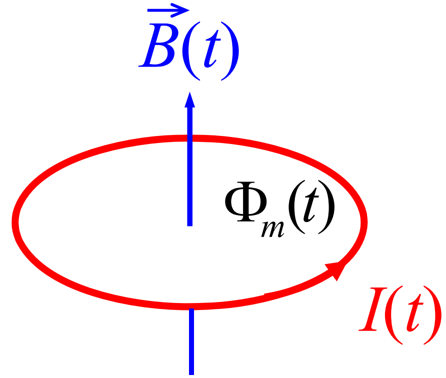
Chapter 28

Inductance, Magnetic Energy Storage

- Self-Inductance
- Mutual Inductance
- Energy Stored in a Magnetic Field
- LR and LC Circuits

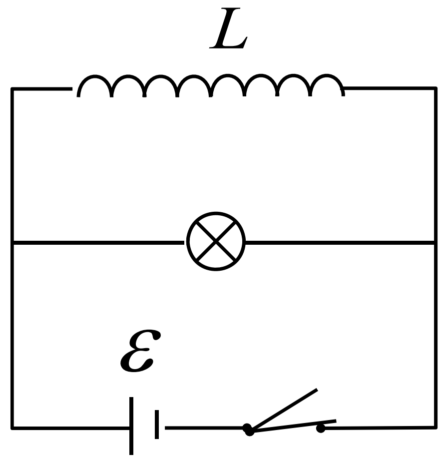
§28-1 Self-Inductance

Self-induction: The change of the current in a circuit causes a change in the magnetic flux through the circuit itself, then generates an EMF in the circuit.



Magnetic flux $\Phi_m \propto$ current I:

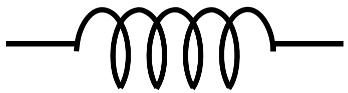
$$N\Phi_m = LI$$



L: self inductance of the coil.

SI unit: Henry (H), 1H =1V·s/A =1Ω· s.

A coil with significant L as a device in a circuit is called inductor:



Note: L depends only on the geometry of the coil, and on the presence of a ferromagnetic material.

Self-induced EMF:

$$\varepsilon_L = -\frac{\mathrm{d}\left(N\Phi_m\right)}{\mathrm{d}t} = -N\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

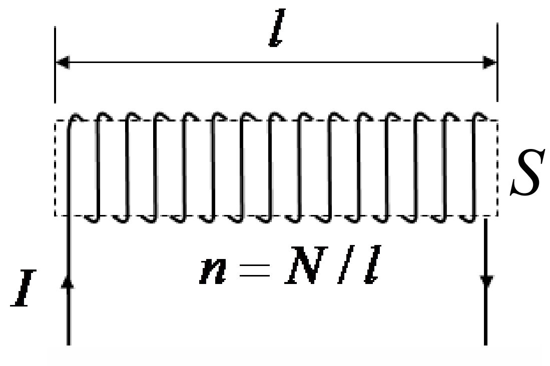
The self-induced EMF always prevents the change in the current of the circuit itself, therefore, L shows the strength of the electromagnetic inertia of a coil.

Inductor in AC circuit → reactance/impedance.

To calculate L:

$$L = \frac{N\Phi_m}{I}, \qquad L = -\frac{\varepsilon_L}{\mathrm{d}I/\mathrm{d}t}$$

Inductance of an infinite long solenoid



Assuming current I passing through a long solenoid, so the magnetic field:

$$B = \mu_0 n I$$

Total magnetic flux:

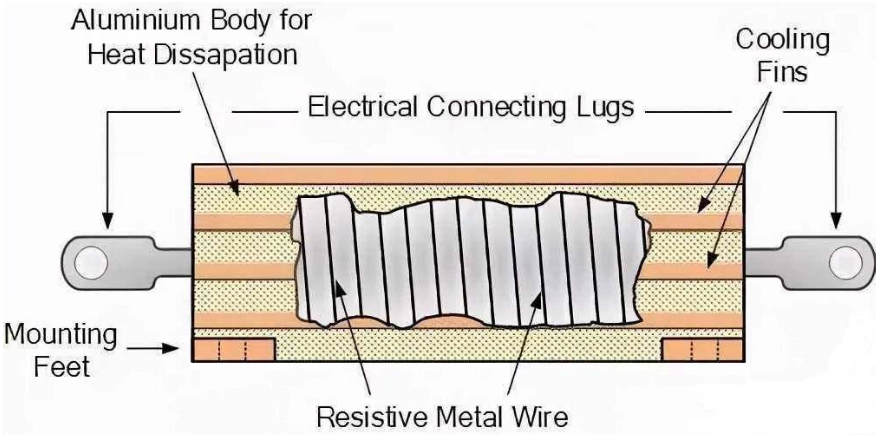
$$N\Phi_B = N \cdot \mu_0 n I \cdot S = \mu_0 n^2 I \cdot Sl = \mu_0 n^2 I \cdot V$$

