FCS Midterm Answers Q2

2. For all nonnegative integers n, 3 divides $n^3 + 2n + 3$.

Proof.

Let $P(n)$ be "$n^3 + 2n + 3$ is divisible by 3."

Base step: $P(0)$ is true because $0^3 + 2 \cdot 0 + 3 = 3$ is divisible by 3.

Inductive step:

Assume that $P(k)$ is true, that is, $k^3 + 2k + 3$ is divisible by 3.

Then

$$\begin{aligned} P(k+1) &= (k+1)^3 + 2(k+1) + 3 \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 + 3 \\ &= k^3 + 3k^2 + 5k + 6 \\ &= (k^3 + 2k + 3) + 3(k^2 + k + 1) \end{aligned}$$

is also divisible by 3, because both terms in this sum are divisible by 3. Given the first part is $P(k)$ which we assume is divisible by 3, and the second part is a multiple of 3.

This completes the inductive step.

We have completed the base and inductive step, so by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, we have proven that $n^3 + 2n + 3$ is divisible by 3 for all nonnegative integers.