Foundation Maths 6th Edition

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Solutions to Challenge Exercises

Chapter 6. Indices

Simplify the following expressions

a) $\sqrt{\left(\frac{27}{y^3}\right)^{-2/3}}$

b) $\left(\frac{48x^7y^{-1}}{3x^{-1}y^{-3}}\right)^{-3/4}$

c) $\sqrt{a\sqrt{\frac{b}{a}}\,\frac{c^2\sqrt{ab}}{b^{-1}c^{-2}}}$

Solution

a)

$$\sqrt{\left(\frac{27}{y^3}\right)^{-2/3}} = \sqrt{\left(\frac{y^3}{27}\right)^{2/3}} = \sqrt{\frac{y^2}{9}} = \frac{y}{3}.$$

b)

$$\left(\frac{48x^7y^{-1}}{3x^{-1}y^{-3}}\right)^{-3/4} = (16x^8y^2)^{-3/4} = \frac{1}{(16x^8y^2)^{3/4}} = \frac{1}{8x^6y^{1.5}}$$

c)

$$a\sqrt{\frac{b}{a}}\,\frac{c^2\sqrt{ab}}{b^{-1}c^{-2}} = ab^2c^4$$

and so

$$\sqrt{a\sqrt{\frac{b}{a}}\,\frac{c^2\sqrt{ab}}{b^{-1}c^{-2}}} = \sqrt{ab^2c^4} = \sqrt{a}bc^2$$

Chapter 7. Simplifying algebraic expressions

Remove the brackets and simplify the following expressions

a) $(x^2+1)(x^2-1)$

b) $(\frac{1}{x}-1)(x^2-x)$

c) $(x^2+x+1)(x^2-x-1)$

Solution

a) $(x^2+1)(x^2-1) = x^4-1$

b) $(\frac{1}{x}-1)(x^2-x) = x-1-x^2+x = -x^2+2x-1$

c)

$$\begin{aligned}(x^2+x+1)(x^2-x-1) &= x^2(x^2-x-1)+x(x^2-x-1)+1(x^2-x-1)\\ &= x^4-x^3-x^2+x^3-x^2-x+x^2-x-1\\ &= x^4-x^2-2x-1\end{aligned}$$

Chapter 8. Factorisation

a) All cubic expressions can be factorised into the product of a linear factor and a quadratic factor. Given $(x-4)$ is a factor of $x^3 - x^2 - 10x - 8$ find a quadratic expression, $Q = ax^2 + bx + c$, which is also a factor. [Hint: Compare coefficients.]

Show that Q factorises and hence state three linear factors of $x^3 - x^2 - 10x - 8$.

b) Factorise (i) $x^2 - 1$ (ii) $x^4 - 1$.

c) Factorise $4x^2 + 4x - 3$. Use this result to deduce the factors of $x^2 + x - \frac{3}{4}$.

Solution

a)

$$x^3 - x^2 - 10x - 8 = (x-4)(ax^2 + bx + c).$$

Equating the coefficient of x^3 on both sides we obtain $a = 1$. Equating the constant term on both sides we obtain $c = 2$. Equating the coefficient of x on both sides we see $-10 = -4b + c$ from which $b = 3$. So

$$x^3 - x^2 - 10x - 8 = (x-4)(x^2 + 3x + 2).$$

We now factorise $x^2 + 3x + 2$ to obtain $(x+1)(x+2)$ and so

$$x^3 - x^2 - 10x - 8 = (x-4)(x+1)(x+2).$$

b) (i) $x^2 - 1 = (x-1)(x+1)$.

(ii) $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2 + 1)$

c)

$$4x^2 + 4x - 3 = (2x-1)(2x+3).$$

$$4x^2 + 4x - 3 = 4(x^2 + x - \frac{3}{4}) = (2x-1)(2x+3) = 2(x - \frac{1}{2})\,2(x + \frac{3}{2})$$

and so

$$x^2 + x - \frac{3}{4} = (x - \frac{1}{2})(x + \frac{3}{2}).$$

Chapter 9. Algebraic fractions

a) Given that $(x-2)$ is a factor of $x^3 - 6x^2 + 12x - 8$ find the other factors.

b) Find the partial fractions of

$$\frac{x^2 - 7x + 12}{x^3 - 6x^2 + 12x - 8}$$

Solution

a) Let $x^3 - 6x^2 + 12x - 8 = (x-2)(ax^2 + bx + c)$. Then examining the coefficients of x^3 we see that $a = 1$. Examining the constant terms on the left and right-hand sides, we see that $-8 = -2c$ and hence $c = 4$. Examining the coefficients of x^2 we see that $-6 = -2a + b$ from which $b = -4$. We use the coefficient of x to check our work. Coefficient of x on the left is 12; coefficient of x on the right is $-2b + c$.

$$x^3 - 6x^2 + 12x - 8 = (x-2)(x^2 - 4x + 4)$$

We now factorise $x^2 - 4x + 4$ to $(x-2)^2$. Hence

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3.$$

b)

$$\frac{x^2 - 7x + 12}{x^3 - 6x^2 + 12x - 8} = \frac{x^2 - 7x + 12}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$
$$x^2 - 7x + 12 = A(x-2)^2 + B(x-2) + C.$$

Equating the coefficient of x^2 we see $A = 1$. By letting $x = 2$ we obtain $2 = C$. Equating the constant terms we obtain $12 = 4A - 2B + C$ from which $B = -3$. Hence

$$\frac{x^2 - 7x + 12}{x^3 - 6x^2 + 12x - 8} = \frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{(x-2)^3}$$

Chapter 10. Transposing formulae

Transpose the following to make x the subject.

a) $x^4 - 2x^2 + 1 = A$, $0 \le A \le 1$

b) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$

Solution

a)

$$\begin{aligned} x^4 - 2x^2 + 1 &= (x^2 - 1)^2 = A \\ x^2 - 1 &= \pm\sqrt{A} \\ x^2 &= 1 \pm \sqrt{A} \\ x &= \pm\sqrt{1 \pm \sqrt{A}}. \end{aligned}$$

b)

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{z} \\ \frac{y+x}{xy} &= \frac{1}{z} \\ z(y+x) &= xy \\ x(y-z) &= yz \\ x &= \frac{yz}{y-z}. \end{aligned}$$

Chapter 11. Solving equations

1. A quadratic curve and a straight line are defined by

$$\begin{aligned} y &= x^2 + 6x - 8 \\ y &= 7x + 12 \end{aligned}$$

Determine the points of intersection of the curve and line.

Solution

Eliminating y from the pair of equations yields

$$x^2 + 6x - 8 = 7x + 12$$

from which

$$\begin{aligned} x^2 - x - 20 &= 0 \\ (x-5)(x+4) &= 0 \\ x &= -4, 5. \end{aligned}$$

When $x = -4$, $y = -16$. When $x = 5$, $y = 47$. The points of intersection are $(-4, -16)$ and $(5, 47)$.

2. Solve the simultaneous equations:

$$\begin{aligned} 2x + y - z &= -2 && (1) \\ 3x - 2y + 3z &= -8 && (2) \\ \frac{x}{2} + 3y - 4z &= 3 && (3) \end{aligned}$$

Solution

We use (1) to eliminate y from (2) and (3). From (1), $y = -2x + z - 2$. Substituting this into (2) and (3) and simplifying produces

$$\begin{aligned} 7x + z &= -12 && (4) \\ \frac{11x}{2} + z &= -9 && (5) \end{aligned}$$

Subtracting (5) from (4) gives

$$\frac{3x}{2} = -3$$

from which $x = -2$. Substituting $x = -2$ into (4) and solving for z gives $z = 2$. Finally, substituting the values of x and z into (1) we find $y = 4$.

