Stochastic Uncertainty Propagation in Power System Dynamics using Measure-valued Proximal Recursions

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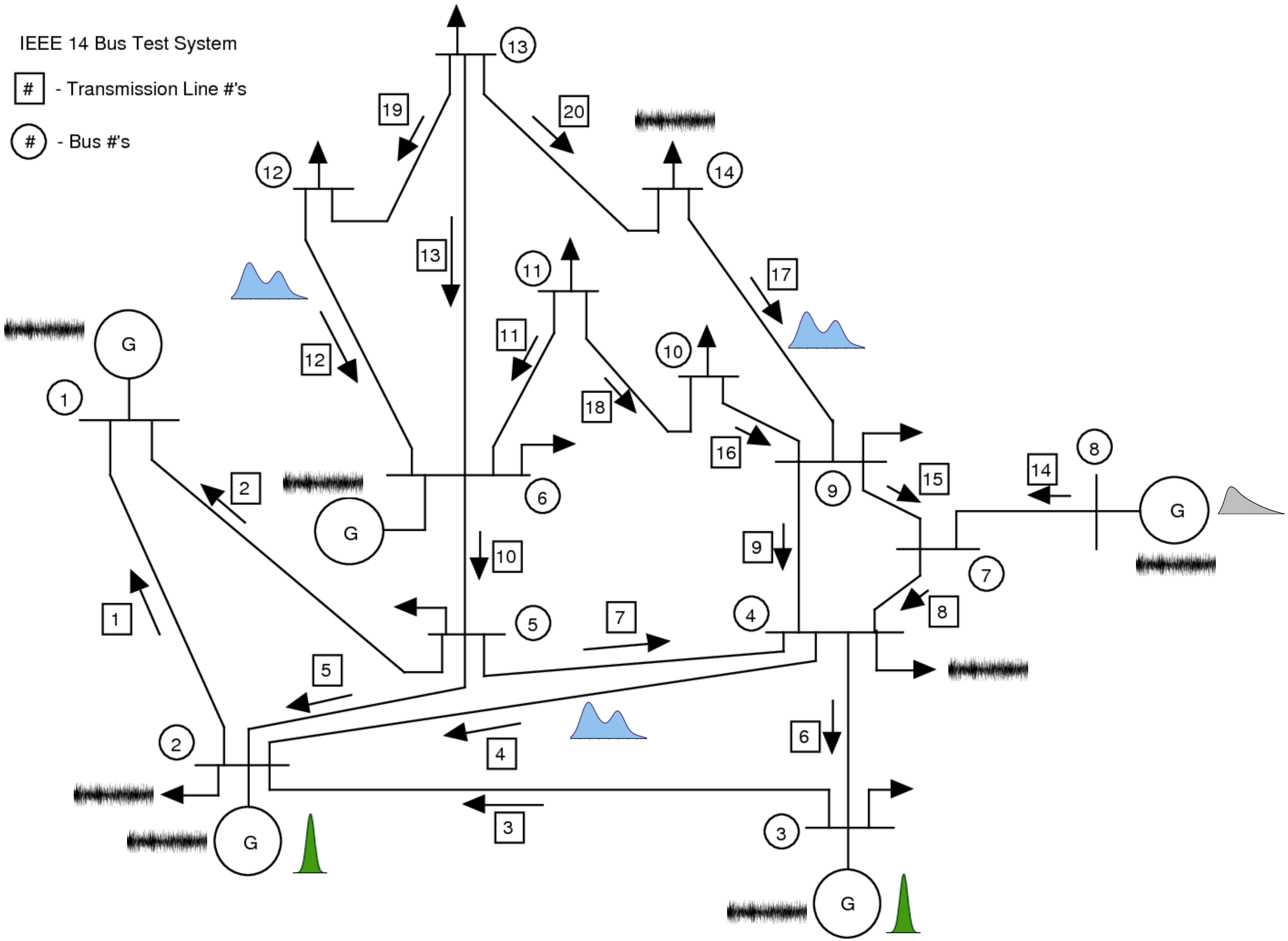
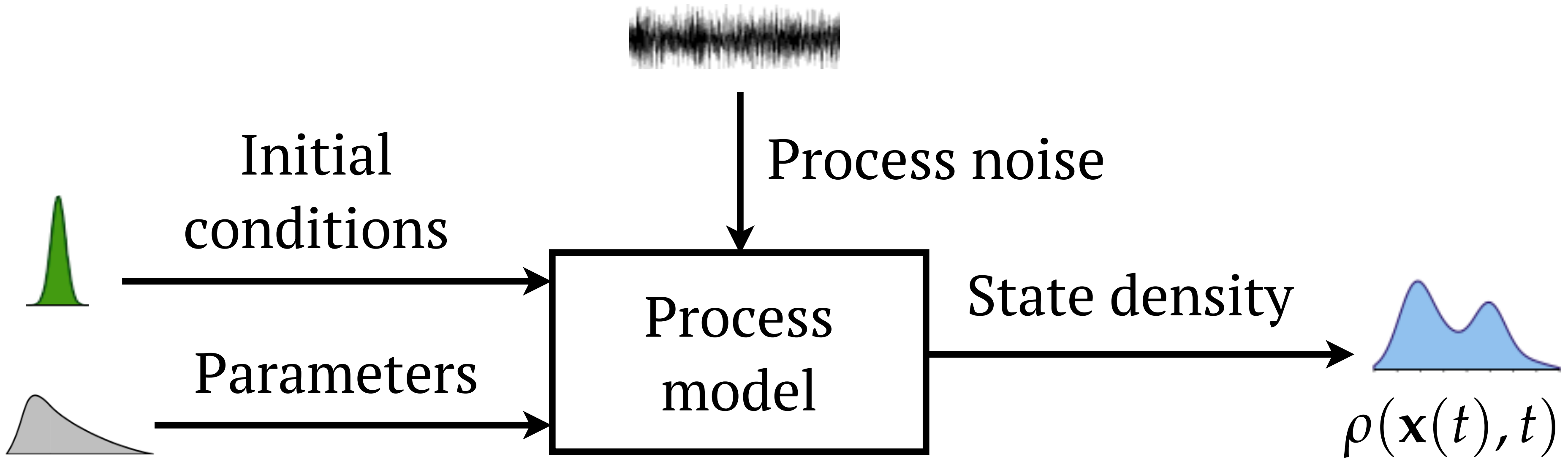
Joint work with K.F. Caluya, P. Ojaghi, X. Geng

Uncertainty propagation in power systems



Propagating joint PDFs



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

Density flow:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}(\rho) := -\nabla \cdot (\rho \mathbf{f}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\mathbf{g} \mathbf{Q} \mathbf{g}^\top \right)_{ij} \rho \right)$$

Long history of methods for solving such PDE

But all need grid

— Monte Carlo methods: p.w. constant approximation

— classical finite element, finite volume methods

— tensor decomposition techniques

— Kernel and other density estimation algorithms

— Recast as NN regression with suitable loss, then do SGD

Can we do gridless computation?

Evolve time-varying weighted point clouds?

We want the transient joint PDF computation be

Non-parametric

— nonlinearity in dynamics destroys Gaussianity

— don't *a priori* know the dimensionality of sufficient statistics

— cannot rigorously justify statistical approximations (exponential family, mixture of Gaussians) for the transient

— don't want to make dynamical approximations (linear, Taylor series)

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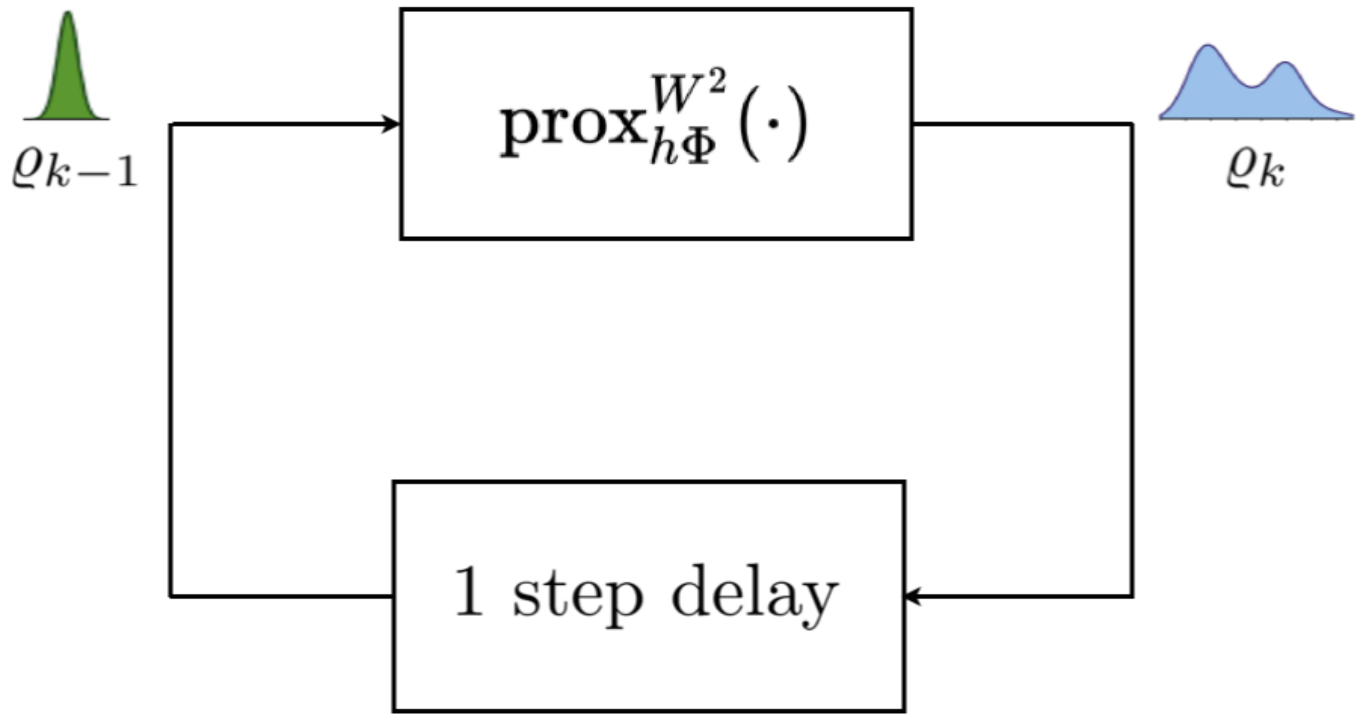
Embrace and make use of

~~Fight~~ exact nonlinearity

What's new?

