Section B

Question 2

(a) Show whether the following series are convergent or divergent. If the series is convergent determine the value of it.

i. $s_n = \dfrac{6+7n^2}{5-3n^2}$

Ans.

$s_0 = \dfrac{6}{5} = 1.2,\ s_{10} = \dfrac{706}{-295} \approx -2.4,\ s_{1000} = \dfrac{700006}{-299995} \approx -2.3.$

Convergent to 2.3 ■

ii. $\Sigma_{n=1}^{\infty}(-1)^n \cos\left(\dfrac{1}{n}\right)$

Ans.

$n = 10 \to 1 \cdot \cos\left(\dfrac{1}{10}\right) = 0.995\ldots$

$n = 100 \to 1 \cdot \cos\left(\dfrac{1}{10}\right) = 0.99995\ldots$

$n = 9 \to 1 \cdot \cos\left(\dfrac{1}{10}\right) = -0.99\ldots$

$n = 99 \to 1 \cdot \cos\left(\dfrac{1}{10}\right) = -0.9999\ldots$

Doesn't converge, hence divergent. ■

(b) Suppose that we have a die which is rolled and a coin which is tossed. What is the probability that the die shows an odd number and the coin shows a head.

Ans.

The probability of rolling an odd number on the die is $\dfrac{3}{6}$ since there are three odd numbers out of the six possible ones.

The probability of tossing a head on a coin is $\dfrac{1}{2}$ since there are two possible outcomes of a coin.

Two events are independent hence the result has to be multiplied.

$\dfrac{3}{6} \cdot \dfrac{1}{2} = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}.$ ■

(c) Find the value of the following, put it in its simplest form.

i. $\cos \dfrac{\pi}{12}$

Ans.

$$\begin{aligned}
\cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

■

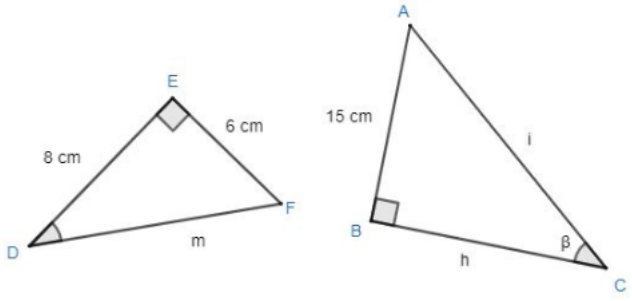
ii. $\log_{x^2} x^3$

Ans.
Let a be the answer.
$(x^2)^a = x^3 \to x^{2a} = x^3 \to 2a = 3 \to a = \dfrac{3}{2}$
■

(d) Consider that the position of an object is given by the following equation: $s(t) = 2te^t$. Will this object stop moving? If so, at which value of t the object will stop moving.

Ans.
No, it just keeps getting faster. ■

(e) If the two triangles given below are similar. Find the length of the side i.



Ans.

$$\frac{6}{15} = \frac{8}{h} = \frac{m}{i}$$
$$\frac{2}{5} = \frac{8}{h} = \frac{10}{i} \quad m = 10 \text{ since } \triangle\text{DEF is a right triangle}$$
$$\frac{2}{5} = \frac{8}{20} = \frac{10}{25} \quad \text{hence } i = 25 \quad \blacksquare$$

Question 4

(a) Point P moves along the x-axis in such a way that its position at time t/s is given by $x = 2t^3 - 15t^2 + 24t$ ft.

i. Find the velocity and acceleration of P at time t

Ans.
$v = x' = 6t^2 - 30t + 24 = 6(t^2 - 5t + 4) = 6(t-4)(t-1)$
$a = x'' = 12t - 30 = 6(2t - 5)$ ■

ii. In which direction and how fast is P moving at $t = 2s$? Is it speeding up or slowing down at that time?

Ans.
$v = 6(-2)(1) = -12$ ft/s, moving at -12 ft/s, negative direction.
$a = 6(2 \cdot 2 - 5) = -6$ ft/s^2, the speed is slowing down. ■

iii. When is P instantaneously at rest? When is its speed instantaneously not changing?

Ans.
At rest means the velocity is zero, if take the velocity equation, we can see that when $t = 1$s and $t = 4$s, the velocity is zero, therefore the answer is $t = 1, 4$s.
Speed not changing means the acceleration is zero, if we take the acceleration equation, we can see that when $t = 2.5$s, the acceleration is zero, therefore the answer is $t = 2.5$s. ■

(b) You want to invite your friends to have a party. You have 15 friends but you have only 7 chairs in your garden.

i. How many different ways do you have for which 7 friends to invite?

Ans.
$$C_7^{15} = \frac{15!}{8!7!} = \frac{15 * 14 * 13 * 12 * 11 * 10 * 9}{2 * 3 * 4 * 5 * 6 * 7} = 6435 \text{ ways } \blacksquare$$

ii. What if you decided not only which friends to invite but also where to seat them along your table? How many different ways do you have?

Ans.
$P_7^{15} = \dfrac{15!}{8!} = 15 * 14 * 13 * 12 * 11 * 10 * 9 = 32432400$ ways ■

(c) Solve the equation $3\log(x+5) = 2\log(7-x)$.

Ans.

$$(x+5)^3 = (7-x)^2$$
$$x^3 + 15x^2 + 75x + 125 = x^2 - 14x + 49$$
$$x^3 + 14x^2 + 89x + 76 = 0$$

$x = 0 \to 76,\ x = 1 \to 180,\ x = -1 \to 0,\ x = -2 \to -54$
$x = -1$ ■

(d) Find the derivatives of the following:

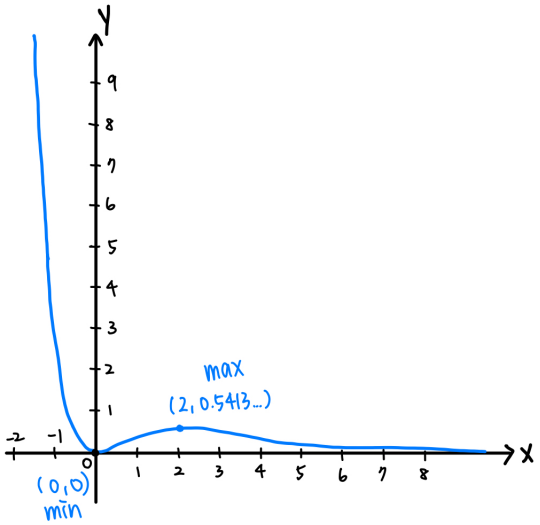
i. $y = e^{x^3 - 3x^2}$

Ans.
$(3x^2 - 6x)e^{x^3-3x^2}$

ii. $y = 10^{5x}$

Ans.
$5 \cdot 10^{5x} \cdot \ln(10)$

(e) Find the critical points (maximum and minimum values) of the function $f(x) = x^2 e^{-x}$. Sketch the graph.



x	0	1	2	3	4
f(x)	0	e^{-1}	$4e^{-2}$	0.4481...	0.2930...
		0.3678...	0.5413...		

$f'(x) = x^2 \cdot (-1)e^{-x} + 2xe^{-x}$
$= e^{-x}(-x^2+2x)$
$= xe^{-x}(-x+2)$

$x = 2 \to$ critical point (max)
$x = 0 \to$ critical point (min)