The solution is $x = -2$, $y = 4$, $z = 2$.

Chapter 12. Sequences and series

1. An arithmetic sequence, x_n, is given by

$$x_n = 2 - 3n, \qquad n = 1, 2, 3, \ldots$$

Show that the sum of the first n terms, S_n, is given by

$$S_n = \frac{n}{2}[1 - 3n].$$

Solution

$x_1 = -1$, $x_2 = -4$ and so $a = -1$, $d = -3$.

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[-2 + (n-1)(-3)] = \frac{n}{2}[1 - 3n].$$

2. A geometric sequence, x_k, $k = 1, 2, 3, \ldots$ has first term a and common ratio r where $|r| < 1$ and sum to infinity, S. Show

$$\sum_{1}^{\infty} (x_k)^2 = \left(\frac{1-r}{1+r}\right) S^2.$$

Solution

We have

$$S = \frac{a}{1-r}.$$

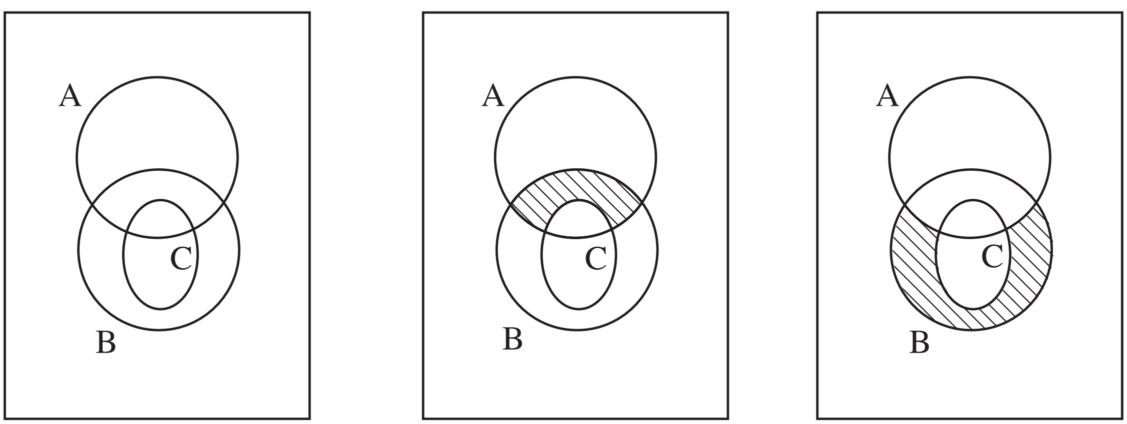
Consider $y_k = (x_k)^2$, $k = 1, 2, 3, \ldots$. Then $y_1 = a^2$, $y_2 = a^2 r^2$, $y_3 = a^2 r^4$, $y_4 = a^2 r^6$ and so on. So y_k is a geometric sequence with first term a^2 and common ratio r^2. Note that $|r^2| < 1$ and so the sum to infinity of y_k exists.

$$\sum_{1}^{\infty} y_k = \sum_{1}^{\infty} (x_k)^2 = \frac{a^2}{1-r^2} = \frac{a^2}{(1-r)(1+r)} = \frac{a^2}{(1-r)^2} \cdot \frac{(1-r)}{(1+r)} = \left(\frac{1-r}{1+r}\right) S^2.$$

Chapter 13. Sets

The Venn diagram in Figure 13.1 shows three intersecting sets; A, B and C.

Figure 13.1 Figure 13.2 Figure 13.3



(a) Write an expression for the area shaded in Figure 13.2.

(b) Write an expression for the area shaded in Figure 13.3.

Solution

(a) $A \cap B \cap \overline{C}$.

(b) $\overline{A} \cap B \cap \overline{C}$.

Chapter 14. Number bases

Write the following decimal numbers in binary, octal and hexadecimal form: a) 0.25 b) 0.75

Solution

Places to the right of a decimal point represent negative powers of the base, for example, 2^{-1}, 2^{-2}, etc in binary; 8^{-1},8^{-2}, etc in octal and 16^{-1}, 16^{-2} etc in hexadecimal.

(a) 0.01, 0.2, 0.4

(b) 0.11, 0.6, 0.C

Chapter 16. Functions

$x(t)$ and $y(t)$ are both linear functions of t, that is, $x(t) = at + b$, $y(t) = ct + d$ where a, b, c and d are constants.

Show that $y^{-1}(x^{-1}(t))$ is identical to the inverse of $x(y(t))$.

Solution

$$x^{-1}(t) = \frac{t-b}{a}, \quad y^{-1}(t) = \frac{t-d}{c}$$

$$y^{-1}(x^{-1}(t)) = y^{-1}\left(\frac{t-b}{a}\right) = \frac{\frac{t-b}{a} - d}{c} = \frac{t-b-ad}{ac}$$

$$x(y(t)) = x(ct+d) = a(ct+d) + b = act + ad + b$$

and so

$$[x(y(t))]^{-1} = \frac{t-b-ad}{ac}$$

Hence $[x(y(t))]^{-1} = y^{-1}(x^{-1}(t))$ for any linear functions.

Chapter 17. Graphs of functions

1.

a) Draw $y = 3x + k$, $-3 \le x \le 3$ for $k = -1, 0, 1, 2$. What do you notice?

b) By drawing graphs of $y = x^2 + 2$ and $y = 3x + k$, $-3 \le x \le 3$ for various values of k, determine the range of values of k for which the simultaneous equations

$$\begin{aligned} x^2 - y + 2 &= 0 \\ 3x - y + k &= 0 \end{aligned}$$

have (i) 2 solutions (ii) no solutions.

c) By deducing a suitable quadratic equation in x and using the quadratic formula, find an algebraic solution to b).

Solution

a)

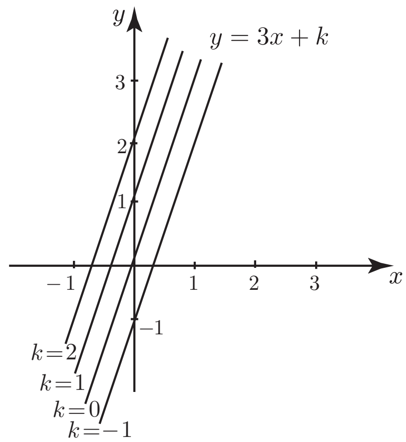


Figure 17.1

The graphs are straight lines with the vertical intercepts being the values of k. The lines are parallel.

b)