Self inductance:

$$L = \mu_0 n^2 V$$

Note: this formula is derived for an idealized infinite long solenoid, so it is valid only for an isolated solenoid.

How to avoid inductance for a wire-wound resistor?



An inductor opposes and suppresses any rapid changes in the current.

lightning strike

fluorescent light tube

Example1 (Inductance of coaxial cable): Determine the inductance per unit length of a coaxial cable with thin conductors.

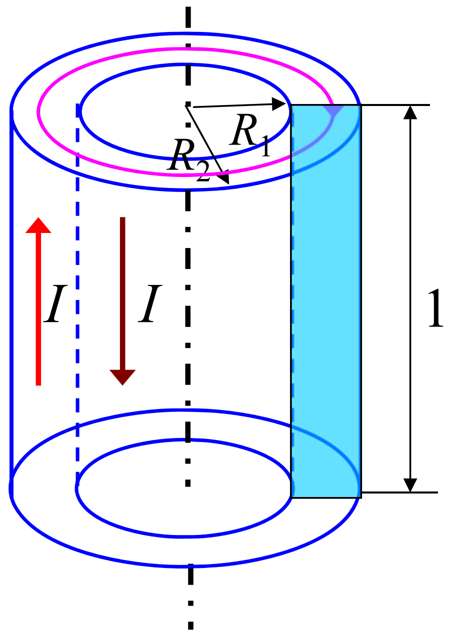
Solution: Magnetic field ?

Ampere's law:

$$B = \frac{\mu_0 I}{2\pi r}, \quad (R_1 < r < R_2)$$

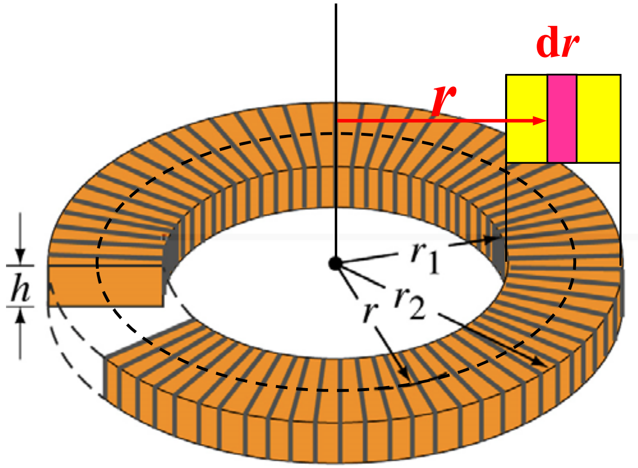
$$\Phi_{\mathrm{m}} = \int \frac{\mu_0 I}{2\pi r} \cdot 1 \cdot \mathrm{d}r = \frac{\mu_0 I}{2\pi} \ln \frac{R_2}{R_1}$$

$$\boxed{L = \frac{\Phi_m}{I} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}}$$



Inductance of toroid

Question: Determine the inductance of a toroid (N loops) with rectangular cross-section.



$$B=\frac{\mu_0 NI}{2\pi r}$$

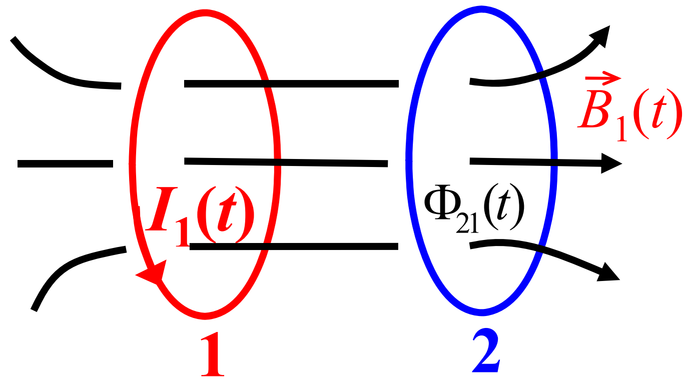
$$\Phi_m=\int_s B\mathrm{d}s\cdot\cos\theta$$

