Ch6.5 Generlized Permutations and Combinations

Permutations with Repetition

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

▼ Example 3

How many ways are there to select 5 bills from a cash box containing $1, $2, $5, $10, $20, $50, and $100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least 5 bills of each type.

Solution: Suppose that a cash box has seven compartments, one to hold each type of bill. These compartments are separated by 6 dividers. The choice of 5 bills corresponds to placing 5 markers in the compartments holding different types of bills.

The number of ways to select 5 bills corresponds to the number of ways to arrange 6 bars and 5 stars in a row with a total of 11 positions. This corresponds to the number of unordered selections of 5 objects from a set of 11 objects, which can be done in $C(11, 5) = \frac{11!}{5!6!} = 462$ ways.

THEOREM 2 There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

▼ Example 4

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(4+6-1, 6) = C(9, 6) = 84$ ways to choose the size cookies.

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

THEOREM 3 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguiushable objects of type 2, ..., and n_k indistinguishable objects of type k, is $\frac{n!}{n_1! n_2! \cdots n_k!}$.

▼ Example 7

How many different strings can be made by reordering the letters of the word $SUCCESS$?

Solution: This word contains three Ss, two Cs, one U, and one E. Note that the three Ss can be placed among hte secen positions in $C(7.3)$ differen ways, leaving four positions free. Then the two Cs can be placed in $C(4,2)$, leaving two free positions. The U can be placed in $C(2,1)$ ways, leaving just one position free. Hence E can be placed in $C(1,1)$ way. Consequently, from the product rule, the number of different strings that can be made is $C(7,3)C(4,2)C(2,1)C(1,1)=\frac{7!}{3!4!}\cdot\frac{4!}{2!2!}\cdot\frac{2!}{1!1!}\cdot\frac{1!}{1!0!}=\frac{7!}{3!2!1!1!}=420$.

Distributing Objects into Boxes

- **distinguishable (labeled)**: different from each other
- **indistinguishable (unlabeled)**: identical

Distunguishable Objects & Distinguishable Boxes

4 **THEOREM 4** The number of ways to distribute n distinguishable objects into k distinguishable boxes so that $n_1$objects are placed into box i, $i=1,2,\dots,k$, equals $\frac{n!}{n_1!n_2!\cdots n_k!}$.

▼ Example 8

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: Note that the first player can be dealt 5 cards in $C(52,5)$ ways. The second player can be dealt 5 cards in $C(47,5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42,5)$ ways. Finally, the fourth player can be dealth 5 cards in $C(37,5)$ ways. Hence, the total number of ways to deal four players 5 cards each is $C(52,5)C(47,5)C(42,5)C(37,5)=\frac{52!}{47!5!}\cdot\frac{47!}{42!5!}\cdot\frac{42!}{37!5!}\cdot\frac{37!}{32!5!}=\frac{52!}{5!5!5!5!32!}$.

Indistinguishable Objects & Distinguishable Boxes

▼ Example 9

How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

The number of ways to place 10 indistinguishable balls into 8 bins equals the number of 10-combinations from a set with 8 elements when repetition is allowed. Consequently, there are $C(8+10-1,10)=C(17,10)=\frac{17!}{10!7!}=19,448$.

⇒There are $C(n+r-1,n-1)$ ways to place r indistinuishable objects into n distinguishable boxes.

Distinguishable Objects & Indistinguishable Boxes

▼ Example 10

How many ways are there to put 4 different employees into 3 indistinguisable offices, when each office can contain any number of employees?

Solution: We will solve this problem by enumerating all the ways these employees can be placed into the offices. We represent the 4 employees by A, B, C, and D.

We can put all 4 employees into one office in exactly one way, represented by {{A, B, C, D}}; 3 employees into 1 office and the fourth employee into a different office in exactly 4 ways, {{A, B, C}, {D}}, {{A, B, D}, {C}}, {{A, C, D}, {B}}, and {{B, C, D}, {A}}; 2 employees into 1 office and two into a second office in exactly 3 ways, {{A, B}, {C, D}}, {{A, C}, {B, D}}, and {{A, D}, {B, C}}; finally, 2 employees into 1 office, and one each into each of the remaining 2 offices in 6 ways, {{A, B}, {C}, {D}}, {{A, C}, {B}, {D}}, {{A, D}, {B}, {C}}, {{B, C}, {A}, {D}}, {{B, D}, {A}, {C}}, and {{C, D}, {A}, {B}}. The total posiibilityes are 14 ways.

Another way to look at this problem is to look at the number of offices into which we put employees. Note that there are 6 ways to put four different employees into 3 indistinguishable offices so that no office is empty, 7 ways to put 4 different employees into 2 indistinguishable offices so that no office is empty, and 1 way to put 4 employees into one office so that it is not empty.

Stirling number of the second kind: Let $S(n,j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty.

Indistinguishable Objects & Indistinguishable Boxes

▼ Example 11

How many ways are there to pack 6 copies of the same book into 4 identical boxes, where a box can contain as many as 6 books?

Solution: We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can peack the books are [6], [5, 1], [4, 2], [4, 1, 1], [3, 3], [3, 2, 1], [3, 1, 1, 1], [2, 2, 2], [2, 2, 1, 1]. (I.E. [4, 1, 1] indicates that 1 box contains 4 books, a second box contains a gingle book, and a third box contains a single book (and the fourth box is empty)). We conclude that there are 9 allowable ways to pack the books.