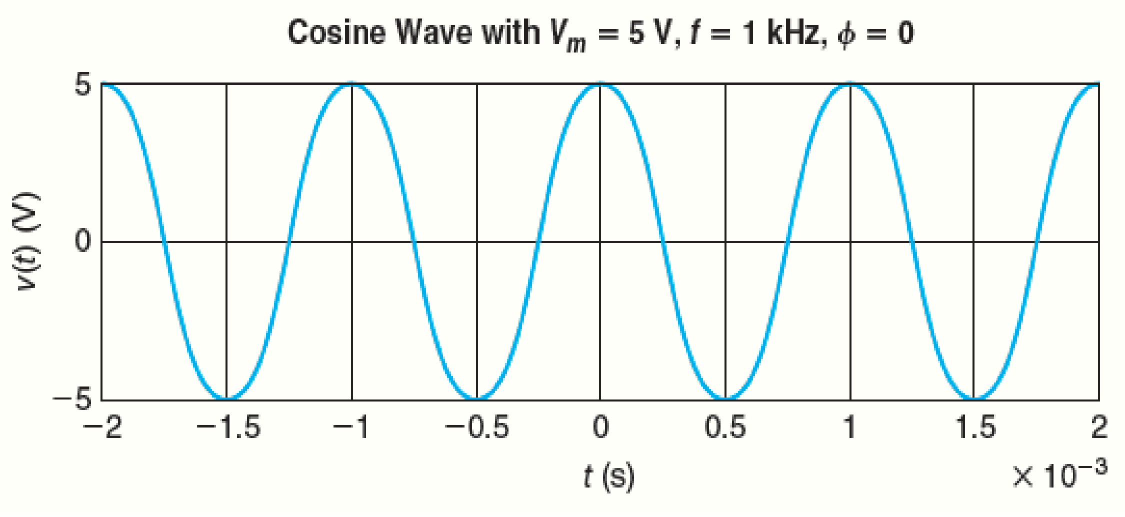
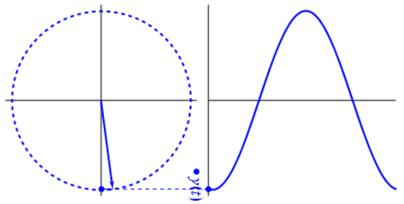


FIGURE 9.1

Sinusoidal Signals (Continued)



- Because the angle changes by 2π radians in one period, and there are f periods in one second, the change in angle in one second is given by

  $$\omega = 2\pi f = 2\pi/T$$

  - The parameter ω is called angular velocity and has a unit of radians per second (*rad/s*). In terms of radian frequency ω, cosine wave becomes

    $$v(t) = V_m \cos(\omega t + \phi)$$

- The phase ϕ of a cosine wave determines its starting value at $t = 0$.
- At $t = 0$, the cosine wave has a value of $V_m \cos(\phi)$. The cosine wave can be rewritten as

$$v(t) = V_m \cos\left(\frac{2\pi}{T}t + \phi\right) = V_m \cos\left(\frac{2\pi}{T}t + \frac{2\pi}{T}\phi\frac{T}{2\pi}\right) = V_m \cos\left[\frac{2\pi}{T}\left(t + \frac{\phi}{2\pi}T\right)\right]$$

- The cosine wave $V_m \cos(2\pi t/T + \phi)$ is a time shift of $V_m \cos(2\pi t/T)$ by $\phi T/(2\pi)$ seconds.
  - If ϕ is positive, the shift is to the left;
  - if ϕ is negative, the shift is to the right.

Sinusoidal Signals (Continued)

- Figure 9.4 shows a cosine wave with three different phases ($\phi = -45^o = -\pi/4$, $0^o = 0$, $45^o = \pi/4$) for $V_m = 1\ V$ and $f = 1\ Hz$ ($T = 1\ s$).
  - When $\phi = 0$, the cosine wave crosses zero at $t = -2/8\ s = -0.25\ s$.
  - When $\phi = \pi/4$, the cosine wave crosses zero at $t = -3/8\ s = -0.375\ s$. This is earlier than $\phi = 0$ by $0.125\ s$ or $T/8$.
  - When $\phi = -\pi/4$, the cosine wave crosses zero at $t = -1/8\ s = -0.125\ s$. This is later than $\phi = 0$ by $0.125\ s$ or $T/8$.

- If we look at the peaks of the three cosine waves around $t = 0$, we arrive at the same conclusion.
  - When $\phi = 0$, the peak of the cosine wave occurs at $t = 0$.
  - When $\phi = \pi/4$, the peak of the cosine wave occurs at $t = -1/8\ s = -0.125\ s$. This is earlier than $\phi = 0$ by $0.125\ s$ or T/8.
  - When $\phi = -\pi/4$, the peak of the cosine wave occurs at $t = 1/8\ s = 0.125\ s$. This is later than $\phi = 0$ by $0.125\ s$ or $T/8$.

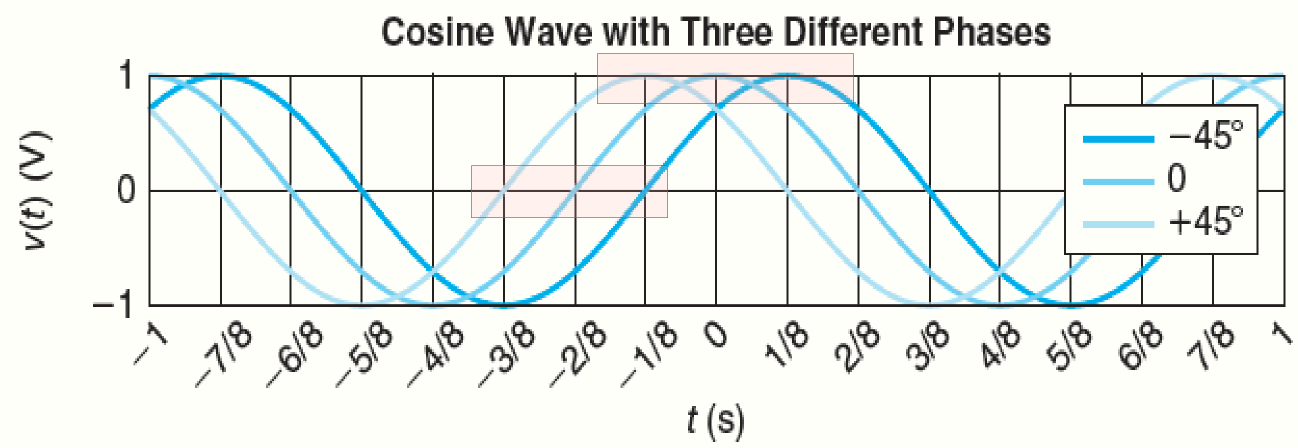


FIGURE 9.4

Sinusoidal Signals (Continued)

- A sine wave $V_m \sin\left(\frac{2\pi}{T}t+\phi\right)$ can be viewed as a time shift of a cosine wave $V_m \cos\left(\frac{2\pi}{T}t+\phi\right)$ by *90°* to the right:

$$V_m \cos\left(\frac{2\pi}{T}t+\phi-\frac{\pi}{2}\right)=V_m \cos\left(\frac{2\pi}{T}t+\phi\right)\cos\left(\frac{\pi}{2}\right)+V_m \sin\left(\frac{2\pi}{T}t+\phi\right)\sin\left(\frac{\pi}{2}\right)$$

$$=V_m \cos\left(\frac{2\pi}{T}t+\phi\right)\times 0+V_m \sin\left(\frac{2\pi}{T}t+\phi\right)\times 1=V_m \sin\left(\frac{2\pi}{T}t+\phi\right)$$

- *90°* to the right is equivalent to *–T/4*. Figure 9.5 shows the cosine wave and the sine wave with the same amplitude (V_m = *1 V*) and frequency (*f = 1 Hz*), and $\phi = 0$.

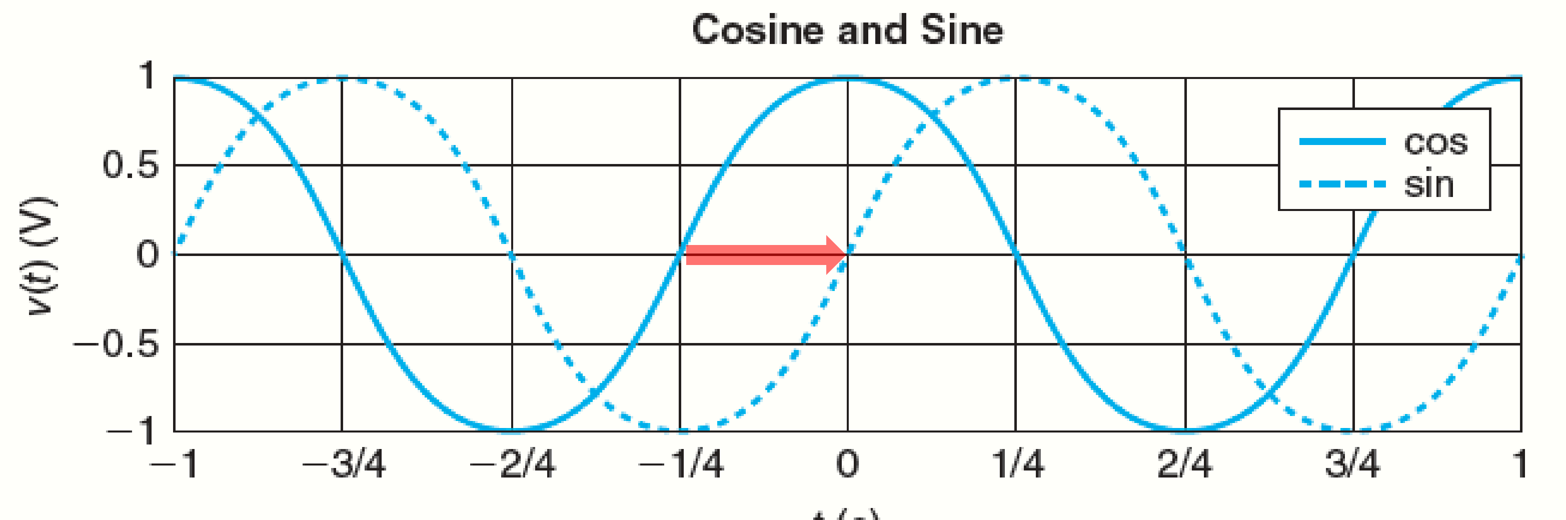


FIGURE 9.5

Root Mean Square (RMS) Value

- A sinusoidal voltage is given by: $v(t) = V_m \cos(\omega t + \theta_v)$
  - The peak amplitude is $V_p = V_m$.
  - The peak-to-peak amplitude is $V_{p\text{-}p} = 2V_m$.
- If the voltage $v(t)$ is squared, we obtain:

$$v^2(t) = V_m^2 \cos^2(\omega t + \theta_v) = \frac{V_m^2}{2} + \frac{V_m^2}{2}\cos(2\omega t + 2\theta_v) \quad (1)$$