Main idea: Solve $\dfrac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}\rho, \;\; \rho(x, t=0) = \rho_0$ **as gradient flow in** $\mathcal{P}_2(\mathcal{X})$

Infinite dimensional variational recursion:



Proximal operator: $\varrho_k = \mathrm{prox}_{h\Phi}^{W^2}(\varrho_{k-1}) := \underset{\varrho \in \mathcal{P}_2(\mathcal{X})}{\arg\inf} \left\{ \dfrac{1}{2} W^2(\varrho, \varrho_{k-1}) + h\Phi(\varrho) \right\}$

Optimal transport cost: $W^2(\varrho, \varrho_{k-1}) := \displaystyle\inf_{\pi \in \Pi(\varrho, \varrho_{k-1})} \int_{\mathcal{X}\times\mathcal{X}} c(x,y)\, \mathrm{d}\pi(x,y)$

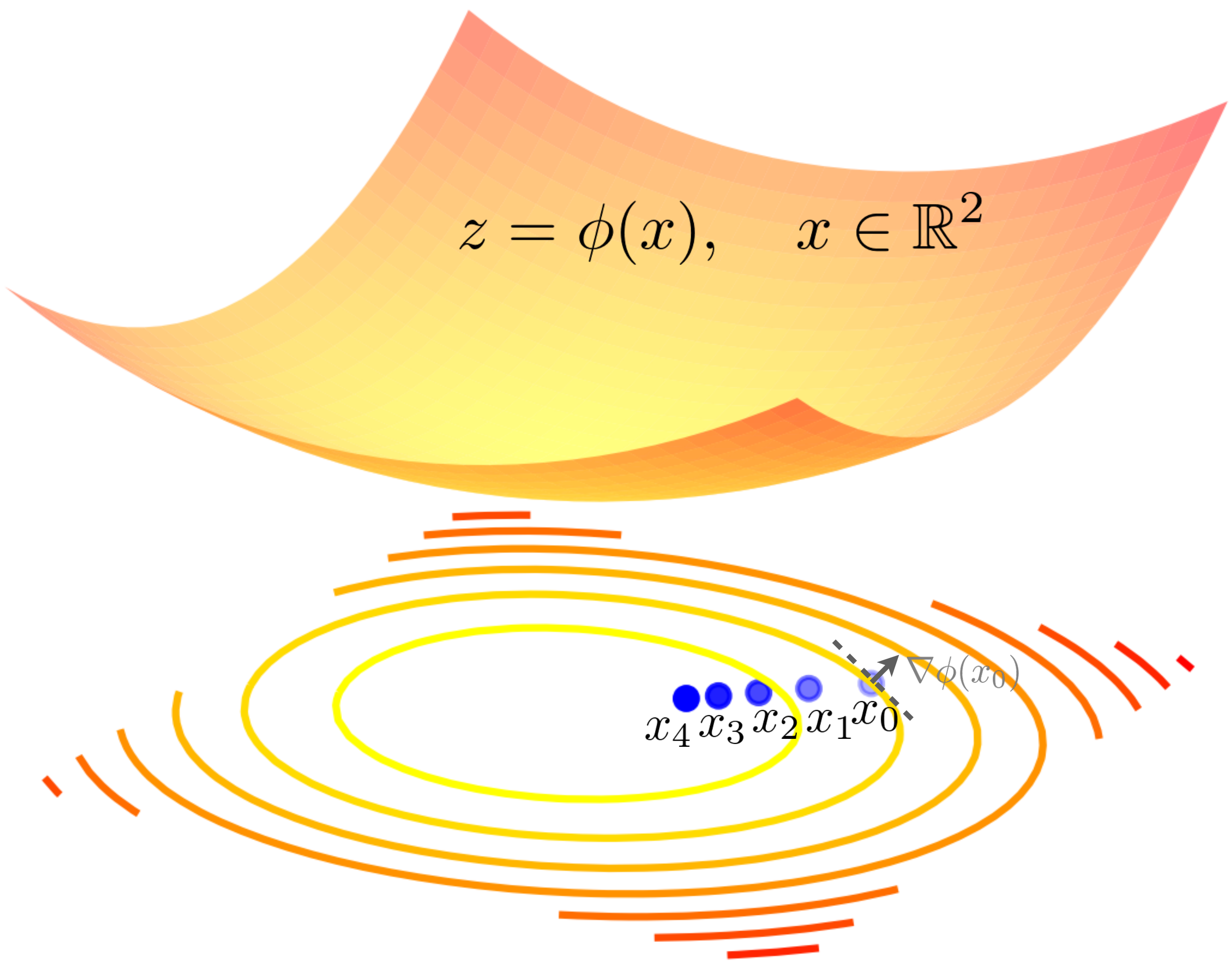
Free energy functional: $\Phi(\varrho) := \displaystyle\int_{\mathcal{X}} \psi\varrho \,\mathrm{d}x + \beta^{-1} \int_{\mathcal{X}} \varrho \log \varrho \,\mathrm{d}x$

Generalized gradient flow

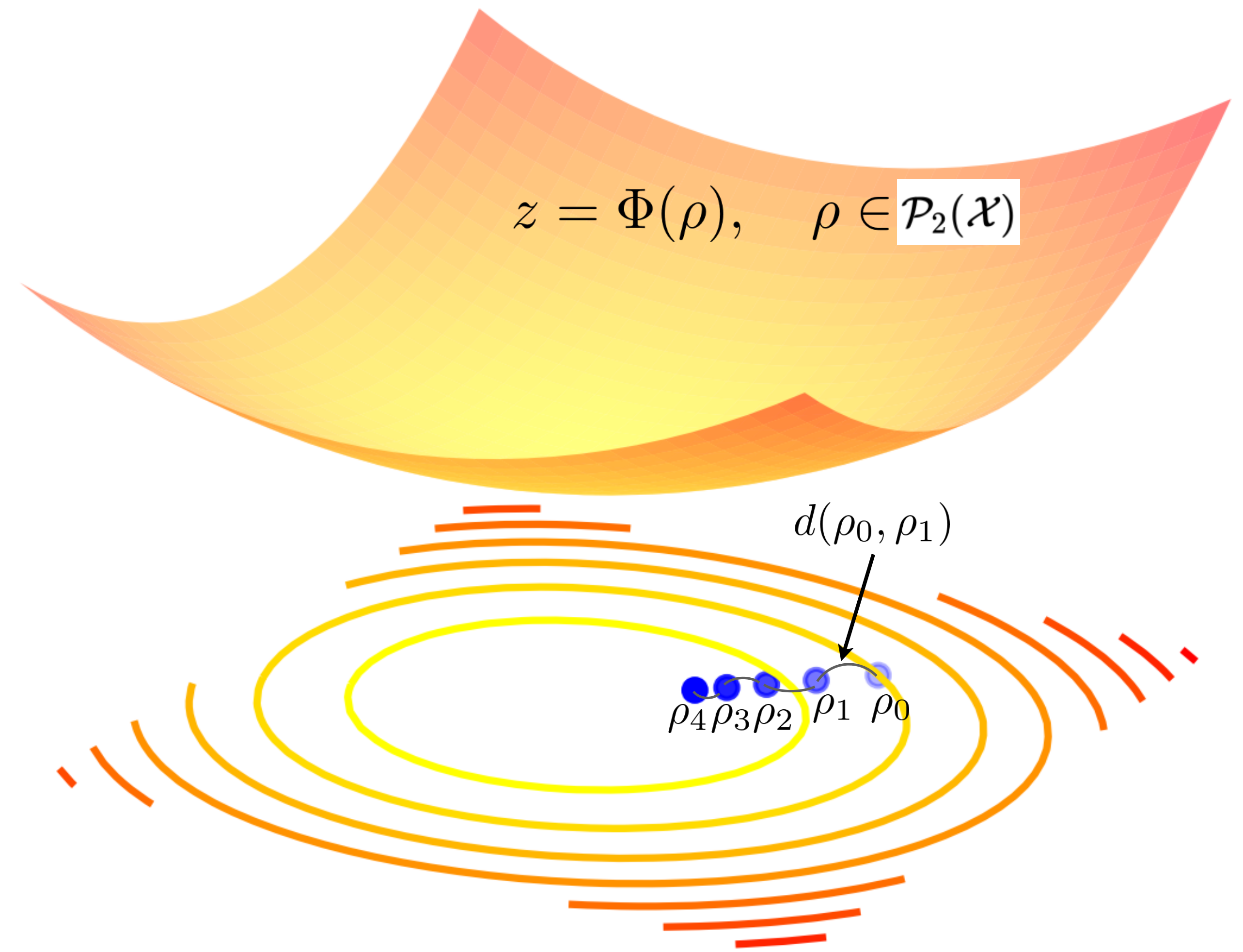
Gradient Flow in $\mathcal{X}$	**Gradient Flow in $\mathcal{P}_2(\mathcal{X})$**
$\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\nabla\varphi(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$	$\dfrac{\partial\rho}{\partial t} = -\nabla^W \Phi(\rho), \quad \rho(\boldsymbol{x},0) = \rho_0$
Recursion:	**Recursion:**
$\boldsymbol{x}_k = \boldsymbol{x}_{k-1} - h\nabla\varphi(\boldsymbol{x}_k)$ $= \underset{\boldsymbol{x}\in\mathcal{X}}{\arg\min}\left\{\frac{1}{2}\Vert\boldsymbol{x}-\boldsymbol{x}_{k-1}\Vert_2^2 + h\varphi(\boldsymbol{x})\right\}$ $=: \mathrm{prox}_{h\varphi}^{\Vert\cdot\Vert_2}(\boldsymbol{x}_{k-1})$	$\rho_k = \rho(\cdot, t = kh)$ $= \underset{\rho\in\mathcal{P}_2(\mathcal{X})}{\arg\min}\left\{\frac{1}{2}W^2(\rho,\rho_{k-1}) + h\Phi(\rho)\right\}$ $=: \mathrm{prox}_{h\Phi}^{W^2}(\rho_{k-1})$
Convergence:	**Convergence:**
$\boldsymbol{x}_k \to \boldsymbol{x}(t = kh) \quad \text{as} \quad h \downarrow 0$	$\rho_k \to \rho(\cdot, t = kh) \quad \text{as} \quad h \downarrow 0$
φ as Lyapunov function:	**Φ as Lyapunov functional:**
$\dfrac{\mathrm{d}}{\mathrm{d}t}\varphi = -\Vert\nabla\varphi\Vert_2^2 \leq 0$	$\dfrac{\mathrm{d}}{\mathrm{d}t}\Phi = -\mathbb{E}_\rho\left[\left\Vert\nabla\dfrac{\delta\Phi}{\delta\rho}\right\Vert_2^2\right] \leq 0$

