Gauss’s Law and its comprehension

Gauss's Law

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q_{in}}{\varepsilon_0}$$

Q1: When can we calculate $\vec{E}$?

In some special cases.

Because $\oint_S \vec{E} \cdot \mathrm{d}\vec{S}$ is a summation, we can not get $\vec{E}$ from the summation. Just like we know $a + b = 8$, but we can not know the value of a and b. You need additional properties of a and b.

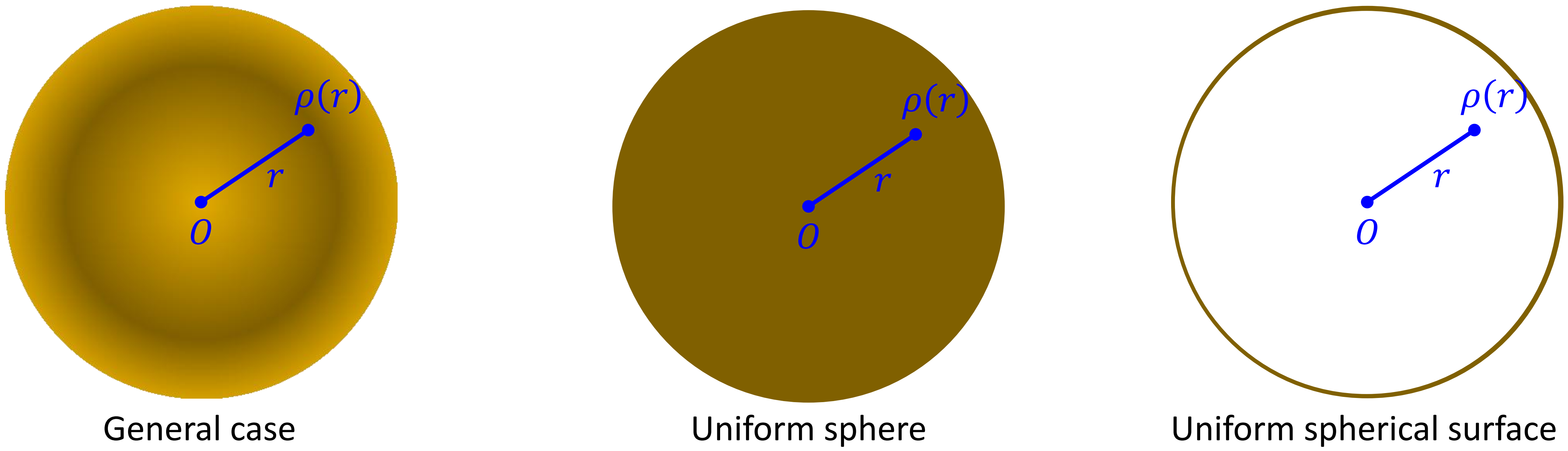
Q2: In which cases we can use Gauss's Law to calculate $\vec{E}$?

- Highly symmetric charge distribution
  - Centro-symmetric
  - Axially symmetric
  - Plane symmetric
- Other special charge distribution
  - Conduction materials, e.g. metal plate, metal sphere, …

Q3: What is centro-symmetric charge distribution? What are its electric field properties?

