Circuit Analysis and Design

Academic Year 2025/2026 – Semester 1

Lecture 10 - Capacitors

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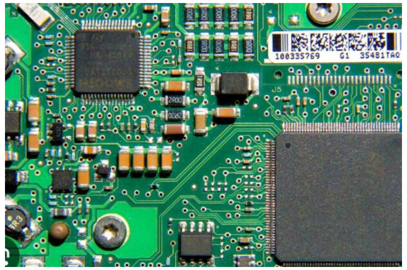
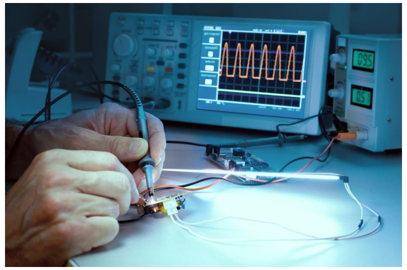




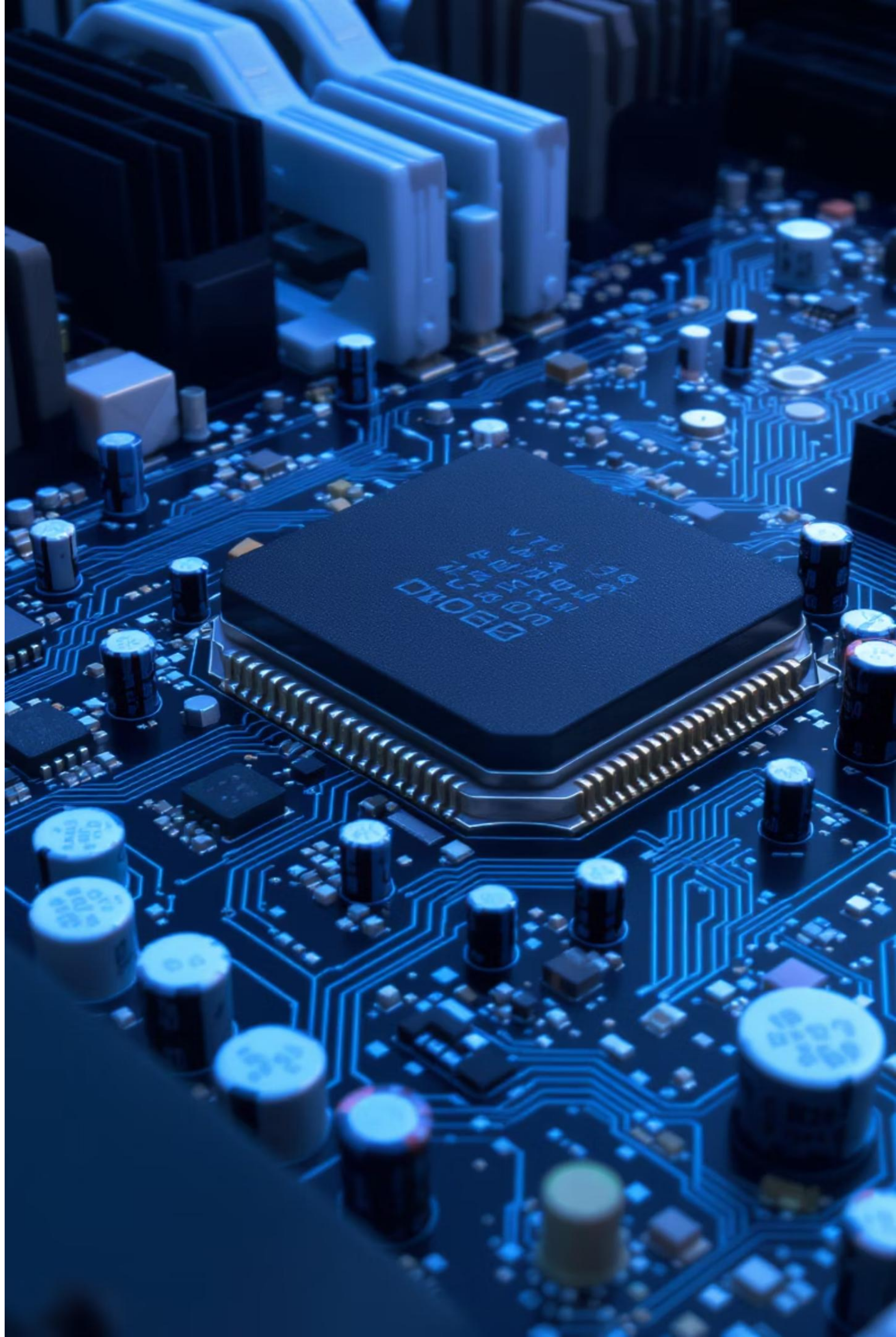
Agenda

1 Introduction to Capacitors

Understanding the fundamental principles of capacitive elements in electrical circuits and their role in energy storage

2 Series and Parallel Connections

Analyzing equivalent capacitance for different circuit configurations and deriving key relationships



Capacitors: Fundamentals

- A capacitor is a passive element that can store and release energy.

- A parallel plate capacitor has two plates filled by dielectric material in between. The positive plate holds positive charges, and the negative plate holds negative charges.

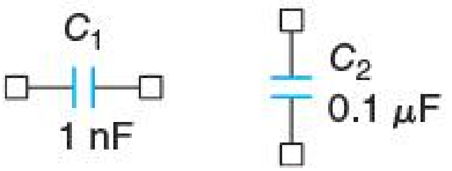
- The capacitance of a capacitor is the amount of charge that the capacitor can hold for the given voltage.

- The energy is stored in the form of an electric field from the positive plate to the negative plate. The amount of energy stored on the capacitor depends on the capacitance and the voltage across the capacitor plates.

- The symbol for capacitors is shown in Figure 6.1.

FIGURE 6.1

Symbol for capacitors.



