UNIVERSITY OF LONDON

CM1015

COMPUTER SCIENCE

Computational Mathematics

2020-2021

INSTRUCTIONS TO STUDENTS:

This paper consists of 5 questions. You should answer **ALL** the questions.

There are 100 marks available on this paper. The marks for each question are indicated at the end of the part in [.] brackets. Full marks will be awarded for complete answers to a total of 5 questions.

All answers need to be written clearly

The point of this assessment is to give you the opportunity to consolidate your learning and to assess your understanding of the topics. You do need to submit your answers as a pdf document (probably a single document is best), or photos of your work, or your work properly formatted using the maths mode of your word processor).

The total work is worth 100 marks distributed as follows:

* 15 marks for topic-1 (Number Bases)

* 30 marks for topic-2 (Sequences, Series and Mathematical induction)

* 15 marks for topic-3 (Modular Arithmetic)

* 20 marks for topic-4 (Angles, Triangles and Trigonometry)

* 20 marks for topic-5 (Graph Sketching and Kinematics)

Question 1

(a) A given number in base x can be converted to any other base y. According to the expansion method, if **abc.de** is any given number in base x, then write its value in base 10. [3]

(b) Convert the following numbers using number system conversions, show your answer in details: [6]

i. $(723)_8$ to hexa decimal system

ii. $(0.ABDF)_{16}$to decimal system

iii. Convert 0.375 to binary system

iv. Which digits from (0,1,2,3,4,5) are not allowed in Quinary system (base 5) representation.

v. $(11010.1011)_2$ to hexadecimal.

vi. $(257)_{10}$ to the binary system.

(c) Consider the binary number 10.0011 [4]

i. Convert the above number to the decimal system

ii. What are the place values of the digits 1 in the number 0.0011_2

iii. what is the sum of (1+1+1+1) in binary system

iv. calculate 101 divided by 10 using long division.

(d) Which one is the correct representation of a binary number from the following? [2]

i. 1101

ii. $(214)_2$

iii. $(0000)_2$

iv. $(11)^2$

Question 2

(a) Is $a_n = \frac{3n+2}{n-4}$ a general term of a sequence? Why? [2]

(b) Which term of the sequence with general term $\frac{3n-1}{5n+7}$ is $\frac{7}{12}$? [2]

(c) An arithmetic sequence has its 4^{th} term equal to 18 and its 12^{th} term equal to 50. Find its 99^{th} term. [4]

(d) State whether the following sequences are arithmetic, geometric or not any of them. Find the common ratio if it is a geometric sequence and find the common difference d if it is an arithmetic sequence. Then, find the next two terms. [6]

i. $-3, 3, -3, 3$

ii. $b_n = n^2 + 3$

iii. $\frac{-1}{2}, \frac{-5}{6}, \frac{-7}{6}$

(e) Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence? [3]

(f) The nineteenth term of a sequence is -52, and the fourth term is -7. The difference between consecutive terms in the sequence is constant. Find the 201^{st} term. [3]

(g) Show whether the following sequence is convergent or divergent.
$lim_{n\to\infty}(\frac{n-1}{n})$ [2]

(h) Is the following numbers 1,-4,9,-16,... represent a sequence, if so, find a formula for the n^{th} term of the sequence.

[2]

(i) Show by mathematical induction that for all positive integers n,
$\frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ [4]

(j) Find the remainder when 3^{123} is divided by 7. [2]

Question 3

(a) State whether the following statements are false or true, explain your answer: [6]

i. Given any integers $a,\ b,\ c$ and any positive integer n
If $a \equiv b\,(mod\,n)$ and $b \equiv c(mod\,n)$, then $a \equiv c(mod\,n)$.

ii. Suppose $a \equiv b(mod\,n)$ and $c \equiv d(mod\,n)$,then
$a + c \equiv b + c(mod\,n)$.

iii. $7x \equiv 12(mod\,7)$.

(b) Find the least positive value of x such that:
$71 \equiv x\,(mod\,8)$ [3]

(c) Calculate the multiplicative inverse of 168 in modulo 83. [3]

(d) Calculate the inverse of 4 modulo 15. Show your steps. [3]

Question 4

(a) A triangle has sides a = 2 and b = 3 and angle C = 60.
Find [4]

i. the length of side c.

ii. Find the sine of angle B using sine rules.

(b) If we have a triangle which has one of its side c = 2 and angles A = $\pi/4$ and B = $\pi/3$. Workout the length a of the side opposite A. [4]

(c) XYZ is a right angled triangle with $Y = 90°$. Given that $y = 85, \sin X = \frac{77}{85}$, find z, $Cos(Z)$ and the angle of Z. [6]

(d) Let g be a function with its domain $(0, \infty)$, defined by $g(x) = \frac{1}{x}$. [6]

i. Sketch the graph of g.

ii. Is g continuous at other points of its domain?

