CM Midterm Answers

Question 1

(a) **A given number in base x can be converted to any other base y. According to the expansion method, if $abc.de$ is any given number in base x, then write its value in base 10.**

▼ Ans

$$ax^2 + bx + c + \frac{d}{x} + \frac{e}{x^2}$$

(b) **Covert the following numbers using number system conversions, show you answer in details:**

i. $(723)_8$ **to hexadecimal system**

▼ Ans

First write the octal number in groups of three using binary system, because three binary digits is all the possible representations in octal system, add zeros on the left if necessary.

Next put them into groups of four, because four binary digits is all the possible representations in hexadecimal system, then turn each group into hexadecimal system.

$(7\ 2\ 3)_8 \ \Rightarrow \ (111\ 010\ 011)_2 \ \Rightarrow \ (0001\ 1101\ 0011)_2 \ \Rightarrow \ (1D3)_{16}$

$(1D3)_{16}$

ii. $(0.ABDF)_{16}$ **to decimal system**

▼ Ans

$$\begin{aligned} &\frac{10}{16} + \frac{11}{16^2} + \frac{13}{16^3} + \frac{15}{16^4} \\ =&\frac{10 \times 16^3 + 11 \times 16^2 + 13 \times 16 + 15}{16^4} \\ =&\frac{43999}{65536} \approx 0.67137 \text{ (to 5 d.p.)} \end{aligned}$$

$(0.67137)_{10}$

iii. **Convert** 0.375 **to binary system**

▼ Ans

Assuming that 0.375 is in the decimal system.

$$\begin{array}{cccccl} 1 & . & 0.5 & 0.25 & 0.125 & \\ 0 & . & 0 & 1 & 1 & \Rightarrow 0.25 + 0.125 = 0.375 \end{array}$$

$(0.011)_2$

iv. **Which digits from** $(0, 1, 2, 3, 4, 5)$ **are not allowed in Quinary system (base 5) representation.**

▼ Ans

5

v. $(11010.1011)_2$ **to hexadecimal.**

▼ Ans

First write the binary number in groups of four, because four binary digits is all the possible representations in hexadecimal system, add zeros on the left if necessary, and then turn each group into hexadecimal system.

$(11010.1011)_2 \;\Rightarrow\; (0001\ 1010\ .\ 1011)_2 \;\Rightarrow\; (1A.B)_{16}$

$(1A.B)_{16}$

vi. $(257)_{10}$ **to binary system.**

▼ Ans

$$\begin{array}{cccccccccl} 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \Rightarrow 256 + 1 = 257 \end{array}$$

$(1\ 0000\ 0001)_2$

(c) **Consider the binary number** 10.0011

i. **Convert the above number to the decimal system**

▼ Ans

$$\begin{array}{ccccccccl} 2 & 1 & . & 0.5 & 0.25 & 0.125 & 0.0625 & \\ 1 & 0 & . & 0 & 0 & 1 & 1 & \Rightarrow 2 + 0.125 + 0.0625 = 2.1875 \end{array}$$

$(2.1875)_{10}$

ii. **What are the place values of the digits** 1 **in the number** 0.0011_2

▼ Ans

The first digit 1 in 0.0011_2 has the value of 0.125 in decimal system, the second digit 1 in 0.0011_2 has the value of 0.0625 in decimal system.

0.125 & 0.0625

iii. **What is the sum of** $(1 + 1 + 1 + 1)$ **in binary system**

▼ Ans

$(100)_2$

iv. **Calculate 101 divided by 10 using long division.**

▼ Ans

$$
\begin{array}{r}
10.1 \\
10\overline{)101} \\
\underline{10} \\
1.0 \\
\underline{1.0} \\
0
\end{array}
$$

$101 \div 10 = 10.1$

(d) **Which one is the correct representation of a binary number from the following?**

▼ Ans

(i.) 1101

Wrong, without the indication subscript, this number could be in many number base systems, and will mostly be recognised as a decimal number.

(ii.) $(214)_2$

Wrong, there's no digits greater than 1 in a binary system.

(iii.) $(0000)_2$

Correct, with all digits allowed in a binary system, as well as the subscript indication.

(iv.) $(11)^2$

Wrong, with the superscript on the top right, which does not indicate number bases.

Question 2

(a) **Is** $a_n = \dfrac{3n+2}{n-4}$ **a general term of a sequence? Why?**

▼ Ans

Without any domain restrictions:

No, because a_4 is undefined, the sequence will break when it reaches $n = 4$.

With domain restrictions:

Yes, if the domain is $n \in \mathbb{N}$ and $n > 4$, $n \in \mathbb{Z}$ and $n \neq 4$, or $n \in \mathbb{Z}$ and $n < 4$ & $n > 4$.

