Ch6.1 The Basics of Counting

Basic Counting Principles

> **THE PRODUCT RULE**   Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$

▼ Example 5

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited)?

Solution: There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ possible license plates.

> **THE SUM RULE**   If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$ when $A_i \cap A_j = \forall i, j.$

▼ Example 13

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is one more then one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are 23+15+19=57 ways to choose a project.

More Complex Counting Problems

▼ Example 15

In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings or two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

Solution: Let V equal the number of different variable names in this version of BASIC. Let V_1 be the number of these that are one character long and V_2 be the number of these that are two characters long. Then by the sum rule, $V = V_1 + V_2$. Note that $V_1 = 26$, because a one-character variable name must be a letter. Furthermore, by the product rule there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $V_2 = 26 \cdot 36 - 5 = 931$. Hence, there are $V = V_1 + V_2 = 26 + 931 = 957$ different names for variables in this version of BASIC.

The Subtraction Rule (Inclusion-Exclusion for Two Sets)

> **THE SUBTRACTION RULE**   If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$

▼ Example 18

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: We can construct a bit string of length either that begins with a 1 in $2^7 = 128$ ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5 = 32$ ways to construct such a string. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals 128+64-32=160.

The Division Rule

> 🗣 **THE DIVISION RULE**   There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

$\rightarrow$ "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$."

▼ Example 20

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neightbor?
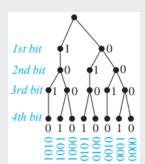
Solution: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note there are four ways to select the person for seat 1, three for seat 2, two for seat 3, and one for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbour. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

Tree Diagrams

▼ Example 21

How many bit strings of length four do not have two consecutive 1s?

Solution: The tree diagram displays all bit strings of length four without two consecutive 1s. We see that there are eight bit strings of length four without two consecutive 1s.