Capacitance

- When a capacitor is connected to a voltage source such as a battery, as shown in Figure 6.2, positive charges accumulate on the plate connected to the positive terminal of the battery, and negative charges accumulate on the plate connected to the negative terminal of the battery.

- The amount of charge Q (in coulombs) on the plates is proportional to the voltage V (in volts) of the voltage source. Let C be the proportionality constant in this linear relation. Then we have

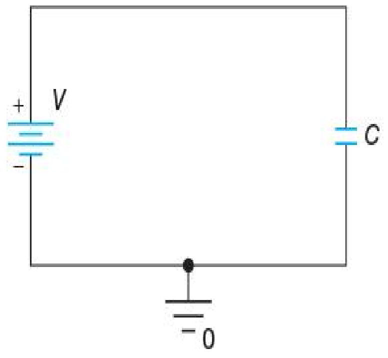
$$Q = CV \quad \Rightarrow C = Q/V$$

- The constant C is called capacitance, measured in farads (F). It is defined as the ratio of the charge stored to the potential difference.

- If the voltage is time varying, the charge can be written as $Q = C\,v(t)$

FIGURE 6.2

A capacitor connected to a battery.



Current-Voltage Relation of a Capacitor

- Let v(t) be the voltage across a capacitor and i(t) be the current through the capacitor, as shown in Figure 6.5.

- Since the current is defined as the time rate of change of the charge, we have

$$i(t) = \frac{dQ(t)}{dt} \quad (1)$$

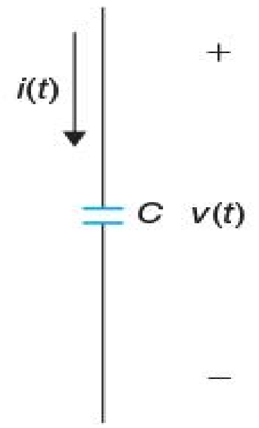
- Since Q(t) = Cv(t), Equation (1) becomes

$$i(t) = C\frac{dv(t)}{dt} \quad (2)$$

- The current through the capacitor is proportional to the time rate of change to the voltage across the capacitor.

FIGURE 6.5

Voltage across and current through a capacitor.



Current-Voltage Relation of a Capacitor (Continued)

- If Equation (2) is integrated, we obtain

$$v(t)=\frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda = v(0)+\frac{1}{C}\int_{0}^{t} i(\lambda)d\lambda$$

where v(0) is the voltage across the capacitor at t = 0.

- The instantaneous power on the capacitor is given by

$$p(t)=v(t)i(t)=v(t)C\frac{dv(t)}{dt}=Cv(t)\frac{dv(t)}{dt}$$

- The energy stored in the capacitor at time t is given by

$$w(t)=\int_{-\infty}^{t} p(\lambda)d\lambda = C\int_{-\infty}^{t} v(\lambda)\frac{dv(\lambda)}{d\lambda}d\lambda = C\int_{-\infty}^{t} v(\lambda)dv(\lambda)=\frac{1}{2}Cv^2(t)$$

- If v(t) = V (constant), w = $0.5CV^2$.

EXAMPLE 6.2: Finding Current, Power & Energy

The voltage across a capacitor with capacitance of 100 μF is given by v(t) and plotted in Figure 6.6.

$$v(t)=\begin{cases} 50t, & 0\le t<1 \\ -50t+100, & 1\le t<3 \\ 50t-200, & 3\le t<4 \\ 0, & \text{otherwise} \end{cases} \text{ V}$$

Calculate and plot:

a. Current i(t) passing through the capacitor.
b. The instantaneous power in the capacitor.
c. The energy stored in the capacitor.

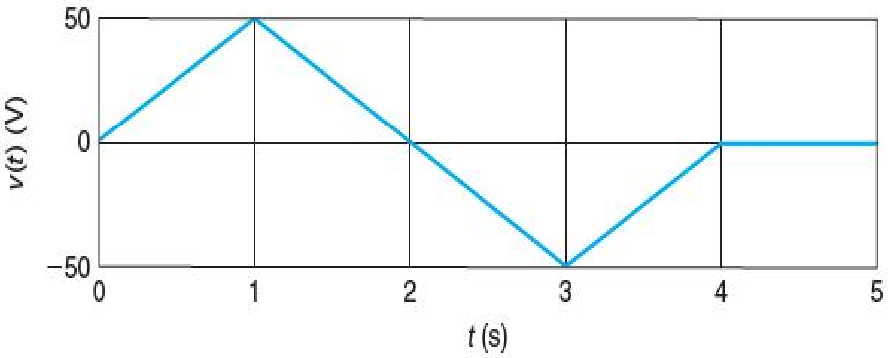


FIGURE 6.6

EXAMPLE 6.2: Finding the Current

- The current through the capacitor is given by

$$i(t) = C\frac{dv(t)}{dt} = \begin{cases} 100\times10^{-6}\times50, & 0\le t<1 \\ 100\times10^{-6}\times(-50), & 1\le t<3 \\ 100\times10^{-6}\times50, & 3\le t<4 \\ 0, & \text{otherwise} \end{cases} \text{A} = \begin{cases} 5, & 0\le t<1 \\ -5, & 1\le t<3 \\ 5, & 3\le t<4 \\ 0, & \text{otherwise} \end{cases} \text{mA}$$

- i(t) is shown in Figure 6.7.