- Centro-symmetric charge distribution

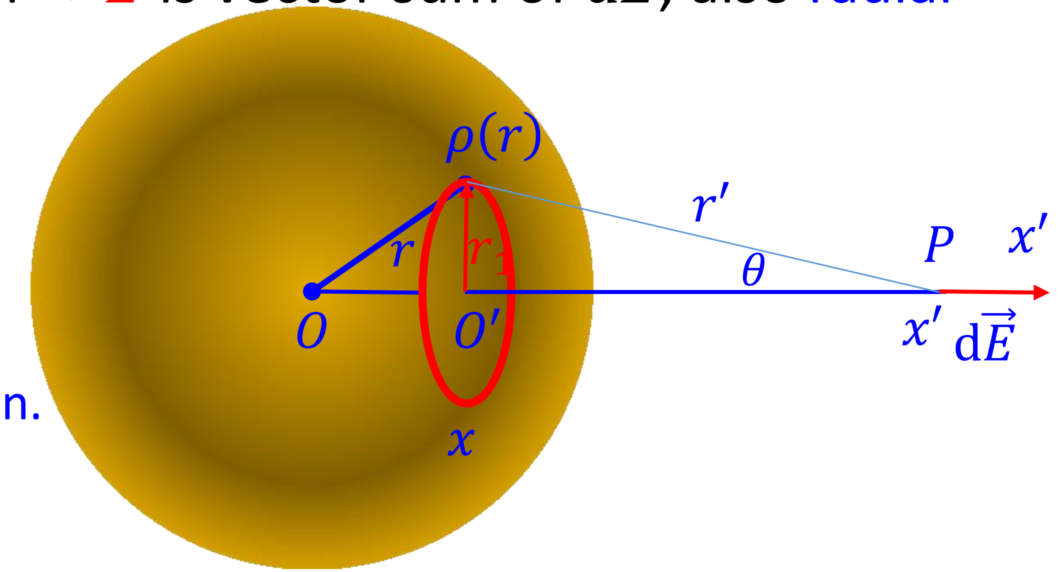
$$\rho = \rho(r)$$



r is the distance from the center!

Q3: What is centro-symmetric charge distribution? What are its electric field properties?

- Magnitude and direction of electric field?
  - Direction?
    - Using the conclude of uniformly charged thin ring
    - Sphere ⇒ thin rings ⇒ $\mathrm{d}\vec{E}$ in radial direction ⇒ $\vec{E}$ is vector sum of $\mathrm{d}\vec{E}$, also radial
  - Magnitude?
    - Method 1: integration in xyz corordinate



$$\mathrm{d}\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z}{r'^2}\vec{r'} \qquad \vec{E} = \int_{Sphere}\mathrm{d}\vec{E}$$

The calculation involving 3-fold integrations in x, y and z direction.

It is extremely complex!

Not suggested.

- Method 2: Using the conclude of uniformly charged thin ring

Between $x \sim x + \mathrm{d}x$, $r_1 \sim r_1 + \mathrm{d}r_1$, take a thin ring.

$$\mathrm{d}E = \mathrm{d}E_x = \frac{1}{4\pi\varepsilon_0}\frac{(\rho\cdot 2\pi r_1\mathrm{d}r_1\mathrm{d}x)\cdot(x'-x)}{[(x'-x)^2+r_1^2]^{\frac{3}{2}}}, \quad E = \int_{-R}^{+R}\left(\int_0^{\sqrt{R^2-x^2}}\frac{1}{4\pi\varepsilon_0}\frac{(\rho\cdot 2\pi r_1\mathrm{d}r_1\mathrm{d}x)\cdot(x'-x)}{[(x'-x)^2+r_1^2]^{\frac{3}{2}}}\mathrm{d}r_1\right)\mathrm{d}x$$