(b) **Which term of the sequence with general term** $\dfrac{3n-1}{5n+7}$ **is** $\dfrac{7}{12}$**?**

▼ Ans

$$\begin{aligned} \frac{3n-1}{5n+7} &= \frac{7}{12} \\ \Rightarrow \ 35n + 49 &= 36n - 12 \quad \text{cross multiplication} \\ \Rightarrow \ n &= 61 \end{aligned}$$

a_{61}

(c) **An arithmetic sequence has its** 4^{th} **term equal to 18 and its** 12^{th} **term equal to 50. Find its** 99^{th} **term.**

▼ Ans

$a_4 = 18$, $a_{12} = 50$, find $a_{99} = ?$

According to the arithmetic sequence formula, $a_n = a_1 + (n-1)d$, where a_n is the n^{th} term in the sequence, a_1 is the first term in the sequence, n is the number of the term, and d is the common difference.

$$\begin{cases} a_1 + (4-1)d = 18 & (1) \\ a_1 + (12-1)d = 50 & (2) \end{cases}$$

$(2) - (1)$
$\Rightarrow 8d = 32 \Rightarrow d = 4$
$\Rightarrow a_1 = 6$

$a_n = 6 + (n-1)4$

$a_{99} = 6 + (99-1) \times 4 = 398$

398

(d) **State whether the following sequences are arithmetic, geometric or not any of them. Find the common ratio if it is a geometric sequence and find the common difference d if it is an arithmetic sequence. Then, find the next two terms.**

i. $-3, 3, -3, 3$

▼ Ans

Geometric sequence, common ratio $= -1$, next two terms: $-3, 3$.

ii. $b_n = n^2 + 3$

▼ Ans

Neither, $b_0 = 3, b_1 = 4, b_2 = 7, b_3 = 12$, no common ratio nor common difference in this sequence.

iii. $-\frac{1}{2}, -\frac{5}{6}, -\frac{7}{6}$

▼ Ans

Arithmetic sequence, common difference $= -\frac{1}{3}$, next two terms: $-\frac{3}{2}, -\frac{11}{6}$.

(e) **Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this seqence?**

▼ Ans

According to the geometric sequence formula, $b_n = b_1 q^{n-1}$, where b_n is the n^{th} term in the sequence, b_1 is the first term in the sequence, q is the common ratio, and n is the number of the term.

First find some pattern using the formula.

$b_1 = \frac{1}{9}, b_2 = \frac{1}{9} \cdot 3^{2-1} = \frac{1}{3}, b_3 = \frac{1}{9} \cdot 3^{3-1} = 1, b_4 = \frac{1}{9} \cdot 3^{4-1} = 3.$

Each sequence is the last term times 3, and since it starts at $\frac{1}{9}$, the denominator is a multiple of 3, hence all values in the sequence are multiple of 3.

Yes, 243 is a term of this sequence, since 243 is a multiple of 3.

(f) **The 19th term of a sequence is -52, and the 4th term is -7. The difference between consecutive terms in the sequence is constant. Find the 201^{st} term.**

▼ Ans

$a_{19} = -52, a_4 = -7, a_{201} = ?$

We know from the question that "the difference between consecutive terms in the sequence is constant", hence this sequence is an arithmetic sequence.

According to the arithmetic sequence formula, $a_n = a_1 + (n-1)d$, where a_n is the n^{th} term in the sequence, a_1 is the first term in the sequence, n is the number of the term, and d is the common difference.

$$\begin{cases} a_1 + (19-1)d = -52 & (1) \\ a_1 + (4-1)d = -7 & (2) \end{cases}$$

$(1) - (2)$

$$\Rightarrow 15d = -45 \Rightarrow d = -3$$
$$\Rightarrow a_1 = 2$$
$$a_n = 2 - 3(n-1)$$
$$a_{201} = 2 - 3 \times 200 = -598$$

-598

(g) **Show whether the following sequence is convergent or divergent.**

$$\lim_{n \to \infty} \left(\frac{n-1}{n} \right)$$

▼ Ans

$$n = 1 \Rightarrow 0$$
$$n = 100 \Rightarrow 0.99$$
$$n = 10000 \Rightarrow 0.9999$$

$\Rightarrow$ Convergent.

(h) **Is the following numbers 1, -4, 9, -16, ... represent a sequence, if so, find a formula for the n^{th} term of the sequence.**

▼ Ans

Not looking at the negatives, this is a sequence of powers of two when the base number is $\mathbb{N}$. However, we need negative values in even terms. Hence we multiple the power of two with the condition to make the sequence negative when it is an even term.