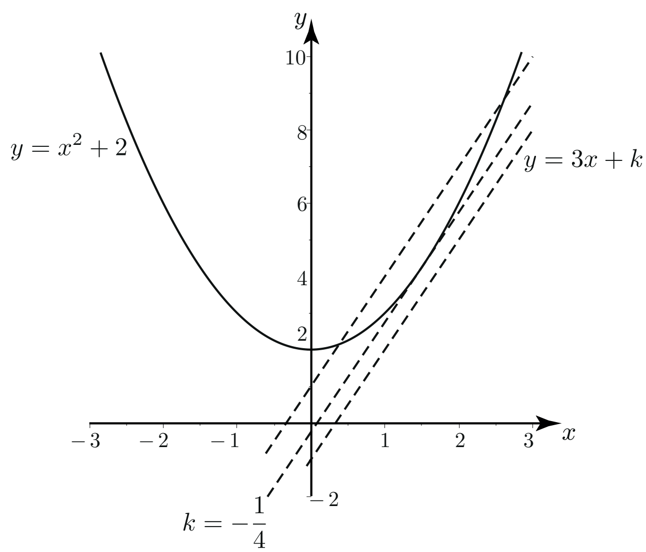


Figure 17.2

The graph of $y = 3x + k$ touches $y = x^2 + 2$ i.e. one solution, when $k = -\frac{1}{4}$. So there are 2 solutions when $k > -\frac{1}{4}$ and no solutions when $k < -\frac{1}{4}$.

c) We solve $y = x^2 + 2$, $y = 3x + k$. Eliminating y produces

$$x^2 - 3x + 2 - k = 0$$

$$x = \frac{3 \pm \sqrt{1 + 4k}}{2}$$

Hence there are 2 roots when $1 + 4k > 0$, that is, when $k > -\frac{1}{4}$, 1 root when $1 + 4k = 0$, that is when $k = -\frac{1}{4}$ and no roots when $1 + 4k < 0$, that is when $k < -\frac{1}{4}$.

2. Given $a > b$ and $x > y$ state which of the following are true and which are false. If it is not possible to state with certainty whether the statement is true or false then state this clearly.

a) $a + x > b + y$

b) $a - x > b - y$

c) $\frac{a}{x} > \frac{b}{y}$

d) $ax > by$

Solution

a) True

b) not possible to state whether True or False.

c) not possible to state whether True or False.

d) not possible to state whether True or False.

Chapter 18. The straight line

Consider the line $y = mx + c$ and the curve $y = \frac{1}{x}$. Show that

a) If $m > -\frac{c^2}{4}$ then the line intersects the curve in two places;

b) If $m > 0$ then one intersection point lies in quadrant 1 and one is in quadrant 3;

c) When $-\frac{c^2}{4} < m < 0$ then both the intersection points are in quadrant 1 if $c > 0$ and both are in quadrant 3 if $c < 0$.

Solution

a) The points of intersection are found by solving the following equations simultaneously

$$\begin{aligned} y &= mx + c \\ y &= \frac{1}{x} \end{aligned}$$

Eliminating y produces $mx^2 + cx - 1 = 0$ from which

$$x = \frac{-c \pm \sqrt{c^2 + 4m}}{2m}$$

If $m > -\frac{c^2}{4}$ then $c^2 + 4m > 0$ and the quadratic formula produces two distinct solutions, that is, the line and curve intersect in two places.

b) We are given $m > 0$. Let the intersection points be (x_1, y_1) and (x_2, y_2). Then

$$x_1 = \frac{-c + \sqrt{c^2 + 4m}}{2m}, \qquad x_2 = \frac{-c - \sqrt{c^2 + 4m}}{2m}$$

Note that $x_1 > 0$, $x_2 < 0$ and since $y = \frac{1}{x}$, then $y_1 > 0$, $y_2 < 0$. So (x_1, y_1) is in quadrant 1 and (x_2, y_2) is in quadrant 3.

c) Suppose $c > 0$. Then, since $m < 0$, both $-c + \sqrt{c^2 + 4m}$, and $-c - \sqrt{c^2 + 4m}$ are negative. Hence $x_1 > 0$, $x_2 > 0$ and $y_1 > 0$, $y_2 > 0$ so both intersection points are in quadrant 1.

Suppose $c < 0$. Then both $-c + \sqrt{c^2 + 4m}$, and $-c - \sqrt{c^2 + 4m}$ are positive so $x_1 < 0$, $x_2 < 0$ and $y_1 < 0$, $y_2 < 0$, that is, both intersection points are in the third quadrant.

Chapter 19. The exponential function

1. y_1 and y_2 are exponential functions given by

$$y_1 = \mathrm{e}^{k_1 x}, \qquad y_2 = \mathrm{e}^{k_2 x}$$

where k_1 and k_2 are constants.

If $y_1\, y_2$ and $\frac{y_1}{y_2}$ approach 0 as x increases indefinitely show that $k_1 < k_2 < -k_1$ and $k_1 < 0$.

Solution

$$y_1 y_2 = \mathrm{e}^{k_1 x}\mathrm{e}^{k_2 x} = \mathrm{e}^{(k_1+k_2)x}$$

and

$$\frac{y_1}{y_2} = \frac{\mathrm{e}^{k_1 x}}{\mathrm{e}^{k_2 x}} = \mathrm{e}^{(k_1-k_2)x}$$

Since $y_1 y_2$ approaches 0 as x increases then $k_1 + k_2 < 0$. (1)

Since $\dfrac{y_1}{y_2}$ approaches 0 as x increases then $k_1 - k_2 < 0$. (2)

Adding (1) and (2) gives $2k_1 < 0$ and so $k_1 < 0$.

From (1), $k_2 < -k_1$. From (2), $k_2 > k_1$. Combining these two results yields $k_1 < k_2 < -k_1$ as required.

2. The function y decays exponentially from a value a to a value b as illustrated in Figure 19.1. State possible forms of the function y.

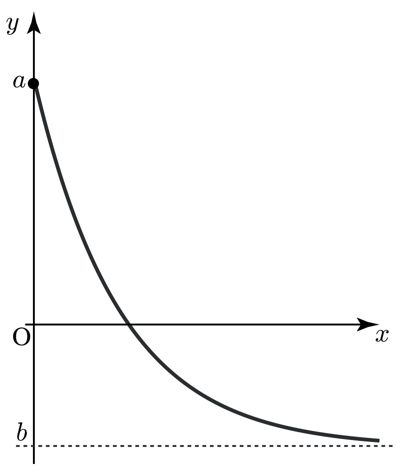


Figure 19.1

Solution

$$y = (a-b)\mathrm{e}^{-kx} + b, \qquad k > 0.$$

Chapter 20. The logarithm function

1. The function, $y(t)$, is given by $y = \mathrm{e}^t$.

a) Calculate values of t such that y is (i) 1, (ii) 2, (iii) 4, (iv) 8.

b) Show that if t is increased by $\ln 2$ then the value of y doubles.

c) A function, $z(t)$,is given by $z = \mathrm{e}^{kt},\ k > 0$. Find the increase in the value of t which will result in the value of z doubling.

Solution

a) (i) 0 (ii) $\ln 2$ (iii) $\ln 4 = 2\ln 2$ (iv) $\ln 8 = 3\ln 2$

b) We are given $y(t) = \mathrm{e}^t$ and so

$$y(t + \ln 2) = \mathrm{e}^{t+\ln 2} = \mathrm{e}^t\mathrm{e}^{\ln 2} = 2\mathrm{e}^t = 2y$$

c) We have $z(t) = \mathrm{e}^{kt}$. Let an increase in the value of t by an amount α lead to the value of z doubling. Then

$$z(t + \alpha) = \mathrm{e}^{k(t+\alpha)} = \mathrm{e}^{kt}\mathrm{e}^{k\alpha} = z(t)\mathrm{e}^{k\alpha} = 2z(t)$$

Hence

$$\begin{aligned} \mathrm{e}^{k\alpha} &= 2 \\ k\alpha &= \ln 2 \\ \alpha &= \frac{\ln 2}{k} \end{aligned}$$

So increasing t by $\frac{\ln 2}{k}$ will result in the value of z being doubled.