Question 5

(a) Consider the function $f : Z \to Z$ given by

$$f(x) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-3, & \text{if } n \text{ is odd} \end{cases}$$

[6]

i. Is f injective? Prove your answer

ii. Is f surjective? Prove your answer

(b) The function $f : R \to R$ is defined as follows:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x \leq 1. \\ 1, & \text{if } x > 1 \end{cases}$$

Plot the function, and say whether it is Bijective function or not. Explain your answer. [5]

(c) The velocity of a particle moving along the x axis varies according to the expression $v_x = 40 - 5t^2$, where v_x is in meters per second and t is in seconds.
Find the average acceleration in the time interval $t = 0$ to $t = 2$ sec. [4]

(d) An object moving with uniform acceleration has a velocity of $12\,cm/s$ in the positive x direction when its x coordinate is $3.00\,cm$. If its x coordinate 2 seconds later is $-5\,cm$, what is its acceleration? [5]

END OF PAPER

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Question 1

a) A given number in base x can be converted to any other base y. According to the expansion method, if abc.de is any given number in base x, then write its value in base 10.

$$ax^2 + bx^1 + cx^0 + dx^{-1} + ex^{-2}$$
$$ax^2 + bx + c + dx^{-1} + ex^{-2}$$

b) Convert the following numbers using number system conversions, show your answer in details:

I) $(723)_8$ to hexadecimal system.

First, $(723)_8$ to base 10

$7 \times 8^2 + 2 \times 8^1 + 3 \times 8^0 = 448 + 16 + 3 = (467)_{10}$

$(723)_8 = (467)_{10}$

Then, $(467)_{10}$ to base 16

Divide 467 by 16 over and over until getting a quotient of 0:

$467/16 = 29$, $R_{emainder} = 3$

$29/16 = 1$, $R_{emainder} = 13 = D$

$1/16 = 0$, $R_{emainder} = 1$

Taking from the last remainder to the first one, I get: 1D3

$(467)_{10} = (1D3)_{16}$

$(723)_8 = (467)_{10} = (1D3)_{16}$

$(723)_8 = (1D3)_{16}$

II) $(0.ABDF)_{16}$ to decimal system.

$0 \times 16^0 + A \times 16^{-1} + B \times 16^{-2} + D \times 16^{-3} + F \times 16^{-4}$

$0 + 0.625 + 0.04296875 + 0.003173828 + 0.000228882$

0.67137146

$(0.ABDF)_{16} = (0.67137146)_{10}$

III) Convert 0.375 to binary system.

Multiply the fractional part by 2 until getting an integer as a result

0.375 x 2 = 0.750

0.750 x 2 = 1.500 $\Rightarrow$ 1.500 – 1 = 0.500

0.500 x 2 = 1.000 = 1

Stop the multiplication as the result yields an integer(1) and take the integer part of each result from top to bottom

$\Rightarrow$ 0.375 = 0.011

$(0.375)_{10} = (0.011)_2$

IV) Which digits from (0, 1, 2, 3, 4, 5) are not allowed in quinary system (base5) representation?

Quinary system is made up of 5 digits, from 0 to 4 (0, 1, 2, 3, 4). Therefore, digit 5 is not allowed.

V) $(11010.1011)_2$ to hexadecimal.

First, $(11010.1011)_2$ to base 10

$1x2^4 + 1x2^3 + 0x2^2 + 1x2^1 + 0x2^0 + 1x2^{-1} + 0x2^{-2} + 1x2^{-3} + 1x2^{-4}$

$16 + 8 +0 + 2 + 0 + 1/2 + 0 + 1/2^3 + 1/2^4$

26 + 0.5 + 0.125 + 0.0625 = 26.6875

$(11010.1011)_2 = (26.6875)_{10}$

Then, $(26.6875)_{10}$ to base 16

Divide the integer part by 16 over and over until getting a quotient of 0:

26/16 = 1, $R_{emainder}$ = 10 = A

1/16 = 0, $R_{emainder}$ = 1

Taking from the last remainder to the first one, I get: 1A

$\Rightarrow (26)_{10} = (1A)_{16}$.....(x)

Then, multiply the fractional part by 16 until getting an integer as a result.

0.6875 x 16 = 11.0 = B

Stop the multiplication as the result yields an integer(11) and take the integer part of the result

$\Rightarrow$ $(0.6875)_{10}$ = 0.11 = $(0.B)_{16}$.....(y)

(x) + (y) = 1A + 0.B = 1A.B

$\Rightarrow$ $(26.6875)_{10}$ = $(1A.B)_{16}$ = $(11010.1011)_2$

$(11010.1011)_2$ = $(1A.B)_{16}$

VI) $(257)_{10}$ to the binary system.

Divide 257 by 2 over and over until getting a quotient of 0:

257/2 = 128, $R_{emainder}$ = 1

128/2 = 64, $R_{emainder}$ = 0

64/2 = 32, $R_{emainder}$ = 0

32/2 = 16, $R_{emainder}$ = 0

16/2 = 8, $R_{emainder}$ = 0

8/2 = 4, $R_{emainder}$ = 0

4/2 = 2, $R_{emainder}$ = 0

2/2 = 1, $R_{emainder}$ = 0

1/2 = 0, $R_{emainder}$ = 1

Taking from the last remainder to the first one, I get: 100000001

$(257)_{10}$ = $(100000001)_2$

c) Consider the binary number 10.0011

I) Convert the above number to the decimal system.

$(10.0011)_2$ to base 10

$1 \times 2^{1} + 0 \times 2^{0} + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4}$

$2 + 1/2^{3} + 1/2^{4}$

$2 + 1/8 + 1/16$

$32/16 + 2/16 + 1/16$

$36/16 = 9/4 = 2.1875$

$(10.0011)_2 = (2.1875)_{10}$

II) What are the place values of the digits 1 in the number 0.0011_2 ?