$n^2 \cdot (-1)^{n+1}$

(i) Show by mathematical induction that for all positive integers n, $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

▼ Ans

Proof.

Let $P(n)$ be "$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$".

Base step: $P(1)$ is true because $\frac{1}{2} = (1 - \frac{1}{2^1})$.

Inductive step:

Assume that $P(k) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$.

Then

$$\begin{aligned} P(k+1) &= \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \\ &= 1 + \frac{-2+1}{2^{k+1}} \\ &= 1 + \frac{-1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \end{aligned}$$

This completes the inductive step.

We have completed the base and inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ for all positive integers.

(j) **Find the remainder when** 3^{123} **is dividied by 7.**

▼ Ans

Since 3^{123} is a very big number, using the property of exponentiation, we break it down and do the modular separately to get the remainder.

$3^{123} = 3^3 \cdot (3^{10})^{12} \Rightarrow 3^{123} \pmod 7 \equiv \{3^3 \pmod 7 \cdot [3^{10} \pmod 7]^{12}\} \mod 7$

$\Rightarrow \{27 \pmod 7 \cdot [59049 \pmod 7]^{12}\} \mod 7 \equiv 6 \cdot (4^{12}) \mod 7 \equiv 6 \cdot 1 \mod 7 = 6$

$3^{123} \pmod 7 = 6$

Question 3

(a) **State whether the following statements are false or true, explain your answer:**

i. **Given any integers** a, b, c **and any positive integer** n**. If** $a \equiv b \pmod{n}$ **and** $b \equiv c \pmod{n}$**, then** $a \equiv c \pmod{n}$**.**

▼ Ans

False, prove given a counterexample.

Let $a = 2, b = 9, c = 16, n = 7$.

$2 \equiv 9 \pmod{7}$
$9 \equiv 16 \pmod{7}$
However, $2 \not\equiv 16 \pmod{7}$.

Hence false.

ii. **Suppose** $a \equiv b \pmod{n}$ **and** $c \equiv d \pmod{n}$**, then** $a + c \equiv b + d \pmod{n}$**.**

▼ Ans

Proof by direct proof.

Because $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, by the following Theorem, there are integers s and t with $b = a + sn$ and $d = c + tn$.

Hence, $b + d = (a + sn) + (c + tn) = (a + c) + n(s + t)$.

Hence, $a + c \equiv b + d \pmod{n}$.

Theorem

> Let n be a positive integer. The integers a and b are congruent modulo n if and only if there is an integer k such that $s = b + kn$.

iii. $7x \equiv 12 \pmod{7}$

▼ Ans

Solution 1:

False, because $7x$ is always going to be a multiple of 7, the congruence should be zero, cannot be equivalent to 5, which is the result of $12 \pmod{7}$.

Solution 2:

Use fractions,

$$7x \equiv 12 \pmod 7$$
$$x \equiv \frac{12}{7} \pmod 7$$

Cancel factors in the numerator and denominator, we obtain $x \equiv \frac{5}{0} \pmod 7$ which is `undefined`

Hence, false.

(b) **Find the least positive value of x such that:** $71 \equiv x \pmod 8$

▼ Ans

$71 - 8 \times 8 = 7$

$71 \equiv 7 \pmod 8$

(c) **Calculate the multiplicative inverse of 168 in modulo 83.**

▼ Ans

$$\frac{1}{168} \pmod{83} \equiv 42$$

(d) **Calculate the inverse of 4 modulo 15. Show your steps.**

▼ Ans

Inverse of $4 \pmod{15}$

$$\begin{array}{ll} 4 \times 0 & 0 \pmod{15} \\ 4 \times 1 & 4 \pmod{15} \\ 4 \times 2 & 8 \pmod{15} \\ 4 \times 3 & 12 \pmod{15} \\ 4 \times 4 & 16 \pmod{15} \quad \equiv 1 \pmod{15} \quad \leftarrow \text{inverse !} \end{array}$$

4

Question 4

(a) **A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find**

i. **the legnth of side c.**

▼ Ans

Using the law of cosine, we obtain:

$$c^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos 60°$$
$$= 13 - 12 \cdot \frac{1}{2} = 7$$
$$c = \sqrt{7}$$

ii. **Find the sine of angle B using sine rules.**

▼ Ans

$$\frac{\sin B}{3} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{7}} = \frac{\sqrt{3}}{2\sqrt{7}}$$
$$2\sqrt{7} \cdot \sin B = 3\sqrt{3}$$
$$\sin B = \frac{3\sqrt{3}}{2\sqrt{7}} = \frac{3\sqrt{21}}{14}$$

$$\frac{3\sqrt{21}}{14}$$

(b) **If we have a triangle which has one of its side $c = 2$ and angles $A = \frac{\pi}{4}$ and $B = \frac{\pi}{3}$. Workout the length a of the side opposite A.**