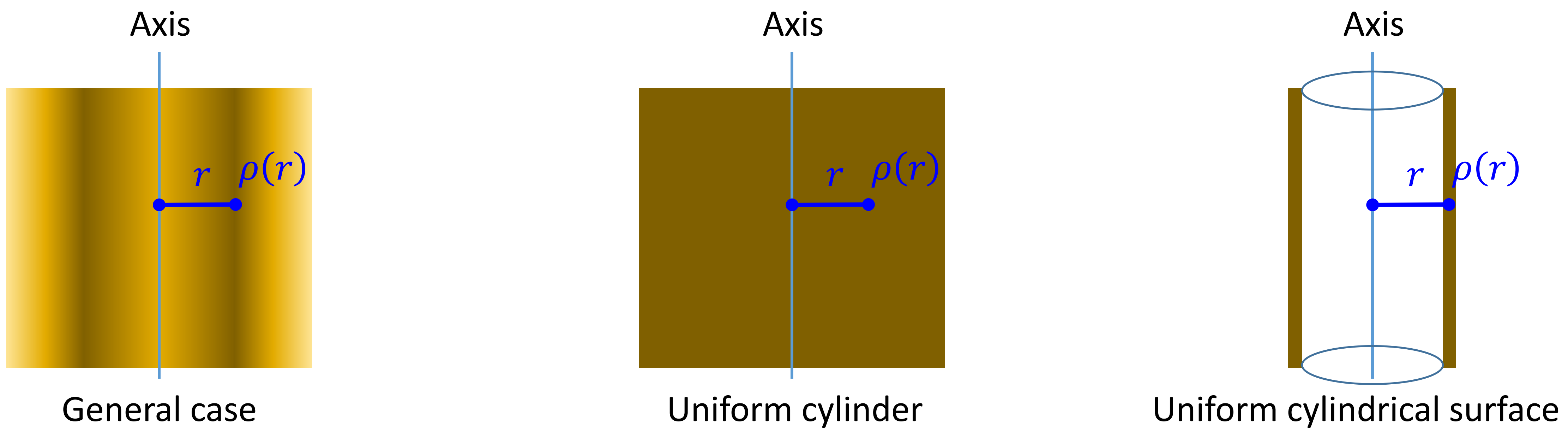
$\mathrm{d}V = \rho\cdot 2\pi r_1\mathrm{d}r_1\mathrm{d}x$ is the volume of thin ring

On the spherical surface with radius r', E is uniform everywhere!

Q4: What is axially symmetric charge distribution? What are its electric field properties?

- Axially symmetric charge distribution

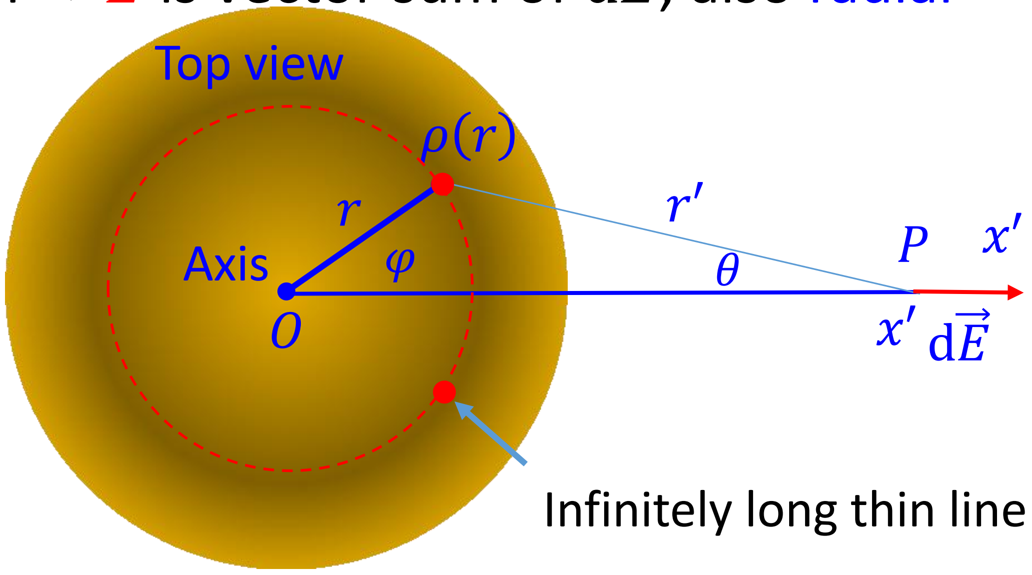
$$\rho = \rho(r)$$



r is the distance from the axis!

Q4: What is axially symmetric charge distribution? What are its electric field properties?

