Lesson 11: Root Finding

In physics we are often faced with the problem of finding the solution (or solutions) of an algebraic equation of the form $f(x) = 0$. These solutions are called the "roots" of the function $f(x)$. In some cases, we can solve for x analytically. In most cases, we cannot. In this lesson you will learn two numerical techniques for solving equations of the form $f(x) = 0$.

1 Projectile motion with linear drag

In Lesson 10 we saw that the force of drag on an object moving through a fluid is

$$\vec{F}_d = -\frac{1}{2}\rho A v C_d \vec{v} \tag{1}$$

where ρ is the fluid density, A is the cross–sectional area of the object, $\vec{v}$ is the velocity with magnitude $v \equiv |\vec{v}|$, and C_d is the drag coefficient. At low velocities, the drag coefficient is proportional to $1/v$.[1] Thus, for low velocities, the drag force can be written as

$$\vec{F}_d = -mk\vec{v} \tag{2}$$

where k is a positive constant and m is the object's mass.

Newton's second law is $\vec{F}_d + \vec{F}_g = m\vec{a}$, where $\vec{F}_g$ is the gravitational force and $\vec{a}$ is the object's acceleration. Thus, we have $m\vec{a} = -mk\vec{v} - mg\hat{y}$ where the unit vector $\hat{y}$ points up and g is the acceleration due to gravity. For a projectile moving in two dimensions, the components of Newton's second law are

$$m\ddot{x} = -mk\dot{x} \tag{3a}$$
$$m\ddot{y} = -mk\dot{y} - mg \tag{3b}$$

Here, $\vec{a} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y}$ is the acceleration vector, with the dots denoting time derivatives.

Equations (3) for projectile motion with linear drag have the analytic solution

$$x(t) = x_0 + \frac{v_{x,0}}{k}\left(1 - e^{-kt}\right) \tag{4a}$$
$$y(t) = y_0 - \frac{gt}{k} + \frac{v_{y,0} + g/k}{k}\left(1 - e^{-kt}\right) \tag{4b}$$

Here, $v_{x,0}$ and $v_{y,0}$ are the components of the initial velocity and x_0 and y_0 are the components of the initial position vector.

[1]The drag coefficient for a sphere is given by the heuristic formula from Lesson 10. For low velocities it is dominated by the first term, $C_d \approx 24/R_e$. The Reynolds number R_e is proportional to v.

How long is the projectile in the air? Let t_L denote the "landing" time. Assuming the ground is level, the landing time is obtained by setting $y(t_L) = y_0$ which gives

$$0 = -\frac{g t_L}{k} + \frac{v_{y,0} + g/k}{k}\left(1 - e^{-k t_L}\right) \tag{5}$$

Note that $t_L = 0$ is a solution of this equation, but we're interested in finding a second solution with $t_L > 0$. This equation *does* have a solution with $t_L > 0$, as we expect from physical reasoning. However, that solution cannot be expressed in terms of the parameters (g, k, and $v_{y,0}$) using standard functions like powers, trig functions, logarithms, *etc.* We must use numerical methods to find the landing time $t_L > 0$.

> Exercise 1: The equation (5) above has the form $f(t_L) = 0$. Write a python function that takes in a time t and computes $f(t)$. Choose $k = 0.02/\mathrm{s}$, $g = 9.8\,\mathrm{m/s^2}$, and $v_{y,0} = 35.0\,\mathrm{m/s}$ for the parameter values. See if you can find t_L to within $\pm 0.01\,\mathrm{s}$ by guess–and–check.

How could you program a computer to systematically search for the solution t_L? One method is to start with an initial guess, then increase (or decrease) your guess in small steps until the function $f(t)$ changes sign. The value of the solution t_L is between the last two guesses. With this method, the accuracy of the answer can be increased by decreasing the step size. However, if you choose a small step size, you are forced to take very many steps (unless the initial guess is really good). Thus, this method of finding a solution is highly inefficient. We will now introduce two methods that are much more efficient, namely, the bisection method and Newton's method.