- The mean square value of $v(t)$ is defined as the average value of $v^2(t)$. If $v(t)$ is periodic with period T, the mean square value of $v(t)$ is given by

$$\begin{aligned}
V_{ms} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v^2(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{V_m^2}{2}dt + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{V_m^2}{2}\cos(2\omega t + 2\theta_v)dt \\
&= \frac{V_m^2}{2} + \frac{V_m^2}{2T}\frac{\sin(2\omega t + 2\theta_v)\Big|_{-\frac{T}{2}}^{\frac{T}{2}}}{2\omega} \\
&= \frac{V_m^2}{2} + \frac{V_m^2}{2T}\frac{\sin\left(2\frac{2\pi}{T}\frac{T}{2} + 2\theta_v\right) - \sin\left(2\frac{2\pi}{T}\frac{-T}{2} + 2\theta_v\right)}{2\omega} \qquad (2) \\
&= \frac{V_m^2}{2} + \frac{V_m^2}{2T}\frac{\sin(2\theta_v) - \sin(2\theta_v)}{2\omega} = \frac{V_m^2}{2} + 0 = \frac{V_m^2}{2}
\end{aligned}$$

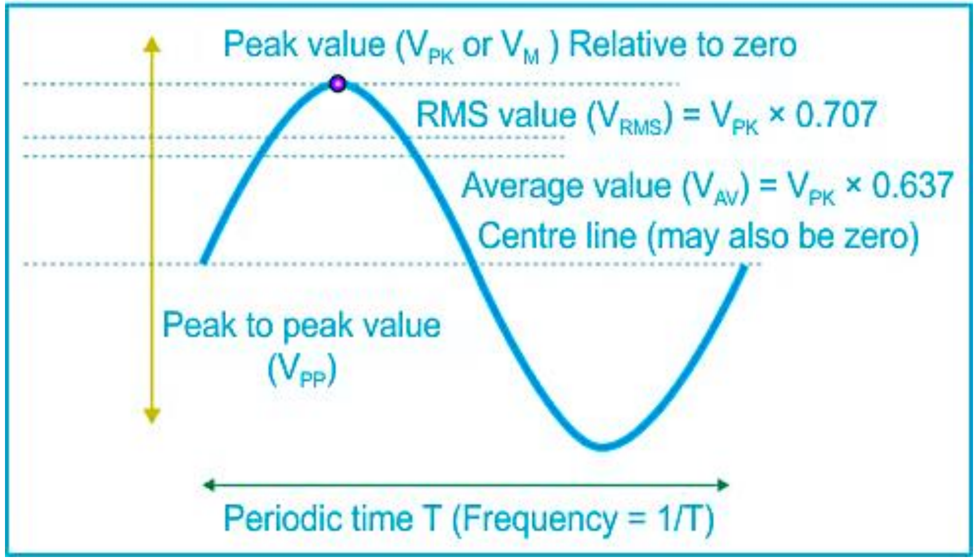
RMS Value (Continued)

- The root mean square (RMS) value is the square root of the mean square value. In general, the RMS value of *v(t)* is defined as

$$V_{rms} = \sqrt{\lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v^2(t)dt}$$

- If *v(t)* is periodic with period *T*, the RMS value of *v(t)* is given by

$$V_{rms} = \sqrt{\frac{1}{T} \int_{-T/2}^{T/2} v^2(t)dt}$$

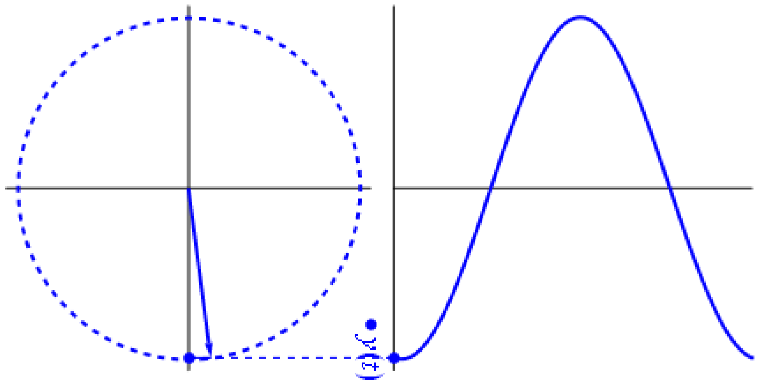


- From equation (2), RMS value can be calculated as

$$V_{rms} = \frac{V_m}{\sqrt{2}} = 0.7071 V_m$$

- RMS value of voltage/current is a useful way of comparing *ac* value to its equivalent *dc* value. It represents the *dc* value that will produce the same heating effect, or power dissipation, as the *ac* value.

Phasors



- Euler's rule says:

  $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

  $Re[e^{j\theta}] = \cos(\theta)$; $\qquad Im[e^{j\theta}] = \sin(\theta)$

- Applying Euler's rule on the point P in two-dimensional space, we have

  $V_m e^{j(\omega t + \phi)} = V_m \cos(\omega t + \phi) + j\, V_m \sin(\omega t + \phi)$

  $Re[V_m e^{j(\omega t + \phi)}] = V_m \cos(\omega t + \phi)$

  $Im[V_m e^{j(\omega t + \phi)}] = V_m \sin(\omega t + \phi)$

- *So*, $V_m e^{j(\omega t + \phi)} = V_m e^{j\phi} e^{j\omega t}$ is a point P in a circle of radius V_m rotating at a constant speed of ω *rad/s* in the counterclockwise direction in the complex plane.

- The cosine wave can be viewed as a projection of P on the circle to the horizontal axis.

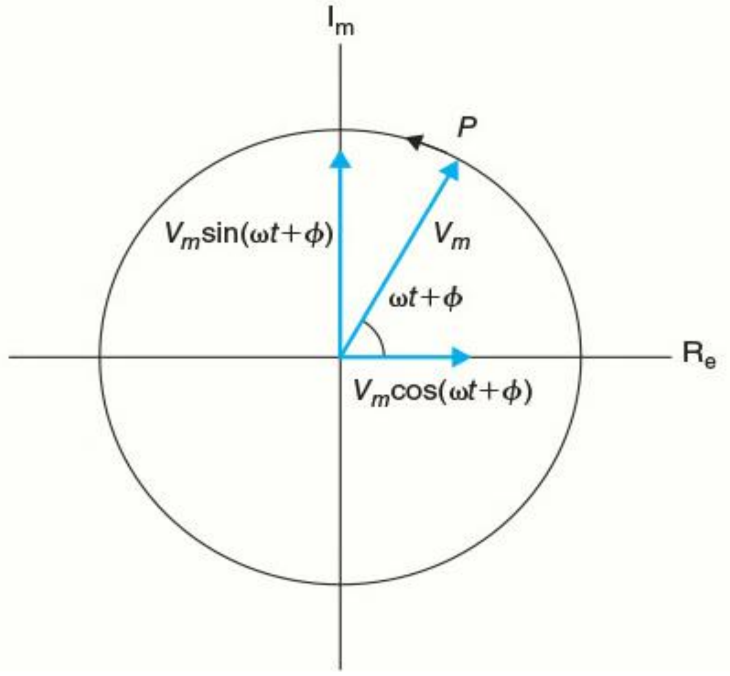


FIGURE 9.14

Rotation of a point P around a circle of radius V_m.

Phasors (Continued)

- **Phasor is the starting point of rotation of the cosine wave around the circle at t = 0.**

  $\boldsymbol{V} = V_m e^{j\phi} = V_m \angle\phi = V_m \cos(\phi) + j\, V_m \sin(\phi)$

  $Re[\boldsymbol{V}] = V_m \cos(\phi)$

  $Im[\boldsymbol{V}] = V_m \sin(\phi)$

- The phasor provides two of the three parameters, *magnitude (V_m)* and *phase (ϕ)* of a sinusoid.

- The third parameter, *angular velocity (ω rad/s)*, determines the speed at which point *P* revolves around the circle.

- For the given value of ω, phasor provides complete information of a sinusoid.

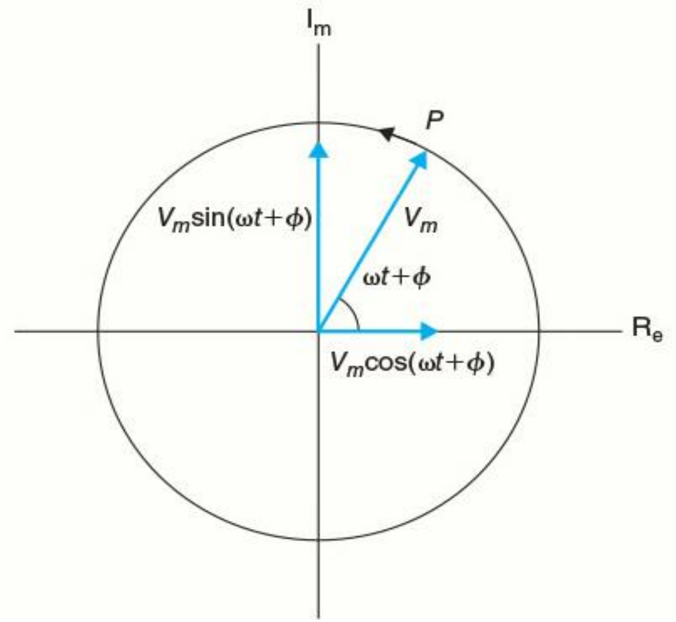
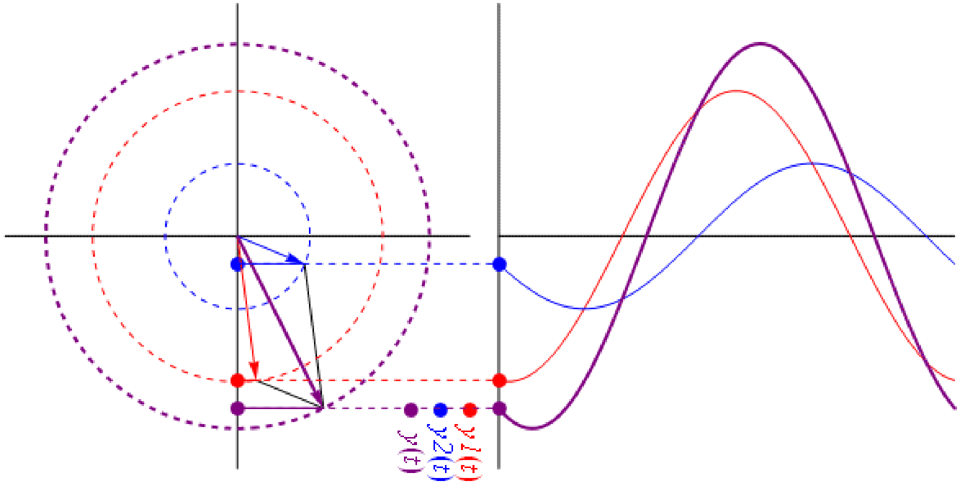


