Circuit Analysis and Design

Academic Year 2025/2026 – Semester 1

Lecture 9 - Circuit Theorems

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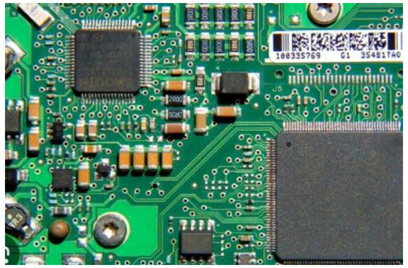
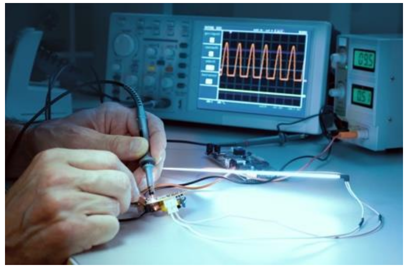




Agenda

- Norton’s theorem
- Maximum power transfer
- Summary

Norton's Theorem

- A circuit looking from terminals *a* and *b* can be replaced by a current source with current I_n and a parallel resistor with resistance R_n, as shown in Figure 4.93.
- This equivalent circuit consisting of a current source and a parallel resistor is called Norton equivalent circuit.
- The current I_n is called Norton equivalent current and the resistance R_n is called Norton equivalent resistance.
- When the terminals *a* and *b* are short-circuited in the Norton equivalent circuit, as shown in Figure 4.94, the short-circuit current I_{sc} is equal to I_n from the current divider rule.
- Thus, the Norton equivalent current can be obtained by finding the short-circuit current.

FIGURE 4.93

A Norton equivalent circuit.

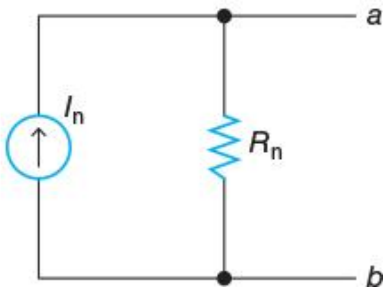
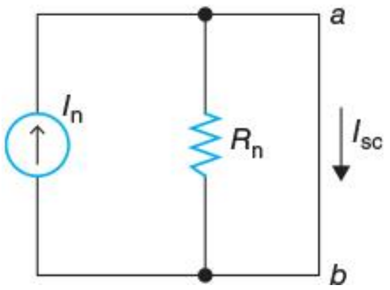


FIGURE 4.94

Short-circuit current.



Finding Norton Equivalent Resistance

Method 1:

- Deactivate all the independent sources by short-circuiting voltage sources and open-circuiting current sources.
- R_n is the equivalent resistance looking into the circuit from terminals *a* and *b.*
- This method can be used if the circuit does not contain dependent sources.

Method 2:

- Find the open-circuit voltage V_{oc} and the short-circuit current I_{sc}.
- The Norton equivalent resistance is given by $R_n = V_{oc}/I_{sc} = V_{oc}/I_n$.

Method 3:

- Deactivate all the independent sources by open-circuiting current sources and short-circuiting voltage sources.
- Apply a test voltage of 1 V (or any other value) between terminals *a* and *b* with terminal *a* connected to the positive terminal of the test voltage.
- Measure the current flowing out of the positive terminal of the test voltage source.
- The Norton equivalent resistance R_n is given by the ratio of the test voltage to the current flowing out of the positive terminal of the test voltage source.
- Alternatively, apply a test current between terminals *a* and *b* after deactivating the independent sources, and measure the voltage across *a* and *b* of the test current source. The Norton equivalent resistance R_n is the ratio of the voltage across *a* and *b* to the test current.

Thévenin and Norton Equivalent Circuits

- Application of source transformation to the Norton equivalent circuit shown in Figure 4.95(a) yields the Thévenin equivalent circuit shown in Figure 4.95(b).
- The Thévenin equivalent voltage is $V_{th} = I_n R_n$ and the Thévenin equivalent resistance is $R_{th} = R_n$.
- The source transformation does not change the resistance value.
- Application of source transformation to the Thévenin equivalent circuit shown in Figure 4.96(a) yields the Norton equivalent circuit, as shown in Figure 4.96(b).
- The Norton equivalent current is

  $I_n = V_{th}/R_{th}$ and the Norton

  equivalent resistance is $R_n = R_{th}$.

FIGURE 4.95

Transformation from Norton equivalent circuit to Thévenin equivalent circuit.

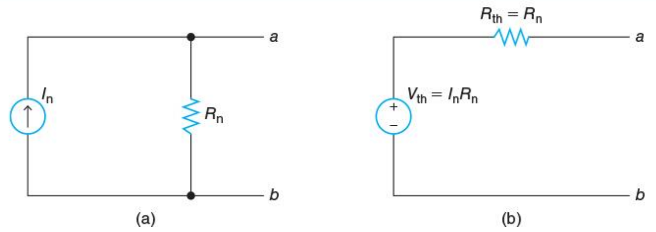
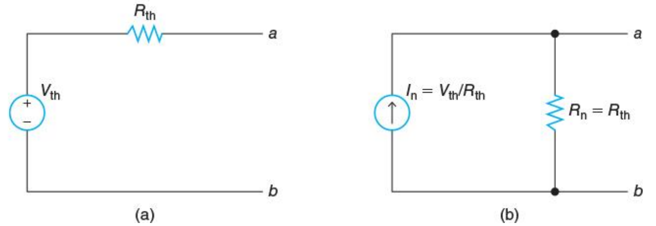


FIGURE 4.96

Transformation from Thévenin equivalent circuit to Norton equivalent circuit.



