# Lesson 14: Numerical Integration I

Throughout physics we encounter the need to evaluate integrals. For example, work is the line integral of a force along a path, Gauss' law relates the surface integral of the electric field to the enclosed charge, the action is the integral of the Lagrangian, *etc.*

## 1 Rectangle Methods

The integral of a function is the "area under the curve".[1] The area under the curve $f(x)$ from $x = a$ to $x = b$ can be approximated as the sum of the areas of the rectangles shown in Fig. 1. The rectangles are formed by dividing the interval $[a, b]$ into $N$ subintervals, from

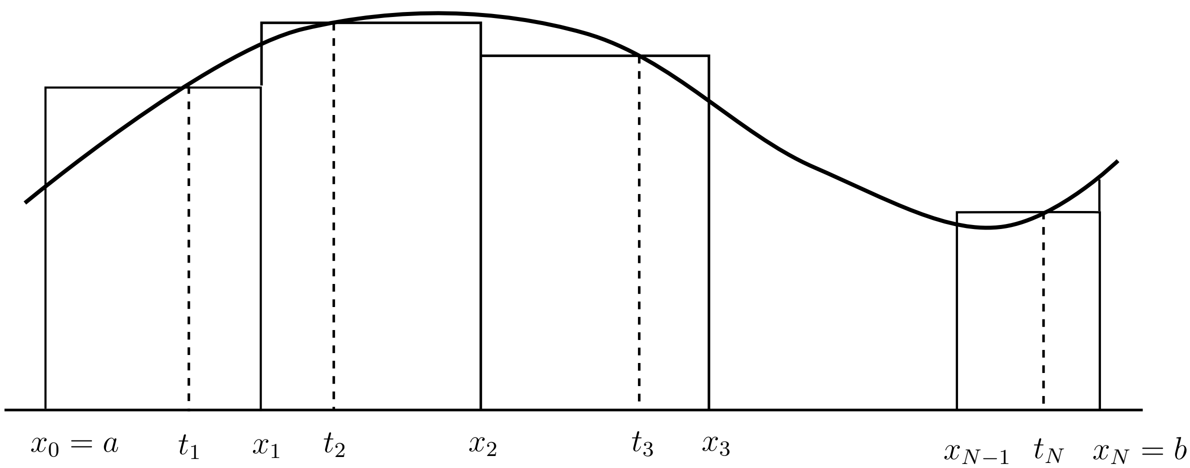


Figure 1: The integral is the area under the curve, approximated by the sum of areas of rectanges.

$[x_0, x_1]$, to $[x_{N-1}, x_N]$. (Note that $x_0 = a$ and $x_N = b$.) The height of each rectangle is the value of the function at some point $t_i$ within the subinterval $[x_{i-1}, x_i]$. The area of the $i$th rectangle is the product of the height $f(t_i)$ and the width $x_i - x_{i-1}$. Then the integral

$$I = \int_a^b f(x)\, dx \tag{1}$$

[1] This is the case for most functions we encounter in physics. Some exotic functions require a more sophisticated definition of the integral.

is approximated by the sum of the areas of the rectangles:

$$I \approx \sum_{i=1}^{N} f(t_i)\,(x_i - x_{i-1}) \tag{2}$$

This approximation is called the Riemann sum. The exact value for $I$ is obtained by taking the limit $N \to \infty$.

In practice we must choose where to place the $x_i$'s and $t_i$'s before using the Riemann sum (2) to approximate the integral $I$. Different choices lead to different numerical integration methods. The simplest choice for the $x_i$'s is to make them equally spaced between the endpoints. That is, let $h = (b - a)/N$ denote the width of each subinterval, with

$$x_i = a + ih \tag{3}$$

for $i = 0, \ldots N$. There are three obvious choices for the $t_i$'s. With $t_i = x_{i-1} = a + ih - h$, the height of each rectangle is the value of the function at the left side of the subinterval. This yields the "left endpoint rule" approximation to $I$:

$$I_L = \sum_{i=1}^{N} f(a + ih - h)\,h \tag{4}$$

With $t_i = x_i = a + ih$, the height of each rectangle is the value of the function at the right side of the subinterval. This is the "right endpoint rule"

$$I_R = \sum_{i=1}^{N} f(a + ih)\,h \tag{5}$$

With $t_i = (x_i + x_{i-1})/2 = a + ih - h/2$, the height of each rectangle is the value of the function at the midpoint of the subinterval. This yields

$$I_M = \sum_{i=1}^{N} f(a + ih - h/2)\,h \tag{6}$$

which is called the "midpoint rule".

> Exercise 1: Write a code to integrate the function $f(x) = \sin x$ between $a = 0$ and $b = \pi/2$ using both the left and right endpoint rules. Structure your code so that it's easy to change $f(x)$, $a$, $b$, and the number of subintervals $N$. Compare your results to each other, and to the correct value.

Your results should become more accurate as you increase the number of subintervals $N$.

> Exercise 2: For the right endpoint rule, compute the error for $N = 2, 4, 8, 16$, *etc.* Make a log–log plot of the error versus $N$. The error is proportional to some power of $N$, given by the slope of the log–log plot. What is that power?

Intuitively, we expect the midpoint rule to give a better approximation to the area under the curve than the left or right endpoint rules.

Homework: Write a code to integrate the function $f(x) = \sin x$ between $a = 0$ and $b = \pi/2$ using the midpoint rule. Compute the error for $N = 2, 4, 8, 16$, *etc*, and make a log–log plot of the absolute value of the error versus $N$. The error is proportional to some power of $N$. What is that power?

## 2 A word of warning

There are many integration methods in addition to the ones discussed here. No matter how sophisticated these methods might be, they will all break down if any of the function evaluations cannot be done. Consider the following integral:

$$I_1 = \int_{-1}^{1} \frac{1}{x^{2/3}} dx \tag{7}$$

This can be evaluated analytically to give $I_1 = 6$. However, the integrand $f(x) = 1/x^{2/3}$ does not exist at $x = 0$. Thus, if you try to evaluate this function at $x = 0$, your code will return an error. This is an example of an "improper" integral—the integral exists even though the integrand is not defined everywhere.

Homework: Use a midpoint rule code to evaluate $I_1$. How does the error depend on $N$, for even values of $N$? For odd values?

There are other integrals that look very similar to the improper integral above; for example,

$$I_2 = \int_{-1}^{1} \frac{1}{x^2} dx \tag{8}$$

This is actually an "impossible" integral—the area under the curve is infinite. This goes to show you that you can run into trouble if you just blindly apply an algorithm without examining the problem at hand.

Homework: Use the midpoint rule to evaluate $I_2$ for various values of $N$. How does the result depend on $N$?