Generalized gradient flow

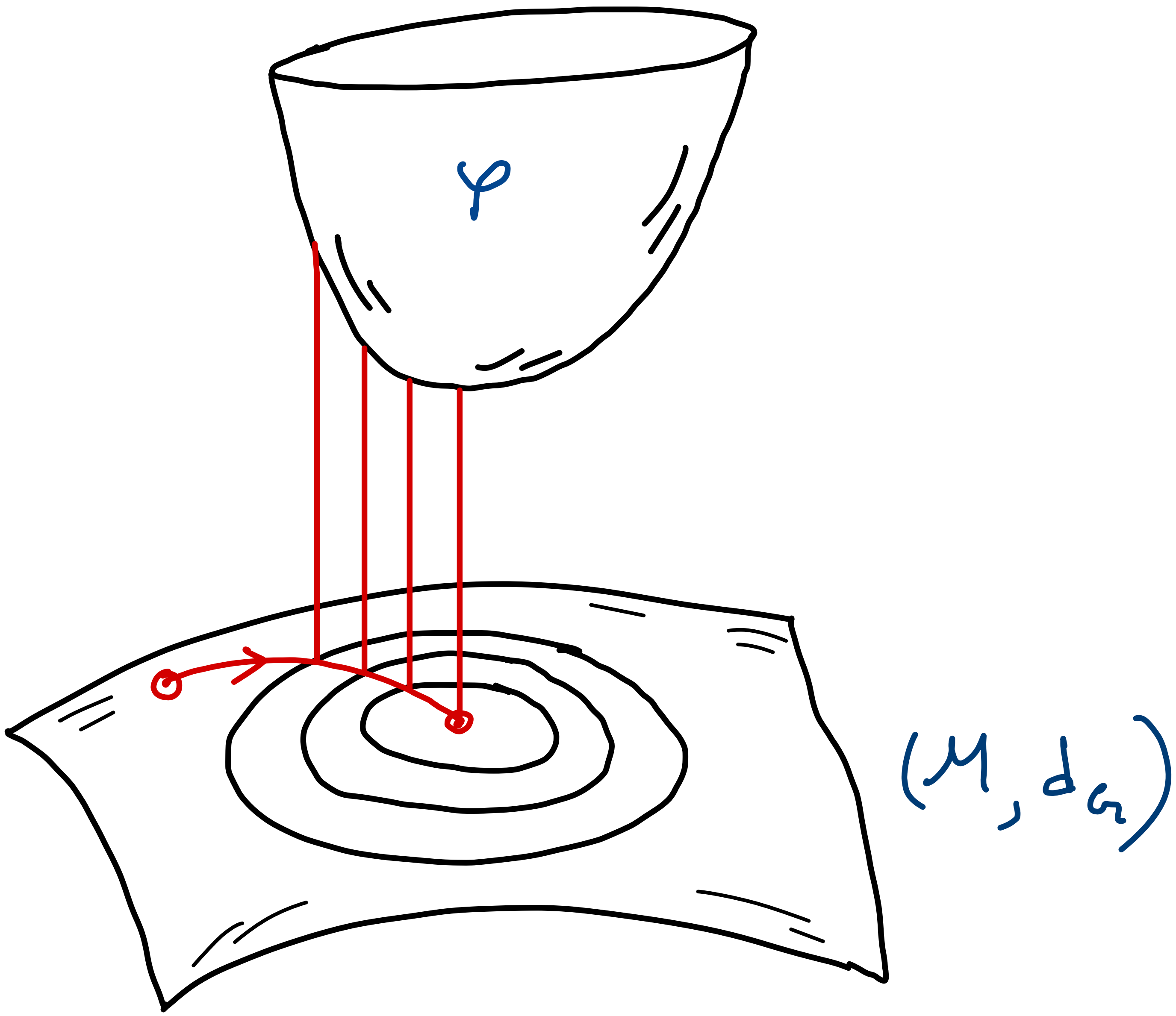
Gradient Flow in $\mathcal{X}$

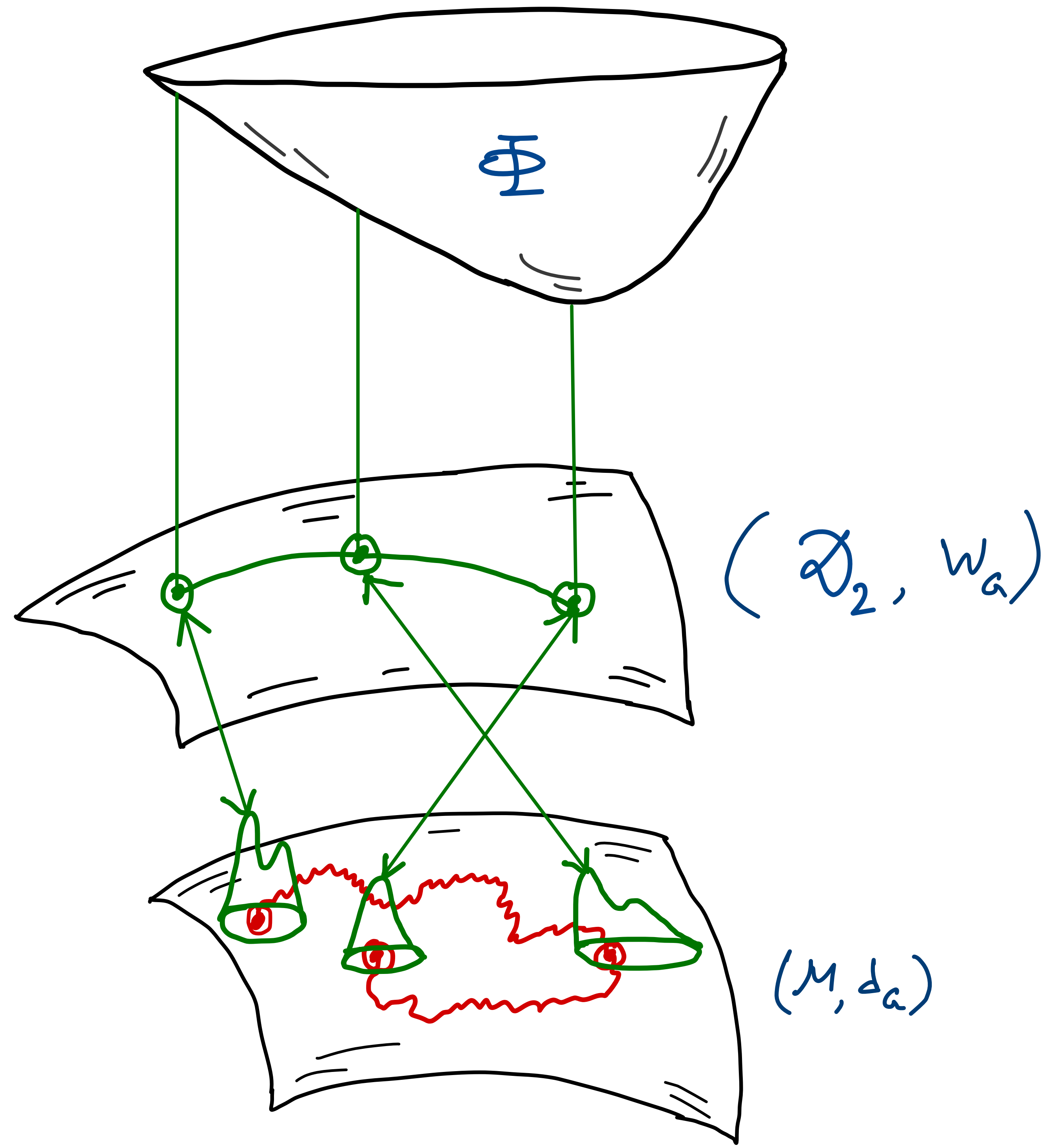


Gradient Flow in $\mathcal{P}_2(\mathcal{X})$



Generalized gradient flow with base manifold

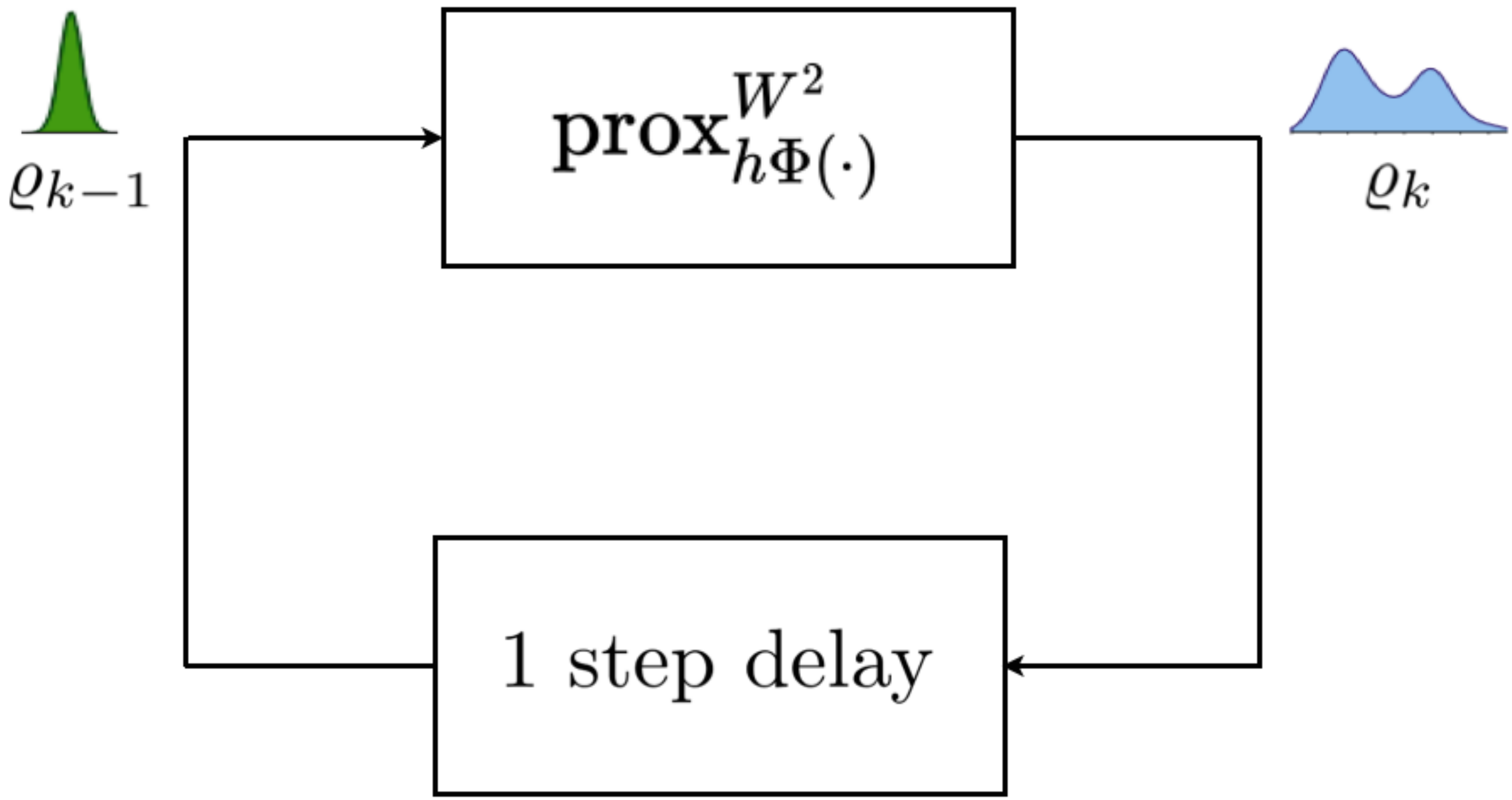




Design pair (W_G, Φ)

Algorithm: gradient ascent on the dual space

Uncertainty propagation via point clouds



No spatial discretization or function approximation

Algorithm: gradient ascent on the dual space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h \, \Phi(\rho) \right\}$$

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$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \underset{\boldsymbol{\varrho}}{\arg\min} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \left\langle \psi_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

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$$\Downarrow \quad \textbf{Entropic Regularization}$$

$$\boldsymbol{\varrho}_k = \underset{\boldsymbol{\varrho}}{\arg\min} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

Algorithm: gradient ascent on the dual space

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$\Updownarrow$ **Dualization**

$$\boldsymbol{\lambda}_0^{\mathrm{opt}}, \boldsymbol{\lambda}_1^{\mathrm{opt}} = \underset{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \geq 0}{\arg\max} \Bigg\{ \langle \boldsymbol{\lambda}_0, \boldsymbol{\varrho}_{k-1} \rangle - F^{\star}(-\boldsymbol{\lambda}_1) - \frac{\epsilon}{h} \left(\exp(\boldsymbol{\lambda}_0^{\top} h/\epsilon) \exp(-\boldsymbol{C}_k/2\epsilon) \exp(\boldsymbol{\lambda}_1 h/\epsilon) \right) \Bigg\}$$

Recursion on the cone

$$\boldsymbol{y} = e^{\frac{\boldsymbol{\lambda}_0^*}{\epsilon}h} \Big\downarrow \qquad \Big\downarrow \boldsymbol{z} = e^{\frac{\boldsymbol{\lambda}_1^*}{\epsilon}h}$$

Coupled Transcendental Equations in *y* and *z*

$$\boldsymbol{\Gamma}_k = e^{\frac{-\boldsymbol{C}_k}{2\epsilon}} \longrightarrow \qquad \varrho_{k-1} \longrightarrow \qquad \boldsymbol{\xi}_{k-1} = \frac{e^{-\beta\boldsymbol{\psi}_{k-1}}}{e} \longrightarrow$$

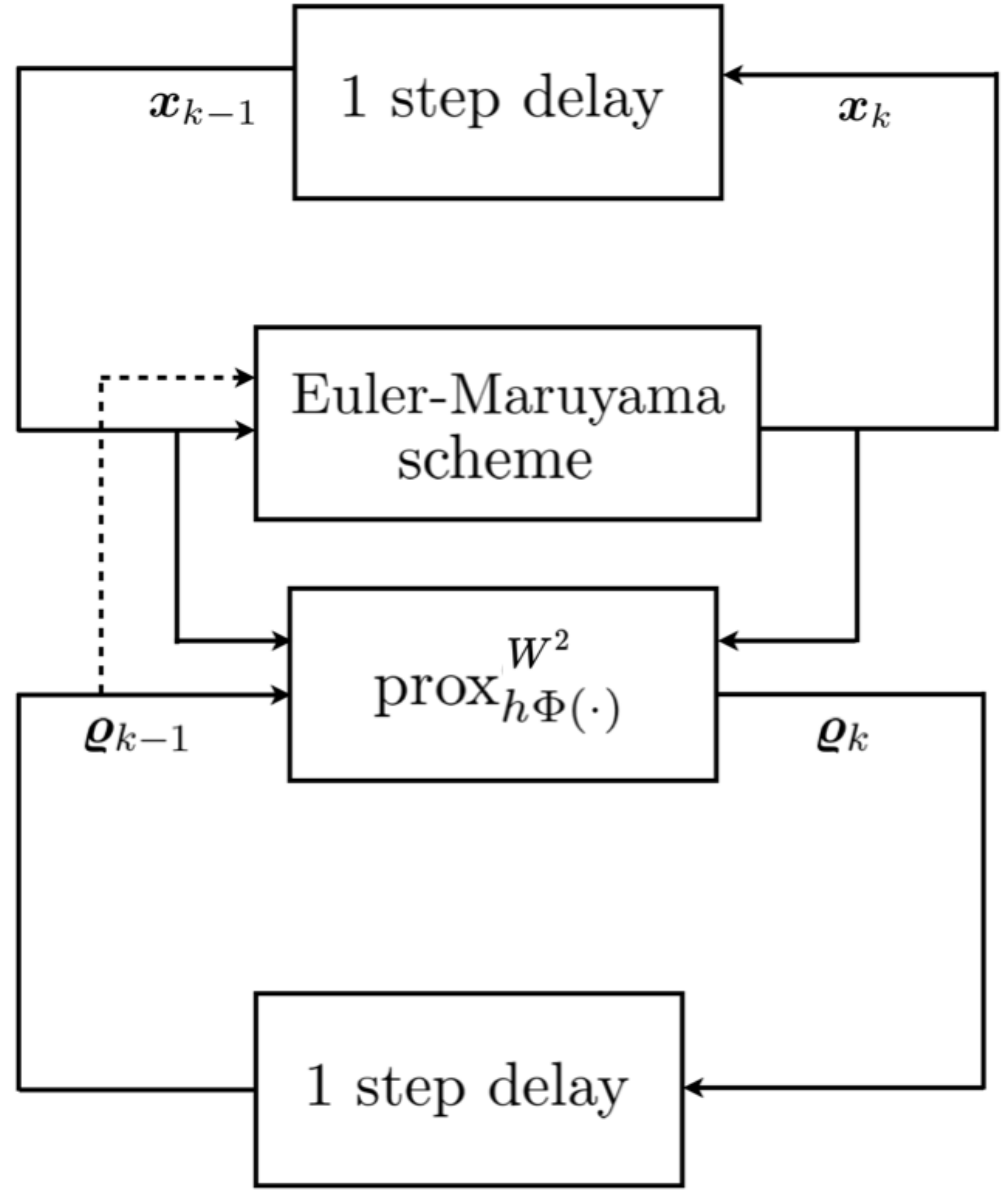
$$\boxed{\begin{aligned} \boldsymbol{y} \odot \boldsymbol{\Gamma}_k \boldsymbol{z} &= \varrho_{k-1} \\ \boldsymbol{z} \odot \boldsymbol{\Gamma}_k^\top \boldsymbol{y} &= \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\beta\epsilon/2h} \end{aligned}} \longrightarrow \varrho_k = \boldsymbol{z} \odot \boldsymbol{\Gamma}_k^\top \boldsymbol{y}$$

Theorem: Consider the recursion on the cone $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$

$$\boldsymbol{y} \odot (\boldsymbol{\Gamma}_k \boldsymbol{z}) = \varrho_{k-1}, \quad \boldsymbol{z} \odot \left(\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}\right) = \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\frac{\beta\epsilon}{h}},$$