Finding I_n and R_n

- We are interested in finding I_n and R_n for the circuit shown in Figure
- To find I_n:
  - To find the short circuit current, we short-circuit *a* and *b* as shown in Figure 4.98. $V_3 = 0$. No current through R_5.
  - Sum the currents leaving node 1: $-0.002+\frac{V_1}{1500}+\frac{V_1-V_2}{1500}=0$
  - Multiply by 1500: $2V_1 - V_2 = 3$ (1)
  - Sum the currents leaving node 2: $\frac{V_2-V_1}{1500}+\frac{V_2-2.5}{1000}+\frac{V_2}{3000}=0$
  - Multiply by 3000: $2V_2 - 2V_1 + 3V_2 - 7.5 + V_2 = 0$
    $\Rightarrow -2V_1 + 6V_2 = 7.5$ (2)
  - Add (1) and (2): $5V_2 = 10.5 \Rightarrow V_2 = 2.1$ V,
    $V_1 = (V_2 + 3)/2 = 2.55$ V
  - $I_n = V_1/R_4 + V_2/R_3 = 1.7$ mA $+ 0.7$ mA $= 2.4$ mA

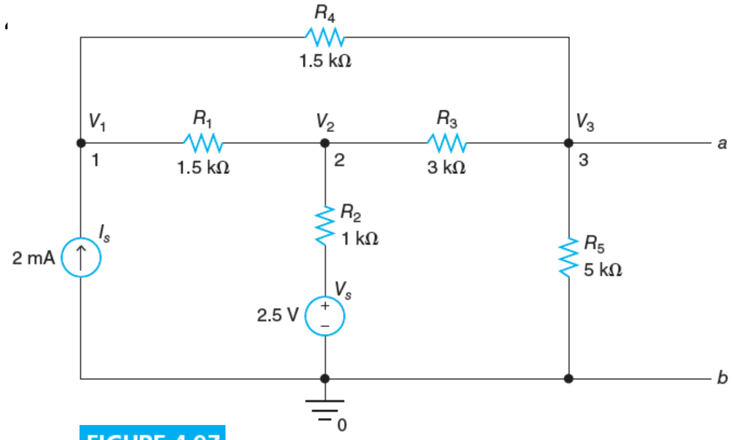


FIGURE 4.97

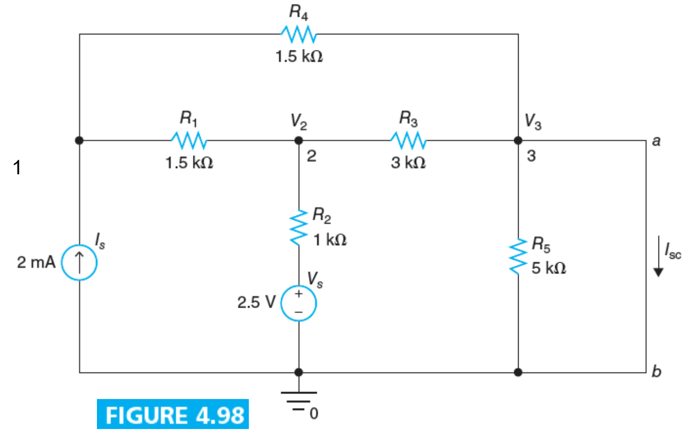


FIGURE 4.98

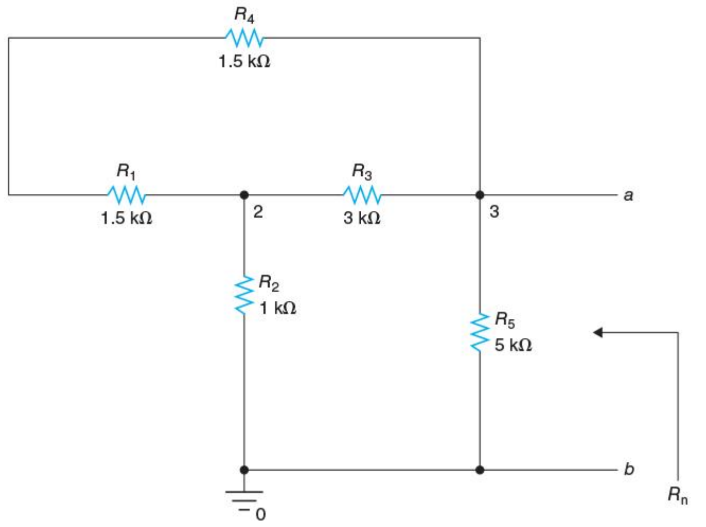
Finding I_n and R_n (Continued)

- To find R_n (**Method 1**):
  - Deactivate V_s and I_s as shown in Figure 4.101.
  - $R_a = R_1 + R_4 = 1.5\ k\Omega + 1.5\ k\Omega = 3\ k\Omega$
  - $R_b = R_3 \parallel R_a = 3 \times 3/(3 + 3)\ k\Omega = 9/6\ k\Omega = 1.5\ k\Omega$
  - $R_c = R_b + R_2 = 1.5\ k\Omega + 1\ k\Omega = 2.5\ k\Omega$
  - $R_n = R_5 \parallel R_c = 5 \times 2.5/(5 + 2.5)\ k\Omega$

    $R_n = (12.5/7.5)\ k\Omega = 1.6667\ k\Omega$

FIGURE 4.101

The circuit from Figure 4.97 with sources deactivated.



Finding I_n and R_n (Continued)

- Find I_n and R_n for the circuit shown in Figure 4.117.
- To find I_n :
  - To find I_{sc}, we short circuited the terminals *a* and *b* (Figure 4.118).
  - Sum the currents leaving node 1:

$$0.001+\frac{V_1}{18000}+\frac{V_1-V_2}{12000}=0$$

  - Multiply by 36000: $36 + 2V_1 + 3V_1 - 3V_2 = 0$

    $\Rightarrow 5V_1 = 3V_2 - 36 \Rightarrow V_1 = 0.6V_2 - 7.2$ (1)
  - Sum the currents leaving node 2:

$$\frac{V_2-V_1}{12000}-0.002+\frac{V_2}{20000}+\frac{V_2}{3000}=0$$

  - Multiply by 60000: $5V_2 - 5V_1 - 120 + 3V_2 + 20V_2 = 0$

    $\Rightarrow 28V_2 - 5V_1 = 120$ (2)
  - Substitute (1) into (2): $28V_2 - 3V_2 = 84 \Rightarrow 25V_2 = 84$