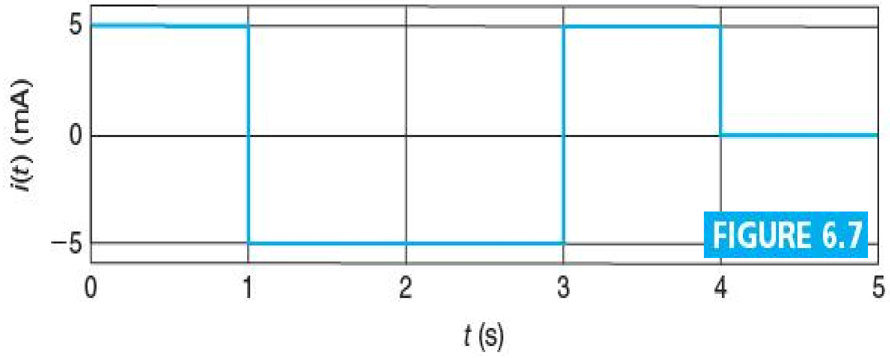
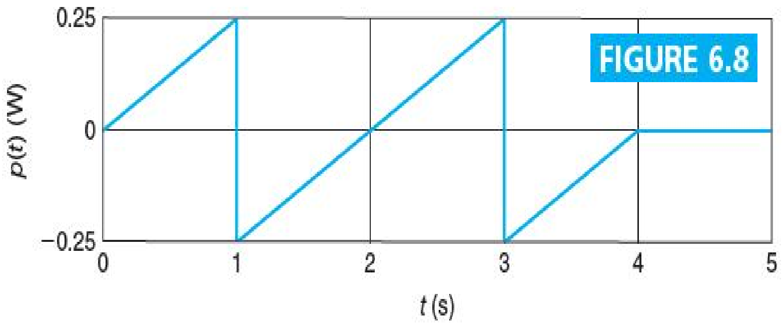


FIGURE 6.7

EXAMPLE 6.2: Finding the Power & Energy

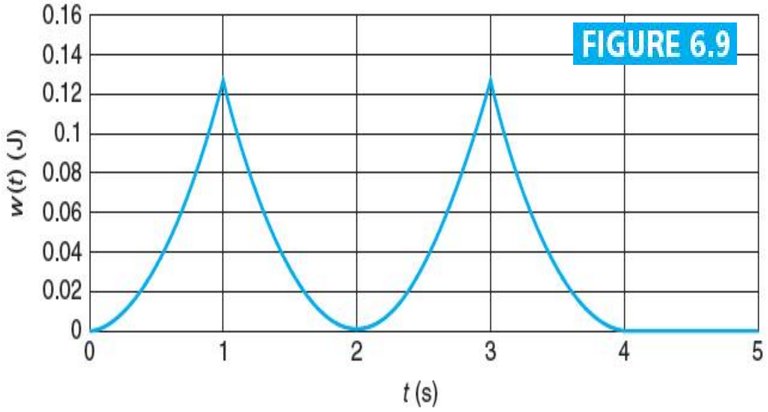
- The instantaneous power absorbed (p(t) > 0) or released (p(t) < 0) is given by

$$p(t) = v(t)i(t) = \begin{cases} 0.25t, & 0 \le t < 1 \\ 0.25t - 0.5, & 1 \le t < 3 \\ 0.25t - 1, & 3 \le t < 4 \\ 0, & \text{otherwise} \end{cases} \text{ W}$$



- The energy stored in the capacitor is given by

$$w(t) = \frac{1}{2} C v^2(t) = \begin{cases} 0.125t^2, & 0 \le t < 1 \\ 0.125(t-2)^2, & 1 \le t < 3 \\ 0.125(t-4)^2, & 3 \le t < 4 \\ 0, & \text{otherwise} \end{cases} \text{ J}$$



- The energy stored in the capacitor increases during intervals 0 < t < 1 s and 2 s ≤ t < 3 s and the energy stored in the capacitor decreases during intervals 1 s ≤ t < 2 s and 3 s ≤ t < 4 s. p(t) in Figure 6.8 and w(t) in Figure 6.9.

EXAMPLE 6.3: When Current & Capacitance are known

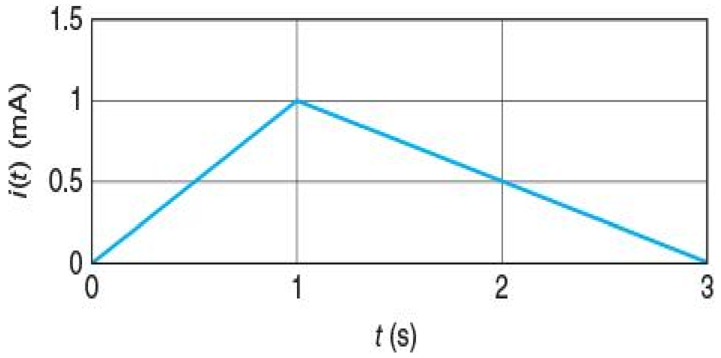
The current through a capacitor with capacitance 100 μF is shown in Figure 6.11. Find the voltage, power, and energy on the capacitor.

- The current through a capacitor is given by

$$i(t)=\begin{cases} t, & 0\le t<1 \\ -\frac{t}{2}+\frac{3}{2}, & 1\le t<3 \text{ mA} \\ 0, & \text{otherwise} \end{cases}$$

FIGURE 6.11

The current through the capacitor.



EXAMPLE 6.3 (Continued)

- The voltage across the capacitor is given by

$$v(t) = \frac{1}{C}\int_0^t i(\lambda)d\lambda$$

$$\text{For } 0 \le t < 1, v(t) = \frac{10^{-3}}{100 \times 10^{-6}}\int_0^t \lambda d\lambda = 5t^2 \text{ V}$$

$$\text{For } 1 \le t < 3, \quad v(t) = 5 + \frac{10^{-3}}{100 \times 10^{-6}}\int_1^t\left(\frac{-\lambda}{2} + \frac{3}{2}\right)d\lambda = 5 + 10\left[\frac{-\lambda^2}{4} + \frac{3}{2}\lambda\right]_1^t$$

$$= 5 + 10\left(\frac{-t^2}{4} + \frac{3}{2}t - \left(\frac{-1}{4} + \frac{3}{2}1\right)\right) = -2.5t^2 + 15t - 7.5 = -2.5(t-3)^2 + 15 \text{ V}$$

Thus,

$$v(t) = \begin{cases} 5t^2 \text{ V}, & 0 \le t < 1 \\ -2.5(t-3)^2 + 15 \text{ V}, & 1 \le t < 3 \text{ V} \\ 0, & otherwise \end{cases}$$