2. Solve for x.

a) $10^x \cdot 9^x = 1000$

b) $10^x \cdot 9^{2x} = 1000$.

Solution

a) $10^x\, 9^x = (10 \times 9)^x = 90^x$.

So

$$\begin{aligned} 90^x &= 1000 \\ x\log 90 &= \log 1000 = 3 \\ x &= \frac{3}{\log 90} = 1.535 \end{aligned}$$

b) $10^x\, 9^{2x} = 10^x\ (9^2)^x = 10^x\, 81^x = (10 \times 81)^x = 810^x$

So

$$\begin{aligned} 810^x &= 1000 \\ x\log 810 &= \log 1000 = 3 \\ x &= \frac{3}{\log 810} = 1.031 \end{aligned}$$

Chapter 21. Measurement

The height of a cylinder is the same as its diameter. If the volume of the cylinder is 1.5 litres, calculate its height.

Solution

Let h be the height of the cylinder, d the diameter, r the radius and V the volume. Then $h = d = 2r$.

$$\begin{aligned} V = \pi r^2 h &= \pi \left(\frac{h}{2}\right)^2 h \\ &= \frac{\pi h^3}{4} \end{aligned}$$

$$h^3 = \frac{4V}{\pi}$$

We are given that the volume is 1500 cm^3 and so

$$h = \sqrt[3]{\frac{6000}{\pi}} = 12.41$$

The height of the cylinder is 12.41cm.

Chapter 22. Introduction to trigonometry

A goblet takes the form of a cone whose apex is fixed to a stem, as shown in Figure 22.1.

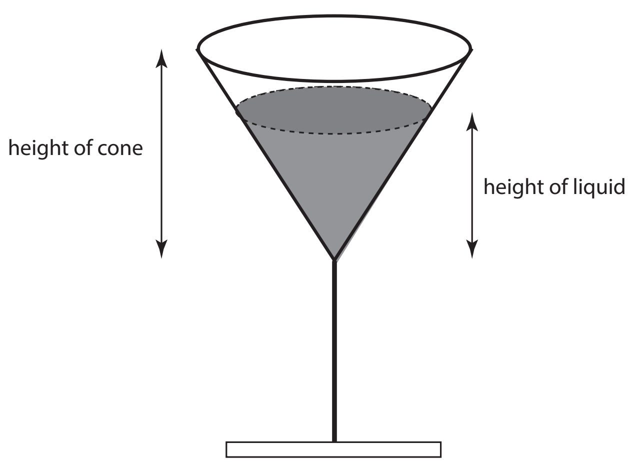


Figure 22.1

Liquid is poured into the goblet until it reaches 90% of the vertical height of the cone. Show that the liquid occupies about 73% of the volume of the goblet.

Solution

Let h be the vertical height of the cone; r the radius of the base of the cone and V the volume of the cone.

Let h_ℓ be the vertical height of the liquid; r_ℓ the radius of the base of the liquid and V_ℓ the volume of liquid.

Let θ be the angle at the apex as shown in Figure 22.2.

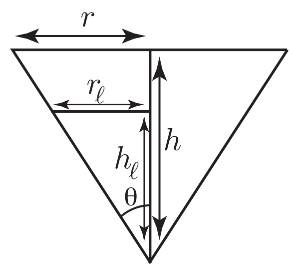


Figure 22.2

Then

$$\tan\theta = \frac{r}{h} = \frac{r_\ell}{h_\ell}$$

and so $\frac{h_\ell}{h} = \frac{r_\ell}{r}$. We are given that $\frac{h_\ell}{h} = 0.9$. Now

$$\frac{V_\ell}{V} = \frac{\frac{1}{3}\pi r_\ell^2 h_\ell}{\frac{1}{3}\pi r^2 h} = \frac{r_\ell^2}{r^2}\frac{h_\ell}{h} = 0.9^3 = 0.729$$

The volume of the liquid is 72.9% of the volume of the goblet.

Chapter 23. The trigonometrical functions and their graphs

Given $0° \leq \theta \leq 360°$ state the quadrants in which θ lies:

a) $\sin\theta = \cos\theta$

b) $\sin\theta = -2\cos\theta$

c) $\tan\theta = -4\sin\theta$

d) $\frac{1}{\tan\theta} = 2\cos\theta$

Solution

a) Using $\frac{\sin\theta}{\cos\theta} = \tan\theta$ we have

$$\tan\theta = 1 > 0$$

and so θ lies in the first and third quadrants.

b)

$$\tan\theta = -2 < 0$$

and so θ lies in the second and fourth quadrants.

c)

$$\frac{\sin\theta}{\cos\theta} = -4\sin\theta$$

from which

$$\cos\theta = -\frac{1}{4} < 0$$

and so θ lies in the second and third quadrants.

d)

$$\frac{\cos\theta}{\sin\theta} = 2\cos\theta$$

$$\sin\theta = \frac{1}{2} > 0$$

and so θ lies in the first and second quadrants.

Chapter 24. Trigonometrical identities and equations

1.

Solve

$$\sin\theta\,\cos\theta\,\tan\theta = 0.25, \qquad 0 \le \theta \le 360^\circ$$

Solution

$$\sin\theta\,\cos\theta\,\tan\theta = 0.25$$

We note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so we have

$$\sin\theta\,\cos\theta\,\frac{\sin\theta}{\cos\theta} = 0.25$$

$$\sin^2\theta = 0.25$$

$$\sin\theta = \pm 0.5$$

$$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

2. A circle has centre O and radius r. A straight line, AB, is drawn, distance d from the centre. If α is the angle at the centre as shown in Figure 24.1, show that the area of the segment $ACBD$ is given by

$$r^2[\alpha - \frac{\sin 2\alpha}{2}]$$

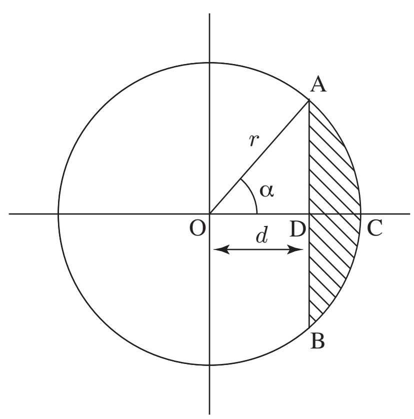


Figure 24.1

Solution

The angle subtended at the centre by AB is 2α. So the area of the sector $OACB$ is $\dfrac{r^2}{2}(2\alpha) = r^2\alpha$.

Area of $\triangle OAD = \frac{1}{2}(OD)(AD) = \frac{1}{2}(r\cos\alpha)(r\sin\alpha) = \frac{r^2\sin 2\alpha}{4}$ using the identity $2\sin\alpha\cos\alpha = \sin 2\alpha$.

So Area of $\triangle OAD + \triangle OBD = \frac{r^2\sin 2\alpha}{2}$.

Area of segment $= r^2\alpha - \frac{r^2\sin 2\alpha}{2} = r^2[\alpha - \frac{\sin 2\alpha}{2}]$

3. a) A straight line makes an angle θ with the positive x axis. If the gradient of the line is m show $m = \tan\theta$.

b) Two straight lines intersect at right angles. Show that the product of their gradients is -1.