    $\Rightarrow V_2 = 84/25 = 3.36$ V
  - $I_n = I_{sc} = V_2/R_4 = 1.12$ mA

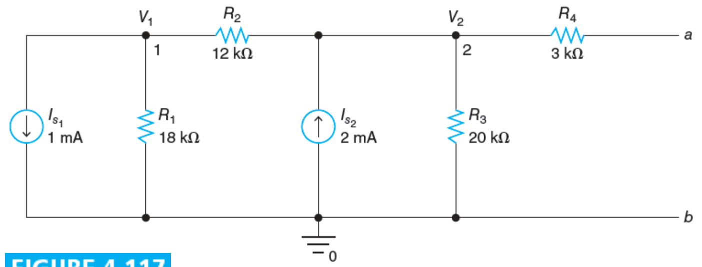


FIGURE 4.117

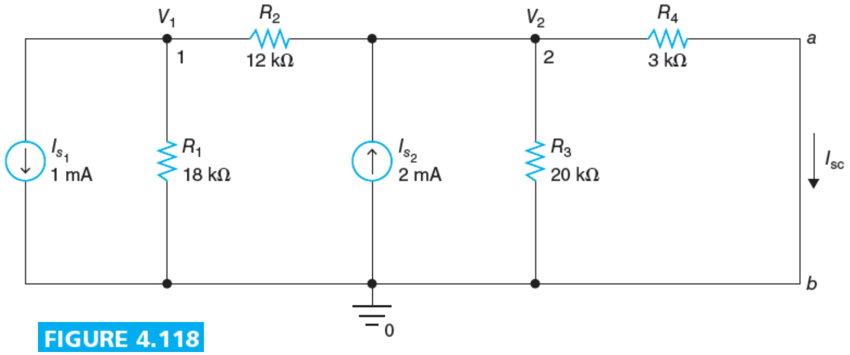


FIGURE 4.118

Finding I_n and R_n (Continued)

- To find R_n (**Method 2**):
  - Sum the currents leaving node 1 (Figure 4.117):

$$0.001+\frac{V_1}{18000}+\frac{V_1-V_2}{12000}=0$$

  - Multiply by 36000: $36 + 2V_1 + 3V_1 - 3V_2 = 0$

    $\Rightarrow 5V_1 = 3V_2 - 36 \Rightarrow V_1 = 0.6V_2 - 7.2$ (1)
  - Sum the currents leaving node 2 :

$$\frac{V_2-V_1}{12000}-0.002+\frac{V_2}{20000}=0$$

  - Multiply by 60000: $5V_2 - 5V_1 - 120 + 3V_2 = 0$ (2)
  - Substitute (1) into (2):

    $8V_2 - 3V_2 + 36 - 120 = 0 \Rightarrow 5V_2 = 84 \Rightarrow V_{oc} = V_2 = 16.8$ V
  - $R_n = V_{oc}/I_{sc} = 16.8$ V/1.12 mA = 15 kΩ
  - The Norton equivalent circuit is shown in Figure 4.119.

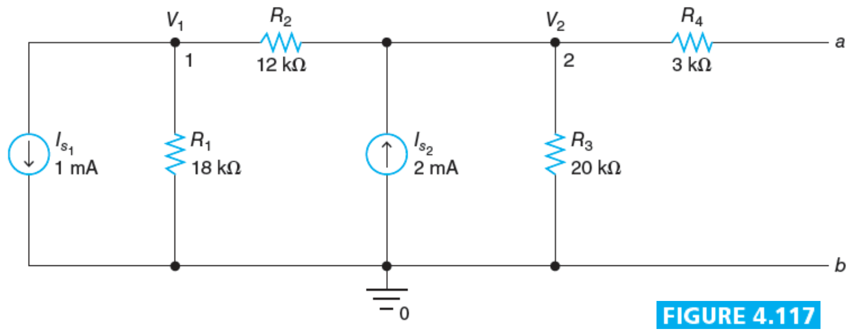


FIGURE 4.117

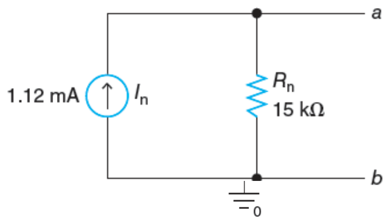


FIGURE 4.119

Finding I_n and R_n (Continued)

- We are interested in finding I_n and R_n for the circuit shown in Figure 4.103.
- To find I_n :
  - To find the short circuit current, we short-circuit *a* and *b* as shown in Figure 4.104. $V_2 = 0$.
  - Sum the currents leaving node 1:

$$-0.003 + \frac{V_1}{3000} + \frac{V_1}{2000} = 0$$

  - Multiply by 6000: $5V_1 = 18$
    $\Rightarrow V_1 = 3.6$ V
  - $i = V_1/3000 = 3.6\text{ V}/3000\ \Omega = 0.0012$ A
  - $V_{CCVS} = 2000i = 2.4$ V
  - $I_{R2} = V_1/R_2 = 1.8$ mA
  - $I_{R3} = V_{CCVS}/R_3 = 2.4$ mA
  - $I_n = I_{R2} + I_{R3} = 1.8\text{ mA} + 2.4\text{ mA} = 4.2$ mA