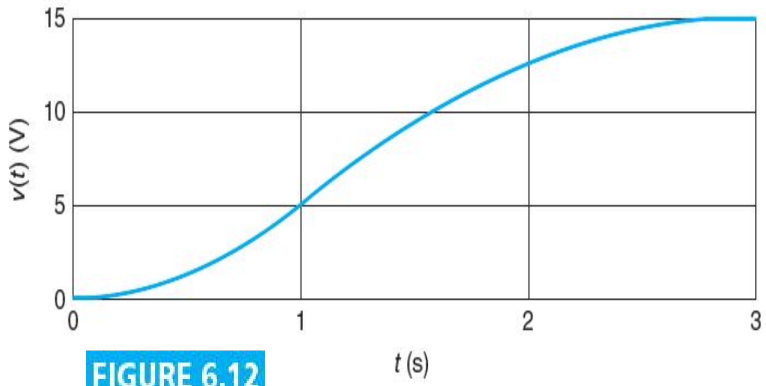
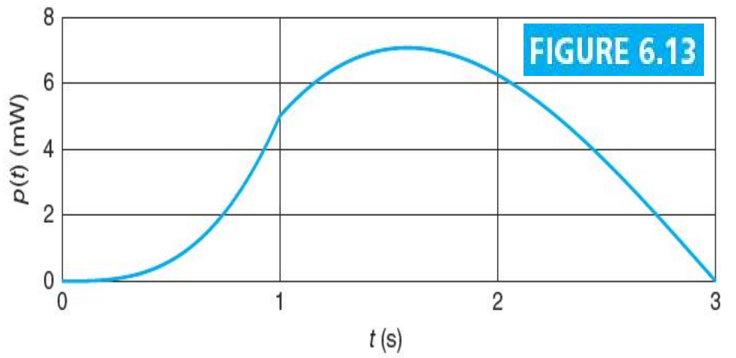


FIGURE 6.12

EXAMPLE 6.3 (Continued)

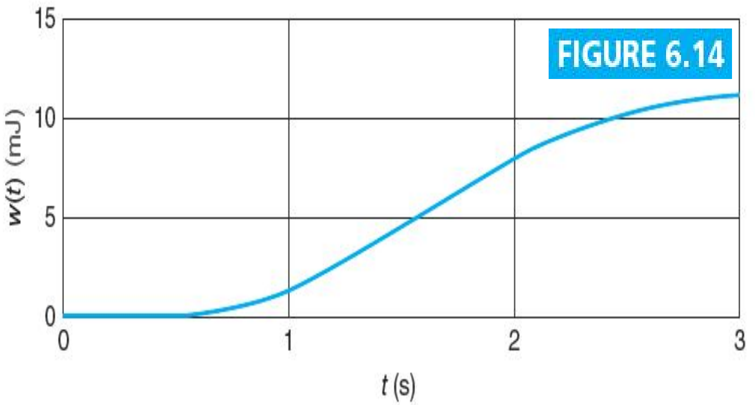
- The power on the capacitor is given by

$$p(t) = v(t)i(t) = \begin{cases} \dfrac{t^3}{200}\,W, & 0 \le t < 1 \\ [(t-3)(t^2-6t+3)]/800\ W, & 1 \le t < 3 \\ 0, & otherwise \end{cases}$$



- The energy stored on the capacitor is given by

$$w(t) = \frac{1}{2}Cv^2(t) = \begin{cases} \dfrac{t^4}{800}\,J, & 0 \le t < 1 \\ [(t-3)^2-6]^2/3200\ J, & 1 \le t < 3 \\ 0, & otherwise \end{cases}$$



Sinusoidal Input: Phase Relationships

When a sinusoidal voltage $v(t) = \cos(2\pi ft)$ is applied to a capacitor, the resulting current exhibits a crucial 90-degree phase lead. This phase relationship is fundamental to AC circuit analysis.

Example Analysis

Given: $v(t) = \cos(20\pi t)$ V applied to $C = 0.01$ F

The current is: $i(t) = C\, dv(t)/dt = 0.01 \times (-20\pi)\sin(20\pi t)$ $i(t) = -0.2\pi \sin(20\pi t)$ A

Using the identity $\sin(\theta) = -\cos(\theta + 90°)$: $i(t) = 0.2\pi \cos(20\pi t + 90°)$ A

The current leads voltage by 90°, meaning current reaches its peak T/4 seconds before voltage (where T = 0.1 s is the period).

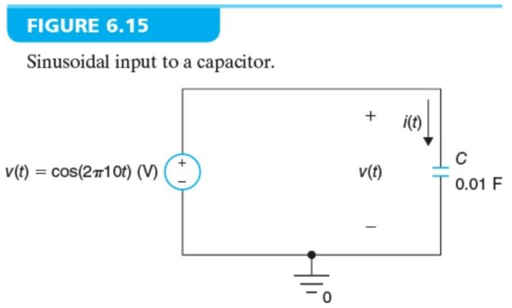


Figure 6.15: Circuit with sinusoidal source

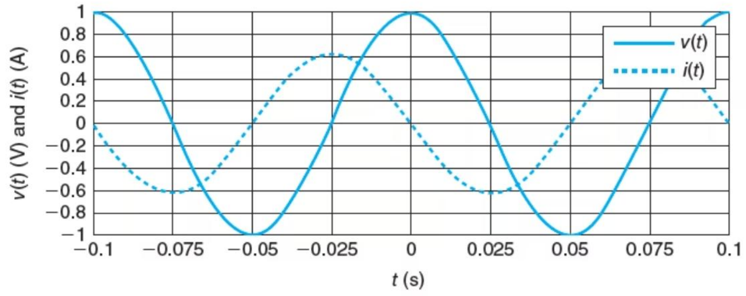


Figure 6.16: Current leads voltage by 90°

Sinusoidal Input: Phase Relationship

When a sinusoidal voltage v(t) = cos(2πft) is applied to a capacitor, the resulting current exhibits a crucial 90-degree phase lead. This phase relationship is fundamental to AC circuit analysis.