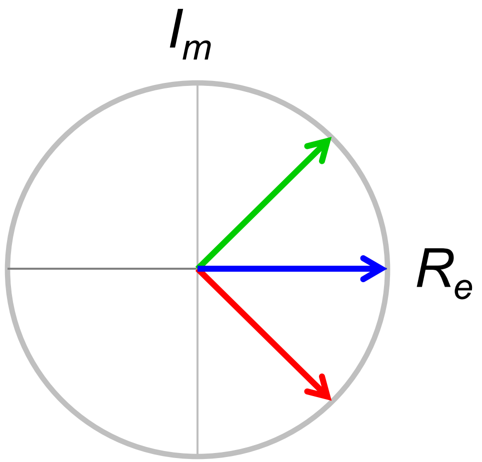
FIGURE 9.14

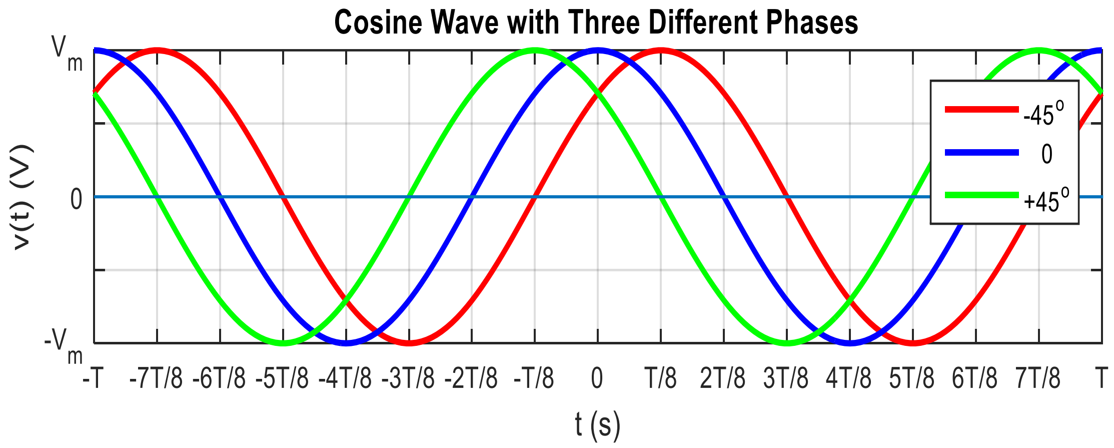
Rotation of a point *P* around a circle of radius V_m.

Phasors (Continued)



- The three phasors $V_m\angle -45^o$, $V_m\angle 0^o$, $V_m\angle 45^o$ and the corresponding cosines in the time domain are shown below.





Phasors (Continued)

❑ Getting correct phasors or Standard Values:

1. The **magnitude is nonnegative**. If the amplitude of a sinusoid is given as a negative number, the negative number is converted to a positive magnitude, and a phase of 180^o or -180^o is added.

$$-\cos(\omega t + \phi) = \cos(\omega t + \phi + 180^o) = \cos(\omega t + \phi - 180^o)$$

2. The **reference for sinusoids is cosine**. If the sinusoid is given as a sine, it should be changed to cosine by subtracting 90^o.

$$-\sin(\omega t + \phi) = \cos(\omega t + \phi - 90^o + 180^o) = \cos(\omega t + \phi + 90^o)$$

EXAMPLE 9.4

- Find the phasors of the following signals, and draw phasors for (a), (b), and (f).

  a. $v(t) = -110\cos(2\pi 60t + 210^o)$ V b. $v(t) = -110\cos(2\pi 60t - 60^o)$ V c. $v(t) = 220\sin(2\pi 50t + 30^o)$ V

  d. $v(t) = -220\sin(2\pi 50t - 120^o)$ V e. $i(t) = 15\sin(2\pi 60t - 60^o)$ A f. $i(t) = -20\sin(2\pi 60t + 120^o)$ A

- The answers we have:

  a. $V = 110\angle 30^o$ V b. $V = 110\angle 120^o$ V c. $\mathbf{V} = 220\angle -60^o$ V

  b. d. $V = 220\angle -30^o$ V e. $\mathbf{I} = 15\angle -150^o$ A f. $\mathbf{I} = 20\angle -150^o$ A

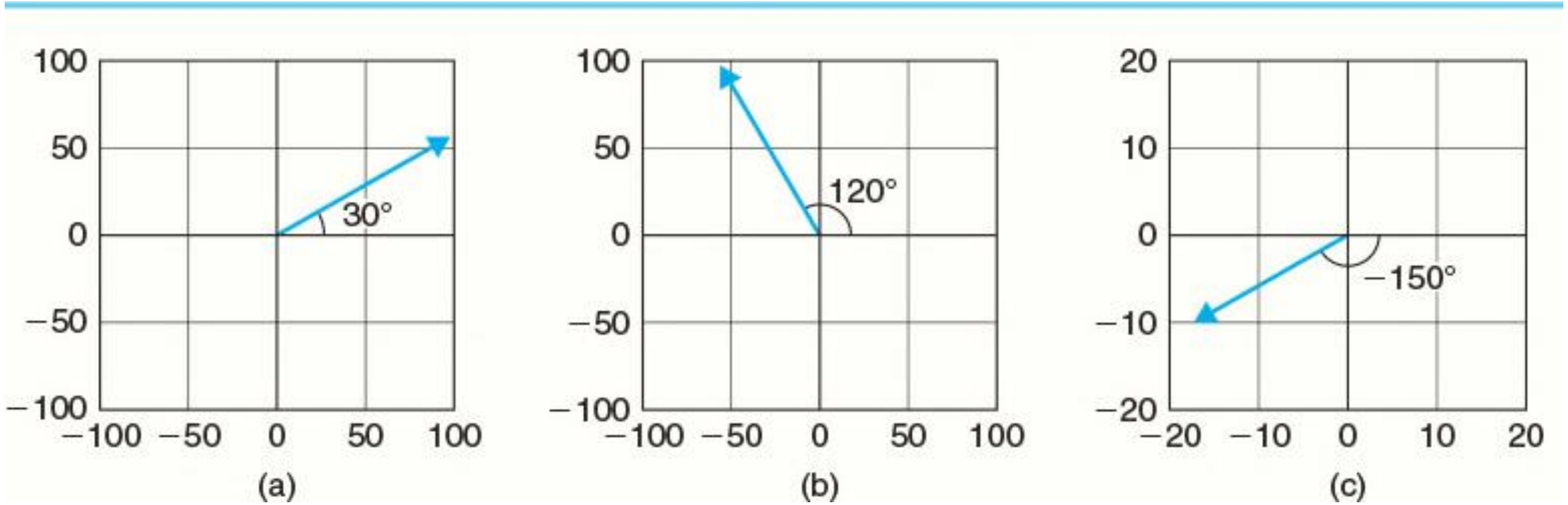


FIGURE 9.16

Phasor diagram for
(a) **V** = 110∠30°
(b) **V** = 110∠120°)
(c) **I** = 20∠−150°.

EXAMPLE 9.5

- Find the waveform for the following phasors when the frequency is *60 Hz*.
  - a. $\boldsymbol{V} = 110\angle 120^o$ V
  - b. $\boldsymbol{V} = 120\angle -30^o$ V

a. $v(t) = 110 \cos(2\pi 60t + 120^o)$ V

b. $v(t) = 120 \cos(2\pi 60t - 30^o)$ V

Conversion between Cartesian and Polar Coordinate Systems

- **Cartesian to Polar:**
  - 1st quadrant: $a > 0,\ b > 0,\ z = a + jb = re^{j\phi} = r\angle\phi$
    $\phi = \tan^{-1}(b/a)$, $0^o < \phi < 90^o$,
    $$r = \sqrt{a^2 + b^2}$$
  - 2nd quadrant: $a > 0,\ b > 0,\ z = -a + jb = re^{j\phi} = r\angle\phi$,
    $\phi = 180^o - \tan^{-1}(b/a)$, $90^o < \phi < 180^o$,
    $$r = \sqrt{(-a)^2 + b^2}$$
  - 3rd quadrant: $a > 0,\ b > 0,\ z = -a - jb = re^{j\phi} = r\angle\phi$,
    $\phi = -180^o + \tan^{-1}(b/a)$, $-180^o < \phi < -90^o$,
    $$r = \sqrt{(-a)^2 + (-b)^2}$$
  - 4th quadrant: $a > 0,\ b > 0,\ z = a - jb = re^{j\phi} = r\angle\phi$,
    $\phi = -\tan^{-1}(b/a)$, $-90^o < \phi < 0^o$,
    $$r = \sqrt{a^2 + (-b)^2}$$
  - $z = a = a\angle 0^o$, $z = -a = a\angle 180^o$, $z = jb = b\angle 90^o$, $z = -jb = b\angle -90^o$
- **Polar to Cartesian:** $z = re^{j\phi} = r\angle\phi = a + jb$
  $a = r\cos(\phi)$, $b = r\sin(\phi)$
- **Degrees to radians:** $\phi_r = \phi_d \times \pi/180$, **Radians to degrees:** $\phi_d = \phi_r \times 180/\pi$