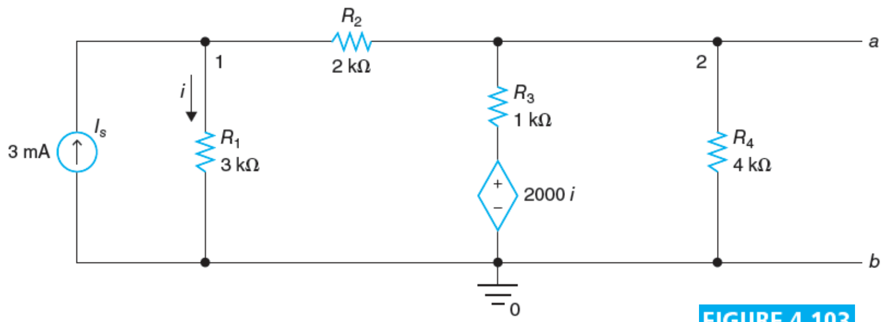


FIGURE 4.103

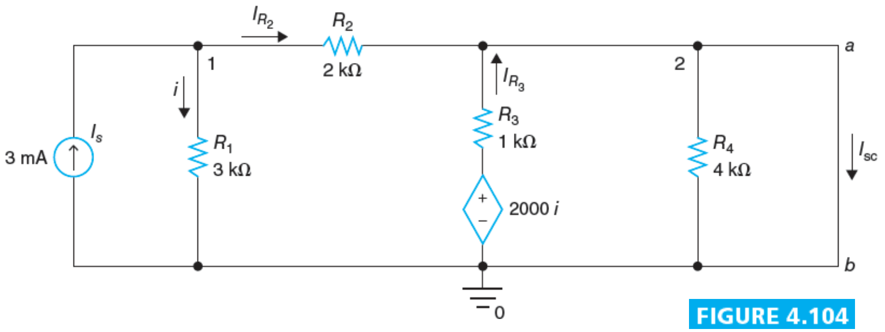


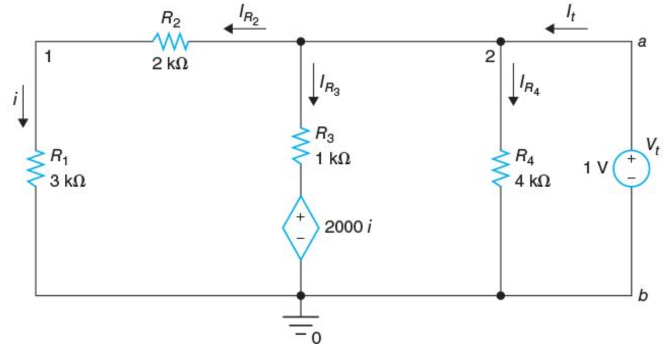
FIGURE 4.104

Finding I_n and R_n (Continued)

- To find R_n (**Method 3**):
  - After deactivating the current source, a test voltage of 1 V is applied across *a* and *b* as shown in Figure 4.105.
  - $V_t = 1\text{ V}$
  - $I_{R2} = i = V_t/(R_1 + R_2) = 1\text{ V}/5\text{ k}\Omega = 0.2\text{ mA}$
  - $V_{CCVS} = 2000i = 0.4\text{ V}$
  - $I_{R3} = (V_t - V_{CCVS})/R_3 = 0.6\text{ mA}$
  - $I_{R4} = V_t/R_4 = 0.25\text{ mA}$
  - $I_t = I_{R2} + I_{R3} + I_{R4} = 0.2\text{ mA} + 0.6\text{ mA} + 0.25\text{ mA} = 1.05\text{ mA}$
  - $R_n = V_t/I_t = 952.381\ \Omega$

FIGURE 4.105

A circuit with a test voltage source.



EXAMPLE 4.14

Find I_n and R_n for the circuit shown in Figure 4.112.

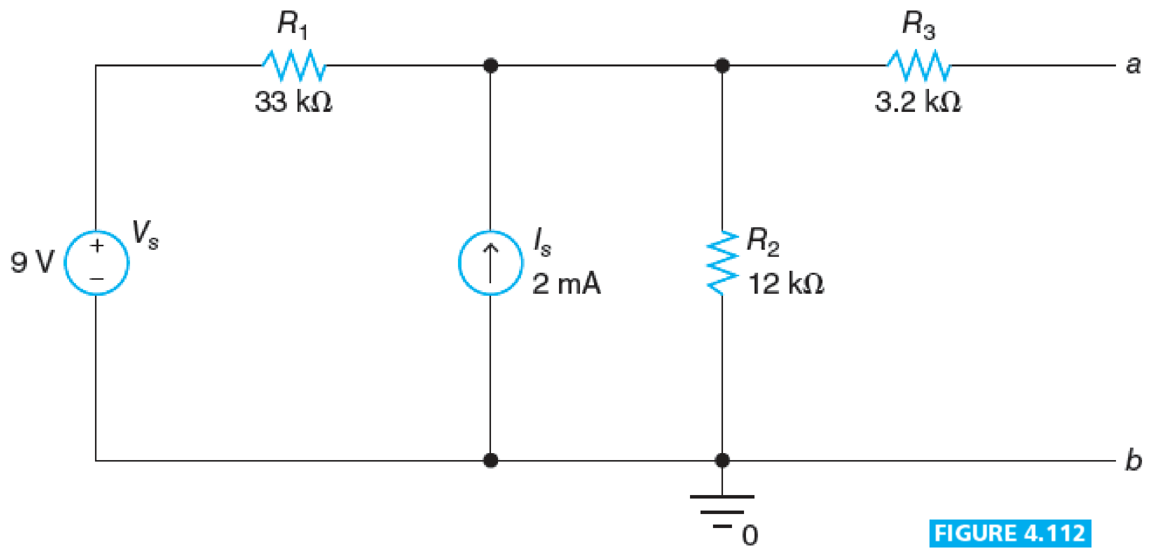


FIGURE 4.112

EXAMPLE 4.17

Find I_n and R_n for the circuit shown in Figure 4.126.