Solution

a) By referring to Figure 24.2 we see that the gradient of the line, m, is given by

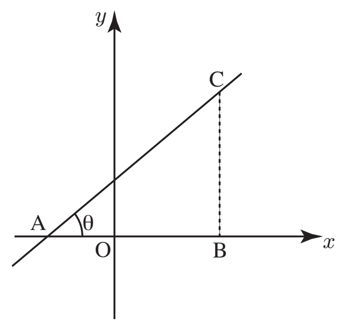


Figure 24.2

$$m = \frac{\text{difference between } y \text{ coordinates}}{\text{difference between } x \text{ coordinates}} = \frac{BC}{AB}$$

By considering $\triangle ABC$, we see that

$$\tan\theta = \frac{BC}{AB}$$

and so $m = \tan\theta$ as required.

b) Line 1 has gradient m_1 and makes an angle θ with the x axis see Figure 24.3. Hence $m_1 = \tan\theta$. Line 2 has gradient m_2 and forms angle α with the x axis. Then $\alpha = \theta + 90°$ and so $m_2 = \tan(\theta + 90°)$.

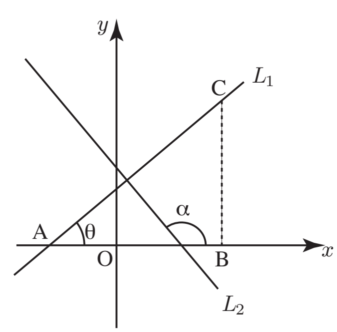


Figure 24.3

Using the identity for $\tan(A+B)$ with $A = \theta$, $B = 90°$ we obtain:

$$m_2 = \frac{\tan\theta + \tan 90°}{1 - \tan\theta \tan 90°} = \frac{\frac{\tan\theta}{\tan 90°} + 1}{\frac{1}{\tan 90°} - \tan\theta} = \frac{-1}{\tan\theta} = \frac{-1}{m_1}$$

and so

$$m_1 m_2 = -1.$$

Chapter 25. Solution of triangles

1. A tower, AT, of unknown height is leaning slightly from the vertical. B is 12.35m from A, C is 11.55m from A, with angles of elevation to the top of the tower of $37°$ and $43°$ respectively. Figure 25.1 illustrates the situation. Calculate

a) The length of the tower, AT.

b) The angle the tower makes with the ground, that is, $\angle BAT$.

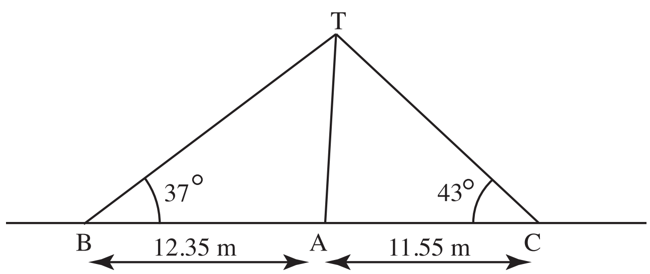


Figure 25.1

Solution

a)

Note that $BC = 12.35 + 11.55 = 23.9$, and

$$\angle BTC = 180° - 37° - 43° = 100°$$

Applying the sine rule to $\triangle BTC$ gives

$$\frac{BT}{\sin 43°} = \frac{BC}{\sin \angle BTC} = \frac{23.9}{\sin 100°}$$

from which

$$BT = \frac{23.9 \sin 43°}{\sin 100°} = 16.5512$$

Now apply the cosine rule to $\triangle BTA$.

$$\begin{aligned} AT^2 &= BT^2 + AB^2 - 2(BT)(AB)\cos 37° \\ &= (16.5512)^2 + (12.35)^2 - 2(16.5512)(12.35)\cos 37° \\ &= 99.97 \end{aligned}$$

Hence $AT = 10.00$ (2 d.p.)

b) Apply the sine rule to $\triangle BTA$.

$$\frac{AT}{\sin B} = \frac{BT}{\sin \angle BAT}$$
$$\frac{10}{\sin 37^\circ} = \frac{16.5512}{\sin \angle BAT}$$

from which $\sin \angle BAT = 1.65512 \sin 37^\circ$ and $\angle BAT = 84.92^\circ$ or 95.08°.

Now, we need to decide if both solutions are acceptable, or only one; and if so, which one.

Let us consider $\angle BAT = 84.92^\circ$. The angles of $\triangle ACT$ can now be calculated as: $\angle TAC = 95.08^\circ$, $\angle ACT = 43^\circ$, $\angle ATC = 40.92^\circ$. However, this leads to a contradictory result. Longer sides of a triangle are always opposite the larger angles. AT, length 10.00m, is opposite $\angle ACT$, an angle of 43°. AC, length 11.55m, is opposite $\angle ATC$, an angle of 40.92°. So here we have the longer side opposite the smaller angle. We conclude that $\angle BAT = 84.92^\circ$ is unacceptable.

We accept the remaining solution; $\angle BAT = 95.08^\circ$. It is easy to check that this solution does not lead to any contradictions and so is acceptable.

2. A metal panel has dimensions as shown in Figure 25.2. It comprises a rectangle, $ABCD$, with the segment of a circle added to the edge BC. The radius of the segment has length BE, where E is the mid-point of AD. Calculate the area of the panel.

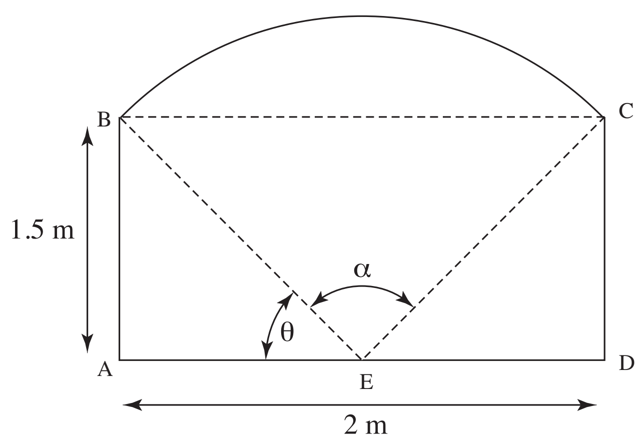


Figure 25.2

Solution

$$BE = \sqrt{1^2 + 1.5^2} = \sqrt{3.25}$$

Let $\angle AEB = \theta$, $\angle BEC = \alpha$. Then $\tan\theta = 1.5$ so that $\theta = 0.9828$ and $\alpha = \pi - 2\theta = 1.1760$.

The area of sector $BEC = \frac{1}{2}r^2\theta = \frac{1}{2}(3.25)(1.1760) = 1.911$.

Area of $\triangle ABE = \frac{1}{2}(1.5) = 0.75$. Similarly, $\triangle DCE$ has area 0.75. So the area of the panel is

Area of panel $= 2(0.75) + 1.911 = 3.411$.

Chapter 26. Vectors

$\underline{a}$ and $\underline{b}$ are two vectors with $|\underline{a}| = |\underline{b}|$. Given that $\underline{a} = 3\underline{i} + 2\underline{j}$ and $\underline{b}$ makes an angle of $50°$ with $\underline{a}$, find $\underline{b}$.

Solution

$$|\underline{a}| = \sqrt{3^2 + 2^2} = \sqrt{13} = |\underline{b}|$$

Figure 26.1 shows possible positions for B such that $\underline{b} = \overrightarrow{OB}$ with the required properties.