EXAMPLE 9.6

- Convert the following numbers in Cartesian coordinates to Polar coordinates:

a. $z = 3 + j4$ b. $z = -4 + j3$ c. $z = -3 - j4$ d. $z = 4 - j3$

a. $z = 3 + j4 = \sqrt{3^2 + 4^2} \angle \tan^{-1}(4/3) = 5\angle 53.1301^o$

b. $z = -4 + j3 = \sqrt{(-3)^2 + (-4)^2} \angle 180^o + \tan^{-1}(3/(-4)) = 5\angle 143.1301^o$

c. $z = -3 - j4 = \sqrt{(-4)^2 + 3^2} \angle -180^o + \tan^{-1}((-4)/(-3)) = 5\angle -126.8699^o$

d. $z = 4 - j3 = \sqrt{4^2 + (-3)^2} \angle \tan^{-1}((-3)/4) = 5\angle -36.8699^o$

EXAMPLE 9.7

- Convert the following Polar to Cartesian coordinates.
  - **a.** $\boldsymbol{V} = 110\angle 120^o$ V b. $\boldsymbol{V} = 240\angle -120^o$ V
  - b. c. $\boldsymbol{V} = 480\angle 150^o$ V d. $\boldsymbol{V} = 880\angle -60^o$ V

a. $\boldsymbol{V} = 110\angle 120^o = 110\cos(120^o) + j110\sin(120^o) = 110\times(-0.5) + j110\times\sqrt{3}/2 = -55 + j95.2626$ V

b. $\boldsymbol{V} = 240\angle -120^o = 240\cos(-120^o) + j240\sin(-120^o) = 240\times(-0.5) - j240\times\sqrt{3}/2 = -120 - j207.8452$ V

c. $\boldsymbol{V} = 480\angle 150^o = 480\cos(150^o) + j480\sin(150^o) = 480\times(-\sqrt{3}/2) + j480\times 0.5 = -415.6922 + j240$ V

d. $\boldsymbol{V} = 880\angle -60^o = 880\cos(-60^o) + j880\sin(-60^o) = 880\times 0.5 - j880\times\sqrt{3}/2 = 440 - j762.1024$ V

Phasor Arithmetic

- **Addition:** The phasors in polar coordinates are converted to Cartesian coordinates before being added.

  $A = 5\angle 60^o = 2.5 + j4.3301$, $B = 10\angle -45^o = 7.0711 - j7.0711$

  $C = A + B = 9.5711 - j2.7409 = 9.9558\angle -15.9804^o$

- **Subtraction:** The phasors in polar coordinates are converted to Cartesian coordinates before being subtracted.

  $D = A - B = -4.5711 + j11.4012 = 12.2834\angle 111.8473^o$

- **Multiplication:** The magnitude of the product of two phasors in polar coordinates is the product of two magnitudes, and the phase of the product is the sum of the phases.

  $E = AB = (5\angle 60^o)(10\angle -45^o) = 50\angle 15^o = 48.2063 + j12.9410$

- **Division:** The magnitude of the division of two phasors in polar coordinates is the division of two magnitudes, and the phase of the division is the difference of the phases.

  $F = A/B = (5\angle 60^o)/(10\angle -45^o) = 0.5\angle 105^o = -0.1294 + j0.4830$

Sum of Sinusoids

- Two or more sinusoids with same frequency can be added using phasors.
- Let

$$v_1(t) = 10\cos(2\pi 100t + 30^o)\ V \Rightarrow \mathbf{V_1} = 10\angle 30^o\ V$$

$$v_2(t) = -5\sin(2\pi 100t - 45^o)\ V \Rightarrow \mathbf{V_2} = 5\angle 45^o\ V$$

- Then, adding phasors, we obtain

$$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} = 10\angle 30^o + 5\angle 45^o = 12.1958 + j8.5355 = 14.8860\angle 34.9872^o\ V$$

- The sum of sinusoids is given by

$$v(t) = v_1(t) + v_2(t) = 14.8860\cos(2\pi 100t + 34.9872^o)\ V$$

Impedance and Admittance

- Impedance and admittance are defined for the resistor, inductor, and capacitor, and are used to analyze *ac* circuits.
- If the input voltage to a resistor, a capacitor, an inductor, or combination of these is a sinusoid, the current through it is also a sinusoid of same frequency.
  - The voltage can be transformed into voltage phasor ***V***, and the current can be transformed into current phasor ***I***.
  - The ratio of ***V*** to ***I*** is defined as impedance ***Z*** of the component, given by

$$\mathbf{Z} = \mathbf{V}/\mathbf{I}$$

  - The SI unit for impedance is *ohm* (Ω).
  - The impedance is similar to resistance, but it is a function of frequency, and in general, is a complex quantity representing both the magnitude and phase. So,

$$\mathbf{Z} = R + jX$$

  - The real part of ***Z***, *R*, is the resistance and the imaginary part of ***Z***, *X*, is the reactance. The SI unit for both *R* and *X* is *ohm* (Ω).

Impedance and Admittance (Continued)

- The equation $\boldsymbol{Z} = \boldsymbol{V}/\boldsymbol{I}$ is the Ohm's law for phasors.

- The ratio of current $\boldsymbol{I}$ to voltage $\boldsymbol{V}$ is defined as the admittance and is denoted as $\boldsymbol{Y}$.

$$\boldsymbol{Y} = \boldsymbol{I}/\boldsymbol{V} = 1/\boldsymbol{Z}$$

- The SI unit for $\boldsymbol{Y}$ is *siemens (S)*.

- The admittance is the inverse of the impedance. In general, it is a complex quantity. So,

$$\boldsymbol{Y} = G + jB$$

- The real part of the admittance is defined as the conductance, G, and the imaginary part of admittance is defined as the susceptance, B. The SI unit for both G and B is *siemens (S)*.

Impedance and Admittance of a Resistor

❑ Find phasor values of **I** and **V** the impendence of a resistor **R** for the circuit of Fig. 9.19 to get Fig. 9.20.

- Ohm's law: $v(t) = R\, i(t)$
- $v(t) = V_m \cos(\omega t + \phi) = Re[V_m e^{j\phi} e^{j\omega t}] = Re[\mathbf{V} e^{j\omega t}]$, $\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$
- $i(t) = v(t)/R = Re[(V_m e^{j\phi}/R) e^{j\omega t}] = Re[\mathbf{I} e^{j\omega t}]$, $\mathbf{I} = (V_m e j^{\phi})/R = \mathbf{V}/R$
- $\boxed{\mathbf{Z} = \mathbf{V}/\mathbf{I} = R \text{ and } \mathbf{Y} = 1/\mathbf{Z} = 1/R = G}$
- $\mathbf{V}$ and $\mathbf{I}$ have same phase ϕ as shown in Figure 9.21 and Figure 9.22.
- And $X = 0$, $B = 0$

FIGURE 9.19

Voltage across and current through a resistor.

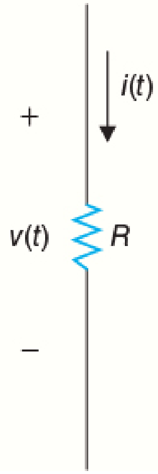


FIGURE 9.20

The impedance of a resistor.

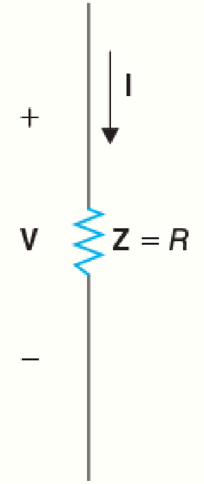


FIGURE 9.21

Phasors **V** and **I** for the resistor.

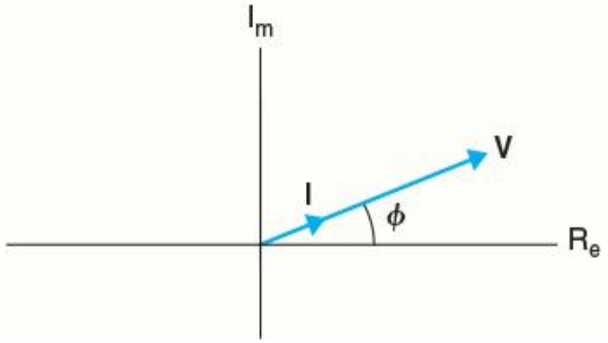
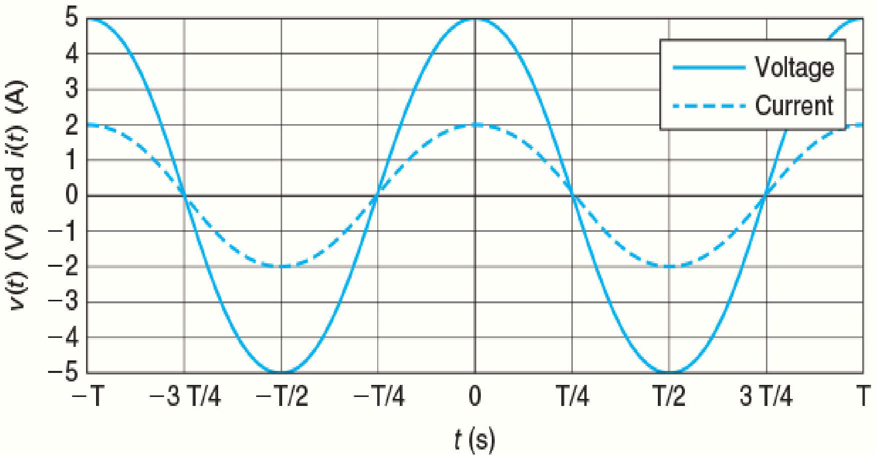


FIGURE 9.22

Waveform for voltage across the resistor and current through the resistor.



Impedance and Admittance of a Capacitor

❑ Find phasor values of **I** and **V** the impendence of a capacitor **C** for the circuit of Fig. 9.23 to get Fig. 9.24.