$0 \times 2^{0} + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4}$

$1 / 2^{3} + 1 / 2^{4}$

$1 / 8 + 1 / 16$

1/8th and 1/16th respectively.

III) What is the sum of (1+1+1+1) in binary system?

$1 + 1 + 1 + 1$

$1 + 1 = 10$

$\Rightarrow 10 + 10 = 100$

$(1 + 1 + 1 + 1)_2 = (100)_2$

IV) Calculate 101 divided by 10 using long division.

```
    10.1
 10|101.0
   -10      10 x 1 = 10
     1
   -  0     10 x 0 = 0
     10
   - 10     10 x 1 = 10
      0
```

$(101)_2 \div (10)_2 = (10.1)_2$

d) Which one is the correct representation of a binary number from the following?

I) 1101 Incorrect, it could be binary, ternary, decimal, and so on.

II) $(214)_2$ Incorrect, digits 2 and 4 don't belong to the binary system.

III) $(0000)_2$ Correct !!

IV) $(11)^2$ Incorrect, that's 11 to the second power.

Question 2

a) Is $a_n = \frac{3n+2}{n-4}$ a general term of a sequence?

- Yes, it is.

Why?

- Because I can find any of the terms of the sequence using the given general term formula, for example:

$a_1 = \frac{3(1)+2}{1-4}$ = -5/3

$a_2 = \frac{3(2)+2}{2-4}$ = -4

$a_3 = \frac{3(3)+2}{3-4}$ = -11

$a_4 = \frac{3(4)+2}{4-4}$ = undefined

$a_5 = \frac{3(5)+2}{5-4}$ = 17

$a_6 = \frac{3(6)+2}{6-4}$ = 10

$a_7 = \frac{3(7)+2}{7-4}$ = 23/3

$a_8 = \frac{3(8)+2}{8-4}$ = 13/2

and the sequence is: -5/3, -4, -11, undefined, 17, 10, 23/3, 13/2, …

b) Which term of the sequence with general term $\frac{3n-1}{5n+7}$ is $\frac{7}{12}$?

$\frac{3n-1}{5n+7} = \frac{7}{12}$

12(3n -1) = 7(5n + 7)

36n – 12 = 35n + 49

36n – 35n = 49 + 32

n = 61st. term

prove:

$\frac{3(61)-1}{5(61)+7} = \frac{183-1}{305+7} = \frac{182}{312} = \frac{7}{12}$

c) An arithmetic sequence has its 4th. term equal to 18 and its 12th. term equal to 50. Find its 99th. term.

a_4 = 18; a_{12} = 50; a_{99} = ?

$a_{12} = a_4 + 8d$

50 = 18 + 8d

50 – 18 = 8d $\Rightarrow$ 32 = 8d $\Rightarrow$ d = 32 / 8

common difference: d = 4

$\Rightarrow a_1 = a_4 - 3d$

$a_1 = 18 - 3(4) \Rightarrow a_1 = 18 - 12$

a_1 = 6, which is the 1st. term

⇒ using the formula for the $n_{th.}$ term of an arithmetic sequence:

$a_n = a_1 + (n - 1)d$

$a_{99} = 6 + (99 - 1)4$

$a_{99} = 6 + (98)4$

$a_{99} = 6 + 392$

$a_{99} = 398$

d) State whether the following sequences are arithmetic, geometric or not any of them. Find the common ratio if it is a geometric sequence and find the common difference d if it is an arithmetic sequence. Then, find the next two terms.

I) $-3, 3, -3, 3$
$a_2 - a_1 = 3 - (-3) = 6$
$a_3 - a_2 = -3 - 3 = -6$
$a_4 - a_3 = 3 - (-3) = 6$
There is not a constant common difference.

$a_2 / a_1 = 3 / -3 = -1$
$a_3 / a_2 = -3 / 3 = -1$
$a_4 / a_3 = 3 / -3 = -1$

There is a constant common ratio = -1

Thus, it is a geometric sequence.

Next two terms are: -3, 3

II) $b_n = n^2 + 3$

$b_1 = 1^2 + 3 = 4$
$b_2 = 2^2 + 3 = 7$
$b_3 = 3^2 + 3 = 12$
$b_4 = 4^2 + 3 = 19$
$b_5 = 5^2 + 3 = 28$

$b_2 - b_1 = 7 - 4 = 3$
$b_3 - b_2 = 12 - 7 = 5$
There is not a constant common difference.

$b_2 / b_1 = 7 / 4 = 1.75$
$b_3 / b_2 = 12 / 7 = 1.71$
There is not a constant common ratio.

Thus, the sequence is neither arithmetic, nor geometric.

III) $\frac{-1}{2}, \frac{-5}{6}, \frac{-7}{6}$

$a_2 - a_1 = \frac{-5}{6} - \frac{-1}{2} = \frac{-1}{3}$

$a_3 - a_2 = \frac{-7}{6} - \frac{-5}{6} = \frac{-1}{3}$

There is a constant common difference $d = \frac{-1}{3}$

Thus, it is an arithmetic sequence.

Next two terms are: $\frac{-3}{2}$, $\frac{-11}{6}$

e) Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence?

