Lesson 13: Linear Algebra II

1 Eigenvalues and Eigenvectors

We previously considered linear algebra problems of the form $Av = r$, where v is the vector of unknowns. Another very important class of problems in linear algebra has the form $Mv = \lambda v$. Here, M is a matrix, λ is a number, and v is a vector. This is the eigenvalue/eigenvector problem.

The eigenvalue/eigenvector equation can be written as $Mv = \lambda I v$ where I is the identity matrix. With this notation the problem can be rewritten as $Av = 0$ where the matrix A is defined by $A \equiv M - \lambda I$. Of course, we know how to solve the system $Av = 0$: simply invert the matrix A and mutiply both sides of the equation by A^{-1}. (Better yet, use the `linalg.solve` command in numpy.) This yields the trivial solution $v = 0$. Note that $v = 0$ is *always* a solution to our original problem, $Mv = \lambda v$. In fact, it is the only solution assuming that the matrix $A \equiv M - \lambda I$ is invertible. However, for certain special values of λ called *eigenvalues*, this matrix will *not* be invertible and the system $Mv = \lambda v$ will have nontrivial solutions for v. The nontrivial vectors v are called *eigenvectors*.

The eigenvalue/eigenvector problem consists of finding all of the eigenvalues λ such that $M - \lambda I$ is not invertible, then for each eigenvalue finding a corresponding eigenvector v that satisfies $Mv = \lambda v$. In python, you can use the numpy function `eig()` in the subpackage `linalg` to solve the eigenvalue/eigenvector problem. More precisely, the command (assuming `numpy` is loaded as `n`)

```
eval,evec = n.linalg.eig(M)
```

will compute the eigenvalues and corresponding eigenvectors of a matrix M. The eigenvalues are placed in an array, called `eval` in this example. The eigenvectors are placed in a matrix, called `evec` in this example. The first column, `evec[:,0]`, is the eigenvector corresponding to the first eigenvalue, `eval[0]`; the second column, `evec[:,1]`, is the eigenvector corresponding to the second eigenvalue, `eval[1]`; *etc.*

Exercise 1: Consider the matrix

$$M = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} . \tag{1}$$

Use the `eig()` command to obtain the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$, and corresponding eigenvectors $v_1 = (-0.707, 0.707)^T$ and $v_2 = (-0.6, -0.8)^T$. Verify by hand calculation that these eigenvalues/eigenvectors satisfy $Mv = \lambda v$.

Note that the eigenvector associated with a given eigenvalue is not unique: If v satisfies the equation $Mv = \lambda v$, then so does any vector proportional to v. Thus, for the problem in Exercise I, we can rescale the eigenvector associated with λ_1 to obtain $v_1 = (1, -1)^T$. Likewise, we can rescale the second eigenvector to $v_2 = (3, 4)^T$. The eigenvectors returned by the `eig()` command are normalized so that the sum of the squares of the components is unity.

2 Small oscillations

Consider two masses moving in one dimension along a frictionless table. The masses, both of mass m, are connected by springs to a fixed wall, as shown in Fig. 4. Each spring has

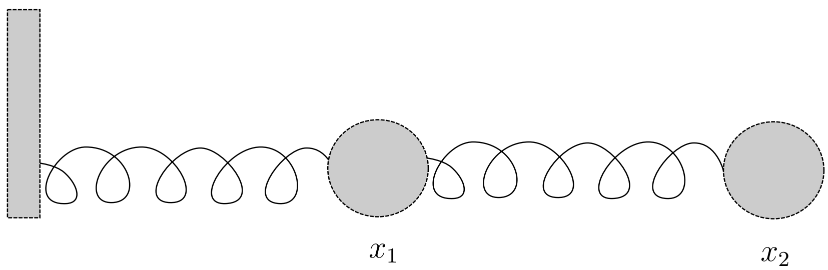

Figure 1: Two masses connected by two springs.

stiffness k and relaxed length ℓ. Let x_1 and x_2 denote the positions of the masses as measured from the wall. Newton's second law, applied to each mass, gives

$$m\ddot{x}_1 = -k(x_1 - \ell) + k(x_2 - x_1 - \ell) \ , \tag{2a}$$

$$m\ddot{x}_2 = -k(x_2 - x_1 - \ell) \ . \tag{2b}$$

Dots denote time derivatives.

The *normal mode* analysis is a powerful technique used to characterize the small amplitude oscillations of a system near equilibrium. A normal mode is a pattern of oscillation in which each part of the system vibrates about its equilibrium position with a common angular frequency ω. For the system (2), the equilibrium configuration is $x_1 = \ell$, $x_2 = 2\ell$. Let $\eta_1 \equiv x_1 - \ell$ and $\eta_2 \equiv x_2 - 2\ell$ denote the displacement of the system away from equilibrium. In terms of these new coordinates, the equations of motion become[1]

$$m\ddot{\eta}_1 = -2k\eta_1 + k\eta_2 \ , \tag{3a}$$

$$m\ddot{\eta}_2 = k\eta_1 - k\eta_2 \ . \tag{3b}$$

[1]These equations are linear in the η's. Many physical systems are described by nonlinear equations for the η's (the displacements from equilibrium). In those cases, we must linearize the system by expanding the equations in a Taylor series and dropping the higher order terms. The discarded terms will be negligible as long as the oscillations are small. This is why the subject is called "small oscillations".

The normal modes have the form

$$\eta_1 = A_1 \cos(\omega t + \phi_1) \tag{4a}$$
$$\eta_2 = A_2 \cos(\omega t + \phi_2) \tag{4b}$$

where A_1, A_2 are amplitudes and ϕ_1, ϕ_2 are phase angles. We can write these modes as

$$\eta_1 = \Re\left(C_1 e^{i\omega t}\right) \tag{5a}$$
$$\eta_2 = \Re\left(C_2 e^{i\omega t}\right) \tag{5b}$$

where $C_1 \equiv A_1 e^{i\phi_1}$, $C_2 \equiv A_2 e^{i\phi_2}$, and $\Re$ denotes the real part of the expression. We can verify that Eqs. (4) and (5) are equivalent by using the identity $e^{i\xi} \equiv \cos\xi + i\sin\xi$.

One can show that the mode (4), or equivalently (5), is a solution of Eq. (3) if and only if

$$\eta_1 = C_1 e^{i\omega t} \tag{6a}$$
$$\eta_2 = C_2 e^{i\omega t} \tag{6b}$$

is a solution. This is the case because Eqs. (3) form a linear system with constant, real coefficients. The analysis of small oscillations is easiest to carry out in terms of the complex modes (6), so we will follow that approach here.

Plugging the mode (6) into the differential equations (3), we find

$$m\omega^2 C_1 = 2kC_1 - kC_2 \tag{7a}$$
$$m\omega^2 C_2 = -kC_1 + kC_2 \tag{7b}$$

In matrix notation,

$$\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = m\omega^2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{8}$$

This is an eigenvalue/eigenvector problem, to be solved for the eigenvalues $\lambda \equiv m\omega^2$ and corresponding eigenvectors $(C_1, C_2)^T$.

Homework: Let $k = 15\,\mathrm{N/m}$ and $m = 0.3\,\mathrm{kg}$. Use Python to solve the eigenvalue/eigenvector problem (8). For each normal mode (that is, each eigenvalue/eigenvector combination) write down the solution in the form of Eq. (4) and plot a graph showing η_1 and η_2 as functions of time. Describe the normal modes qualitatively. The most general motion of the system is a linear combination of normal modes.