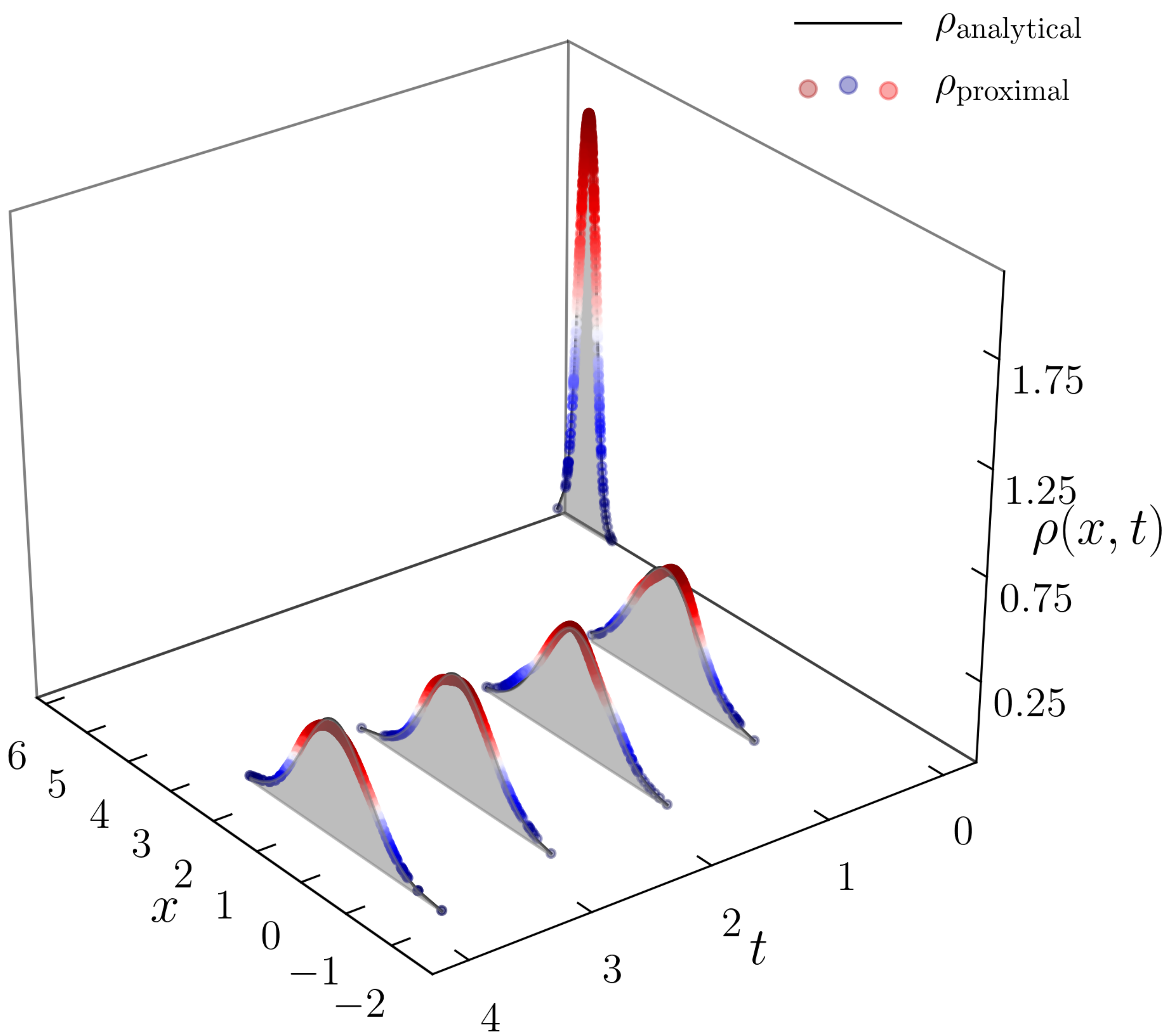
Then the solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ gives the proximal update $\varrho_k = \boldsymbol{z}^* \odot (\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}^*)$

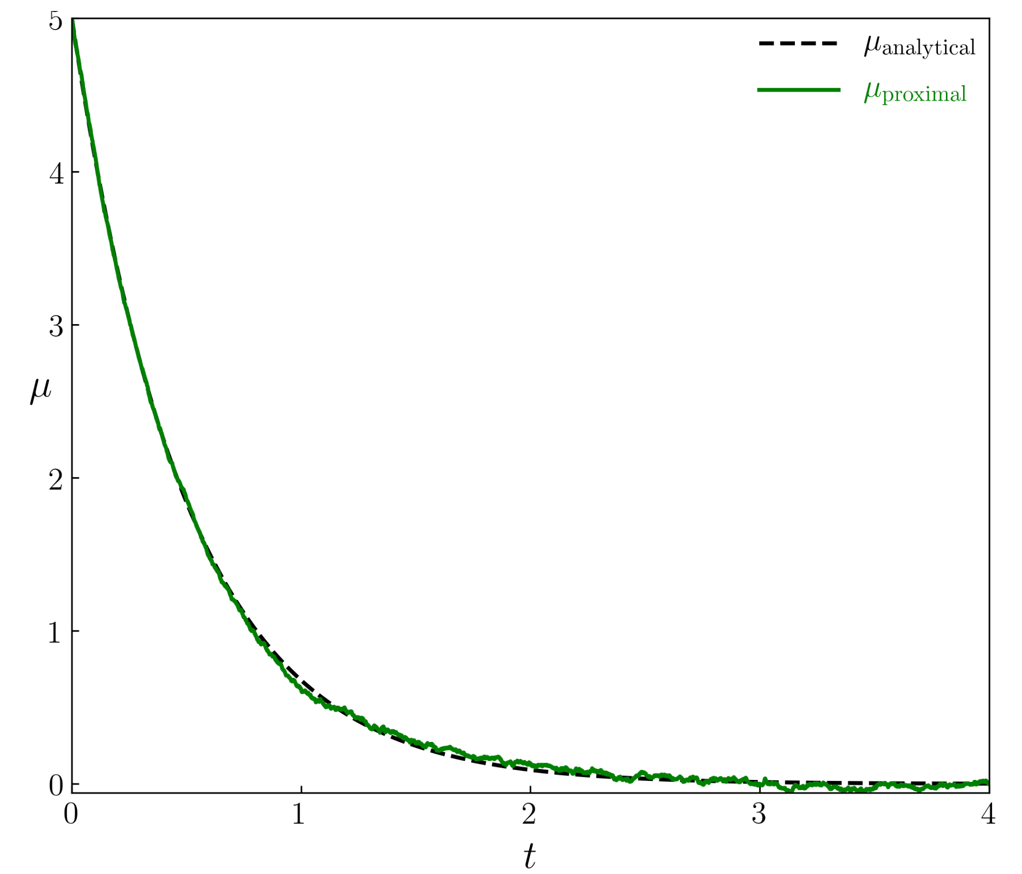
Algorithmic setup

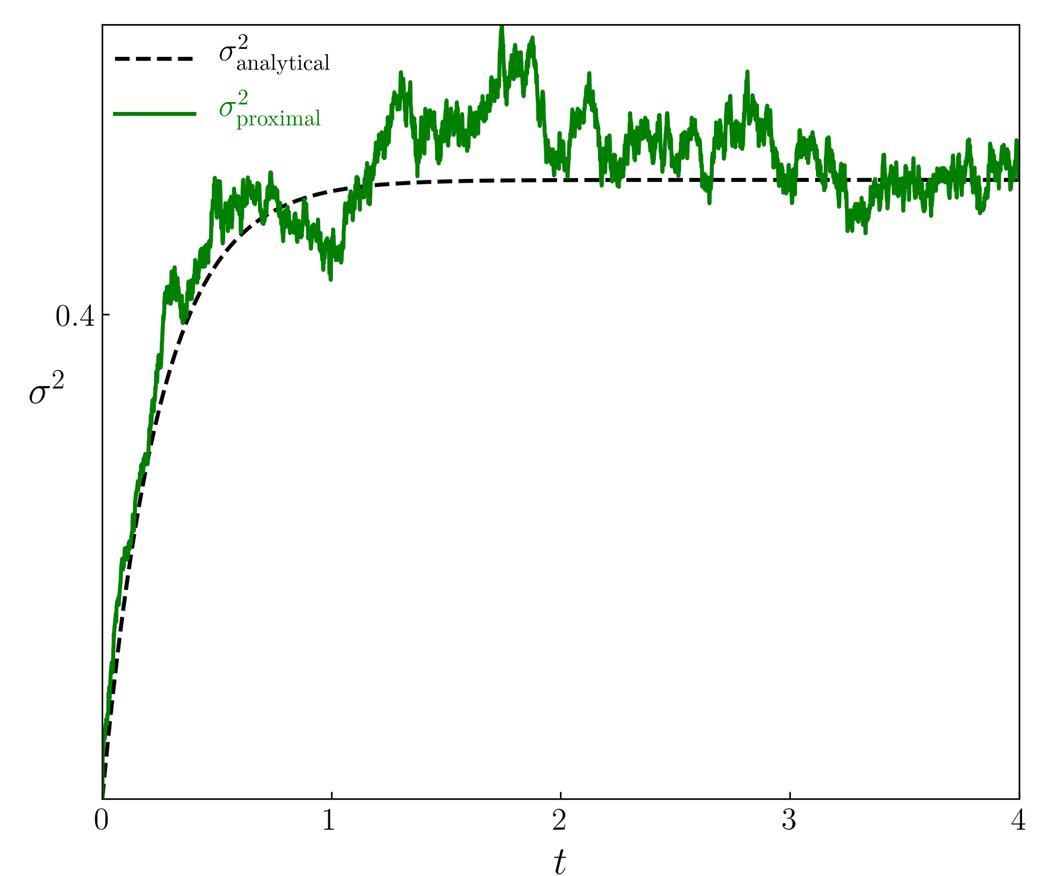


Theorem: Block co-ordinate iteration of $(\mathbf{y}, \mathbf{z})$ recursion is contractive on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