- Example: A sinusoidal voltage, v(t) = cos(2π10t) is applied to a capacitor with capacitance 0.01 F as shown in Figure 6.15.
- The current through the capacitor is given by $i(t) = C\dfrac{dv(t)}{dt} = 0.01 \times (-1) \times (2\pi 10) \times \sin(2\pi 10t)$

$$= -0.6283\sin(2\pi 10t) = 0.6283\cos(2\pi 10t + 90^o)\,\mathrm{A}$$

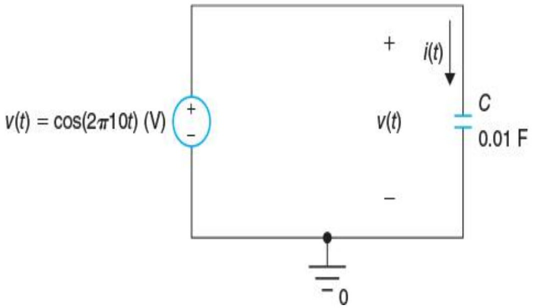
Using the identity sin(θ) = −cos(θ + 90°): i(t) = 0.2π cos(20πt + 90°) A

- The phase of current is 90°, compared to 0° for the voltage. The current leads the voltage by 90°.

- The current crosses zero T/4 s earlier than voltage as shown in Figure 6.16. T is a period given by 0.1 s. T/4 s is equivalent to 360°/4 = 90°

FIGURE 6.15

Sinusoidal input to a capacitor.



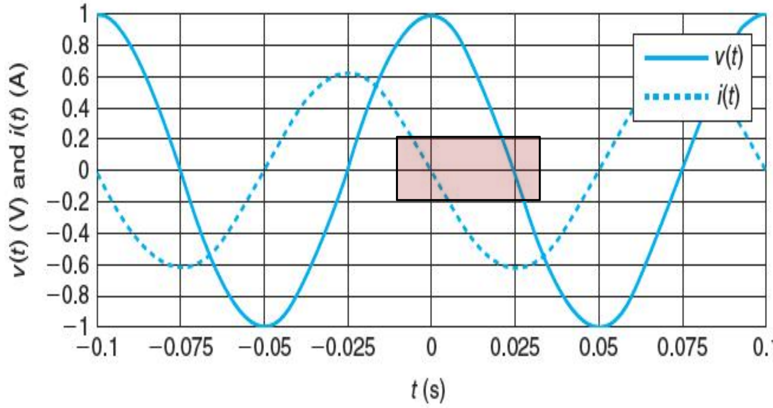


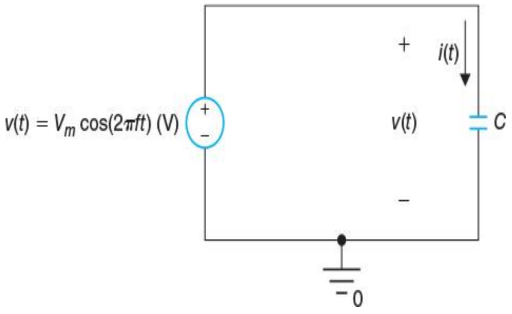
FIGURE 6.16

Sinusoidal Input: Frequency Dependent Behavior

- A sinusoidal voltage, $v(t) = V_m \cos(2\pi ft)$ is applied to a capacitor with capacitance C as shown in Figure 6.17.

FIGURE 6.17

Sinusoidal input to a capacitor.

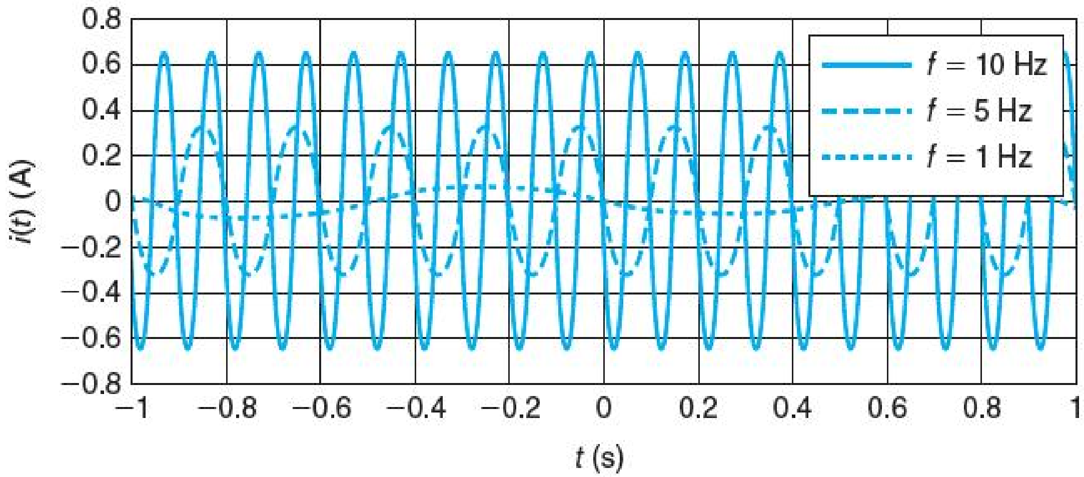


- The current through the capacitor is given by

$$i(t) = C\frac{dv(t)}{dt} = CV_m \times (-1) \times (2\pi f) \times \sin(2\pi ft) = -CV_m 2\pi f \times \sin(2\pi ft)$$

- The amplitude of the current is proportional to the frequency of the voltage applied. As the frequency decreases, the amplitude decreases as shown in Figure 6.18.

FIGURE 6.18



- For a dc voltage (f = 0), the current through the capacitor is zero in the steady state.

- The capacitor acts as an open circuit for a dc voltage, and acts as a short circuit for high frequency voltage.

Series Connection of Capacitors

- Figure 6.19(a) shows *n* capacitors connected in series. The current through the *n* capacitors is i(t).