- $i(t) = C\,dv(t)/dt$
- $v(t) = V_m \cos(\omega t + \phi) = Re[V_m e^{j\phi} e^{j\omega t}] = Re[\mathbf{V} e^{j\omega t}]$, $\mathbf{V} = V_m e^{j\phi} = V_m\angle\phi$
- $i(t) = C\,dv(t)/dt = Re[(j\omega C V_m e^{j\phi}) e^{j\omega t}] = Re[\mathbf{I} e^{j\omega t}]$, $\mathbf{I} = j\omega C V_m e^{j\phi} = (j\omega C)\mathbf{V}$
- $\boxed{\mathbf{Z} = \mathbf{V}/\mathbf{I} = 1/(j\omega C) = -j/(\omega C) = 1/(\omega C)\angle -90^o}$ and $\mathbf{Y} = j\omega C = \omega C\angle 90^o$
- $\mathbf{I}$ leads $\mathbf{V}$ by 90^o as shown in Figure 9.25 and Figure 9.26.
- $R = 0,\ X = -1/(\omega C),\ G = 0,\ B = \omega C$

FIGURE 9.23

Voltage across and current through a capacitor.

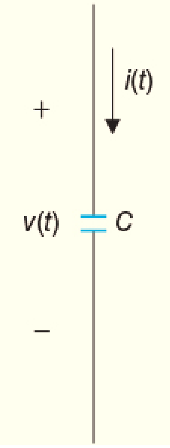


FIGURE 9.24

The impedance of a capacitor.

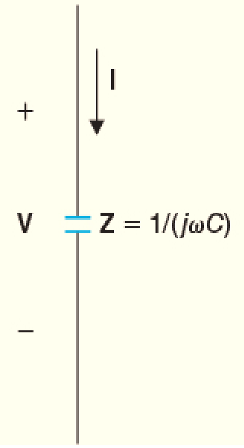


FIGURE 9.25

Phasors V and I for the capacitor.

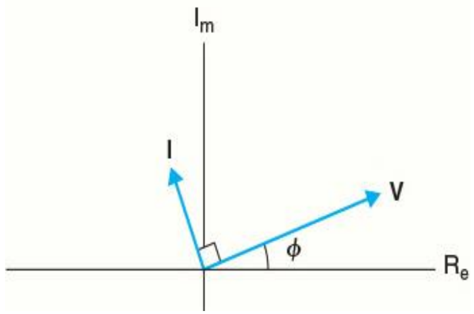
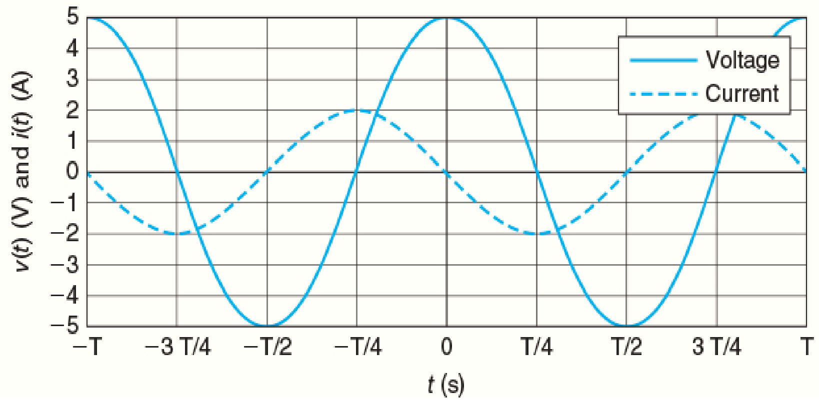


FIGURE 9.26

Waveform for voltage across the capacitor and current through the capacitor.



Impedance and Admittance of an Inductor

- Find phasor values of **I** and **V** the impendence of an inductor **L** for the circuit of Fig. 9.27 to get Fig. 9.28.

$$i(t)=\frac{1}{L}\int_{-\infty}^{t} v(\lambda)d\lambda=\frac{1}{L}\int_{-\infty}^{t}\mathrm{Re}\left[V_m e^{j\phi}e^{j\omega\lambda}\right]d\lambda=\mathrm{Re}\left[V_m e^{j\phi}\frac{1}{j\omega L}e^{j\omega t}\right]=\mathrm{Re}\left[\mathbf{I}e^{j\omega t}\right]$$

- $v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}[V_m e^{j\phi} e^{j\omega t}] = \mathrm{Re}[\mathbf{V} e^{j\omega t}]$, $\mathbf{V} = V_m e^{j\phi} = V_m\angle\phi$
- $\mathbf{I} = V_m e^{j\phi}/(j\omega L) = \mathbf{V}/(j\omega L)$
- $\boxed{Z = V/I = j\omega L = \omega L\angle 90^o}$
- $Y = 1/(j\omega L) = -j/(\omega L) = 1/(\omega L)\angle -90^o$
- $\mathbf{I}$ lags $\mathbf{V}$ by 90^o as shown in Figure 9.29 and Figure 9.30.
- $R = 0,\ X = \omega L,\ G = 0,\ B = -1/(\omega L)$

FIGURE 9.27

Voltage across and current through an inductor.

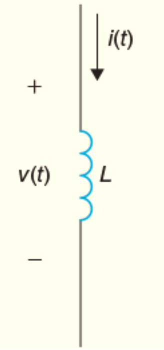
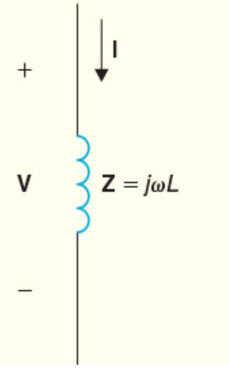


FIGURE 9.28

The impedance of an inductor.



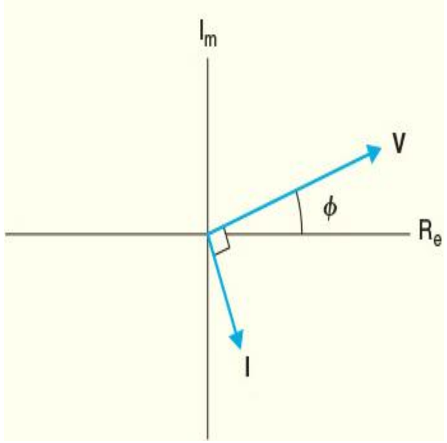


FIGURE 9.29

Phasors **V** and **I** for inductor.

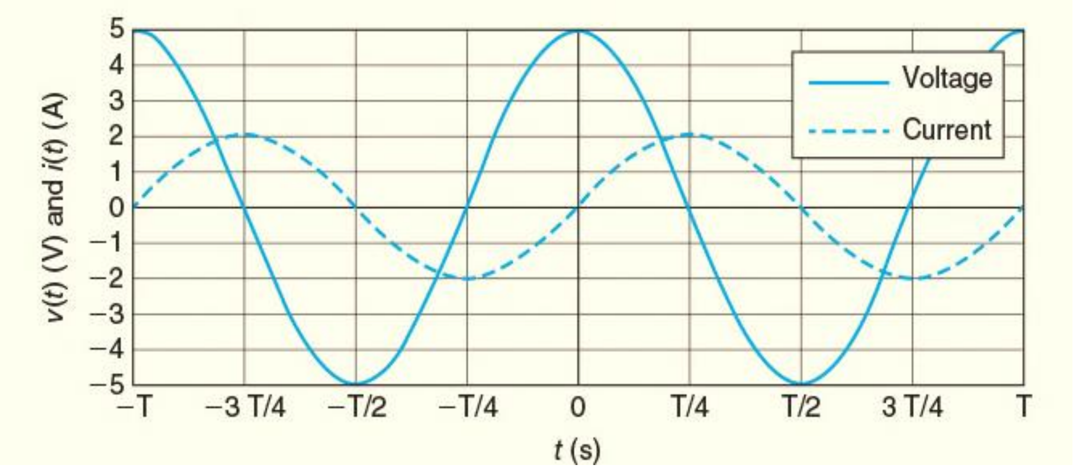


FIGURE 9.30

Waveform for voltage across the inductor and current through the inductor.

Phasor-Transformed Circuit

- If a circuit is driven by a sinusoid (ac), the steady-state response of the circuit can be found by transforming the circuit to the **phasor domain** and then applying the circuit laws and theorems.
- Circuit elements can be transformed into the impedances, voltages to voltage phasors and currents to current phasors
- Circuits consisting of voltage phasors, current phasors, and impedances are called **phasor-transformed circuits.**
- All of the circuit laws and theorems for the resistive circuits can be applied to phasor-transformed circuits, including Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL), the voltage divider rule, the current divider rule, nodal analysis, mesh analysis, source transformation, Thévenin's theorem, and Norton's theorem.
- The unknown voltages and currents in the phasor-transformed circuit can be found by applying these circuit laws and theorems.

EXAMPLE 9.9

❑ Draw the phasor-transformed circuit for the circuit shown in Figure 9.31. The AC voltage source is given by $v_s(t) = 150\cos(2\pi 60t + 60^o)$ V.

- $\boldsymbol{V_s} = 150\angle 60^o$ V
- $\boldsymbol{Z_{R1}} = R_1 = 55\ \Omega$
- $\boldsymbol{Z_{R2}} = R_2 = 105\ \Omega$
- $\boldsymbol{\omega = 2\pi 60 = 376.9911}$ **rad/s**
- $\boldsymbol{Z_{L1}} = j\omega L_1 = j2\pi 60\times 65\times 10^{-3} = j24.5044\ \Omega$
- $\boldsymbol{Z_{L2}} = j\omega L_2 = j2\pi 60\times 210\times 10^{-3} = j79.1681\ \Omega$
- $\boldsymbol{Z_{C1}} = 1/(j\omega C_1) = -j75.7881\ \Omega$
- $\boldsymbol{Z_{C2}} = 1/(j\omega C_2) = -j106.1033\ \Omega$

❑ The phasor-transformed circuit is shown in Figure 9.32.