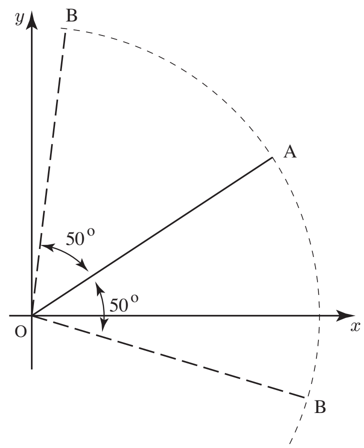


Figure 26.1

Let $\underline{a}$ make an angle θ with the x axis. Then $\tan\theta = \frac{2}{3}$ and so $\theta = 33.69°$.

<u>Case 1</u>

$\underline{b}$ makes an angle of $\theta + 50° = 83.69°$ with the x axis. If $\underline{b} = (b_1, b_2)$ then

$$b_1 = \sqrt{13}\cos 83.69° = 0.3963, \quad b_2 = \sqrt{13}\sin 83.69° = 3.5837$$

and so $\underline{b} = 0.3963\underline{i} + 3.5837\underline{j}$.

<u>Case 2</u>

$\underline{b}$ makes an angle $\theta - 50° = -16.31°$ with the x axis. Then

$$b_1 = \sqrt{13}\cos(-16.31°) = 3.4605, \qquad b_2 = \sqrt{13}\sin(-16.31°) = -1.10126$$

so

$$\underline{b} = 3.4605\underline{i} - 1.0126\underline{j}$$

Chapter 27. Matrices

1.

Although we have looked at the inverses only of 2×2 matrices, the concept of an inverse can be applied to other larger matrices.

Show that any matrix that has an inverse must be square.

Solution

Suppose the matrix A has an inverse, A^{-1}. Let A be $m \times n$. We need to show that A is square, that is, $m = n$.

Since the products AA^{-1} and $A^{-1}A$ both exist then A^{-1} is $n \times m$ [See Self Assessment 27.2, Q4].

We see that AA^{-1} is a $m \times m$ matrix and that $A^{-1}A$ is a $n \times n$ matrix. However from the property of inverses we know that

$$AA^{-1} = A^{-1}A$$

and so $m = n$. Hence A is a square matrix.

2. A and B are matrices such that A^{-1}, B^{-1} and AB all exist.

a) Show that $B^{-1}A^{-1}$ exists.

b) Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution

a) The matrix A has an inverse and so it must be square [See Challenge Exercise 1]. Suppose A is $m \times m$. Then from consideration of the products AA^{-1} and $A^{-1}A$ we see that A^{-1} is also $m \times m$.

Similarly B has an inverse and so let B and its inverse, B^{-1}, be $n \times n$ matrices.

We know that AB exists and so $m = n$, that is, A, B, A^{-1} and B^{-1} are all $m \times m$ matrices. Hence the product $B^{-1}A^{-1}$ exists.

b)

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

So the inverse of AB is $B^{-1}A^{-1}$.

Chapter 28. Complex numbers

1. In this exercise we seek all the cube roots of 1. We make use of De Moivre's Theorem (see Exercise 28.6 Q5):

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta).$$

a) Show that 1 may be expressed in the form $\mathrm{e}^{\mathrm{i}(2n\pi)}$ for $n = 0, 1, 2, 3, 4, \ldots$.

b) Hence show that the cube roots of 1 are given by

$$1^{1/3} = \left(\mathrm{e}^{\mathrm{i}(2n\pi)}\right)^{1/3}, \quad n = 0, 1, 2, 3, 4, \ldots$$

Use the laws of indices to simplify this expression.

c) By expressing your simplified version from (b) in polar form and noting the periodic nature of sine and cosine, show that there are 3 distinct cube roots of 1. Express these roots in both polar and Cartesian forms.

Solution

a) Consider $\mathrm{e}^{\mathrm{i}(2n\pi)}$ for $n = 0, 1, 2, 3, 4, \ldots$

$\underline{n = 0}$

$\mathrm{e}^{\mathrm{i}(2n\pi)} = \mathrm{e}^{\mathrm{i}\,0} = \mathrm{e}^0 = 1$

$\underline{n = 1}$

$\mathrm{e}^{\mathrm{i}(2n\pi)} = \mathrm{e}^{\mathrm{i}(2\pi)} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1 + \mathrm{i}\,0 = 1$

$\underline{n = 2}$

$\mathrm{e}^{\mathrm{i}(2n\pi)} = \mathrm{e}^{\mathrm{i}(4\pi)} = \cos 4\pi + \mathrm{i}\sin 4\pi = 1$

$\underline{n = 3}$

$\mathrm{e}^{\mathrm{i}(2n\pi)} = \mathrm{e}^{\mathrm{i}(6\pi)} = \cos 6\pi + \mathrm{i}\sin 6\pi = 1$

Clearly similar results hold true for any integer value of n. So

$$1 = \mathrm{e}^{\mathrm{i}(2n\pi)}$$

b) Raising a number to the power of $\frac{1}{3}$ is equivalent to finding the cube root of that number. So the cube root of 1 is $1^{1/3}$. Using the result of (a) we have:

$$1^{1/3} = \left(\mathrm{e}^{\mathrm{i}(2n\pi)}\right)^{1/3} = \mathrm{e}^{\mathrm{i}(2n\pi/3)} \qquad n = 0, 1, 2, 3, \ldots$$

c) Now

$$1^{1/3} = \mathrm{e}^{\mathrm{i}(2n\pi/3)} = \cos\frac{2n\pi}{3} + \mathrm{i}\sin\frac{2n\pi}{3} \qquad n = 0, 1, 2, 3, \ldots$$

Applying values of n gives

$\underline{n = 0}$

$$1^{1/3} = \cos 0 + \mathrm{i}\sin 0 = 1$$

$\underline{n = 1}$

$$1^{1/3} = \cos(\frac{2\pi}{3}) + \mathrm{i}\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$\underline{n = 2}$

$$1^{1/3} = \cos(\frac{4\pi}{3}) + \mathrm{i}\sin(\frac{4\pi}{3}) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

$\underline{n = 3}$

$$1^{1/3} = \cos(\frac{6\pi}{3}) + \mathrm{i}\sin(\frac{6\pi}{3}) = \cos 2\pi + \mathrm{i}\sin 2\pi = 1$$

$\underline{n = 4}$

$$1^{1/3} = \cos(\frac{8\pi}{3}) + \mathrm{i}\sin(\frac{8\pi}{3}) = \cos(\frac{2\pi}{3}) + \mathrm{i}\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

Because sine and cosine are periodic with a period of 2π, we see that the results are simply being repeated as the value of n increases. Hence there are 3 distinct cube roots of 1.

2. Find all fourth roots of i.

Solution

The modulus is 1; the argument is $\frac{\pi}{2}$. So

$$\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$$

Adding multiples of 2π to an argument does not alter the complex number because both sine and cosine are periodic with period 2π. Hence

$$\mathrm{i} = \mathrm{e}^{\mathrm{i}\left(2n\pi + \frac{\pi}{2}\right)}, \qquad n = 0, 1, 2, 3, 4, \ldots$$

Raising a number to the power of $\frac{1}{4}$, results in the fourth root of that number. Now

$$\mathrm{i}^{\frac{1}{4}} = \left(\mathrm{e}^{\mathrm{i}\left(2n\pi + \frac{\pi}{2}\right)}\right)^{\frac{1}{4}}, \qquad n = 0, 1, 2, 3, 4, \ldots$$

We now substitute in values for n.