- Let $v_1(t)$ be the voltage across C_1, $v_2(t)$ be the voltage across C_2, and $v_n(t)$ be the voltage across C_n. The total voltage v(t) across all *n* capacitors is

$$v(t) = v_1(t) + v_2(t) + \ldots + v_n(t) \quad (1)$$

- Substitution of the i-v relation to Equation (1) yields $v_k(t) = \frac{1}{C_k}\int_{-\infty}^{t} i(\lambda)d\lambda,\ 1 \le k \le n$

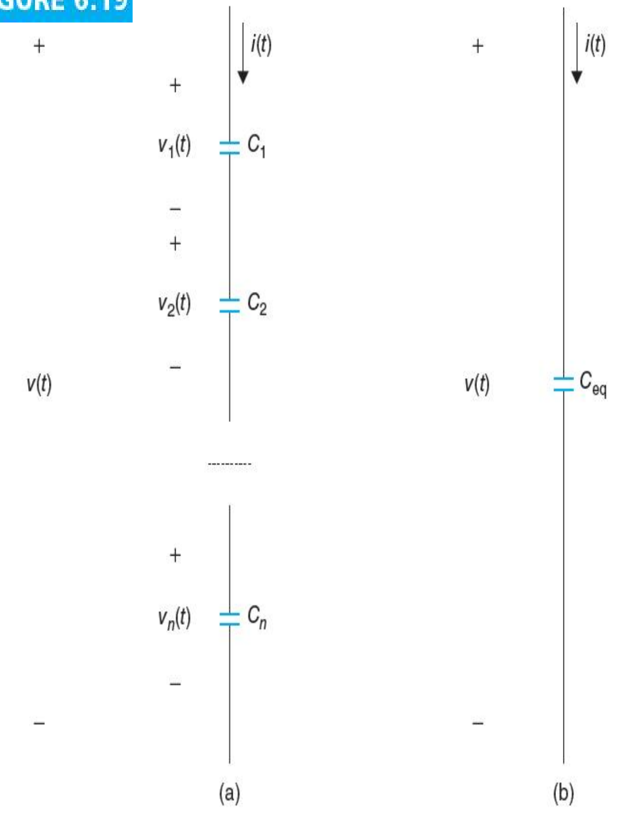
$$v(t) = \frac{1}{C_1}\int_{-\infty}^{t} i(\lambda)d\lambda + \frac{1}{C_2}\int_{-\infty}^{t} i(\lambda)d\lambda + \ldots + \frac{1}{C_n}\int_{-\infty}^{t} i(\lambda)d\lambda$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}\right)\int_{-\infty}^{t} i(\lambda)d\lambda = \frac{1}{C_{eq}}\int_{-\infty}^{t} i(\lambda)d\lambda$$

where equivalent capacitance, C_{eq} is

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}}$$

FIGURE 6.19



Series Connection of Capacitors (Continued)

- Notice that the equation for the equivalent capacitance of the series connected capacitors is similar to the equivalent resistance of parallel connected resistors.

- The equivalent capacitance of two capacitors with capacitances C_1 and C_2 connected in series is given by

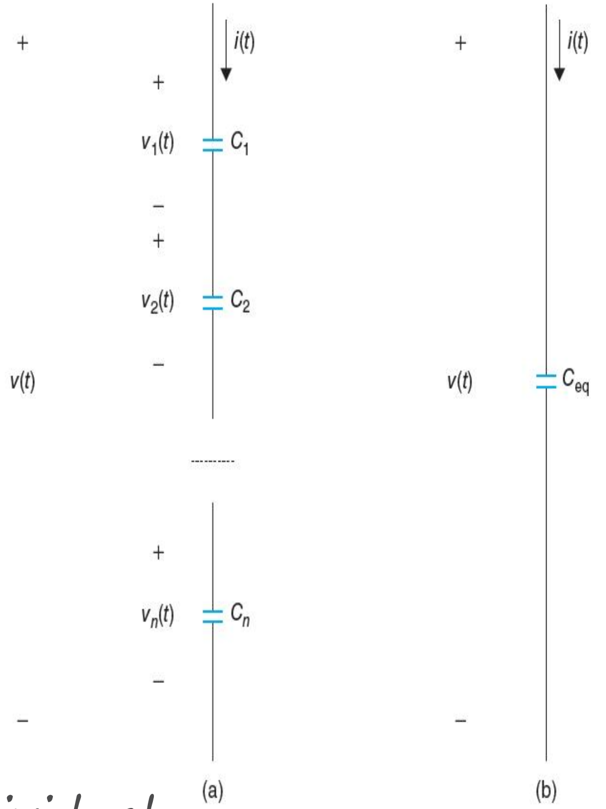
$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$$

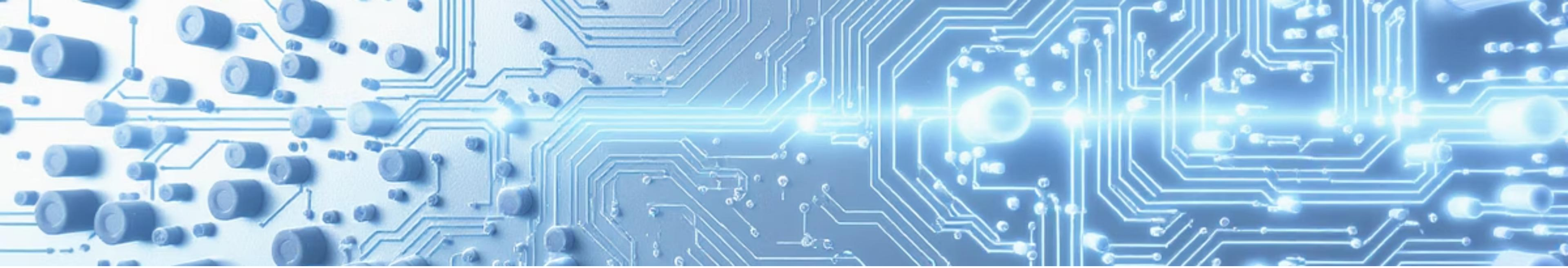
- The equivalent capacitance of three capacitors with capacitances C_1, C_2, and C_3 connected in series is given by