$$\Phi_m=\int_{r_1}^{r_2}\frac{\mu_0 NI}{2\pi r}h\mathrm{d}r=\frac{\mu NIh}{2\pi}\ln\frac{r_2}{r_1}$$

$$L=\frac{N\Phi_m}{I}=\frac{\mu_0 N^2 h}{2\pi}\ln\frac{r_2}{r_1}$$

§28-2 Mutual Inductance

Mutual induction: The change of the current in a coil causes a change in the magnetic flux through the other coil, then generates an EMF in that coil.



Magnetic flux Φ_{21} through coil 2 due to I_1 is proportional to I_1:

$$N_2\Phi_{21} = M_{21}I_1$$

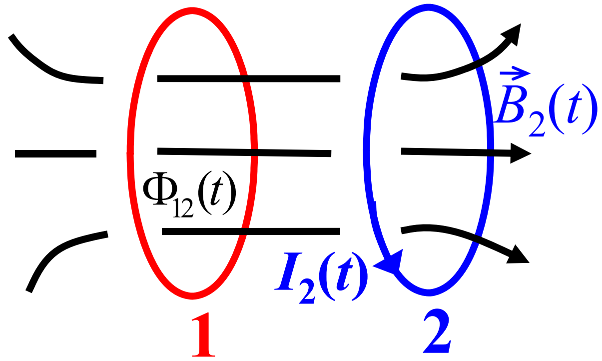
M_{21}: mutual inductance of coil 2 with respect to coil 1; SI unit: Henry (H).

EMF in coil 2:

$$\varepsilon_{21} = N_2\frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t} = M_{21}\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

The change of the current in coil 2 causes a change in the magnetic flux through coil 1:

$$N_1\Phi_{12} = M_{12}I_2$$



EMF in coil 1:

$$\boldsymbol{\varepsilon_{12} = N_1 \frac{d\Phi_{12}}{dt} = M_{12}\frac{dI_2}{dt}}$$

It can be proved that:

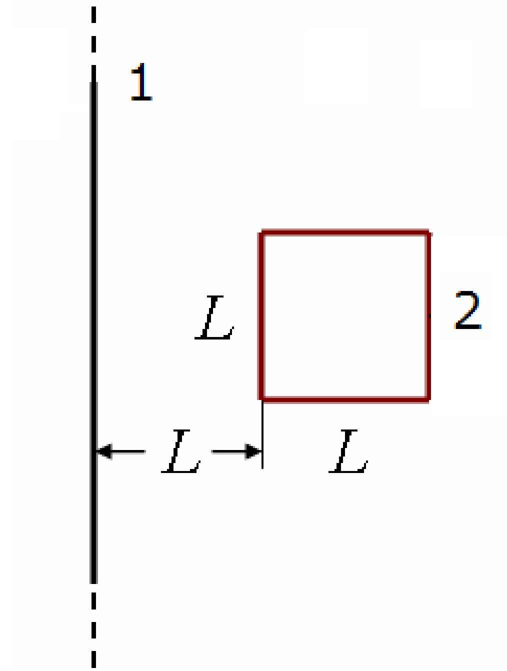
$$M_{12} = M_{21} = M$$

Note: ***M* depends only on the geometry, relative position of the two coils, and on the presence of a ferromagnetic material.**

To calculate M:

$$M = \frac{N_2\Phi_{21}}{I_1} \qquad M = -\frac{\varepsilon_{21}}{dI_1/dt}$$

Example2 (Straight wire and coil): A long straight wire and a square coil lie in the same plane. If the current in coil is $I_2=I_0\cos\omega t$, what is the EMF on straight wire?