- To find I_{sc}, *a* and *b* are short-circuited as shown in Figure 4.127.
- Sum the currents leaving node 1:

$$\frac{V_1 - 0.6}{1100} + \frac{V_1}{2700} + \frac{V_1}{1800} + 0.005V_1 = 0$$

- Multiply by 59400: $54V_1 - 32.4 + 22V_1 + 33V_1 + 297V_1 = 0$

  $\Rightarrow 406V_1 = 32.4 \Rightarrow V_1 = 32.4/406 = 0.079803$ V
- $I_n = I_{sc} = V_1/R_3 + 0.005V_1 = 443.3498\ \mu$A
- To find R_n, a test voltage of 1 V is applied after short-circuiting V_s as shown in Figure 4.128.
- Sum the currents leaving node 1:

$$\frac{V_1}{1100} + \frac{V_1}{2700} + \frac{V_1 - 1}{1800} + 0.005V_1 = 0$$

- Multiply by 59400: $54V_1 + 22V_1 + 33V_1 + 297V_1 = 33$

  $\Rightarrow V_1 = 33/406$ V $= 0.0813$ V
- $I_t = (1 - V_1)/R_3 - 0.005V_1 = 103.9956\ \mu$A
- $R_n = V_t/I_t = 9.6158$ kΩ

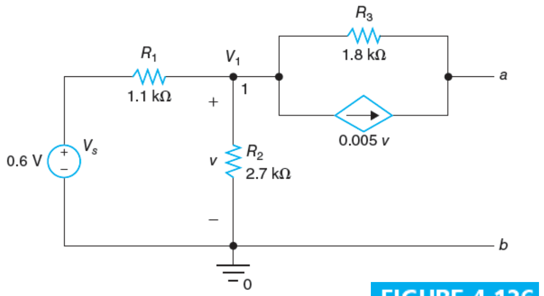


FIGURE 4.126

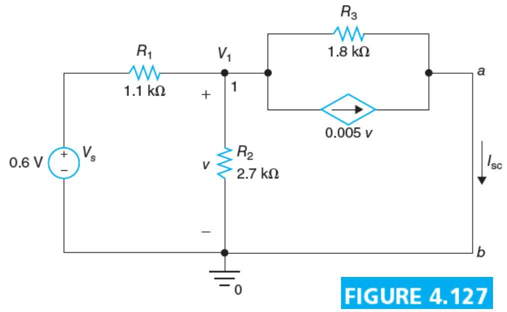


FIGURE 4.127

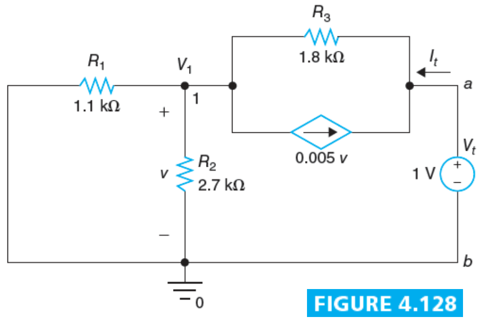


FIGURE 4.128

EXAMPLE 4.18

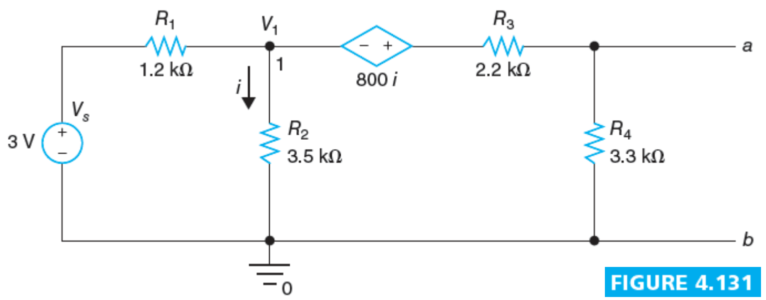


FIGURE 4.131

Find I_n and R_n for the circuit shown in Figure 4.131.

- Sum the currents leaving node 1:

$$\frac{V_1-3}{1200}+\frac{V_1}{3500}+\frac{V_1+800\frac{V_1}{3500}}{5500}=0$$

- Multiply by 46,200: $38.5V_1 - 115.5 + 13.2V_1 + 8.4V_1 + 1.92V_1 = 0 \Rightarrow 62.02V_1 = 115.5$
  $\Rightarrow V_1 = 115.5/62.02 = 1.8623$ V, $V_{oc} = (V_1 + 800 \times V_1/R_2) \times R_4/(R_3 + R_4) = 1.3728$ V
- To find I_{sc}, *a* and *b* are short-circuited as shown in Figure 4.132.
- Sum the currents leaving node 1 of Figure 4.132:

$$\frac{V_1-3}{1200}+\frac{V_1}{3500}+\frac{V_1+800\frac{V_1}{3500}}{2200}=0$$

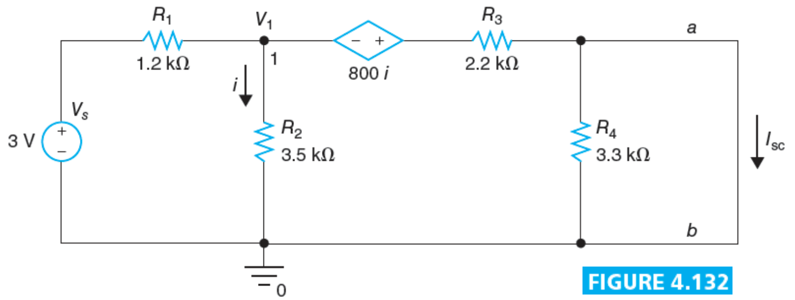


FIGURE 4.132

- Multiply by 46,200: $38.5V_1 - 115.5 + 13.2V_1 + 21V_1 + 4.8V_1 = 0$
  $\Rightarrow V_1 = 115.5/77.5 = 1.4903$ V
- $I_n = I_{sc} = (V_1 + 800\times V_1/R_2)/R_3 = 832.2581\ \mu A$
- $R_n = V_{oc}/I_{sc} = 1.6495\ k\Omega$

Maximum Power Transfer

- Suppose that a load with resistance R_L is connected to a circuit between terminals *a* and *b*.
- We are interested in finding the power p_L delivered to the load and finding the load resistance R_L that maximizes the power delivered to the load.
- We first find the Thévenin equivalent circuit with respect to the terminals *a* and *b.*
- Let V_{th} be the Thévenin equivalent voltage and R_{th} be the Thévenin equivalent resistance. With the original circuit replaced by the Thévenin equivalent circuit, we obtain the circuit shown in Figure 4.135.
- The current through the load resistor is given by

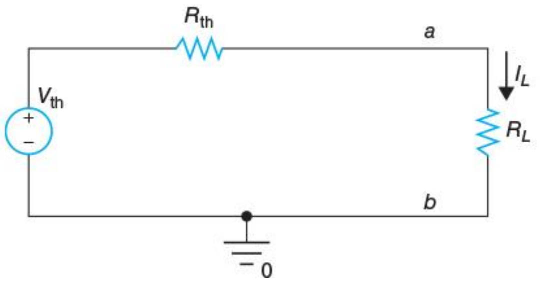
$$I_L = \frac{V_{th}}{R_{th} + R_L}$$

- The voltage across the load resistor is given by

$$V_L = R_L I_L = \frac{R_L V_{th}}{R_{th} + R_L}$$

FIGURE 4.135

A load connected to the Thévenin equivalent circuit.