Using the formula for the $n_{th.}$ term of a geometric sequence:

$a_n = a_1.r^{n-1}$

and finding the general term formula for the sequence:

$b_n = b_1.q^{n-1} \Rightarrow b_n = \frac{1}{9}.3^{n-1}$

$b_n = \frac{1}{3}.\frac{1}{3}.3^{n-1} \Rightarrow b_n = 3^{-1}.3^{-1}.3^{n-1} \Rightarrow b_n = 3^{-1 + -1 + n-1}$

$b_n = 3^{n-3}$, so this is the general term formula for the sequence.

$243 = 3^{n-3}$

$\log(243) = \log(3^{n-3})$

$\log(243) = (n-3).\log(3)$

$$n-3 = \frac{\log(243)}{\log(3)}$$

$n-3 = 5 \Rightarrow n = 5 + 3$

$n = 8$

Proving by using the general term formula:

$b_n = 3^{n-3}$

$b_8 = 3^{8-3} \Rightarrow b_8 = 3^5$

$b_8 = 243$

Yes, indeed 243 is a term of the sequence.

f) The nineteenth term of a sequence is -52, and the fourth term is -7. The difference between consecutive terms in the sequence is constant. Find the 201st. term.

$a_{19} = -52$
$a_4 = -7$
d = constant, so this is an arithmetic sequence.
$a_{201} =?$

19 - 4 = 15, so between a_4 and a_{19} there are 15 terms having a constant difference.

Then, $a_{19} - a_4$

$-52 - (-7) = -45$

$\Rightarrow$ -45/15 = -3, which is de common difference d $\Rightarrow$ d = 3

Using the formula for the $n_{th.}$ term of an arithmetic sequence to find

a_1:

$a_n = a_1 + (n -1)d$

$a_4 = a_1 + (4 -1)d$

$-7 = a_1 + (4 -1).-3$

$-7 = a_1 + (3).-3$

$-7 = a_1 - 9$

$-7 + 9 = a_1$

$a_1 = 2$, which is the 1st. term.

Using again the formula for the $n_{th.}$ term of an arithmetic sequence to find a_{201}:

$a_n = a_1 + (n -1)d$

$a_{201} = 2 + (201 -1).-3$

$a_{201} = 2 + (200).-3$

$a_{201} = 2 + (-600)$

$a_{201} = -598$, which is the 201st. term.

g) Show whether the following sequence is convergent or divergent.

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)$$

$$n = 1; \frac{1-1}{1} = \frac{0}{1} = 0$$

$$n = 2; \frac{2-1}{2} = \frac{1}{2} = 0.5$$

$$n = 3; \frac{3-1}{3} = \frac{2}{3} = 0.666$$

$$n = 4; \frac{4-1}{4} = \frac{3}{4} = 0.75$$

$$n = 5; \frac{5-1}{5} = \frac{4}{5} = 0.8$$

$$n = 100; \frac{100-1}{100} = \frac{99}{100} = 0.99$$

$$n = 1000; \frac{1000-1}{1000} = \frac{999}{1000} = 0.999$$

$$n = 10000; \frac{10000-1}{10000} = \frac{9099}{10000} = 0.9999$$

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = 1$$

As n tends to infinity, the limit of the sequence is 1. So, the sequence is convergent.

h) Is the following numbers 1,-4,9,-16,... represent a sequence, if so, find a formula for the $n^{th.}$ term of the sequence.

1, -4, 9, -16 is a sequence of consecutives $n^{th.}$ terms to the power of 2; however, the $n^{th.}$ terms are alternately positive and negative values. So it looks like each of them have a coefficient of 1 and -1 respectively.

but:
$1.n^2$ will make all terms positives and,
$-1.n^2$ will make all terms negatives.

What about using coefficient -1 to the power of an even and odd number alternately using (n+1):

So, if:
$-1^{\text{even number}}.n^2$, I'll get a positive term.

and if:
$-1^{\text{odd number}}.n^2$, I'll get a negative term.

So, I would have the general term formula: $-1^{n+1}.n^2$

Let's check it:
$n = 1, \Rightarrow -1^{1+1}.1^2 = 1$
$n = 2, \Rightarrow -1^{2+1}.2^2 = -4$
$n = 3, \Rightarrow -1^{3+1}.3^2 = 9$
$n = 4, \Rightarrow -1^{4+1}.4^2 = -16$

Indeed, the general term formula is: n^{th} term $= -1^{n+1}.n^2$

i) Show by mathematical induction that for all positive integers n,

$$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$, for all positive integers n

Givens: let the given statement be P(n)

P(n): $\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$, for all positive integers n

Base case: let's proof by plugin in P(2) = $\frac{1}{2}+\frac{1}{2^2}=1-\frac{1}{2^2}$

$$\frac{1}{2}+\frac{1}{4}=1-\frac{1}{4}$$
$$\frac{2}{4}+\frac{1}{4}=\frac{4}{4}-\frac{1}{4}$$
$$\frac{3}{4}=\frac{3}{4}$$

let's observe that for P(2) the proof yields a value of true; but let's see if the conclusion is also true for any positive integer n.