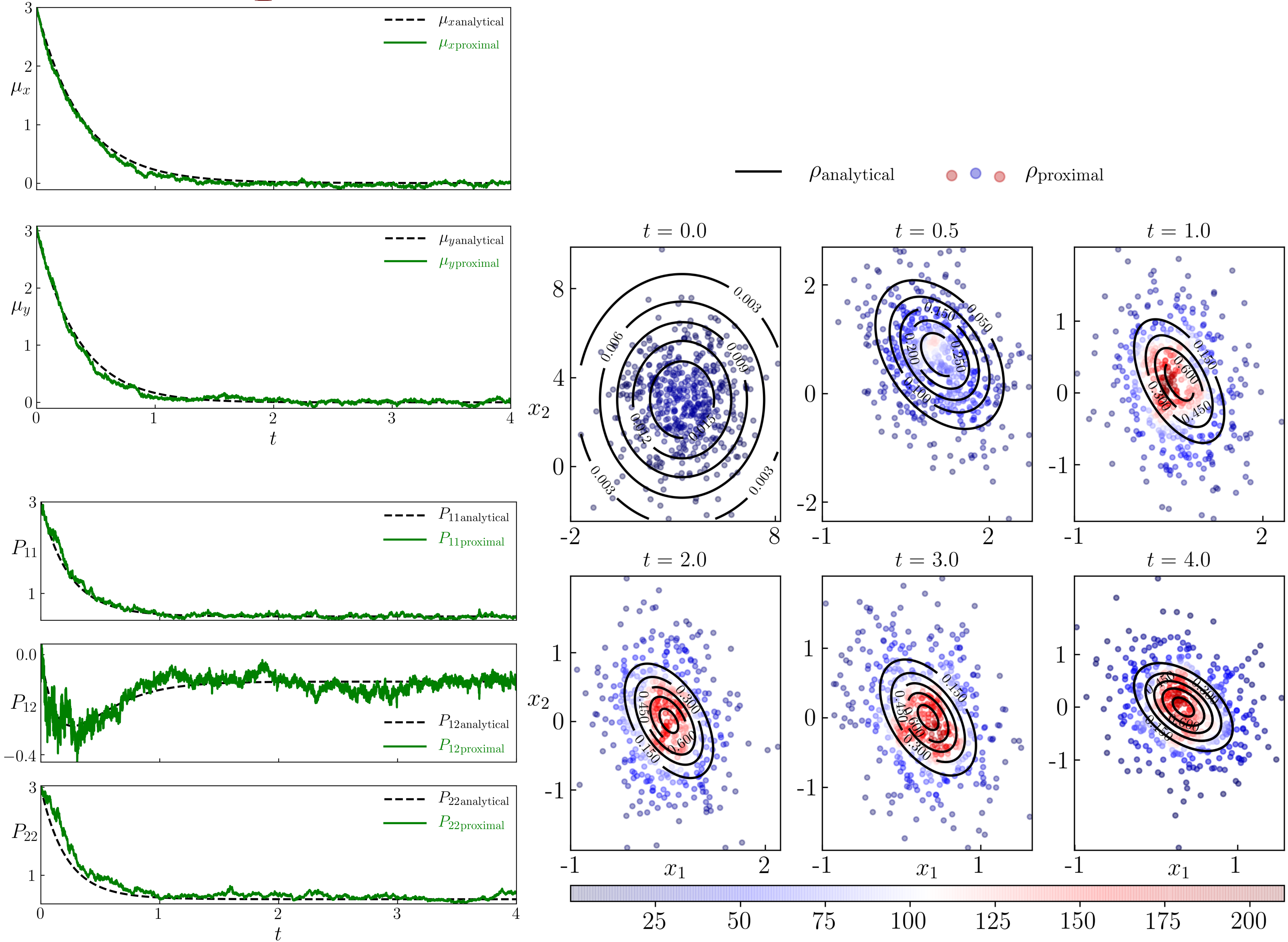
Proximal prediction: 1D linear Gaussian



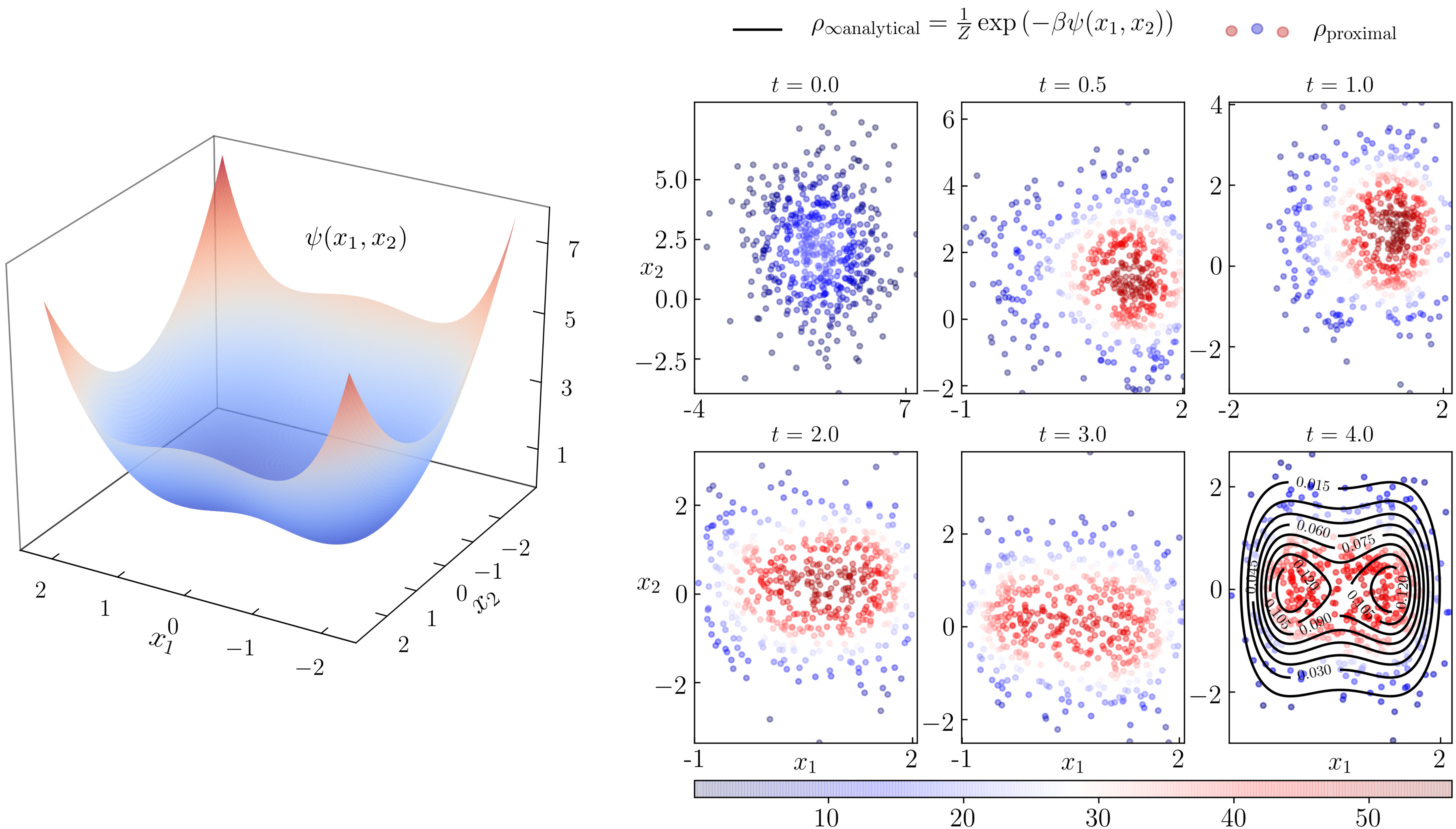




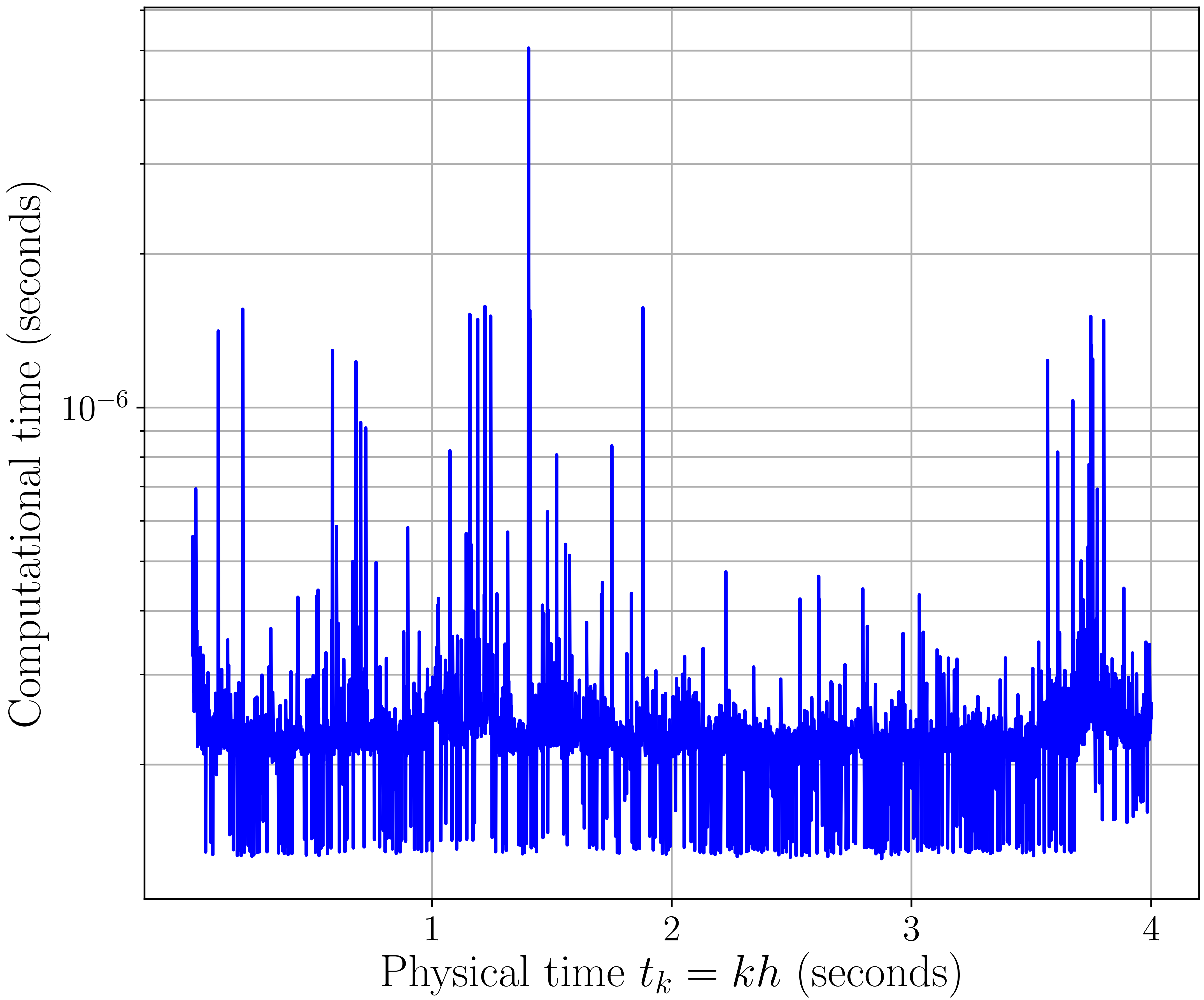
Proximal prediction: 2D linear Gaussian



Proximal prediction: nonlinear non-Gaussian



Computational time: nonlinear non-Gaussian



Details and more numerical case studies

K.F. Caluya, and A. H., Gradient flow algorithms for density propagation in stochastic systems , *IEEE Trans. Automatic Control*, 65(10), pp. 3991–4004, 2019.