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} = \frac{C_1 C_2 C_3}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

- *The series equivalent capacitance is **always smaller** than the smallest individual capacitance.*

FIGURE 6.19





Series Capacitance: Key Formulas

General Formula

For n capacitors in series:

$1/C_{eq} = \Sigma(1/C_i)$

Similar to parallel resistors

Two Capacitors

Simplified product-over-sum form:

$C_{eq} = (C_1 \ C_2)/(C_1 + C_2)$

Most commonly used formula

Three Capacitors

Extended formula:

$1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3$

Calculate reciprocals then invert

Important: Series connection *reduces* total capacitance. The equivalent capacitance is always less than the smallest capacitor in the series string. This is opposite to the behavior of series resistors.

Parallel Connection of Capacitors

- Figure 6.20(a) shows *n* capacitors connected in parallel. The voltage across the *n* capacitor is v(t).

- Let $i_1(t)$ be the current through C_1, $i_2(t)$ be the current through C_2, . . ., $i_n(t)$ be the current through C_n, then the total current i(t) through all *n* capacitors is

$$i(t) = i_1(t) + i_2(t) + \ldots + i_n(t) \quad (1)$$

- Substitution of the i-v relation to Equation (1) yields

$$i_k(t) = C_k \frac{dv(t)}{dt}, \quad 1 \le k \le n$$

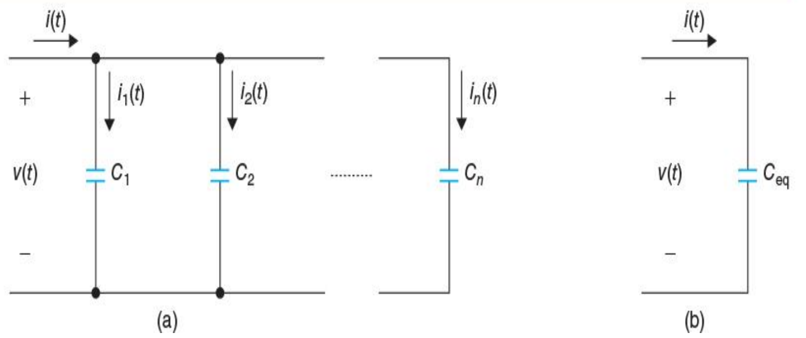
$$i(t) = i_1(t) + i_2(t) + \ldots + i_n(t) = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + \ldots + C_n \frac{dv(t)}{dt} = (C_1 + C_2 + \ldots + C_n)\frac{dv(t)}{dt} = C_{eq}\frac{dv(t)}{dt}$$

where the equivalent capacitance, C_{eq}, is given by

$$C_{eq} = C_1 + C_2 + \ldots + C_n$$

FIGURE 6.20

(a) Parallel connection of capacitors. (b) Equivalent capacitor.

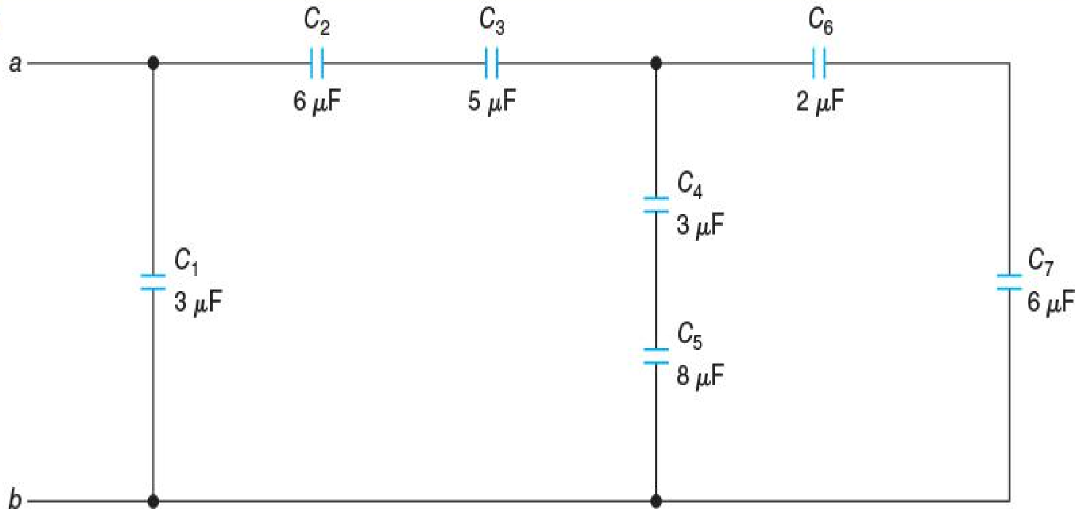


EXAMPLE 6.4

Find the equivalent capacitance between a and b for the circuit shown in Figure 6.21.

FIGURE 6.21

Circuit for EXAMPLE 6.4.



EXAMPLE 6.4

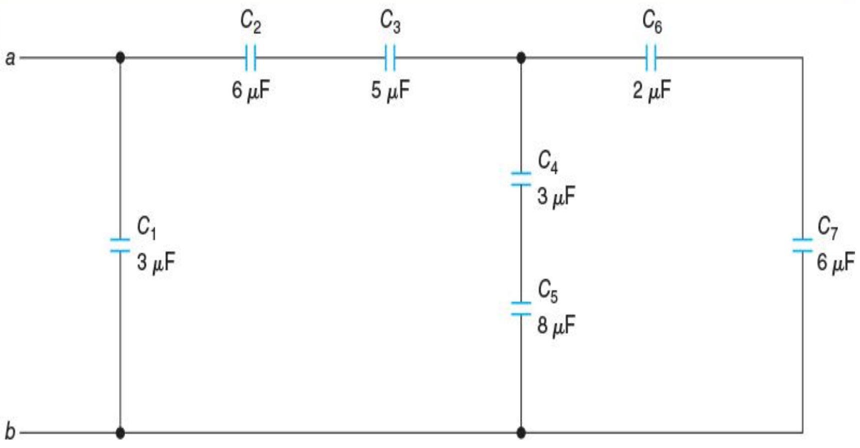
Find the equivalent capacitance between a and b for the circuit shown in Figure 6.21.