Induction: assuming that P(n) is true for some $k \in \mathbb{N} : k \in \mathbb{N}$

P(k): $\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$

Let's prove that P(k+1) is true:

adding $\frac{1}{2^{k+1}}$ to both sides, I get:

$$\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

Solving for $1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$,

$$\frac{2^k}{2^k}-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$\frac{2^{k+1}-1}{2^{k+1}}$$

$$\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$1-\frac{1}{2^{k+1}}$$

Thus, P(k+1) is true, whenever P(k) is true.
Hence, by the principle of mathematical induction, P(n) is true for all positive integers n or $\forall n \in \mathbb{N}[P(n) \rightarrow P(n+1)]$

j) Find the remainder when 3^{123} is divided by 7.

Let's then find $3^{123} (\mathrm{mod}\, 7)$

$3^1 \equiv 3 (\mathrm{mod}\, 7)$

$3^2 \equiv 2 (\mathrm{mod}\, 7)$

$3^3 \equiv 3^2 \text{ x } 3 \equiv 2 \text{ x } 3 \equiv 6 (\mathrm{mod}\, 7)$

$3^4 \equiv (3^2)^2 \equiv 2^2 \equiv 4 (\mathrm{mod}\, 7)$

$$3^5 \equiv 3^3 \text{ x } 3^2 \equiv 6 \text{ x } 2 \equiv 12 \equiv 5 \pmod 7$$
$$3^6 \equiv (3^2)^3 \equiv 2^3 \equiv 1 \pmod 7$$

Getting 1 mod 7 means that, I now can do powers and powers of powers of each 6 and powers of 1 will always will be 1 mod 7. So that means that from here onwards, any power of 3 is going to be 1 mod 7. The above is a clear reference to an extension of Fermat's Little Theorem which is condensed in the following formula:

$R\left[\frac{A^{(P-1)K}}{P}\right] = 1$ where A and P are coprime.

A = 3; P = 7 and K is any natural number.

Then,

$$3^{123} \pmod 7$$
$$(3^6)^{20} \equiv 1^{20} \equiv 1 \pmod 7$$
$$3^{123} \equiv 3^{120} x 3^3 \equiv 1 x 3^3 \equiv 1 x 27 \equiv 6 \pmod 7$$

So, $3^{123} \pmod 7 = 6$ or the remainder of 3^{123}(mod 7) = 6

Using Fermat's Little Theorem formula it will yield the same result:

$$R\left[\frac{3^{(7-1)20}}{7}\right] = 1$$

$$\Rightarrow R\left[\frac{3^{(6)20}}{7}\right] x R\left[\frac{3^3}{7}\right]$$

$$R\left[\frac{3^{120}}{7}\right] x R\left[\frac{3^3}{7}\right]$$

$$1 x R\left[\frac{3^3}{7}\right]$$

$$1xR\left[\frac{27}{7}\right]$$

$$1x6=6$$

Indeed, I get a remainder of 6 for 3^{123}(mod 7)

Question 3

a) State whether the following statements are false or true, explain your answer:

I) Given any integers a, b, c and any positive integer n.
If a ≡ b (mod n) and b ≡ c(mod n), then a ≡ c(mod n).

a ≡ b (mod n) and b ≡ c(mod n), then a ≡ c(mod n) is true

The congruence relation has properties in the equality relation.

Transitive: if a=b and b=c, then a=c, $\forall a,b,c \in \mathbb{Z}$

The congruence module has the same transitive property.

Transitive: if a ≡ b (mod n) and b ≡ c(mod n), then a ≡ c(mod n), $\forall a,b,c \in \mathbb{Z}$

if a ≡ b (mod n) and b ≡ c(mod n), then a ≡ c(mod n), then

a - b = nK…..(1)
b - c = nK'…..(2)

(1) + (2)

a – b + b - c = nK + nK'

a – c = n(k + K')

a ≡ c(mod n).

Then, the statement a ≡ c(mod n) is true.

II) Suppose $a \equiv b(\text{mod } n)$ and $c \equiv d(\text{mod } n)$,then
$a + c \equiv b + d(\text{mod } n)$

if $a \equiv b(\text{mod } n)$ and $c \equiv d(\text{mod } n)$, then $a + c \equiv b + d(\text{mod } n)$

By the definition of congruence, there are integer y and z such
$a - b = yn$…..(1) and
$c - d = zn$…..(2)

then, (1) + (2)
$a - b + c - d = yn + zn$
$a - b + c - d = n(y + z)$

adding b + d to both sides of the equation:
$a - b + c - d + b + d= n(y + z) + b +d$
$a + c = b + d(y + z)$

hence, $a + c \equiv b + d(\text{mod } n)$ is true.

III) $7x \equiv 12(\text{mod } 7)$.

$ax \equiv b(\text{mod } n)$

finding the GCD of a and n = GCD(7, 7) = 7

testing b to see if it can be evenly divided by the GCD
$12 / 7 = 1$, $R_{emainder} = 5$
12 is not exactly divisible by 7, so there is not solution.

Therefore, $7x \equiv 12(\text{mod } 7)$ is false.

b) Find the least positive value of x such that:
$71 \equiv x(\text{mod } 8)$

Applying the symmetric property:
$x \equiv 71(\text{mod } 8) \Rightarrow x = 71(\text{mod } 8)$

x = remainder of 71 / 8
$x = 71 / 8 = 8$, $R_{emainder} = 7$; $x = 7$

Also, $71 \equiv 7(\text{mod } 8)$

Then, 7 is the least positive value.

c) Calculate the multiplicative inverse of 168 in modulo 83.