K.F. Caluya, and A. H., Proximal recursion for solving the Fokker-Planck equation, *Proc. American Control Conference*, pp. 4098–4103, 2019.

Network reduced power system model

Noisy nonuniform Kuramoto model

$$m_i\ddot{\theta}_i + \gamma_i\dot{\theta}_i = P_i - \sum_{j=1}^{n} k_{ij}\sin(\theta_i - \theta_j - \varphi_{ij}) + \sigma_i \times \text{stochastic forcing}$$

Mixed Conservative-Dissipative SDE over state variables $(\boldsymbol{\theta}, \boldsymbol{\omega}) \in \mathbb{T}^n \times \mathbb{R}^n$

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\theta} \\ \mathrm{d}\boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ -\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) - \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \boldsymbol{0}_{n\times n} \\ \boldsymbol{M}^{-1}\boldsymbol{\Sigma} \end{pmatrix} \mathrm{d}\boldsymbol{w}$$

Potential function $V : \mathbb{T}^n \mapsto \mathbb{R}$

$$V(\boldsymbol{\theta}) := -\sum_{i=1}^{n} P_i\theta_i + \sum_{i<j} k_{ij}(1 - \cos(\theta_i - \theta_j - \varphi_{ij}))$$

Network reduced power system model

Kron reduced admittance matrix

$$\boldsymbol{Y} = [Y_{ij}]_{i,j=1}^{n} \in \mathbb{C}^{n\times n}, \quad Y_{ij} = Y_{ji}$$

Internal voltage and current for generator i

$$E_i, I_i \in \mathbb{C}$$

Parameters in the dynamics

$$P_i = P_i^{\mathrm{mech}} - P_i^{\mathrm{load}} - |E_i|^2 \Re(Y_{ii}) + \Re(E_i \cdot I_i^*),$$

$$\varphi_{ij} = \begin{cases} -\arctan\left(\frac{\Re(Y_{ij})}{\Im(Y_{ij})}\right), & \text{if } i \neq j, \\ 0, & \text{otherwise}, \end{cases}$$

$$k_{ij} = \begin{cases} |E_i||E_j||Y_{ij}|, & \text{if } i \neq j, \\ 0, & \text{otherwise}, \end{cases}$$

Not straightforward to apply the prox recursion

Kinetic Fokker-Planck PDE

$$\frac{\partial \rho}{\partial t} = -\langle \boldsymbol{\omega}, \nabla_\theta \rho \rangle + \nabla_{\boldsymbol{\omega}} \cdot \left(\rho \left(\boldsymbol{M}^{-1} \boldsymbol{\Gamma} \boldsymbol{\omega} + \boldsymbol{M}^{-1} \nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \frac{1}{2} \boldsymbol{M}^{-1} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^\top \boldsymbol{M}^{-1} \nabla_{\boldsymbol{\omega}} \log \rho \right) \right)$$

Einstein relation does not hold ...

$$\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top = \beta^{-1}\left(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^\top\right) \quad \text{for some } \beta > 0$$

Consider change of variable $\begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} := \boldsymbol{\Psi} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}$

where invertible linear map $\boldsymbol{\Psi} := \boldsymbol{I}_2 \otimes \left(\boldsymbol{M}\boldsymbol{\Sigma}^{-1}\right)$

From anisotropic to isotropic diffusion

Transformed SDE

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\xi} \\ \mathrm{d}\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\Upsilon}\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \nabla_{\boldsymbol{\eta}} F(\boldsymbol{\eta}) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \mathbf{0}_{n\times n} \\ \boldsymbol{I}_n \end{pmatrix} \mathrm{d}\boldsymbol{w}$$

Set up

$$\begin{aligned} \tilde{\varrho}_k &= \mathrm{prox}_{h\widetilde{\Phi}}^{\widetilde{W}}(\tilde{\varrho}_{k-1}) \\ &\equiv \underset{\tilde{\rho}\in\mathcal{P}_2}{\arg\inf} \frac{1}{2}\widetilde{W}^2(\tilde{\varrho}, \tilde{\varrho}_{k-1}) + h\widetilde{\Phi}(\tilde{\rho}), \qquad \tilde{\varrho}_0 := \tilde{\rho}_0, \end{aligned}$$

where

$$\widetilde{\Phi}(\tilde{\rho}) := \int_{\mathbb{T}^n\times\mathbb{R}^n} \left(F(\boldsymbol{\eta}) + \frac{1}{2}\log\tilde{\rho} \right) \tilde{\rho}\,\mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}$$

From anisotropic to isotropic diffusion

Consistency guarantee

$$\tilde{\varrho}_k(\boldsymbol{\xi},\boldsymbol{\eta}) \xrightarrow{h\downarrow 0} \tilde{\varrho}(t=kh,\boldsymbol{\xi},\boldsymbol{\eta}) \text{ in } L^1(\mathbb{T}^n \times \mathbb{R}^n)$$

Evolve weighted point cloud $\left\{\tilde{\boldsymbol{x}}_k^i,\, \tilde{\varrho}_k^i\right\}_{i=1}^N$

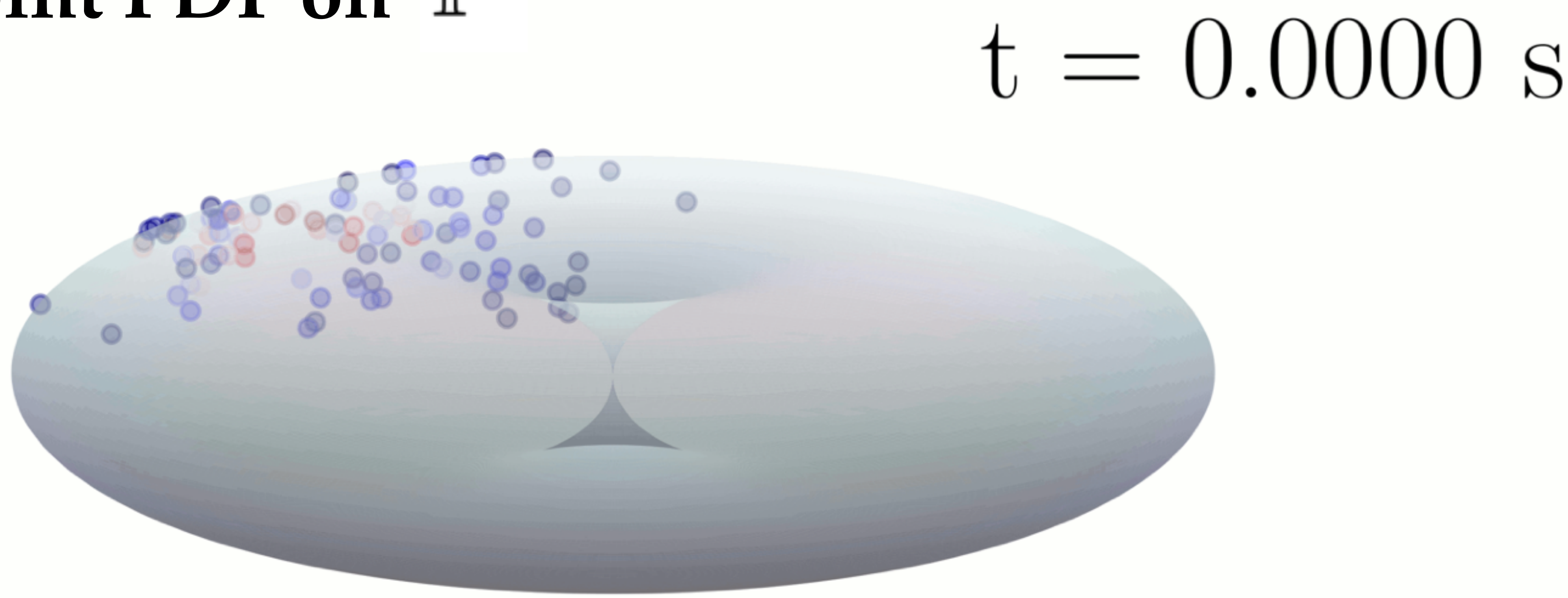
where

$$\tilde{\boldsymbol{x}}_k^i := \left(\boldsymbol{\xi}_k^i, \boldsymbol{\eta}_k^i\right)^\top, \quad i=1,\ldots,N, \quad \boldsymbol{k} \in \mathbb{N}$$