- Magnitude and direction of electric field?
  - Direction?
    - Using the conclude of uniformly charged infinitely long thin line
    - Cylinder $\Rightarrow$ thin lines $\Rightarrow$ $\mathrm{d}\vec{E}$ in radial direction $\Rightarrow$ $\vec{E}$ is vector sum of $\mathrm{d}\vec{E}$, also radial $\Rightarrow$ perpendicular to the axis
  - Magnitude?
    - Using the conclude of uniformly charged infinitely long thin line



In cross-sectional plane, at polar coordinates (r,φ), take an An infinitesimal area $dS = r\mathrm{d}r\mathrm{d}\varphi$, which is the cross-sectional area of long thin line.

$$\mathrm{d}E_x = \frac{\lambda}{2\pi\varepsilon_0}\cos\theta\,,\quad E = E_x = \int_0^R\left(\int_0^{2\pi}\frac{\lambda}{2\pi\varepsilon_0}\cos\theta\,\mathrm{d}\varphi\right)\mathrm{d}r$$

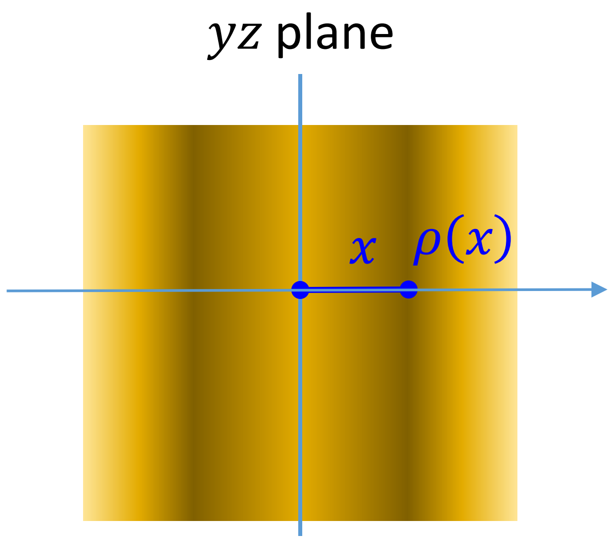
$\lambda = \rho\cdot(1\cdot r\mathrm{d}r\mathrm{d}\varphi)$ is the charge on thin line of length 1.

On the cylindrical surface with radius r', E is uniform everywhere!

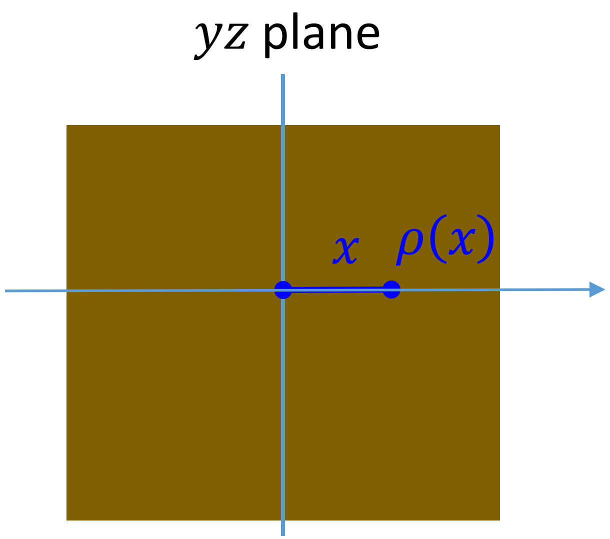
Q5: What is plane-symmetric charge distribution? What are its electric field properties?