Concept:
given: a(mod b)
the multiplicative inverse is the number that multiplies a and then divided by b yields a remainder of 1

168 ÷ 83 yields a remainder of 2 $\Rightarrow$ $168 \equiv 2 \pmod{83}$

$168^{-1} \equiv 2^{-1} \equiv x \pmod{83}$

$2x = 83 + 1 \rightarrow$ the 1 to be the remainder
$2x = 84$
$x = 42$

$42 \times 168 \equiv 1 \pmod{83}$
$168^{-1} \equiv 2^{-1} \equiv 42 \pmod{83}$

prove:
$42 \times 168 = 7056$
$7056 / 83 = 85$, $R_{emainder} = 1$

Therefore, 42 is the multiplicative inverse of 168 mod 83.

d) Calculate the inverse of 4 modulo 15. Show your steps.

the multiplicative inverse is going to be a number that multiplied by 4 and then divided by 15 will yield a remainder of 1

4 ÷ 15 yields a remainder of 4 $\Rightarrow$ $4 \equiv 4 \pmod{15}$

$4^{-1} \equiv 4 \equiv x \pmod{15}$

$4x = 15 + 1 \rightarrow$ the 1 to be the remainder
$4x = 16$
$x = 4$

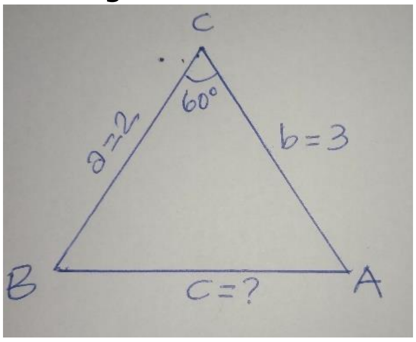
$4 \times 4 \equiv 1 \pmod{15}$
$4^{-1} \equiv 4^{-1} \equiv 4 \pmod{15}$

prove:
4 x 4 = 16
16 / 15 = 1, $R_{emainder}$ = 1

Therefore, 4 is the multiplicative inverse of 16[illegible] mod 15.

Question 4

a) A triangle has sides a = 2 and b = 3 and angle C = 60.



Find:

I) The length of side c.

$c^2 = a^2 + b^2 - 2ab.Cos(C)$
$c^2 = 2^2 + 3^2 - 2(2)(3).Cos(60°)$
$c^2 = 4 + 9 - 12 \times 0.5$
$c^2 = 13 - 6$
$c^2 = 7$
$c = \sqrt{7}$
$c = 2.6458$ linear units.

II) Find the sine of angle B using sine rules.

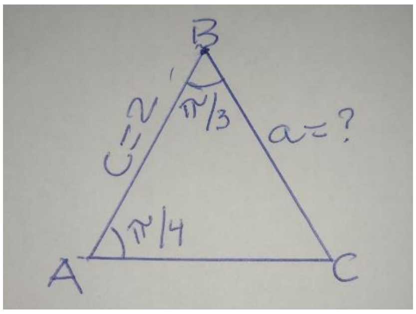
$$\frac{c}{Sin(C)} = \frac{b}{Sin(B)}$$

$$\frac{2.6458}{\frac{\sqrt{3}}{2}} = \frac{3}{Sin(B)} \Rightarrow Sin(B) = \frac{\frac{3\sqrt{3}}{2}}{2.6458} = \frac{2.5981}{2.6458} = 0.98197$$

$Sin(B) = 0.981971426$

$$bonus:$$
$$Sin^{-1}(0.981971426) = 79.1039$$
$$B = 79.1039^{o}$$

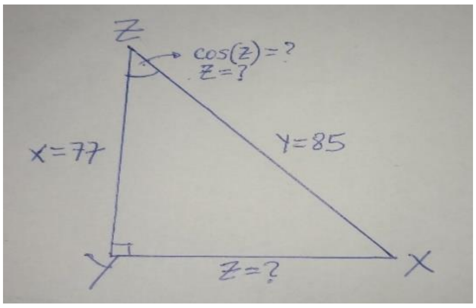
b) If we have a triangle which has one of its side c = 2 and angles A = π/4 and B = π/3. Workout the length a of the side opposite A



$$A = \frac{\pi}{4} = 45^{o}$$
$$B = \frac{\pi}{3} = 60^{o}$$
$$C = 180^{o} - A - B$$
$$C = 180^{o} - 45^{o} - 60^{o}$$
$$C = 75^{o} = \frac{5\pi}{12}$$
$$\frac{a}{Sin(A)} = \frac{c}{Sin(C)}$$

$$a = \frac{Sin(A)\times c}{Sin(C)} = \frac{Sin(45^{\circ})\times 2}{Sin(75^{\circ})} = \frac{0.7071\times 2}{0.9659} = \frac{1.4142}{0.9659}$$
$a = 1.4641$ linear units

c) XYZ is a right angled triangle with Y = 90◦. Given that y = 85, sin X = 77/85, find z, Cos(Z) and the angle of Z.



