Lesson 19: Driven Damped Pendulum and Chaos

Use your RK4 code to investigate the dynamics of the damped driven pendulum. Remember: the number of timesteps needed for a given level of accuracy depends *strongly* on both the total run time and the parameter values. As a general rule, you need to run each simulation two or more times, using different resolutions, to make sure your results are accurate.

1 Phase space

Continue to use $\omega = 2\pi$, $\omega_0 = 1.5\omega$, and $\beta = \omega_0/4$. With small γ, say, $\gamma \lesssim 1.0$, you should see the following results: After a few drive periods, the effects of the initial conditions decay away. The pendulum motion is determined entirely by the driving force. In particular, we expect the pendulum to execute periodic motion with period $T = 1$.

> Exercise 1: Carry out two simulations using $\gamma = 0.8$, but with different initial conditions. For $0 \le t \le 5$, plot a graph of θ vs. t showing the results of the two simulations.

Your results should confirm that, at least for small γ, the behavior of the pendulum after a few drive periods is independent of initial conditions. Moreover, you should see that the motion of the pendulum is periodic with period $T = 1$. The periodicity can been seen clearly in a plot of θ vs. Ψ. The space of coordinates and velocities is called *phase space*.[1] If you exclude the early times to allow the pendulum to settle into a nice periodic motion, your phase space plot should show a simple "orbit" that repeats once for each drive period.

> Exercise 2: For $\gamma = 0.8$ and any initial conditions, create a phase space plot for $10 \le t \le 50$. Does the graph indicate that the motion is periodic?

2 Multi-period solutions and chaos

As γ is increased, the system exhibits a variety of interesting behaviors. The motion depends strongly on initial conditions.

[1] Phase space is sometimes called *state space*.

Homework: Carry out numerical experiments to determine the distinct types of motion for $\gamma = 0.6$, 1.073 and 1.077, using the initial conditions $\theta(0) = 0.0$ and $\Psi = 0.0$. (Keep the parameters unchanged: $\omega = 2\pi$, $\omega_0 = 1.5\omega$, and $\beta = \omega_0/4$. You should run your simulations out to at least $t = 30.0$.) For each case:

- Make plots of θ vs. t and θ vs. Ψ excluding early t. Label the axes appropriately.
- What is the highest resolution used that you used? Approximately how accurate is your final value of θ?
- Describe the motion qualitatively. What is the period of the motion? (Remember, the period of the driving force is $T = 1$.)

Homework: Carry out numerical experiments to determine the distinct types of motion for $\gamma = 1.08$, 1.1 and 1.2 and 1.25, using the initial conditions $\theta(0) = -\pi/2$ and $\Psi = 0.0$. (Keep the parameters unchanged.) For each case:

- Make plots of θ vs. t and θ vs. Ψ excluding early t.
- Describe the motion qualitatively. Is the motion periodic? If so, what is the period?

If the motion is not periodic, it is chaotic. Chaotic motion can be hard to simulate because numerical errors tend to be large. In other words, it can take a very large number of timesteps to obtain an accurate simulation of chaotic motion.

3 Initial conditions

The behavior of a chaotic system can be extremely sensitive to initial conditions. Consider a choice of parameters that results in chaotic motion. If you run your code twice, with slightly different initial conditions, the results might appear nearly identical for some period of time. Eventually, however, the motions will diverge from one another. In this sense, the motion of a chaotic system is unpredictable. In other words, we would need to know the initial conditions to arbitrarily high accuracy to predict the exact motion of a chaotic system for any extended time.

Homework: Make a version of your code to run two simulations using nearly identical initial conditions. Compute the difference in angles as a function of time, $\Delta\theta(t)$. (Choose an initial angle difference of, say, $\Delta\theta(0) \approx 0.0001$.) Plot a graph of $\log|\Delta\theta(t)|$ vs. t for $0 \leq t \leq 15$. Show the results for two cases, one chaotic and another not chaotic. How does $\Delta\theta$ change with time? Does it increase or decrease? Is the increase or decrease an exponential function of time?