$\underline{n = 0}$

$$\mathrm{i}^{\frac{1}{4}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{8}} = \cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}$$

$\underline{n = 1}$

$$\mathrm{i}^{\frac{1}{4}} = \mathrm{e}^{\mathrm{i}\frac{5\pi}{8}} = \cos\frac{5\pi}{8} + \mathrm{i}\sin\frac{5\pi}{8}$$

$\underline{n = 2}$

$$\mathrm{i}^{\frac{1}{4}} = \mathrm{e}^{\mathrm{i}\frac{9\pi}{8}} = \cos\frac{9\pi}{8} + \mathrm{i}\sin\frac{9\pi}{8}$$

$\underline{n = 3}$

$$\mathrm{i}^{\frac{1}{4}} = \mathrm{e}^{\mathrm{i}\frac{13\pi}{8}} = \cos\frac{13\pi}{8} + \mathrm{i}\sin\frac{13\pi}{8}$$

$\underline{n = 4}$

$$\mathrm{i}^{\frac{1}{4}} = \mathrm{e}^{\mathrm{i}\frac{17\pi}{8}} = \cos\frac{17\pi}{8} + \mathrm{i}\sin\frac{17\pi}{8} = \cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}$$

using the periodicity of the trigonometrical functions.

We can see that the values of $\mathrm{i}^{\frac{1}{4}}$ are beginning to repeat as the values of n increase. Hence there are 4 distinct fourth roots of i:

$$\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8},\ \cos\frac{5\pi}{8}+\mathrm{i}\sin\frac{5\pi}{8},\ \cos\frac{9\pi}{8}+\mathrm{i}\sin\frac{9\pi}{8},\ \cos\frac{13\pi}{8}+\mathrm{i}\sin\frac{13\pi}{8}$$

Chapter 29. Tables and charts

Chapter 30. Statistics

1. Show that

$$\sum(x_i-\bar{x})^2=\sum x_i^2-n\bar{x}^2$$

Solution

$$\begin{aligned}\sum(x_i-\bar{x})^2 &= \sum x_i^2-2x_i\bar{x}+\bar{x}^2=\sum x_i^2-2\bar{x}\sum x_i+\sum\bar{x}^2\\ &= \sum x_i^2-2\bar{x}(n\bar{x})+n\bar{x}^2=\sum x_i^2-n\bar{x}^2\end{aligned}$$

2. The data set, X, comprises n items $x_1, x_2, x_3, \ldots x_n$ with mean $\bar{x}$ and standard deviation σ_x. Each data point, x_i, is linearly transformed to produce a new data point, y_i, where

$$y_i=a+bx_i, \qquad i=1,2,3,\ldots n$$

where a and b are constants.

a) If $\bar{y}$ is the mean of the new data set show that

$$\bar{y}=a+b\bar{x}$$

b) If σ_y is the standard deviation of the new data set show that

$$\sigma_y=b\sigma_x$$

Solution

a)

$$\bar{y}=\frac{\sum y_i}{n}=\frac{\sum a+bx_i}{n}=\frac{na+b\sum x_i}{n}=a+b\bar{x}.$$

b)

$$\sigma_x=\sqrt{\frac{\sum(x_i-\bar{x})^2}{n}}, \qquad \sigma_y=\sqrt{\frac{\sum(y_i-\bar{y})^2}{n}}$$

Now

$$y_i-\bar{y}=a+bx_i-(a+b\bar{x})=b(x_i-\bar{x})$$

and so

$$\sum(y_i-\bar{y})^2=\sum b^2(x_i-\bar{x})^2$$

Hence

$$\sigma_y=\sqrt{\frac{\sum b^2(x_i-\bar{x})^2}{n}}=b\sqrt{\frac{\sum(x_i-\bar{x})^2}{n}}=b\sigma_x$$

Chapter 31. Probability

A particular athlete knows that the probability he will win a road race is 0.23. Calculate the probability he will win:

a) three consecutive races

b) two of four races

c) at least one race from five.

Solution

Let W represent win a race and L represent lose a race. Then $p(W) = 0.23$ and $p(L) = 1 - 0.23 = 0.77$.

a)

$$p(3\text{ wins}) = p(WWW) = (0.23)^3 = 0.0122$$

b) There are 6 ways in which the 2 wins can be distributed amongst the 4 races ($WWLL$, $WLWL$, $WLLW$, $LWWL$, $LWLW$, $LLWW$).

Each of these has a probability $(0.23)^2(0.77)^2$.

So the probability of 2 wins from 4 races is $6(0.23)^2(0.77)^2 = 0.1882$.

c)

$$p(\text{ at least 1 win from 5 races}) = 1 - p(\text{ all 5 races lost}) = 1 - (0.77)^5 = 0.7293.$$

Chapter 32. Correlation

Two variables, x_i and y_i are related by

$$y_i = mx_i + c, \qquad i = 1, 2, , 3, \ldots n$$

where m and c are constants, that is, there is a linear relationship between x_i and y_i. Show that the product-moment correlation coefficient, r, is ± 1.

Solution

We know from Challenge Exercise, Chapter 30, that $\bar{y} = m\bar{x} + c$. So

$$y_i - \bar{y} = mx_i + c - [m\bar{x} + c] = m(x_i - \bar{x})$$

Then

$$\sum (y_i - \bar{y})^2 = \sum m^2 (x_i - \bar{x})^2 = m^2 \sum (x_i - \bar{x})^2$$

So

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} = \frac{m \sum (x_i - \bar{x})^2}{\sqrt{\sum (x_i - \bar{x})^2 m^2 \sum (x_i - \bar{x})^2}} = \frac{\sum (x_i - \bar{x})^2}{\sqrt{\left(\sum (x_i - \bar{x})^2\right)^2}} = \pm 1$$

Chapter 33. Regression

Chapter 34. Gradients of curves

A function, $y(x)$, is defined by

$$y(x) = x^3 - x^2 - x - 2$$

a) Determine the range of values of x for which y is decreasing. b) Locate the point(s) at which this decrease is most rapid.

Solution

a) y decreases when $y' < 0$. Now

$$y' = 3x^2 - 2x - 1 = (3x+1)(x-1)$$

So $y' < 0$ when either A) $(3x+1) < 0$ and $(x-1) > 0$, or B) $(3x+1) > 0$ and $(x-1) < 0$. There are no values of x which satisfy the conditions in A). From B) we see that y is decreasing when $-\frac{1}{3} < x < 1$.

b) The most rapid decrease occurs when y' is most negative; that is, when y' is minimised. So we need to find where the minimum value of y' occurs.

Let $z = y' = 3x^2 - 2x - 1$. Then

$$z' = 6x - 2, \quad z'' = 6$$

So $z' = 0$ when $x = \frac{1}{3}$ and $z'' > 0$ i.e. minimum. The minimum value of y' occurs when $x = \frac{1}{3}$. When $x = \frac{1}{3}$, $y = -\frac{65}{27}$. Hence y is decreasing most rapidly at $(\frac{1}{3}, -\frac{65}{27})$.

Chapter 35. Techniques of differentiation

Consider the function $y(x) = x^n \mathrm{e}^x$, where n is a positive integer. Show that

a) when $n = 1$ there is a minimum point

b) when n is odd and greater than or equal to 3, y has a minimum point and a point of inflexion

c) when n is even, there is a minimum point and a maximum point.