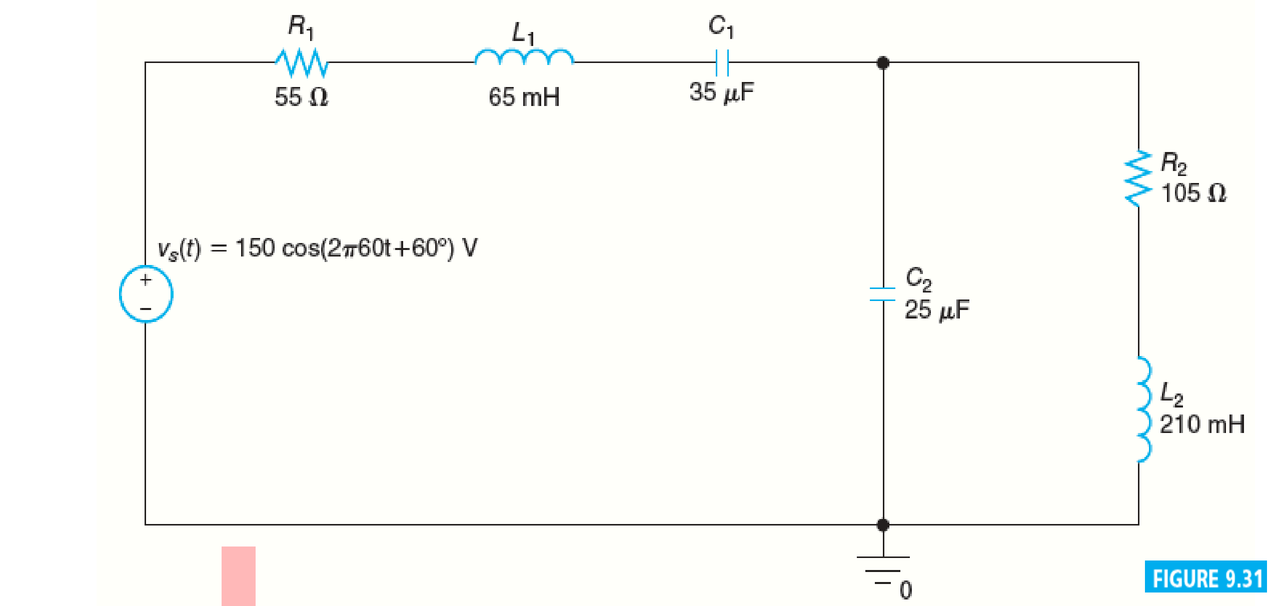


FIGURE 9.31

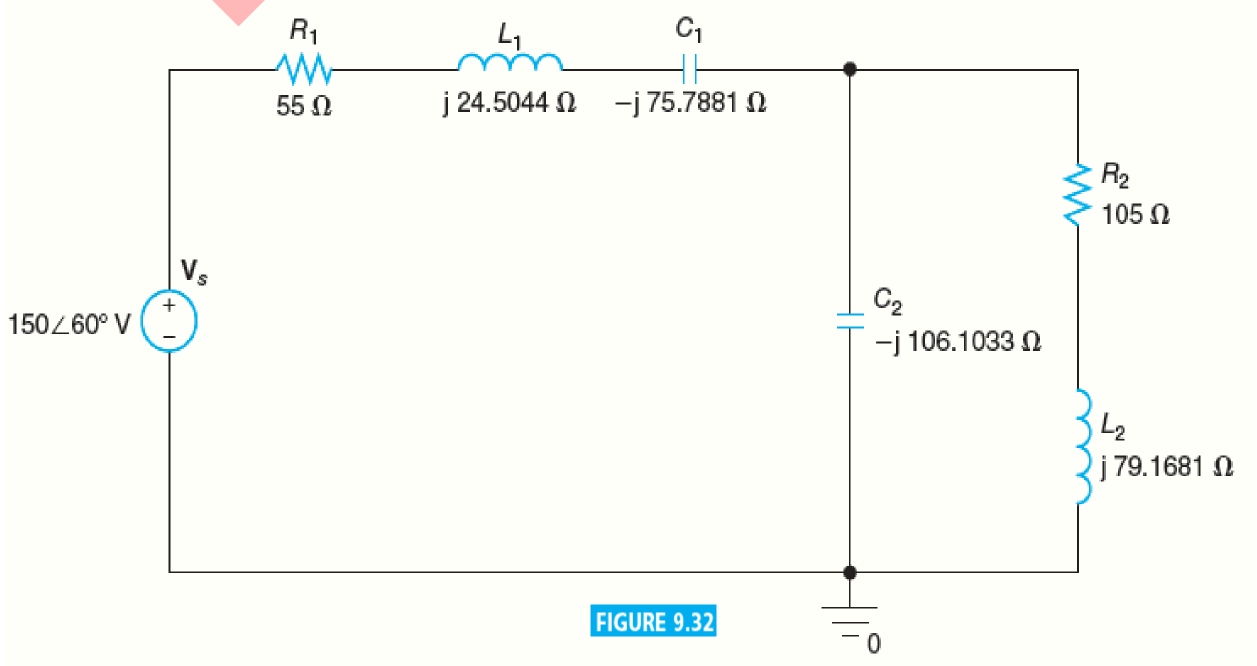


FIGURE 9.32

EXAMPLE 9.10

- Draw the phasor-transformed circuit for the circuit shown in Figure 9.35, and find $\mathbf{I}$, $\mathbf{V_w}$, $\mathbf{V_o}$, $i(t)$, $v_w(t)$, $v_o(t)$.
  - $v_s(t) = 200\cos(2\pi 60t)$ V
  - $\mathbf{V_s} = 200\angle 0^o$ V
  - $\mathbf{Z_{R1}} = R_1 = 50\ \Omega$, $\mathbf{Z_{R2}} = R_2 = 100\ \Omega$
  - $\omega = 2\pi 60 = 376.9911$ rad/s
  - $\mathbf{Z_{L1}} = j\omega L_1 = j2\pi 60\times 30\times 10^{-3} = j11.3097\ \Omega$
  - $\mathbf{Z_{L2}} = j\omega L_2 = j2\pi 60\times 350\times 10^{-3} = j131.9469\ \Omega$
- The phasor-transformed circuit is shown in Figure 9.36.
  - $\mathbf{Z} = \mathbf{Z_{R1}} + \mathbf{Z_{L1}} + \mathbf{Z_{R2}} + \mathbf{Z_{L2}} = 150 + j143.2566$
    $= 207.4186\angle 43.6827^o\ \Omega$
  - $\mathbf{I} = \mathbf{V_s}/\mathbf{Z} = 0.9642\angle -43.6827^o$ A
  - $i(t) = 0.9642\cos(2\pi 60t - 43.6827^o)$ A
  - $\mathbf{V_w} = \mathbf{I}\times(\mathbf{Z_{R1}} + \mathbf{Z_{L1}}) = 49.4297\angle -30.9372^o$ V
  - $\mathbf{V_o} = \mathbf{I}\times(\mathbf{Z_{R2}} + \mathbf{Z_{L2}}) = 159.6382\angle 9.1595^o$ V
  - $v_w(t) = 49.4297\cos(2\pi 60t - 30.9372^o)$ V
  - $v_o(t) = 159.6382\cos(2\pi 60t + 9.1595^o)$ V

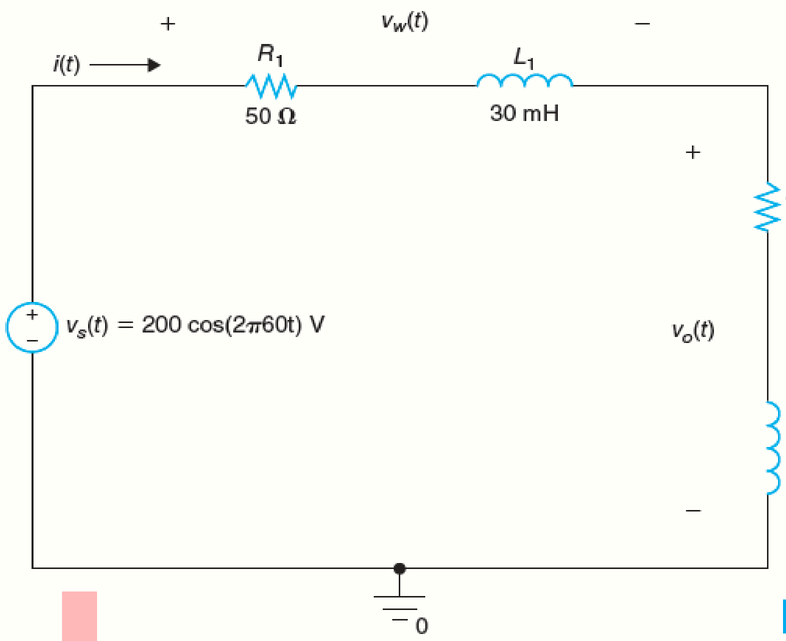


FIGURE 9.35

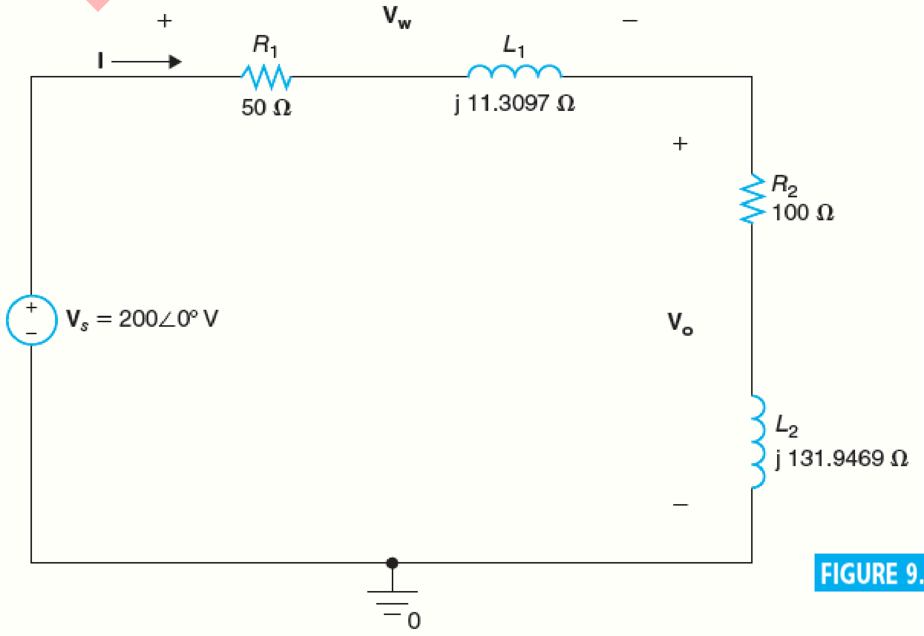


FIGURE 9.36

EXAMPLE 9.11

- Find the currents $\boldsymbol{I_1}$, $\boldsymbol{I_2}$, and $\boldsymbol{I}$ in the circuit shown in Figure 9.39.