Maximum Power Transfer (Continued)

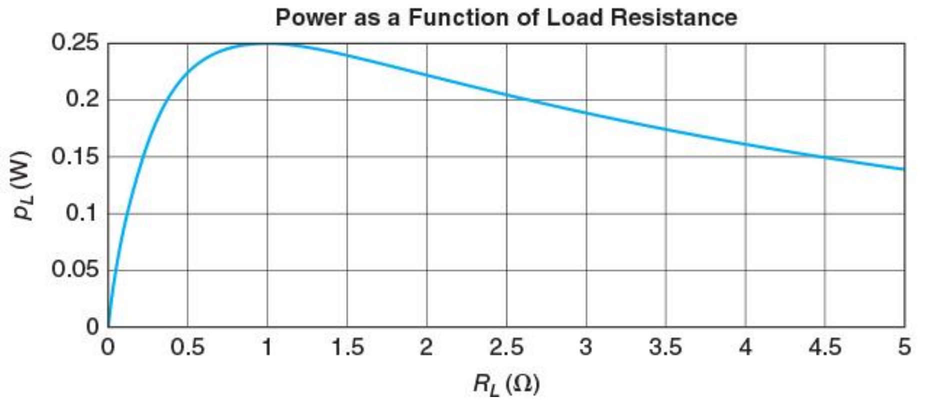
- The power delivered to the load is

$$p_L = I_L V_L = \frac{R_L V_{th}^2}{\left(R_{th} + R_L\right)^2} \quad (1)$$

- When $R_L = 0$, $p_L = 0$; and when $R_L = \infty$, $p_L = 0$.
- The power delivered to the load p_L must peak at a certain value.
- The plot shown in Figure 4.136 shows p_L as a function of R_L for $0 \le R_L \le 5R_{th}$ (V_{th} = 1 V, R_{th} = 1Ω).

FIGURE 4.136

Plot of the power on the load as a function of load resistance.



Maximum Power Transfer (Continued)

- The load resistance value for the maximum power transfer can be found by differentiating Equation (1) with respect to R_L and setting that equal to zero using

$$\frac{d}{dt}\left(\frac{u(t)}{v(t)}\right) = \frac{v(t)\dfrac{du(t)}{dt} - u(t)\dfrac{dv(t)}{dt}}{v^2(t)}$$

$$\frac{dp_L}{dR_L} = \frac{d}{dR_L}\left(\frac{R_L V_{th}^2}{(R_{th}+R_L)^2}\right) = \frac{(R_{th}+R_L)^2\dfrac{dR_L}{dR_L} - R_L\dfrac{d(R_{th}+R_L)^2}{dR_L}}{(R_{th}+R_L)^4}V_{th}^2 = \frac{(R_{th}+R_L)^2\times 1 - R_L 2(R_{th}+R_L)}{(R_{th}+R_L)^4}V_{th}^2$$

$$\frac{(R_{th}+R_L)[(R_{th}+R_L) - 2R_L]}{(R_{th}+R_L)^4}{V_{th}}^2 = \frac{[(R_{th}+R_L) - 2R_L]}{(R_{th}+R_L)^3}{V_{th}}^2 = 0$$

- The answer is $R_L = R_{th}$. Thus, the load resistance that maximizes the power transfer to load is given by

$R_L = R_{th}$ (2)

Maximum Power Transfer (Continued)

- The maximum power delivered to the load when the load resistance is $R_L = R_{th}$ is obtained by using Equation 2 in Equation 1

$$p_{L,\max} = \frac{R_{th}V_{th}^2}{\left(R_{th} + R_{th}\right)^2} = \frac{V_{th}^2}{4R_{th}} = \frac{V_{th}^2}{4R_L} \quad (3)$$

- When a load resistor is connected to a Norton equivalent circuit as shown below, it can be shown that the load resistance value that provides maximum power to the load is given by

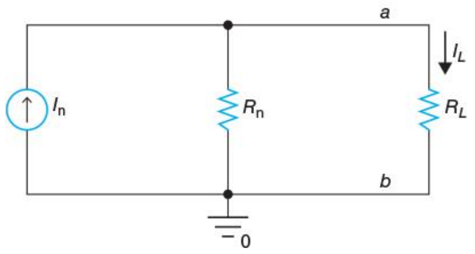
$$R_L = R_n \quad (4)$$

- The maximum power delivered to the load when $R_L = R_n$ is given by

$$p_{L,\max} = \frac{I_n^2 R_n}{4} = \frac{I_n^2 R_L}{4} \quad (5)$$

FIGURE 4.137

A Norton equivalent circuit with load R_L.



EXAMPLE 4.19

Find the load resistance value R_L that maximizes the power transfer to load for the circuit shown in Figure 4.138. Also find the maximum power delivered to load.