Solution: First determine M

Suppose current I in straight wire: $B=\dfrac{\mu_0 I}{2\pi r}$

$$\Rightarrow \Phi_m = \int_L^{2L} \frac{\mu_0 I}{2\pi r} L\mathrm{d}r = \frac{\mu_0 IL}{2\pi}\ln 2$$

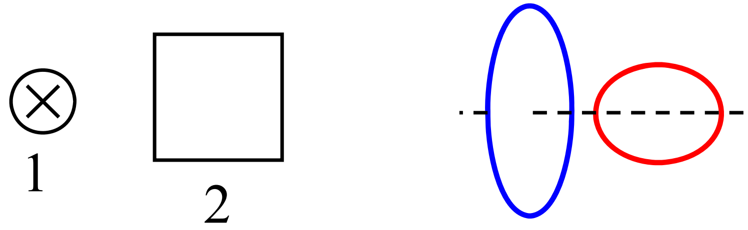
$$\therefore\ M = \frac{\Phi_B}{I} = \frac{\mu_0 L}{2\pi}\ln 2$$

EMF on straight wire:

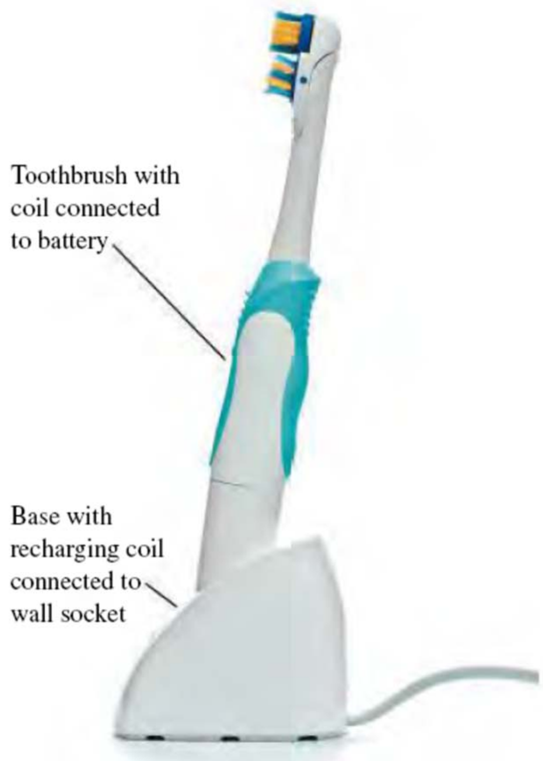
$$\varepsilon_1 = -M\frac{\mathrm{d}I_2}{\mathrm{d}t} = \frac{\mu_0 L}{2\pi}\ln 2\cdot I_0\omega\sin\omega t$$

How to minimize the mutual inductance?

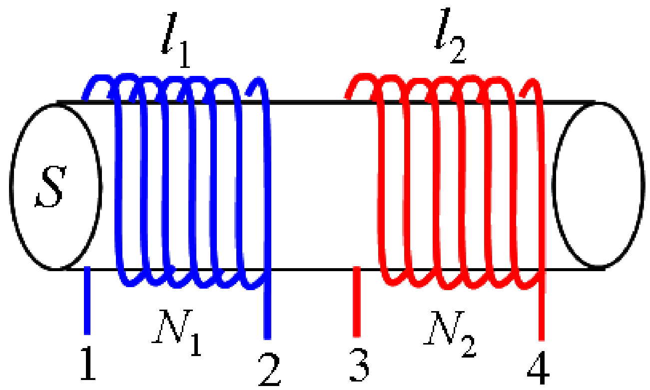
Minimize Φ_m :



Application of the mutual inductance



Example3 (Two coils): Determine the mutual inductance of two **ideal coupling** coils (L_1, L_2).



Solution: Current I_1 in coil 1:

$$B_1 = \mu n_1 I_1$$

$$\therefore\ M = \frac{N_2 \Phi_{21}}{I_1} = \mu n_1 S N_2$$

$$= \mu n_1 \frac{N_1}{l_1} S l_1 \frac{N_2}{N_1} \quad = \mu n_1^2 V_1 \frac{N_2}{N_1} \qquad\qquad (L_1 = \mu n_1^2 V_1)$$

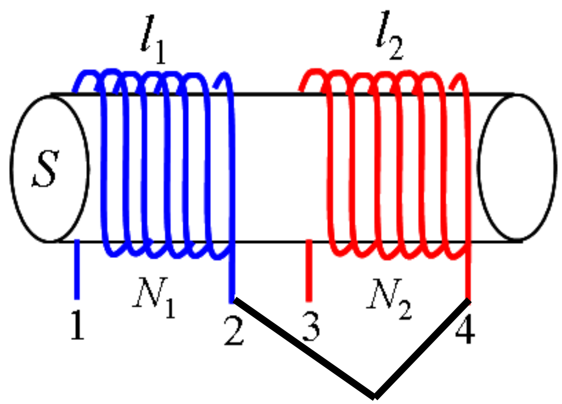
$$= L_1 \cdot \frac{N_2}{N_1}$$

Current I_1 in coil 1: $M = L_1 \cdot \dfrac{N_2}{N_1}$

Current I_2 in coil 2: $M = L_2 \cdot \dfrac{N_1}{N_2}$

$$\left.\begin{array}{l} M = L_1 \cdot \dfrac{N_2}{N_1} \\ M = L_2 \cdot \dfrac{N_1}{N_2} \end{array}\right\} \Longrightarrow M = \sqrt{L_1 L_2}$$