- $\boldsymbol{Z_1} = \boldsymbol{Z_{R1}} + \boldsymbol{Z_L} = 125 + j180 = 219.1461\angle 55.2222^o\ \Omega$
- $\boldsymbol{Z_2} = \boldsymbol{Z_{R2}} + \boldsymbol{Z_C} = 200 - j260 = 328.0244\angle - 52.4314^o\ \Omega$
- $\boldsymbol{I_1} = \boldsymbol{V_s}/\boldsymbol{Z_1} = 2.5097\angle 64.7778^o = 1.0695 + j2.2705\ A$
- $\boldsymbol{I_2} = \boldsymbol{V_s}/\boldsymbol{Z_2} = 1.6767\angle 172.4314^o = -1.6621 + j0.2208\ A$
- From KCL:

  $\boldsymbol{I} = \boldsymbol{I_1} + \boldsymbol{I_2} = -0.5926 + j2.4913\ A = 2.5608\angle 103.3806^o$

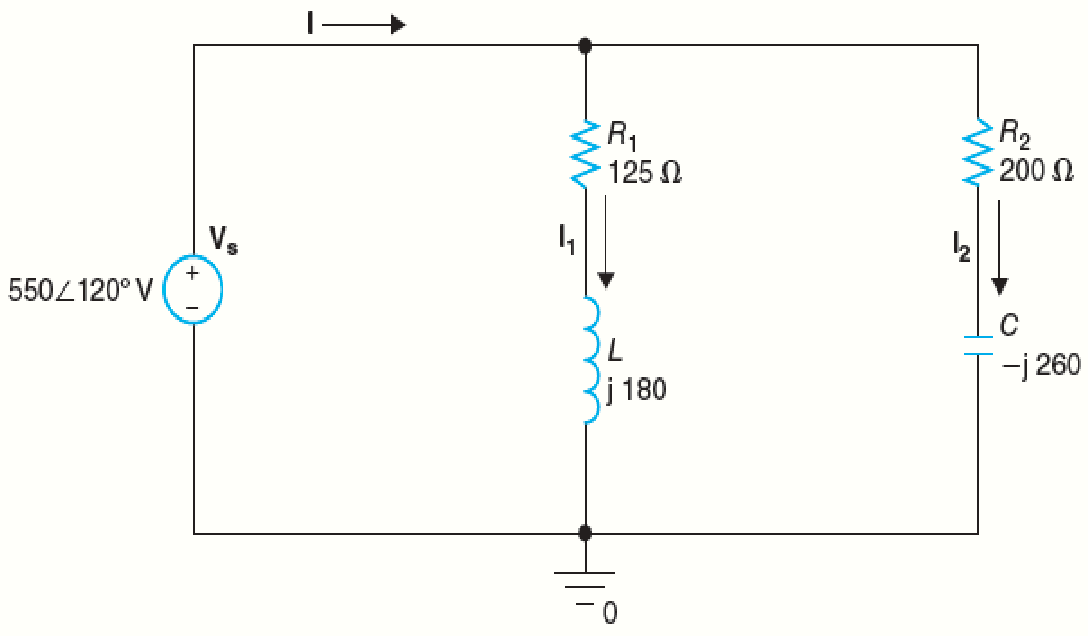


FIGURE 9.39

EXAMPLE 9.12

- Find ***I***, $\boldsymbol{V_1}$, $\boldsymbol{V_2}$, $\boldsymbol{V_3}$, $\boldsymbol{V_4}$, and $\boldsymbol{V_5}$ in the circuit shown in Figure 9.41.

- From KVL:

  $-50\angle{-120^o} + 5\boldsymbol{I} + j6\boldsymbol{I} + (-j3\boldsymbol{I}) + 10\boldsymbol{I} + j5\boldsymbol{I} = 0$

So,

- $\boldsymbol{I} = (50\angle{-120^o})/(5 + j6 - j3 + 10 + j5) = 2.9412\angle{-148.0725^o} = -2.4962 - j1.5554$ A
- $\boldsymbol{V_1} = 5\times\boldsymbol{I} = 14.7059\angle{-148.0725^o}$ V
- $\boldsymbol{V_2} = j6\times\boldsymbol{I} = 17.6471\angle{-58.0725^o}$ V
- $\boldsymbol{V_3} = (-j3)\times\boldsymbol{I} = 8.8235\angle{121.9275^o}$ V
- $\boldsymbol{V_4} = 10\times\boldsymbol{I} = 29.4118\angle{-148.0725^o}$ V
- $\boldsymbol{V_5} = (j5)\times\boldsymbol{I} = 14.7059\angle{-58.0725^o}$ V

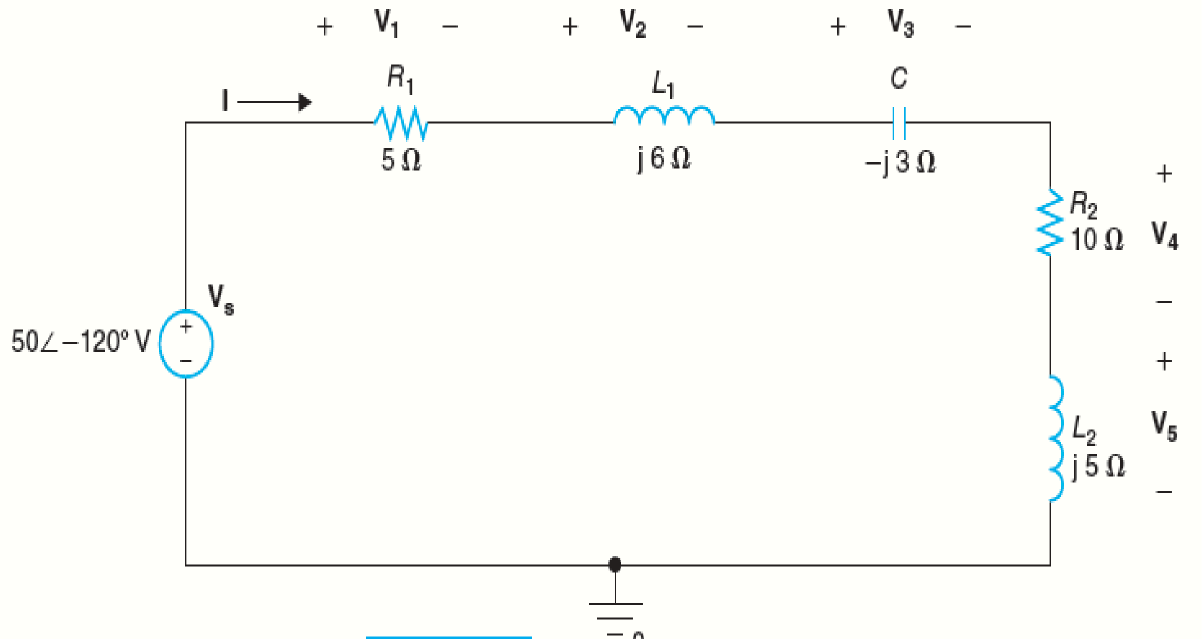


FIGURE 9.41

Series Connection of Impedances

- If n impedances are connected in series as shown in Figure 9.43(a), the current through all the impedances is **I**. Thus, according to KVL, the voltage across all the impedances is given by

$$V = V_1 + V_2 + \ldots + V_n = Z_1 I + Z_2 I + \ldots + Z_n I = (Z_1 + Z_2 + \ldots + Z_n)I = Z_{eq} I$$

$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_n$$

- The impedance Z_{eq} is the equivalent impedance of the n impedances connected in series.

- Figure 9.43(b) shows the equivalent circuit with single impedance Z_{eq}.

- For two impedances Z_1 and Z_2 in series,

$$Z_{eq} = Z_1 + Z_2$$

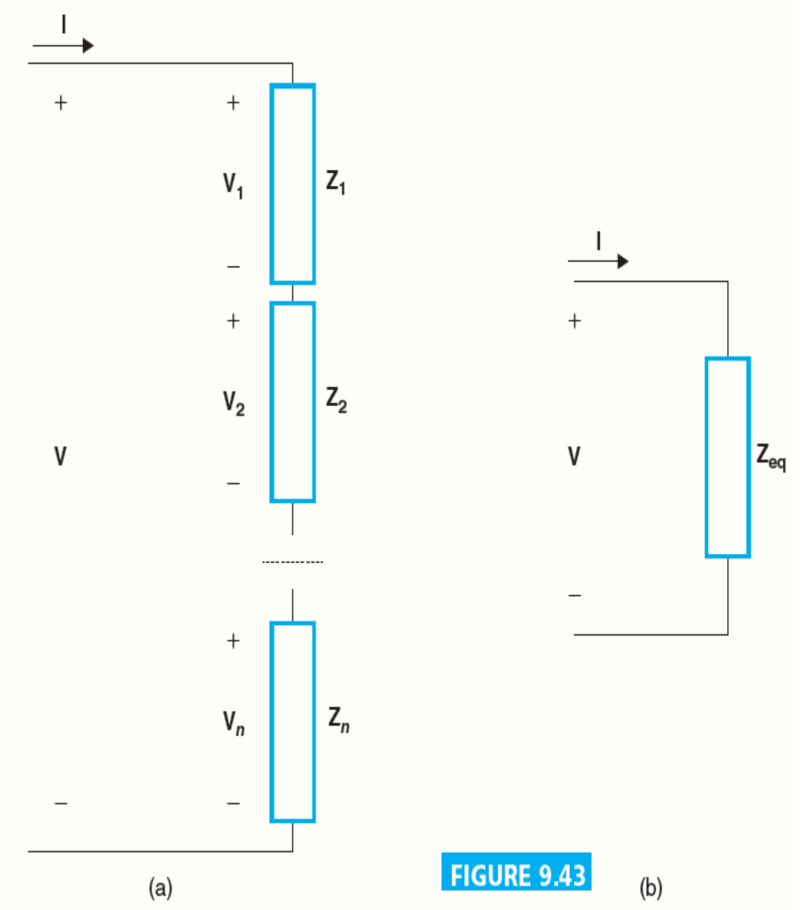


FIGURE 9.43

Parallel Connection of Impedances

- If n impedances are connected in parallel as shown in Figure 9.44(a), the voltage across all the impedances is $\boldsymbol{V}$. Thus, according to KCL, the current through all the impedances is given by

$$\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \ldots + \boldsymbol{I}_n = \boldsymbol{V}/\boldsymbol{Z}_1 + \boldsymbol{V}/\boldsymbol{Z}_2 + \ldots + \boldsymbol{V}/\boldsymbol{Z}_n = \boldsymbol{V}/(1/(1/\boldsymbol{Z}_1 + 1/\boldsymbol{Z}_2 + \ldots + 1/\boldsymbol{Z}_n)) = \boldsymbol{V}/\boldsymbol{Z}_{eq}$$

$$\boldsymbol{Z}_{eq} = 1/(1/\boldsymbol{Z}_1 + 1/\boldsymbol{Z}_2 + \ldots + 1/\boldsymbol{Z}_n)$$