$$z^2 = 85^2 - 77^2$$

$$z^2 = 7225 - 5929$$

$$z^2 = 1296$$

$$z = \sqrt{1296}$$

$z = 36$ linear units

$$Cos(Z) = \frac{77}{85}$$

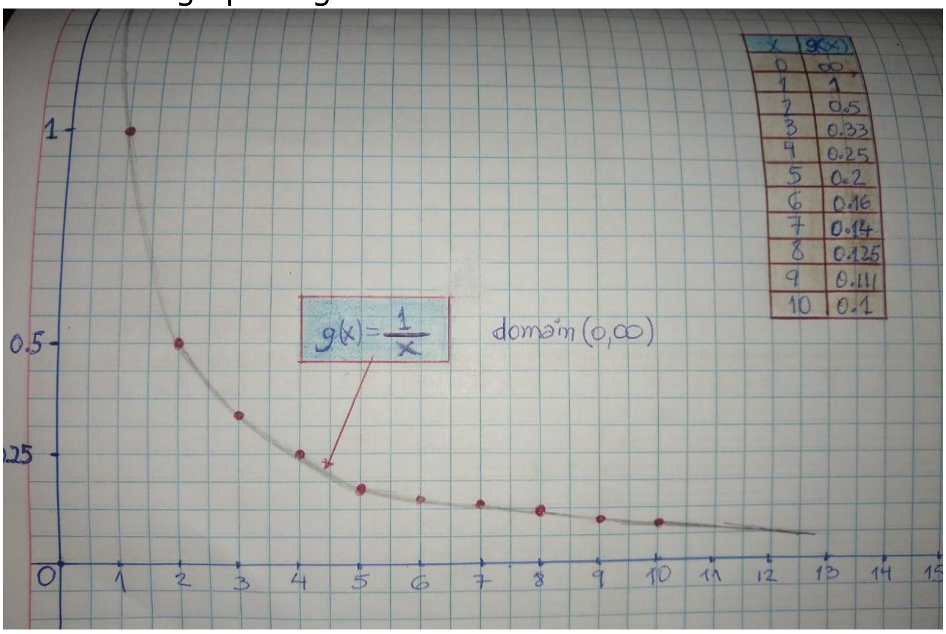
$$Cos(Z) = 0.9058$$

$$Z = Cos^{-1}(0.9058)$$

$$Z = 25.0576^{o}$$

d) Let g be a function with its domain (0, ∞), defined by g(x) = 1/x

I) Sketch the graph of g.



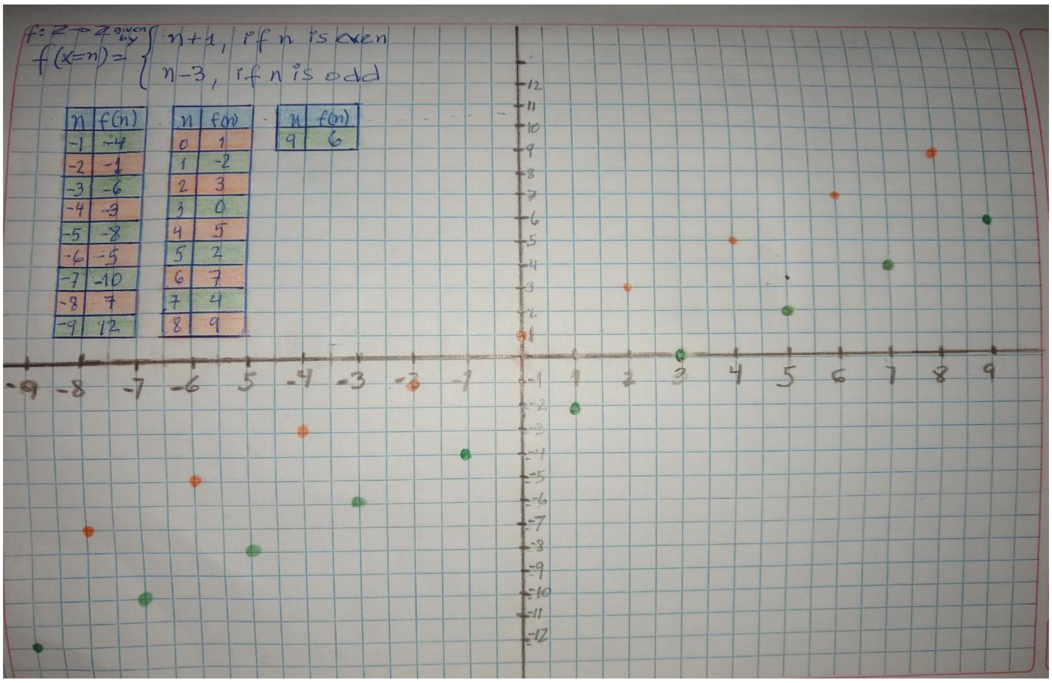
II) Is g continuous at other points of its domain?

Yes, g(x) = 1/x is continuous in its domain (0, ∞)

Question 5

a) Consider the function $f:\mathbb{Z}\rightarrow\mathbb{Z}$ given by

$$f(\mathrm{n}=\mathrm{x})=\begin{cases} n+1, & if\ n\ is\ even \\ n-3, & \text{if } n\ is\ odd \end{cases}$$



I) Is f injective? Prove your answer

f(n) is injective because $f(n)=f(y),\mathrm{n}=\mathrm{y}$

Example: $f(n)=n+1, if\ n\ is\ even$ from the set of integer numbers $\mathbb{Z}\ to\,\mathbb{Z}$ is an injective function.