- C_8 = Equivalent capacitance of C_6 and C_7 = $2\times6/8$ μF = $12/8$ μF = 1.5 μF.
- C_9 = Equivalent capacitance of C_4 and C_5 = $3\times8/11$ μF = $24/11$ μF = 2.1818 μF.
- C_{10} = Equivalent capacitance of C_8 and C_9 = $C_8 + C_9$ = 3.6818 μF.
- C_{11} = Equivalent capacitance of C_2, C_3, and C_{10}

$$C_{11} = \frac{1}{\frac{1}{C_2}+\frac{1}{C_3}+\frac{1}{C_{11}}} = \frac{1}{\frac{1}{6}+\frac{1}{5}+\frac{1}{3.6818}}\mu\text{F} = 1.5667\mu\text{F}$$

- The equivalent capacitance is $C_{eq} = C_1 + C_{11} = 4.5667$ μF.

FIGURE 6.21
Circuit for EXAMPLE 6.4.



EXAMPLE 6.5

You have three capacitors with capacitance values of $C_1 = 0.1\ \mu F$, $C_2 = 0.22\ \mu F$, and $C_3 = 0.47\ \mu F$, respectively. List all the capacitance values that you can get from these three capacitors.

- **a**. Using one capacitor: $0.1\ \mu F$, $0.22\ \mu F$, $0.47\ \mu F$
- **b**. Using two capacitors:
  - In parallel: $0.1\ \mu F + 0.22\ \mu F = 0.32\ \mu F$, $0.1\ \mu F + 0.47\ \mu F = 0.57\ \mu F$, $0.22\ \mu F + 0.47\ \mu F = 0.69\ \mu F$
  - In series: $C_1C_2/(C_1 + C_2) = 0.06875\ \mu F$, $C_1C_3/(C_1 + C_3) = 0.0825\ \mu F$, $C_2C_3/(C_2 + C_3) = 0.1499\ \mu F$
- **c**. Using three capacitors:
  - All three in parallel: $C_1 + C_2 + C_3 = 0.79\ \mu F$
  - All three in series: $C_1C_2C_3/(C_1C_2 + C_1C_3 + C_2C_3) = 0.06\ \mu F$
  - Two in parallel, series with third: $0.32 \times 0.47/(0.32 + 0.47)\ \mu F = 0.1904\ \mu F$
    $0.57 \times 0.22/(0.57 + 0.22)\ \mu F = 0.1587\ \mu F$, $0.69 \times 0.1/(0.69 + 0.1)\ \mu F = 0.0873\ \mu F$
  - Two in series, parallel with third: $0.06875\ \mu F + 0.47\ \mu F = 0.5387\ \mu F$,
    $0.0825\ \mu F + 0.22\ \mu F = 0.3025\ \mu F$, $0.1499\ \mu F + 0.1\ \mu F = 0.2499\ \mu F$

Example 6.5: All Possible Combinations

Given three capacitors ($C_1 = 0.1$ μF, $C_2 = 0.22$ μF, $C_3 = 0.47$ μF), find all achievable capacitance values

Using One Capacitor

- 0.1 μF
- 0.22 μF
- 0.47 μF

Using Two Capacitors

Parallel: 0.32 μF, 0.57 μF, 0.69 μF

Series: 0.06875 μF, 0.0825 μF, 0.1499

Using Three Capacitors

All parallel: 0.79 μF

All series: 0.06 μF

Two parallel + one series: 0.1904 μF, 0.1587 μF, 0.0873 μF

Two series + one parallel: 0.5387 μF, 0.3025 μF, 0.2499 μF

Total unique values: 18 different capacitance values can be achieved through various series-parallel combinations of these three capacitors.

Summary

- **Energy Storage Device:** A capacitor is a passive element that can store energy. The energy is stored in the form of an electric field from the positive plate to the negative plate. The amount of energy stored on the capacitor depends on the capacitance and the voltage across the capacitor plates.
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- **Charge-Voltage Dynamics**: In a parallel plate capacitor, the amount of charge Q (in coulombs) on the plates is proportional to the voltage V (in volts) across the plates $Q = CV$

- The **current** through a capacitor is proportional to the time rate of change of the voltage across the capacitor.

$$i(t) = C\frac{dv(t)}{dt}$$

- The **voltage** across a capacitor is given by $$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda$$

- The **energy** stored in the capacitor is given by $$w(t) = \frac{1}{2}Cv^2(t)$$

Summary (Continued)

- The equivalent capacitance of *n* capacitors with capacitances C_1, C_2, …, C_n connected in series is given by

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}}$$

- The equivalent capacitance of *n* capacitors with capacitances C_1, C_2, …, C_n connected in parallel is given by

$$C_n = C_1 + C_2 + \ldots + C_n$$

Key Insights from the Series and Parallel Formulas

Series Connection

For n capacitors C_1, C_2, ..., C_n in series:

$$1/C_{eq} = 1/C_1 + 1/C_2 + \ldots + 1/C_n$$

- Same current through all
- Different voltages across each
- Equivalent capacitance is **smaller** than smallest capacitor
- Similar to parallel resistors

Parallel Connection

For n capacitors C_1, C_2, ..., C_n in parallel:

$$C_{eq} = C_1 + C_2 + \ldots$$

- Same voltage across all
- Different currents through each
- Equivalent capacitance is **larger** than largest capacitor
- Similar to series resistors

Next lecture: We'll explore inductors and complete our study of fundamental passive circuit elements.