- The impedance $\boldsymbol{Z}_{eq}$ is the equivalent impedance of the n impedances connected in parallel.
- Figure 9.44(b) shows the equivalent circuit with single impedance $\boldsymbol{Z}_{eq}$.
- For two impedances $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ in parallel,

$$\boldsymbol{Z}_{eq} = 1/(1/\boldsymbol{Z}_1 + 1/\boldsymbol{Z}_2) = \boldsymbol{Z}_1 \times \boldsymbol{Z}_2/(\boldsymbol{Z}_1 + \boldsymbol{Z}_2)$$

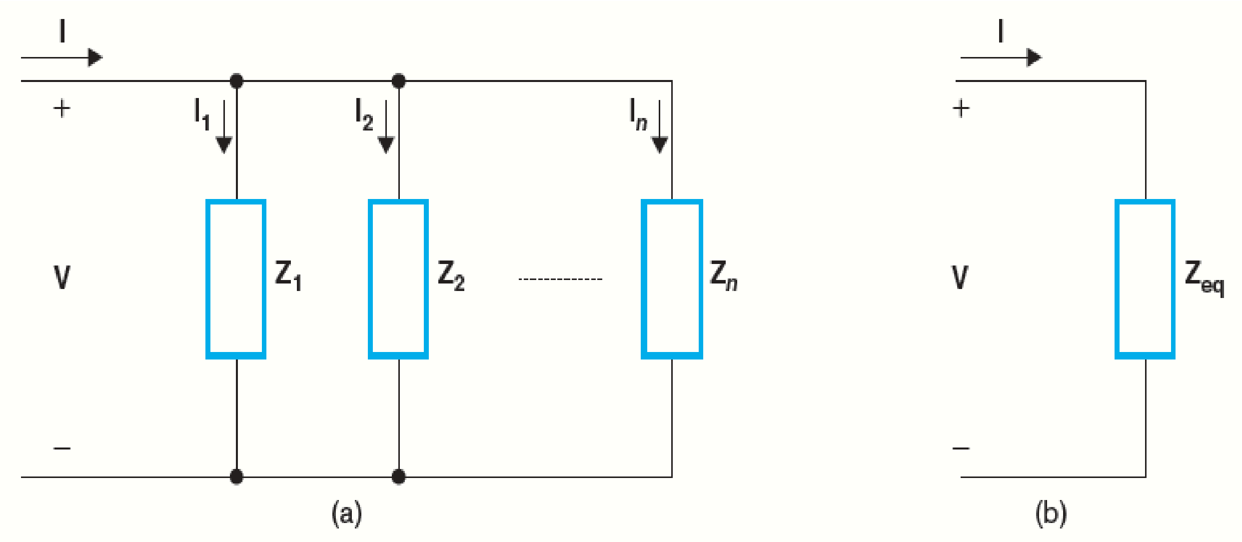


FIGURE 9.44

EXAMPLE 9.13

- Find $v_a(t)$ and $v_b(t)$ in the circuit shown in Figure 9.45.

- $\mathbf{Z}_{L1} = j\omega L_1 = j26.3894\ \Omega$
- $\mathbf{Z}_{L2} = j\omega L_2 = j37.7\ \Omega$
- $\mathbf{Z}_{C1} = 1/(j\omega C_1) = -j106.1033\ \Omega$
- $\mathbf{Z}_{C2} = 1/(j\omega C_2) = -j88.4194\ \Omega$
- $\mathbf{Z}_a = \mathbf{Z}_{C2} \,||\, (R_4 + \mathbf{Z}_{L2}) = 69.7059 - j44.2256\ \Omega$
- $\mathbf{Z}_b = (R_2 + \mathbf{Z}_{L1}) \,||\, (R_3 + \mathbf{Z}_a) = 74.1614 - j6.7508\ \Omega$
- $\mathbf{Z}_t = R_1 + \mathbf{Z}_{C1} + \mathbf{Z}_b = 104.1614 - j112.8540\ \Omega$
- $\mathbf{I} = \mathbf{V}_s/\mathbf{Z}_t = 0.8833 + j0.9579\ A$
- $\mathbf{V}_b = \mathbf{Z}_b \times \mathbf{I} = 96.9786\angle 42.0926^o\ V$
- $\mathbf{I}_{R3} = \mathbf{V}_b/(R_3 + \mathbf{Z}_a) = 0.3439 + j0.6185\ A$
- $\mathbf{V}_a = \mathbf{Z}_a \times \mathbf{I}_{R3} = 58.4199\angle 28.5269^o\ V$
- $v_a(t) = 58.4199 \cos(2\pi 60t + 28.5269^o)\ V$
- $v_b(t) = 96.9786 \cos(2\pi 60t + 42.0926^o)\ V$

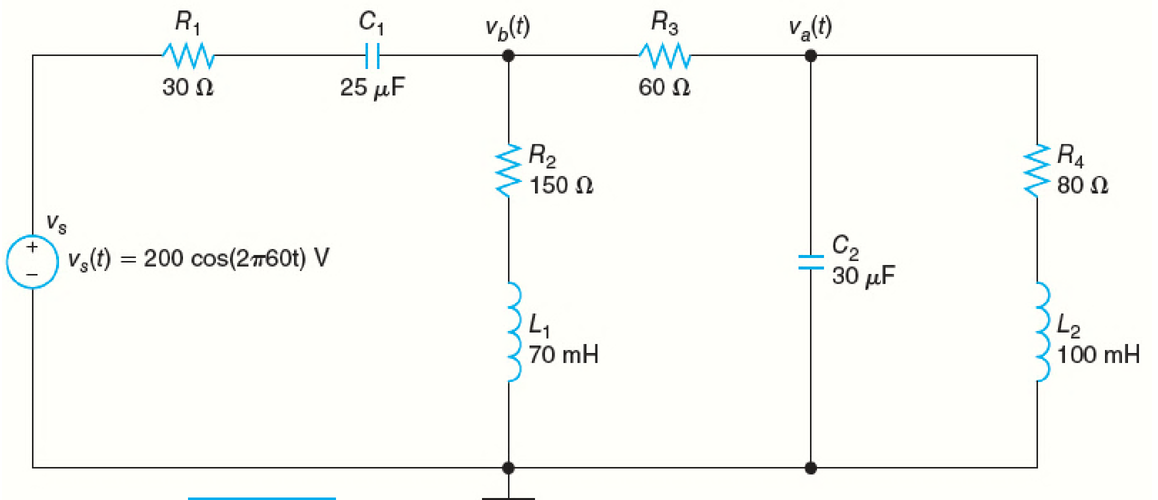


FIGURE 9.45

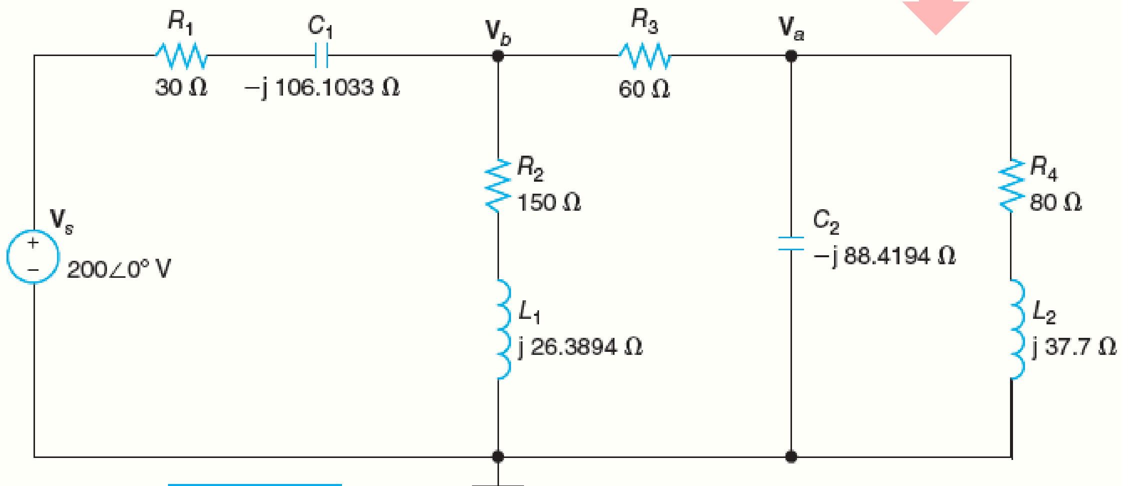


FIGURE 9.46

Summary

- Phasor represents the magnitude and phase of a sinusoid. Phasors are useful in finding voltages and currents in the steady state when the input signals are sinusoids.

- The ratio of the voltage phasor **V** to the current phasor **I** is defined as the impedance **Z** of circuit elements. The impedance is similar to resistance, but the impedance is a function of frequency, and is a complex quantity representing both the magnitude and phase of the sinusoid.

- The admittance **Y** is defined as the ratio of the current phasor **I** to the voltage phasor **V**.

- If voltage sources and current sources are sinusoids, these sources can be transformed to voltage phasors and current phasors, respectively. Circuit elements can be transformed to impedances. Circuits consisting of voltage phasors, current phasors, and impedances are called phasor transformed circuit.

- Circuit laws and theorems for resistive circuits can be applied to phasor transformed circuits.

- Voltage phasors and current phasors can be transformed to voltages and currents in the time domain.