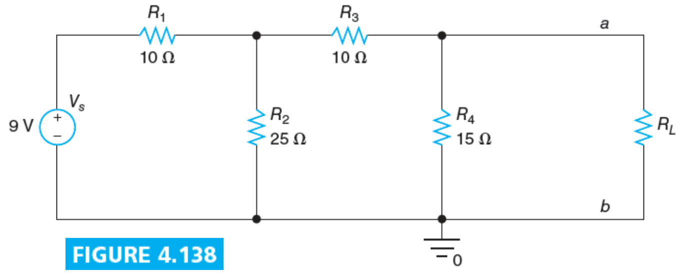


FIGURE 4.138

- Figure 4.139 shows circuit without R_L. Summing currents at node 1:

$$\frac{V_1 - 9}{10} + \frac{V_1}{25} + \frac{V_1 - V_2}{10} = 0$$

- Multiplying by 50: $5V_1 - 45 + 2V_1 + 5V_1 - 5V_2 = 0$

  $\Rightarrow 12V_1 - 5V_2 = 45 \quad (1)$

- Summing currents at node 2:

$$\frac{V_2 - V_1}{10} + \frac{V_2}{15} = 0$$

- Multiplying by 30: $5V_1 - 45 + 2V_1 + 5V_1 - 5V_2 = 0$

  $\Rightarrow 5V_2 - 3V_1 = 0 \Rightarrow V_2 = 3/5V_1 \quad (2)$

- Substituting (2) in (1): $9V_1 = 45 \Rightarrow V_1 = 5$ V
- $V_2 = V_{th} = 3/5 \times (5) = 3$ V

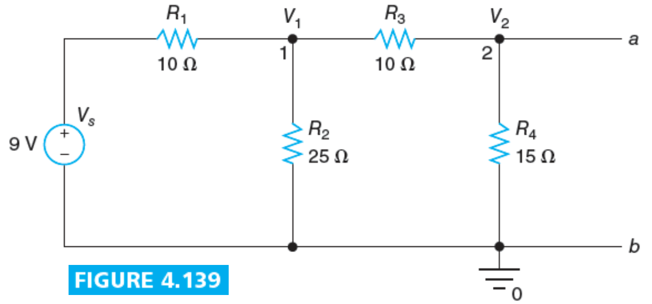


FIGURE 4.139

EXAMPLE 4.19 (Continued)

- We now find R_{th} using Method 1. Figure 4.140 shows the circuit.
- $R_a = R_1 \parallel R_2 = 250/35\ \Omega = 50/7\ \Omega$
- $R_b = R_3 + R_a = 120/7\ \Omega$
- $R_{th} = R_4 \parallel R_b = 1800/225\ \Omega = 8\ \Omega$
- $R_L = R_{th} = 8\ \Omega$
- The maximum power would be:
- $p_{L,max} = V_{th}^2/(4R_L) = 9/32\ W = 281.25\ mW$

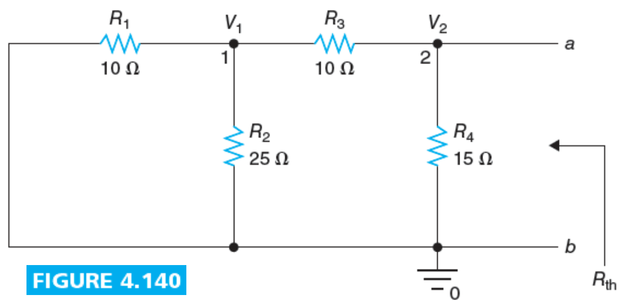
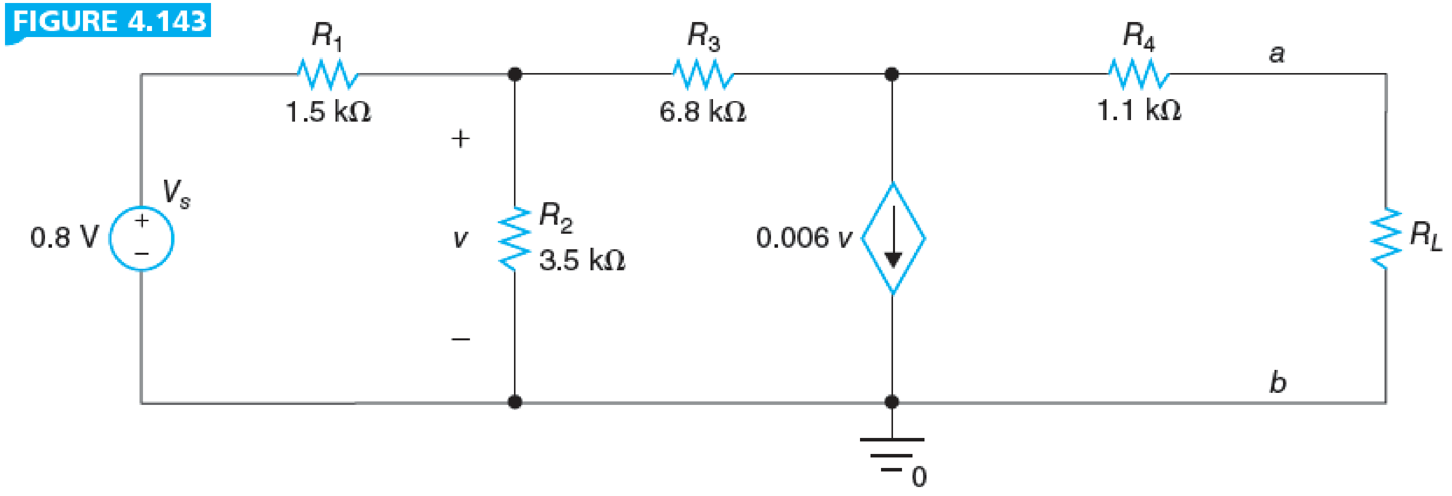


FIGURE 4.140

EXAMPLE 4.20

Find the load resistance value R_L that maximizes the power transfer to load for the circuit shown in Figure 4.143. Also find the maximum power delivered to load.

FIGURE 4.143



EXAMPLE 4.21

Find the load resistance value R_L that maximizes the power transfer to load for the circuit shown in Figure 4.147. Also find the maximum power delivered to load.

