Lesson 10: Fluid Drag on a Sphere

1 Drag equation and drag coefficient

The drag equation says that the drag force on an object moving through a fluid (gas or liquid) is

$$\vec{F}_d = -\frac{1}{2}\rho A v C_d \vec{v} \tag{1}$$

where ρ is the fluid mass density, A is the cross–sectional area of the object, and $\vec{v}$ is the object's velocity relative to the fluid. The magnitude of $\vec{v}$ is $v \equiv |\vec{v}|$. The dimensionless factor C_d is the *drag coefficient*, which depends on the shape of the object, the fluid properties, and v. In elementary treatments of drag at low speeds, C_d is often assumed to be proportional to $1/v$. This makes the drag force proportional to $-\vec{v}$. In elementary treatments of drag at high speeds, C_d is often assumed to be constant, independent of v. This makes the drag force proportional to $-v\vec{v}$.

The actual dependence of C_d on v is complicated. At low velocities the fluid flow around the object can be smooth and steady. At higher velocities the flow can produce vorticies, and eventually become turbulent. The dependence of C_d on v must be determined by experiment. The results for a perfect sphere are well–approximated by the following heuristic formula:

$$C_d(R_e) = \frac{24}{R_e} + \frac{2.6(R_e/5.0)}{1+(R_e/5.0)^{1.52}} + \frac{0.411(R_e/263000)^{-7.94}}{1+(R_e/263000)^{-8.00}} + \frac{(R_e)^{0.80}}{461000} \tag{2}$$

Here,

$$R_e \equiv \frac{\rho v D}{\mu} \tag{3}$$

is the *Reynolds number*, where D is the sphere's diameter and μ is the viscosity of the fluid. The Reynolds number is a dimensionless quantity proportional to the magnitude of velocity v. This heuristic formula for $C_d(R_e)$ is considered to be reasonably accurate for values of Reynolds number up to at least 10^6.

Homework I: Write a python program to make a log–log plot of C_d versus R_e, in the domain $0.1 \leq R_e \leq 10^7$. Make sure your plot is correctly labeled.

Hint: If you choose R_e values that are spread out evenly (linearly) across this domain, for example $R_e = 0.1, 0.2, \ldots, 10^7$, the graph will require about 10^8 evaluations of the function $C_d(R_e)$. You should, instead, choose R_e values that are distributed logarithmically. That is, choose $R_e = 10^{-1}, 10^{-0.9}, \ldots, 10^7$. This will take about 80 evaluations of the function. You can create a numpy array of R_e values spread out logarithmically by using the `logspace` command (a cousin of the `linspace` command).

2 Motion of a shot pellet

A shotgun is designed to shoot round pellets, called "shot", typically made of lead. A size #F shot pellet has a mass of $m = 1.04\,\mathrm{g}$ and a diameter of $D = 5.59\,\mathrm{mm}$. The muzzle velocity of a shotgun is about $500\,\mathrm{m/s}$ (relatively low compared to other rifles). If a shotgun is shot straight up into the air, how high will the shot pellets rise? When will they land? How fast will they be going when they land?

> Exercise: The density of air is about $\rho = 1.28\,\mathrm{kg/m^3}$ and the viscosity is about $\mu = 1.83 \times 10^{-5}\,\mathrm{kg/(m \cdot s)}$. Find the Reynolds number for a shot pellet as it emerges from the shotgun. Is it within the range that is well approximated by the heuristic formula for C_d?

Let y denote the height of the shot pellet above ground. Newton's second law says $\vec{F}_g + \vec{F}_d = m\vec{a}$, where $\vec{F}_g = -mg\hat{y}$ is the gravitational force and $\vec{a}$ is the acceleration. For this one–dimensional problem, the velocity is $\vec{v} = v_y\hat{y}$, with $v_y = (dy/dt)$. The *magnitude* of velocity $\vec{v}$ is $v = |v_y|$. Thus, the drag force from Eq. (1) is $\vec{F}_d = -\rho A|v_y|C_d v_y\,\hat{y}/2$ where C_d is given as a function of the Reynolds number $R_e = \rho|v_y|D/\mu$ by Eq. (2). The y–component of Newton's second law becomes

$$-mg - \frac{1}{2}\rho A|v_y|C_d v_y = m\frac{d^2y}{dt^2}$$

where $\vec{a} = (d^2y/dt^2)\hat{y}$ is the acceleration of the shot pellet.

The second–order differential equation above can be written as a system of two first–order differential equations by using $v_y = dy/dt$. This gives

$$\frac{dy}{dt} = v_y \tag{4}$$

$$\frac{dv_y}{dt} = -g - \frac{\rho A|v_y|C_d(R_e)v_y}{2m} \tag{5}$$

where C_d is determined as a function of $R_e = \rho|v_y|D/\mu$ by Eq. (2). These equations can be solved numerically by applying the second–order Runge–Kutta method:

$$y_h = y_i + v_{y,i}\,dt/2 \tag{6}$$

$$v_{y,h} = v_{y,i} + \left[-g - \frac{\rho A|v_{y,i}|C_d(R_{e,i})v_{y,i}}{2m}\right]dt/2 \tag{7}$$

$$y_{i+1} = y_i + v_{y,h}\,dt \tag{8}$$

$$v_{y,i+1} = v_{y,i} + \left[-g - \frac{\rho A|v_{y,h}|C_d(R_{e,h})v_{y,h}}{2m}\right]dt \tag{9}$$

Here, $R_{e,i}$ is the Reynolds number evaluated at the initial time; that is, $R_{e,i} = \rho|v_{y,i}|D/\mu$. Likewise, $R_{e,h}$ is the Reynolds number evaluated at the half–timestep; thus, $R_{e,h} = \rho|v_{y,h}|D/\mu$.

Homework II: Write a python program to solve for the motion of the shot pellet using RK2. Use a function routine to compute C_d. Plot two graphs, height y versus time and velocity v_y versus time. Give approximate answers these questions: How high do the shot pellets rise? When do they land? How fast are they going when they land?