2 The bisection method

The bisection method is a simple, robust method of root finding. For a continuous function $f(x)$, we start by assuming a root exists between points $x = a$ and $x = b$. If $f(a)$ and $f(b)$ have opposite signs, then there must be at least one root between a and b. If a and b are sufficiently close together, then the midpoint $c = (a + b)/2$ is a good approximation to the root. If a and b are not particularly close, then we use the midpoint $c = (a + b)/2$ to replace either a (if c is bigger than the root) or b (if c is smaller than the root). We repeat this process until a and b are close enough that the root is accurately determined by the midpoint.

Here is the step–by–step procedure:

1. if the sign of $f(a)$ is different from $f(b)$, then let $c = (a + b)/2$
2. if $\mathrm{sign}(f(c)) == \mathrm{sign}(f(a))$, c is smaller than the root and we replace the value of a with c
3. if $\mathrm{sign}(f(c)) == \mathrm{sign}(f(b))$, c is larger than the root and we replace the value of b with c
4. repeat until $\mathrm{abs}(a - b) <= \mathrm{tolerance}$

The calculation guarantees that the root is within ±tolerance of the final values of a and b.

The bisection method is robust, meaning it almost always succeeds in finding a root. With each cycle through the step–by–set procedure, the error in the answer is cut in half. This is an improvement over the method described at the end of the last section. Can you see why this is called the "bisection method"?

> Homework: Use the bisection method to find the landing time t_L for the projectile with linear drag. Your answer should be accurate to within $\pm 0.001\,\mathrm{s}$. Experiment with different initial values for a and b.

3 Newton's method

Newton's method (aka the Newton–Raphson method) is much more efficient than the bisection method—it reaches the answer to within a comparable error in fewer steps. However, Newton's method is less robust than bisection, and it can be more difficult to apply because it requires the derivative of the function $f(x)$.

Let x_1 denote an initial guess for a root of $f(x)$. Consider the Taylor expansion of $f(x)$ about x_1:

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \cdots \tag{6}$$

The unwritten terms are proportional to $(x - x_1)^2$, $(x - x_1)^3$, *etc.* Let's assume for the moment that x is the root we're looking for; that is, $f(x) = 0$. Now, if the guess x_1 is close to the actual root x, then the unwritten terms in Eq. (6) will be small. Dropping these terms, the resulting equation becomes

$$0 = f(x_1) + f'(x_1)(x - x_1) \tag{7}$$

This can be solved for the root x. Note, however, that this equation is only approximate because we have dropped the higher order terms in Eq. (6). Thus, this value x will not be the exact root. We can consider it to be our second guess. Instead of x, let's call it x_2. Then the result for x_2 is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{8}$$

In most cases, x_2 will be a closer approximation to the root than x_1.

Clearly we can iterate this process:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{9}$$

With each iteration, the approximation x_{i+1} should be closer to the actual root of $f(x)$ than the previous approximation x_i.

> Exercise 2: Write a code applying Newton's method to find the landing time t_L for a projectile with linear drag. Experiment with different values for the initial guess, and different values for the tolerance.

4 Rocket equation

Here is another problem that involves root finding. The equation governing the motion of a rocket moving in one dimension (along the x–axis) in outer space is

$$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt} \tag{10}$$

Here, v is the rocket's velocity with respect to a fixed inertial frame, and m is the mass of the rocket (including the unburned fuel it contains). Also, u is the speed of the exhaust gas relative to the rocket. Assuming u is constant in time, this equation can be integrated to give

$$v(t) = v_0 - u\log(m(t)/m_0) \tag{11}$$

where v_0 and m_0 are the velocity and mass at $t = 0$. If the rocket burns fuel at a constant rate, then $m(t) = m_0 - \rho t$, where $\rho = -dm/dt$ is the rate of fuel consumption. Setting $v = dx/dt$ and integrating again, we find

$$x(t) = x_0 + (u + v_0)t + u(m_0/\rho - t)\log(1 - \rho t/m_0) \tag{12}$$

where x_0 is the initial position along the x–axis.

Homework: A rocket whose initial mass is 5000 kg burns fuel at a rate of 200 kg/s. The relative speed of the exhaust gas is 2000 m/s. Write a computer code to determine the time required for the rocket, starting from rest, to travel 4 km. Your answer should be accurate to within 0.001 s. What are the velocity and mass of the rocket at this time?