Discussion:



(1) Generally: $M = k \cdot \sqrt{L_1 L_2}$, $(0 < k < 1)$

(2) Total inductance? (connect 2 and 3)

$$L = L_1 + L_2 + 2M \neq L_1 + L_2$$

(3) Total inductance? (connect 2 and 4)

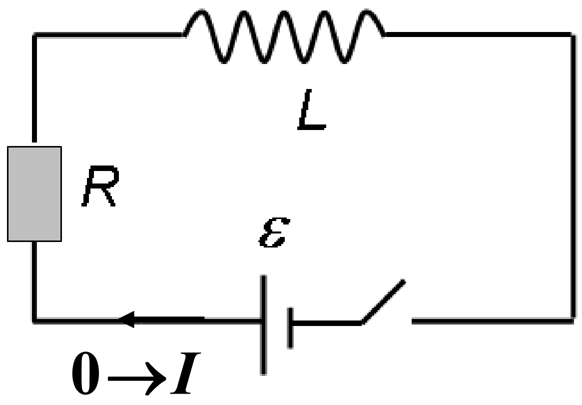
§28-3 Energy Stored in a Magnetic Field

1. Current carrying inductor stores magnetic energy

After turning on the switch, current in the circuit increases from 0 to I, at time t during this process:

$$\varepsilon + \varepsilon_L = IR$$

$$\Rightarrow \quad \varepsilon - L\frac{\mathrm{d}I}{\mathrm{d}t} = IR$$



Multiplying $I\mathrm{d}t$ for both sides:

$$\boldsymbol{\varepsilon I \mathrm{d}t = I^2 R \mathrm{d}t + L I \mathrm{d}I}$$

When current reaches stable:

$$\int_0^t I\varepsilon \mathrm{d}t = \int_0^t I^2 R \mathrm{d}t + \int_0^I LI \mathrm{d}I$$ **Conservation of energy**

$\int_0^t I\varepsilon \mathrm{d}t$: output energy of the power source;

$\int_0^t I^2 R \mathrm{d}t$: energy dissipated on the resistor R;

$\int_0^I LI \mathrm{d}I$: work done by the power source to oppose the EMF of self-inductance;

This work done is equal to the energy stored in the inductor:

$$\boxed{U = \int_0^I LI \mathrm{d}I = \frac{1}{2} LI^2}$$

单选题 10分

设置

Q: What is the inductance of an inductor if it has a stored energy of 1.5 J when there is a current of 2.5 A in it?

A 0.48 H

B 1.2 H

C 2.1 H

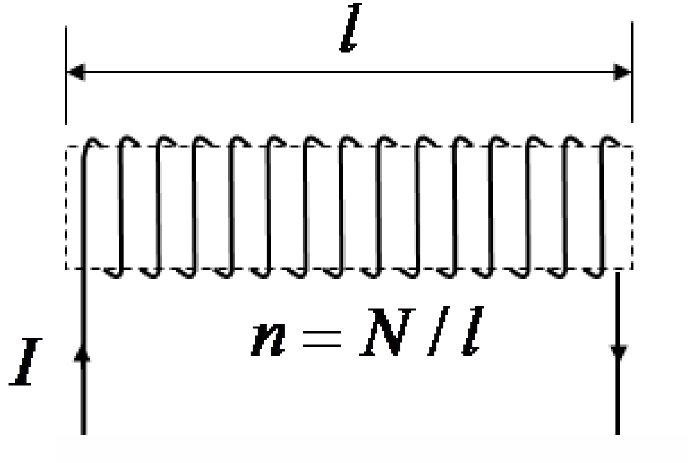
D 4.7 H

E 19 H

提交

2. Energy stored in a magnetic field

For infinite long solenoid with self-inductance L:



$$B=\mu_0 nI, \qquad L=\mu_0 n^2 I\cdot V$$

$$U=\frac{1}{2}LI^2=\frac{1}{2}\mu_0 n^2 V\cdot I^2=\frac{1}{2}\frac{B^2}{\mu_0}\cdot V$$