Let's say $n=4$, then $f(n)=5$

Now, I say that $f(\mathrm{y})=5$, what's the value of y?

$$f(\mathrm{n})=f(y)=5$$

$$5=y+1$$

$$y=4\Rightarrow y=n$$

Also, the function passes the horizontal line test to prove that each value of y has a unique correspondence with its correspondent n value.

II) Is f surjective? Prove your answer

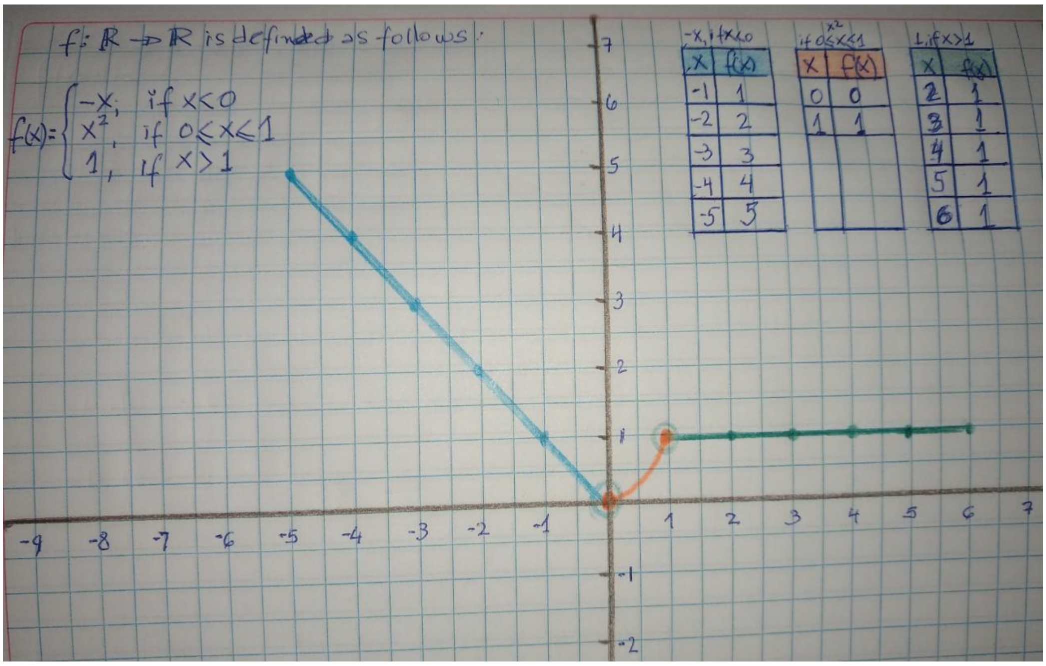
The function f(n) is also surjective because there is at least one n such that f(n) = y
Example:
if n = -5, f(n) = -8
if n = 5, f(n) = 2
if n = -4, f(n) = -3
if n = 4, f(n) = 5

So, being the function injective and surjective at the same time, I can say that the function is also bijective.

b) The function f : R → R is defined as follows:

$$f(x) = \begin{cases} -x, & if\ x \prec 0 \\ x^2, & if\ 0 \leq x \leq 1 \\ 1, & if\ x \succ 1 \end{cases}$$

Plot the function, and say whether it is bijective function or not. Explain your answer.



Knowing that a function is bijective when it is injective and surjective at the same time.

I do the horizontal line test to check whether f(x) is injective at
$f(\mathrm{x}) = \{x^2, \quad if\ 0 \le x \le 1$

$$if \begin{cases} y = 1, & x = -1 \\ y = 1, & x = 1 \end{cases}$$

So, at y=1, I have 2 values for x; x=-1 and x=1 respectively. Since the fact that I have 2 values of x for one value of y, within the domain [0, 1], the function is not injective, therefore the function is not bijective either.

c) The velocity of a particle moving along the x axis varies according to the expression $v_x = 40 - 5t^2$, where v_x is in meters per second and t is in seconds.
Find the average acceleration in the time interval t = 0 to t = 2 sec.

$V_1 = 40 - 5(t_1)^2$

$V_1 = 40 - 0$

$V_1 = 40$ m/sec

$V_2 = 40 - 5(t_2)^2$

$V_2 = 40 - 20$

$V_2 = 20$ m/sec

$dt = 2 - 0$

$dt = 2$ secs.

$a = \frac{dv}{dt} = \frac{20-40}{2}$

$a = \frac{-20}{2}$

$a_{verage}\ a_{cceleration} = -10$ m/sec^2

d) An object moving with uniform acceleration has a velocity of 12 cm/s in the positive x direction when its x coordinate is 3.00 cm. If its x coordinate 2 seconds later is −5 cm, what is its acceleration?

$u = 12$ cm/sec

$t = 2$ secs

$ds = s2 - s1$

$s = -3 - 5 = -8$ cm

$$s = ut + \frac{at^2}{2}$$

$$-8 = 12 \times 2 + \frac{a \times 2^2}{2}$$

$$-8 = 24 + \frac{a \times 4}{2}$$

$$-8 = 24 + 2a$$

$$-32 = 2a$$

$a = -16$ cm/sec^2

$a_{cceleration} = -0.16$ m/sec^2