Solution

a) When $n = 1$ we have:

$$y = xe^x, \quad y' = \mathrm{e}^x(x+1), \quad y'' = \mathrm{e}^x(x+2)$$

Solving $y' = 0$ yields $x = -1$. Evaluating y'' when $x = -1$ produces

$$y''(x = -1) = \mathrm{e}^{-1}(-1+2) = \mathrm{e}^{-1} > 0.$$

Hence there is a minimum point.

b) We now consider $y = x^n \mathrm{e}^x$ for $n = 3, 5, 7, 9, \ldots$. Then use of the product rule shows that $y' = \mathrm{e}^x x^{n-1}(x+n)$ and hence $y' = 0$ when $x = 0, -n$.

The second derivative can be shown to be $y'' = e^x x^{n-2}[(x+n)^2 - n]$.

At $x = 0$, $y'' = 0$, and so the second derivative test is inconclusive. We consider the sign of y' just to the left and just to the right of $x = 0$, that is for x small and negative and then for x small and positive.

When x is small and negative then, noting that $n-1$ is even, we see that $y' > 0$. When x is small and positive then $y' > 0$ and so there is a point of inflexion when $x = 0$.

At $x = -n$, then

$$y'' = \mathrm{e}^{-n}(-n)^{n-2}[-n] = \mathrm{e}^{-n}(-n)^{n-1} > 0.$$

Hence there is a minimum when $x = -n$.

c) We now consider the case in which n is even. As in b), $y' = 0$ when $x = 0, -n$.

At $x = 0$, $y'' = 0$ and so the second derivative test is inconclusive so we consider the sign of y' for small values of x just to the left and just to the right of $x = 0$.

To the left of $x = 0$, that is for small negative values of x, then x^{n-1} is negative since $n - 1$ is odd and so y' is negative. For small positive values of x, then a similar analysis shows that y' is positive. Since the sign of y' changes from negative to positive there must be a minimum when $x = 0$.

At $x = -n$ we have

$$y'' = \mathrm{e}^{-n}(-n)^{n-2}[-n] = \mathrm{e}^{-n}(-n)^{n-1} < 0$$

And so there is a maximum when $x = -n$.

Chapter 36. Integration and areas under curves

Figure 36.1 shows $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi$. Calculate the shaded area.

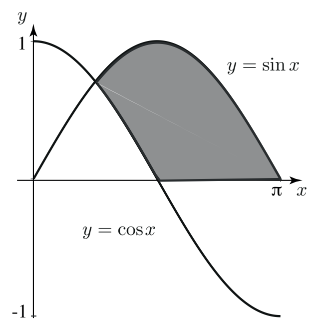


Figure 36.1

Solution

The curves intersect where $\sin x = \cos x$, that is where $\tan x = 1$. So the curves intersect where $x = \frac{\pi}{4}$. Hence the area, A, is given by

$$\begin{aligned}
A &= \int_{\pi/4}^{\pi} \sin x \, \mathrm{d}x - \int_{\pi/4}^{\pi/2} \cos x \, \mathrm{d}x \\
&= [-\cos x]_{\pi/4}^{\pi} - [\sin x]_{\pi/4}^{\pi/2} \\
&= [-(-1) + \cos(\pi/4)] - [\sin(\pi/2) - \sin(\pi/4)] \\
&= \cos(\pi/4) + \sin(\pi/4) = \sqrt{2}
\end{aligned}$$

Chapter 37. Techniques of integration

A circle, centre the origin and radius, r, is given by the equation

$$x^2 + y^2 = r^2.$$

a) If $P(x, y)$ is any point on the circumference with coordinates x and y, show that

$$x = r\cos\theta, \quad y = r\sin\theta$$

where θ is the angle between the positive x axis and the radius arm to P. Figure 37.1 illustrates the situation.

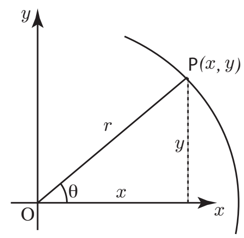


Figure 37.1

b) The vertical line, $x = d$, is drawn as illustrated in Figure 37.2. The shaded area, A, is given by

$$A = 2\int_d^r y\,\mathrm{d}x$$

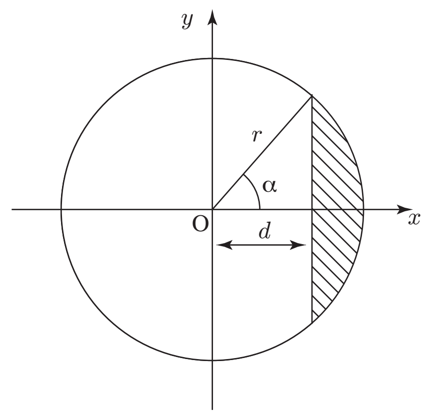


Figure 37.2

By means of the substitution, $x = r\cos\theta$, $y = r\sin\theta$, show that A may be expressed as

$$A = 2\int_\alpha^0 r\sin\theta\,(-r\sin\theta)\mathrm{d}\theta$$

where $\alpha = \cos^{-1}\left(\dfrac{d}{r}\right)$.

c) Use the trigonometrical identity

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

to simplify A and hence show that

$$A = r^2[\alpha - \frac{\sin 2\alpha}{2}]$$

Solution

a) Using Figure 37.1 and the trigonometrical ratios, $\sin\theta$ and $\cos\theta$, we have

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}$$

from which

$$x = r\cos\theta, \quad y = r\sin\theta$$

b) The shaded area, A, is now examined. We have from a), $x = r\cos\theta$ from which

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -r\sin\theta, \qquad \text{and so} \qquad \mathrm{d}x = -r\sin\theta\,\mathrm{d}\theta$$

We also know $y = r\sin\theta$.

Consider the limits. When $x = d$, then $d = r\cos\theta$, that is, $\theta = \cos^{-1}\frac{d}{r} = \alpha$; see Figure 37.2. When $x = r$, $\theta = 0$. Hence, using the substitutions for y and $\mathrm{d}x$, and the relationship between the x and θ limits we have

$$A = 2\int_d^r y\,\mathrm{d}x = 2\int_\alpha^0 r\sin\theta(-r\sin\theta)\,\mathrm{d}\theta.$$

c) We are now able to simplify the integral expression for A.

$$\begin{aligned} A = -2r^2\int_\alpha^0 \sin^2\theta\,\mathrm{d}\theta &= -r^2\int_\alpha^0 1-\cos 2\theta\,\mathrm{d}\theta \\ &= -r^2\left[\theta - \frac{\sin 2\theta}{2}\right]_\alpha^0 \\ &= r^2\left[\alpha - \frac{\sin 2\alpha}{2}\right] \end{aligned}$$

Chapter 38. Functions of more than one variable and partial differentiation

Show that $z = \ln(x^2+y^2)$ is a solution of the equation

$$z_{xx} + z_{yy} = 0$$

Solution

Differentiate z partially with respect to x and y to produce

$$z_x = \frac{2x}{x^2+y^2}, \qquad z_y = \frac{2y}{x^2+y^2}$$

Differentiate again to yield

$$z_{xx} = 2\left(\frac{(x^2+y^2) - x[2x]}{(x^2+y^2)^2}\right) = 2\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

Similarly

$$z_{yy} = 2\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

So

$$z_{xx} + z_{yy} = 2\left(\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) + \left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)\right) = 0$$

as required.