Energy per unit volume / energy density:

$$\boxed{u_m=\frac{1}{2}\frac{B^2}{\mu_0}} \qquad U_E=\frac{1}{2}CV^2,\ u_E=\frac{1}{2}\varepsilon_0 E^2$$

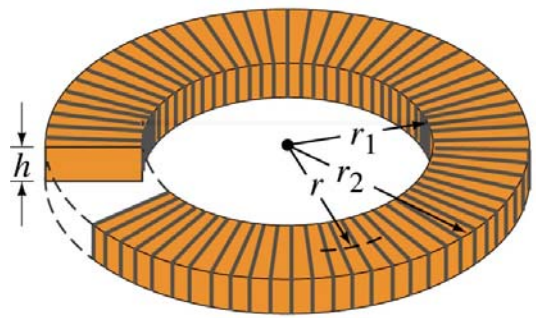
Energy stored in a magnetic field:

$$U_m=\int_V u_m \mathrm{d}V$$

Example4 (Energy in toroid): Determine the total energy stored in the toroid (N loops with current I)

Solution:

$$B = \frac{\mu_0 NI}{2\pi r}$$



$$u = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\mu_0 N^2 I^2}{8\pi^2 r^2}$$

$$U = \int u \mathrm{d}V = \int_{r_1}^{r_2} \frac{\mu_0 N^2 I^2}{8\pi^2 r^2} \cdot h \cdot 2\pi r \mathrm{d}r$$

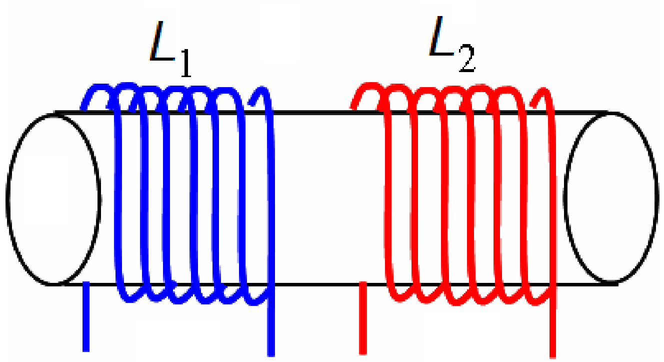
$$(L = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{r_2}{r_1})$$

$$= \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\frac{r_2}{r_1} = \frac{1}{2} L I^2$$

Challenging problem: Show that $M_{12}=M_{21}$ by using the expression of magnetic energy stored in two coils.

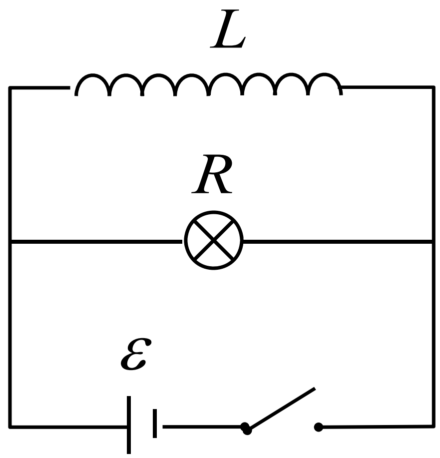
$$U=\frac{1}{2}L_1I_1^2+\frac{1}{2}L_2I_2^2+M_{12}I_1I_2$$

$$=\frac{1}{2}L_1I_1^2+\frac{1}{2}L_2I_2^2+M_{21}I_1I_2$$



§28-4 LR Circuits and LC Circuits

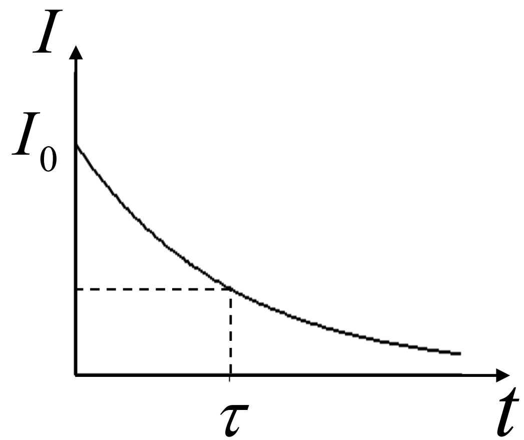
1. LR circuits



After the switch is turned off:

$$\varepsilon_L = -L\frac{\mathrm{d}I}{\mathrm{d}t} = RI$$

$$\Rightarrow I = I_0 e^{-t/\tau}$$



Time constant: $\tau = \dfrac{L}{R}$

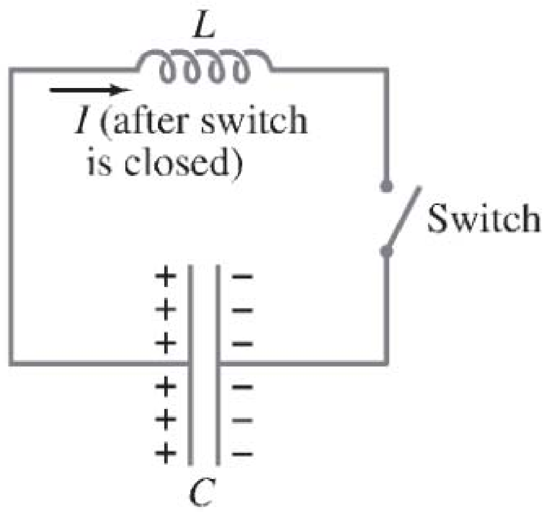
An inductor can act as a "surge protector" for sensitive electronic equipment that can be damaged by high currents.

2. LC circuits

After switch is closed:

$$\varepsilon_L = -L\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{Q}{C}, \qquad I = -\frac{\mathrm{d}Q}{\mathrm{d}t}$$

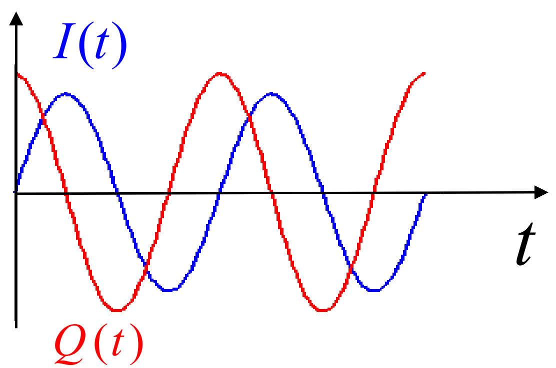
$$\Rightarrow \quad \frac{\mathrm{d}^2 Q}{\mathrm{d}^2 t} + \frac{Q}{LC} = 0$$



Solve this second order differential equation:

$$Q = Q_0 \cos(\omega t + \varphi) \qquad \omega = \sqrt{\frac{1}{LC}}$$

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \omega Q_0 \sin(\omega t + \varphi)$$



Q and I oscillate periodically (LC oscillator or electromagnetic oscillation).

Summary

1. Self-Inductance

$$L = \frac{N\Phi_m}{I}, \qquad \varepsilon_L = -N\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

SI unit: Henry (H), 1H =1V·s/A =1Ω· s.

2. Inductance of an infinite long solenoid

$$\boldsymbol{L} = \boldsymbol{\mu_0 n^2 V}$$

3. Mutual induction

$$M = \frac{N_2\Phi_{21}}{I_1} \qquad \boldsymbol{\varepsilon_{21} = -N_2\frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t} = -M_{21}\frac{\mathrm{d}I_1}{\mathrm{d}t}}$$

4. mutual inductance of two coupling coils

$$M = k\cdot\sqrt{L_1L_2}, \ (0 \le k \le 1)$$

5. Current carrying inductor stores magnetic energy

$$U = \int_0^I LI\mathrm{d}I = \frac{1}{2}LI^2$$

6. Energy stored in a magnetic field

$$\boldsymbol{u_m = \frac{1}{2}\frac{B^2}{\mu_0}} \qquad U_m = \int_V u_m \mathrm{d}V = \frac{1}{2}LI^2$$

7. LR circuits

$$\varepsilon_L = -L\frac{\mathrm{d}I}{\mathrm{d}t} = RI \qquad I = I_0 e^{-t/\tau} \qquad \text{Time constant:} \qquad \tau = \frac{L}{R}$$

8. LC circuits

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}^2 t} + \frac{Q}{LC} = 0 \qquad Q = Q_0\cos(\omega t + \varphi) \qquad I = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \omega Q_0 \sin(\omega t + \varphi) \qquad \omega = \sqrt{\frac{1}{LC}}$$

LC oscillator or electromagnetic oscillation