- Plane-symmetric charge distribution

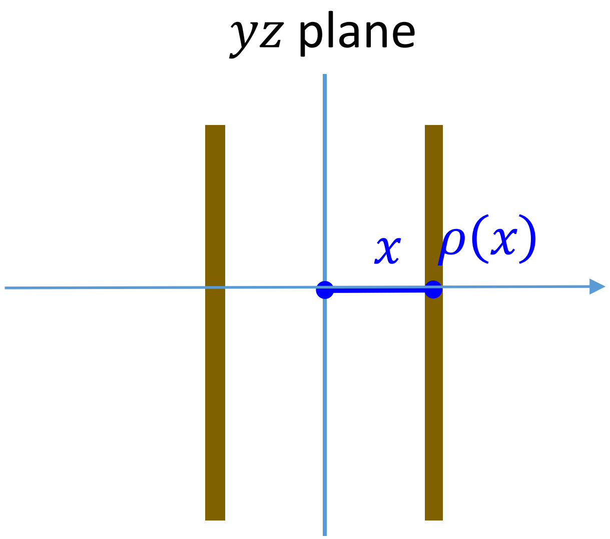
$$\rho = \rho(|x|) = \rho(x) = \rho(-x)$$



General case



Uniform thick plate

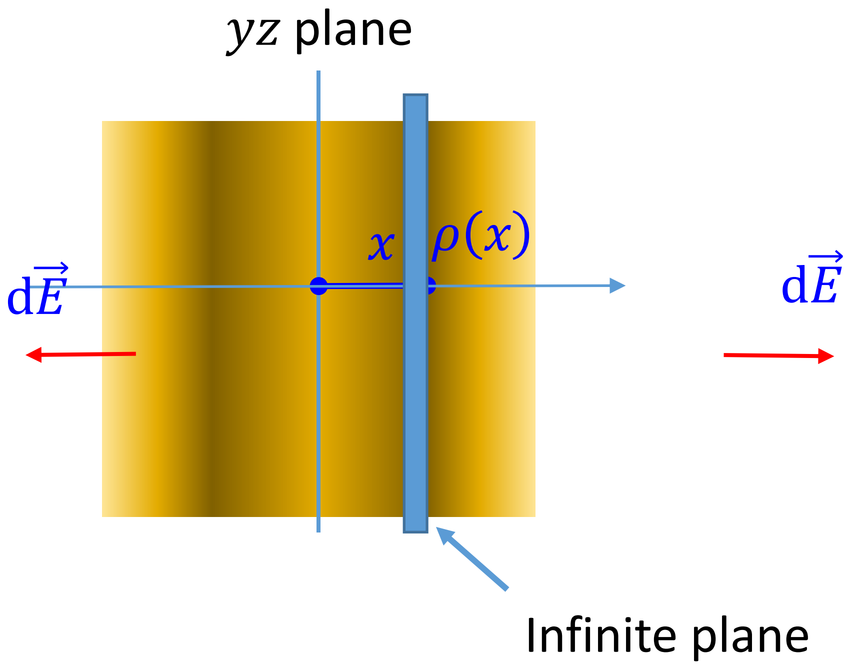


Two uniform surface

$|x|$ is the distance from the yz plane!

Q5: What is plane-symmetric charge distribution? What are its electric field properties?

- Magnitude and direction of electric field?
  - Direction?
    - Using the conclude of uniformly charged infinite plane
    - Thick plate ⇒ thin plate ⇒ infinite plane ⇒ uniform $\mathrm{d}\vec{E}$ in normal direction ⇒ $\vec{E}$ is uniform and in perpendicular to the plate
  - Magnitude?
    - Using the conclude of uniformly charged infinite plane



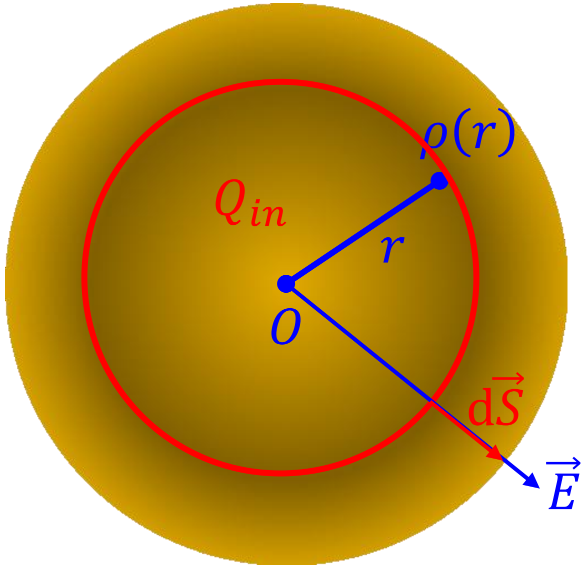
Between planes perpendicular to x axis at x and $x + \mathrm{d}x$,