▼ Ans

Angle A = 180/4 = 45 degrees, angle B = 180/3 = 60 degrees, hence angle C = 180 - 45 - 60 = 75 degrees.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\Rightarrow \frac{\sin 45°}{a} = \frac{\frac{1}{\sqrt{2}}}{a} = \frac{\sin 75°}{2}$$
$$a \cdot \sin 75° = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$$
$$\Rightarrow a = \frac{\sqrt{2}}{\sin 75°} = 1.464 \text{ to 3 d.p.}$$

1.464

(c) XYZ **is a right angled triangle with** $Y = 90°$**. Given that** $y = 85, \sin X = \frac{77}{85}$**, find** $z, \cos Z$ **and the angle of** Z**.**

▼ Ans

Since XYZ is a right angled triangle, and y = 85, from $\sin X = \frac{77}{85}$, we obtain:

$z = \sqrt{85^2 - 77^2} = 36$.

Therefore, $\cos Z = \frac{77}{85}$.

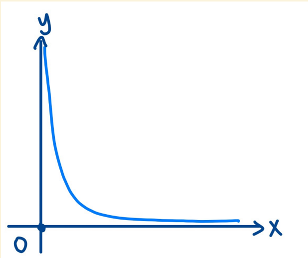
$\cos^{-1}(\frac{77}{85}) = 25.0576$ to 4 d.p.

Therefore, $z = 25.0576°$.

(d) **Let** g **be a function with its domain** $(0, \infty)$**, defined by** $g(x) = \frac{1}{x}$**.**

i. **Sketch the graph of** g**.**

▼ Ans



ii. **Is** g **continuous at other points of its domain?**

▼ Ans

Yes, g is continuous at all points in the domain $(0, \infty)$.

Question 5

(a) **Consider the function** $f : \mathbb{Z} \to \mathbb{Z}$ **given by** $f(x) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-3, & \text{if } n \text{ is odd} \end{cases}$.

i. **Is** f **injective? Prove your answer.**

▼ Ans

If x & y are odd

$$\begin{aligned} f(x) &= f(y) \\ x-3 &= y-3 \\ x &= y \end{aligned}$$

Therefore, *injective*.

If x & y are even

$$\begin{aligned} f(x) &= f(y) \\ x+1 &= y+1 \\ x &= y \end{aligned}$$

Therefore, *injective*.

Even though the odd and even separates the function, because the domain and subdomain is $\mathbb{Z}$, every input still maps to a single output, hence still *injective*.

ii. **Is** f **surjective? Prove your answer.**

▼ Ans

If x & y are odd

$$\begin{aligned} y &= x-3 \\ y+3 &= x \end{aligned}$$

For every $y \in \mathbb{Z}$, x has an element maps to y in $\mathbb{Z}$.

Therefore, *surjective*.

If x & y are even

$$\begin{aligned} y &= x+1 \\ y-1 &= x \end{aligned}$$

For every $y \in \mathbb{Z}$, x has an element maps to y in $\mathbb{Z}$.

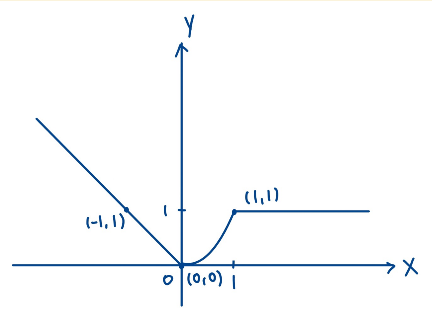
Therefore, *surjective*.

(b) **The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows:**

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x^2, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

Plot the function, and say whether it is bijective function ro not. Explain your answer.

▼ Ans



Injective: Yes.

Using the Vertical Line Test, we can see that throughout the domain $\mathbb{R}$, every x value maps to a single y value.

Surjective: No.

The codomain is $\mathbb{R}$, however, none of the negative values have a x that maps to it.

Bijective: No.

(c) **The velocity of a particle moving along the x axis varies according to the expression$v_x = 40 - 5t^2$, where v_x is in meters per second and t is in seconds.**

Find the average acceleration in the time interval $t = 0$ to $t = 2$ sec.

▼ Ans

$$a_{\text{average}} = \frac{20 - 40}{2 - 0} = -10\ m/sec^2$$

(d) **An object moving with uniform acceleration has a velocity of $12cm/s$ in the positive x direction when its x coordinate is $3.00cm$. If its x coordinate 2 seconds later is $-5cm$, what is its acceleration?**

▼ Ans

$a = -16\ cm/s^2$