Equation Derivation

A point in 3D space, P, is projected onto a point, p, on an image plane by:

$$\boldsymbol{p} = M\boldsymbol{P}$$

Where **p** and **P** are in the homogenous coordinate system, i.e.:

$$P = (X, Y, Z, 1)^T;\ p = (u, v, 1)^T$$

The first equation can be expanded as follows:

$$u = \frac{\boldsymbol{m_1 P}}{\boldsymbol{m_3 P}}$$

$$v = \frac{\boldsymbol{m_2 P}}{\boldsymbol{m_3 P}}$$

Where $\boldsymbol{m_1}, \boldsymbol{m_2}$ and $\boldsymbol{m_3}$ are the rows of M**.** Expanding these two equations we can get the following:

$$\boldsymbol{m_1 P} - \boldsymbol{u m_3 P} = \boldsymbol{m_1 P} + \boldsymbol{0 m_2} - \boldsymbol{u m_3 P} = \boldsymbol{0}$$

$$\boldsymbol{m_2 P} - \boldsymbol{v m_3 P} = \boldsymbol{0 P} + \boldsymbol{m_2 P} - \boldsymbol{v m_3 P} = \boldsymbol{0}$$

Where **0** is a vector of zeros. If we have n correspondences from **P** to **p,** we can write this as a system of equations as follows:

$$\begin{bmatrix} \boldsymbol{P_1} & \boldsymbol{0} & -u_1\boldsymbol{P}_1 \\ \boldsymbol{0} & \boldsymbol{P_1} & -v_1\boldsymbol{P}_1 \\ \vdots & \vdots & \vdots \\ \boldsymbol{P_n} & \boldsymbol{0} & -u_n\boldsymbol{P}_1 \\ \boldsymbol{0} & \boldsymbol{P_n} & -v_n\boldsymbol{P}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{m_1} \\ \boldsymbol{m_2} \\ \boldsymbol{m_3} \end{bmatrix} = AM = \boldsymbol{0}$$

M is 3-by-4, so we need at least 6 point correspondences to solve these equations. Here we can use homogeneous linear least squares. To do this we simply take the eigenvector of $\mathbf{A}^\mathbf{T}\mathbf{A}$ that corresponds to the smallest eigenvalue. **A** is 2n-by-12, so $\mathbf{A}^\mathbf{T}\mathbf{A}$ will be 12x12 and we will get 12 eigenvectors each with 12 elements.

Experiment

For this I used an image of a calibration grid which can be seen in Figures 1 and 2. This is a grid that has 3 flat faces corresponding to the xy, xz, and yz planes. Each corner of the grid is 5cm from the previous corner, and the origin was placed in the corner of the three planes. 13 grid corner points were chosen at random. At least 6 points are required but it is important to make sure that the span of the vectors that construct the points span the entire space of R^3. Once these 3D points are selected and the corresponding 2D locations in the image are found, it is simple to build the matrix **A** and subsequently solve for M. Once M is found, the 3D points can be reprojected onto the image plane. Figure 2 below shows both the original points and the reprojected points.

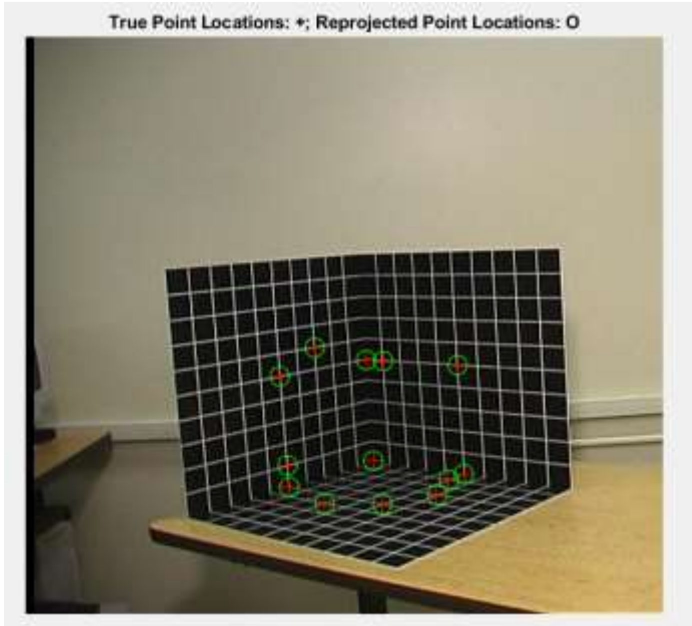

Figure 1: Original and Reprojected Points

Note that both the original and reprojected points are very close to each other. Similarly, the grid lines can be reprojected onto the image plane as shown in Figure 2 below. Some error can be found in the xz plane (red lines). This is most likely due to human error in choosing the points in the image. Nonlinear methods can be used to find a better approximate which takes into account all parameters such as image skew.

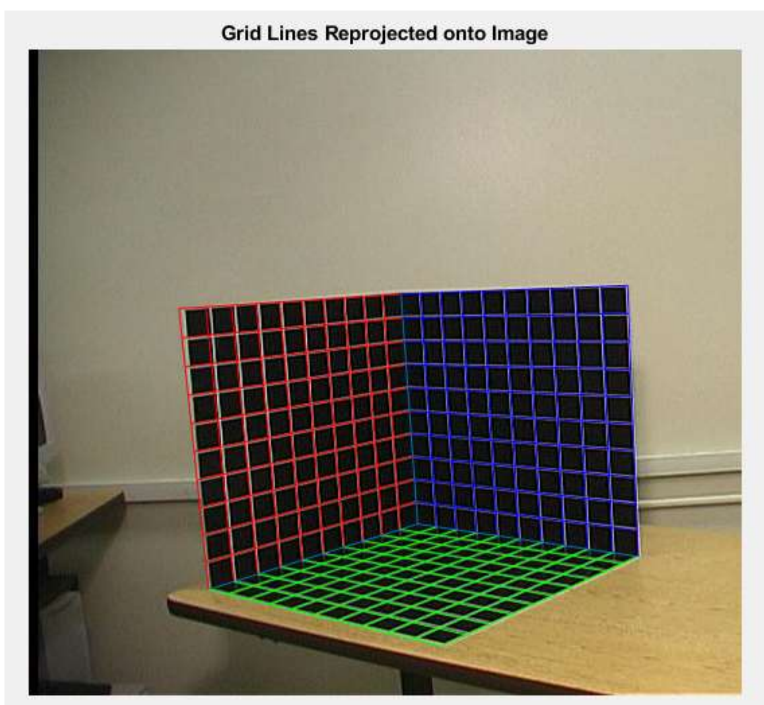

Figure 2: Grid Lines Reprojected