$$\mathrm{d}E_x = \frac{\sigma}{2\varepsilon_0}, \quad E = E_x = \int_{-x_0}^{x_0} \frac{\sigma}{2\varepsilon_0}\mathrm{d}x$$

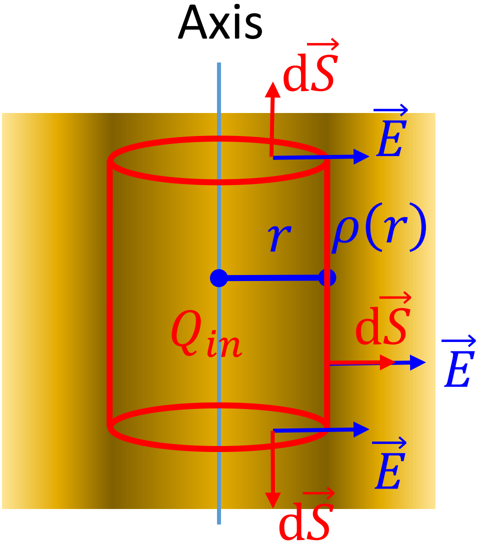
$\sigma = \rho \cdot (1 \cdot 1 \cdot \mathrm{d}x)$ is the charge on unit area.
x_0 is half-thickness.

On the plane surface perpendicular to x axis, E is uniform everywhere!

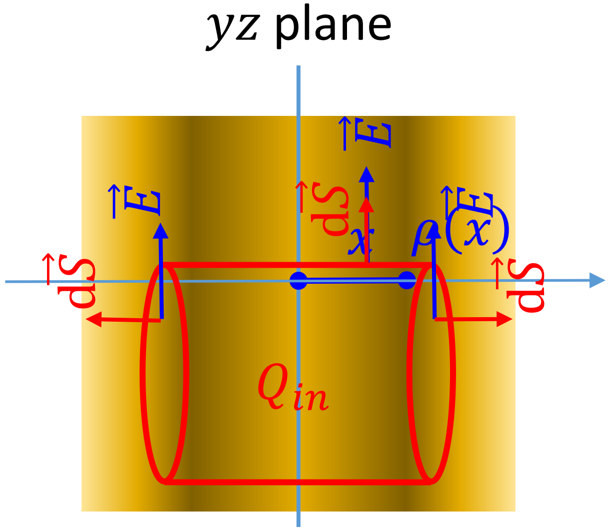
Q6: How to choose Gaussian surface? Why?



Centro-symmetry



Axial symmetry



Plane-symmetry

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \oint_S E \cdot dS$$
$$= E \cdot \oint_S dS = E \cdot 4\pi r^2$$

Spherical Gaussian surface

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \oint_{flats} \vec{E} \cdot d\vec{S} + \oint_{side} \vec{E} \cdot d\vec{S}$$
$$= \oint_S E \cdot dS = E \cdot \oint_S dS = E \cdot 2\pi r \cdot l$$

Cylindrical Gaussian surface

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \oint_{flats} \vec{E} \cdot d\vec{S} + \oint_{side} \vec{E} \cdot d\vec{S}$$
$$= \oint_{flats} \vec{E} \cdot d\vec{S} = 2E \cdot \oint_S dS = 2E \cdot \pi r^2$$

Cylindrical Gaussian surface

In these special cases, it’s convenient to calculate electric flux!