- Figure 4.148 shows circuit without R_L. No current through R_4.
- Sum the currents leaving node 1:

$$-0.001 + V_1/4500 + V_1/3900 = 0$$

$$\Rightarrow V_1 = 0.001/(1/4500 + 1/3900) = 2.0893 \text{ V}$$

- $i = V_1/R_1 = 0.4642857$ mA
- $V_{th} = V_{oc} = V_1 \times R_3/(R_2 + R_3) + 2500i = 2.6071$ V

FIGURE 4.147

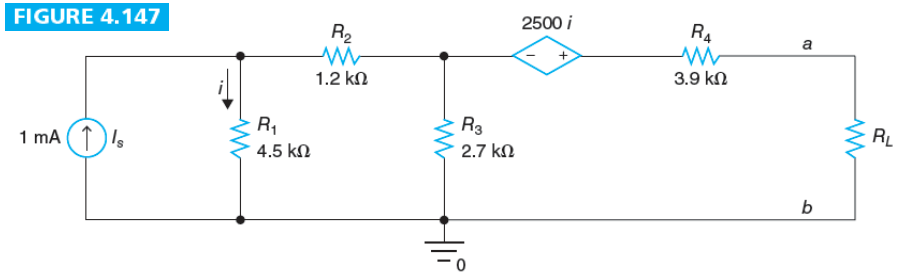
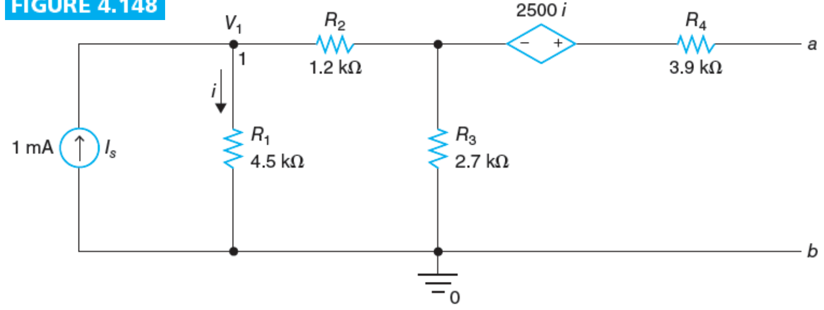


FIGURE 4.148



EXAMPLE 4.21 (Continued)

- To find the short-circuit current, *a* and *b* are short-circuited as shown in Figure 4.149.
- Sum the currents leaving node 1:

$$-0.001+\frac{V_1}{4500}+\frac{V_1-V_2}{1200}=0$$

- Multiply by 18000: $4V_1 + 15V_1 - 15V_2 = 18 \Rightarrow 19V_1 - 15V_2 = 18$ (1)
- Sum the currents leaving node 2:

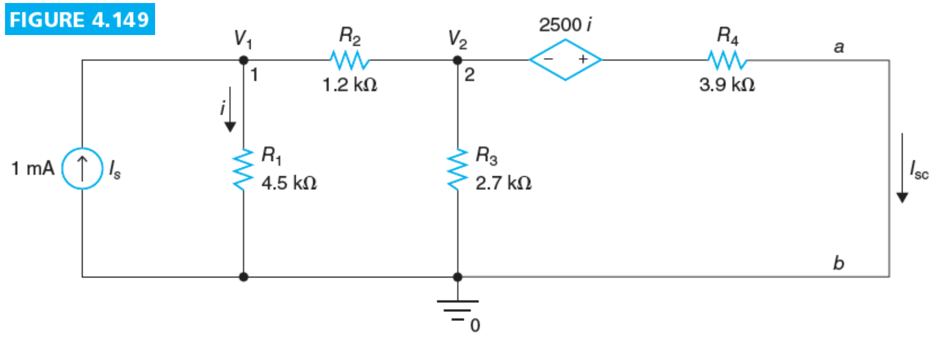
$$\frac{V_2-V_1}{1200}+\frac{V_2}{2700}+\frac{V_2+2500\dfrac{V_1}{4500}}{3900}=0$$

- Multiply by 14040: $11.7V_2 - 11.7V_1 + 5.2V_2 + 3.6V_2 + 2V_1 = 0$

$\Rightarrow -9.7V_1 + 20.5V_2 = 0 \Rightarrow V_1 = (205/97)V_2$ (2)

- Substitute (2) into (1): $V_2 = 0.71557377$ V, $V_1 = 1.5123$ V
- $I_{sc} = (V_2 + 2500 \times V_1/R_1)/R_4 = 0.3989$ mA
- $R_{th} = V_{oc}/I_{sc} = 6.5357\ k\Omega = R_L$
- $p_{L,max} = V_{th}^2/(4R_L) = 0.26$ mW

FIGURE 4.149



Summary

- Norton's Theorem: A circuit looking from terminals *a* and *b* can be replaced by a current source with current I_n and a parallel resistor with resistance R_n. This equivalent circuit consisting of a current source and a parallel resistor is called Norton equivalent circuit. The current I_n is called Norton equivalent current and the resistance R_n is called Norton equivalent resistance.
- Finding Norton equivalent resistance:
  - Method 1: Deactivate all the independent sources by short-circuiting voltage sources and open-circuiting current sources. Find the equivalent resistance looking into the circuit from terminals *a* and *b.*
  - Method 2: Short-circuit terminals *a* and *b*. Find the short-circuit current I_{sc}. The Norton equivalent resistance is given by $R_n = V_{oc}/I_{sc} = V_{oc}/I_n$.
  - Method 3: Deactivate all the independent sources. Apply a test voltage of 1 V between terminals *a* and *b* with terminal *a* connected to the positive terminal of the test voltage. Measure the current flowing out of the positive terminal of the test voltage source. The Norton equivalent resistance is the ratio of the voltage to current. Test current can be used also.

Summary (Continued)

- Maximum Power Transfer

  Suppose that a load with resistance R_L is connected to a circuit between terminals *a* and *b*. The load resistance that maximizes the power transfer to the load is given by:

  $$R_L = R_{th}$$

  where R_{th} is the Thévenin equivalent resistance when the circuit between terminals *a* and *b* looking from the load is replaced by Thévenin equivalent circuit.

- The maximum power delivered to the load when the load resistance is $R_L = R_{th}$ is given by:

  $$p_{L,\max} = \frac{V_{th}^2}{4R_{th}} = \frac{V_{th}^2}{4R_L}$$

- What will we study in next lecture.