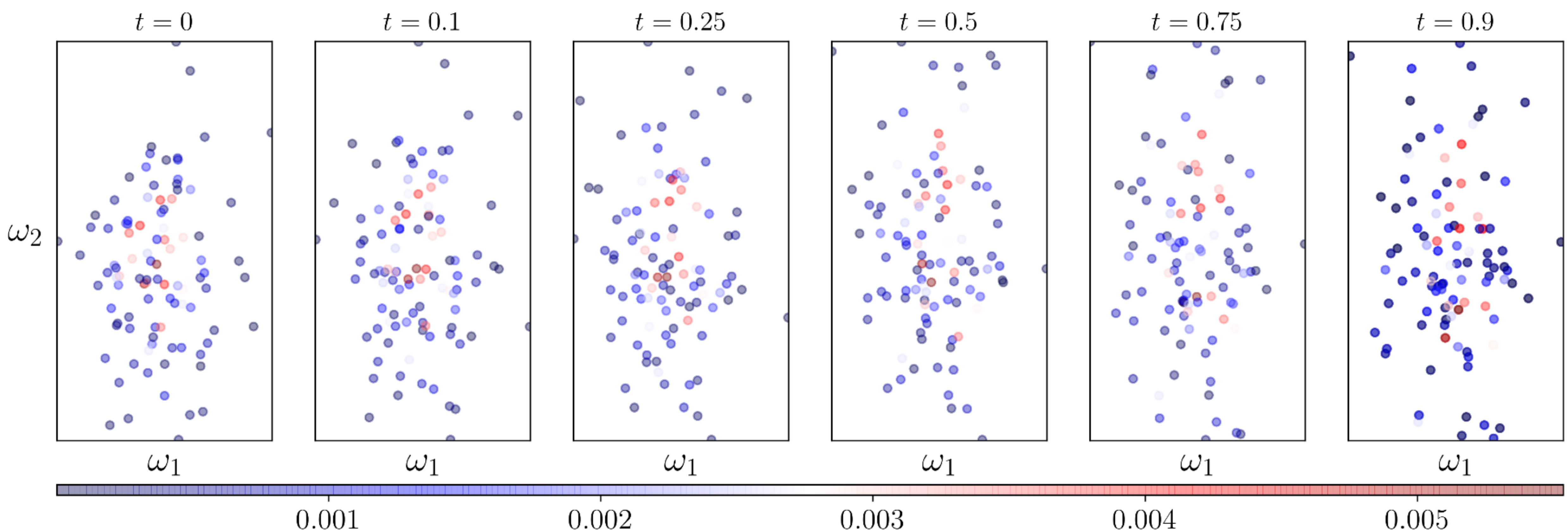
and then push forward via inverse map to come back to $(\boldsymbol{\theta},\boldsymbol{\omega})$ coordinate

Proximal Prediction: Power System with $n = 2$

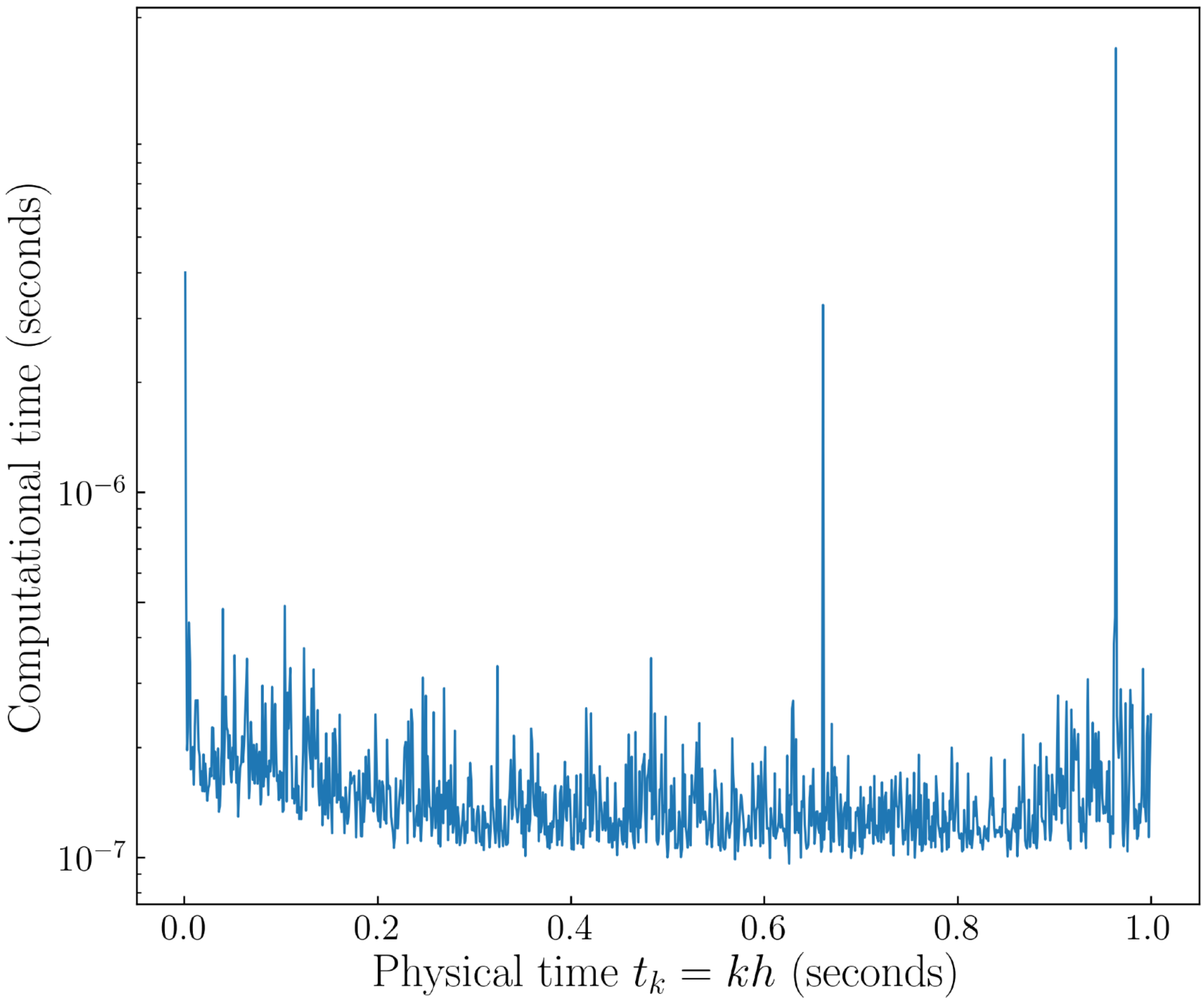
Projection of the joint PDF on $\mathbb{T}^2$



Projection of the joint PDF on $\mathbb{R}^2$



Computational Time: Power System with $n = 2$



IEEE 14 bus case study

Parameters from MATPOWER and ANDES

Nominal case (Case I): $P_i^{\text{mech}}, P_i^{\text{load}}$ from steady state power flow

Post-contingency case (Case II): Line 13 fails at $t = 0$

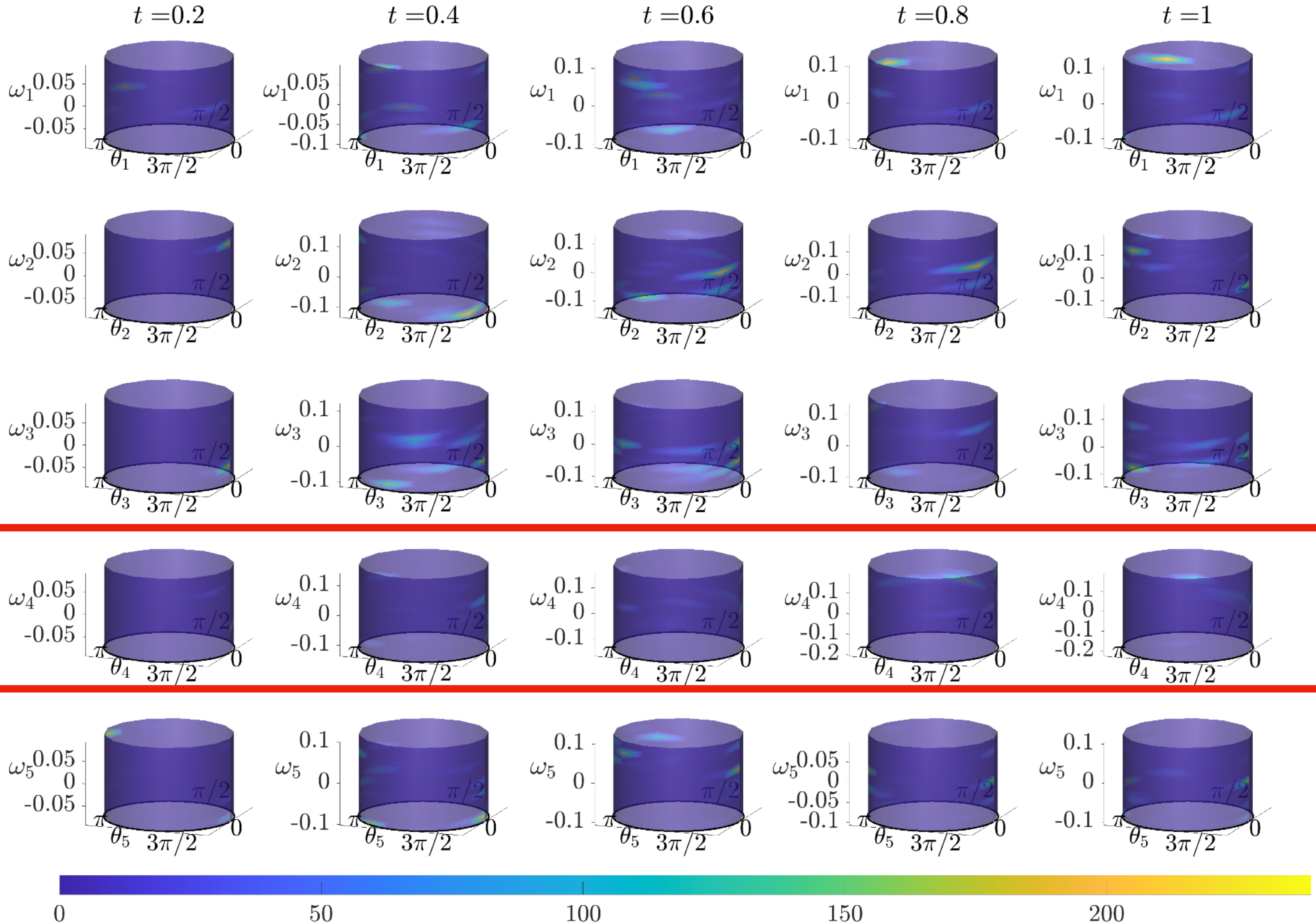
State space: $\mathbb{T}^5 \times \mathbb{R}^5$

Can account time-varying $P_i^{\text{mech}}, P_i^{\text{load}}$

Initial $\mathbb{T}^n$ marginal as product von Mises with mean angles from the steady state AC power flow

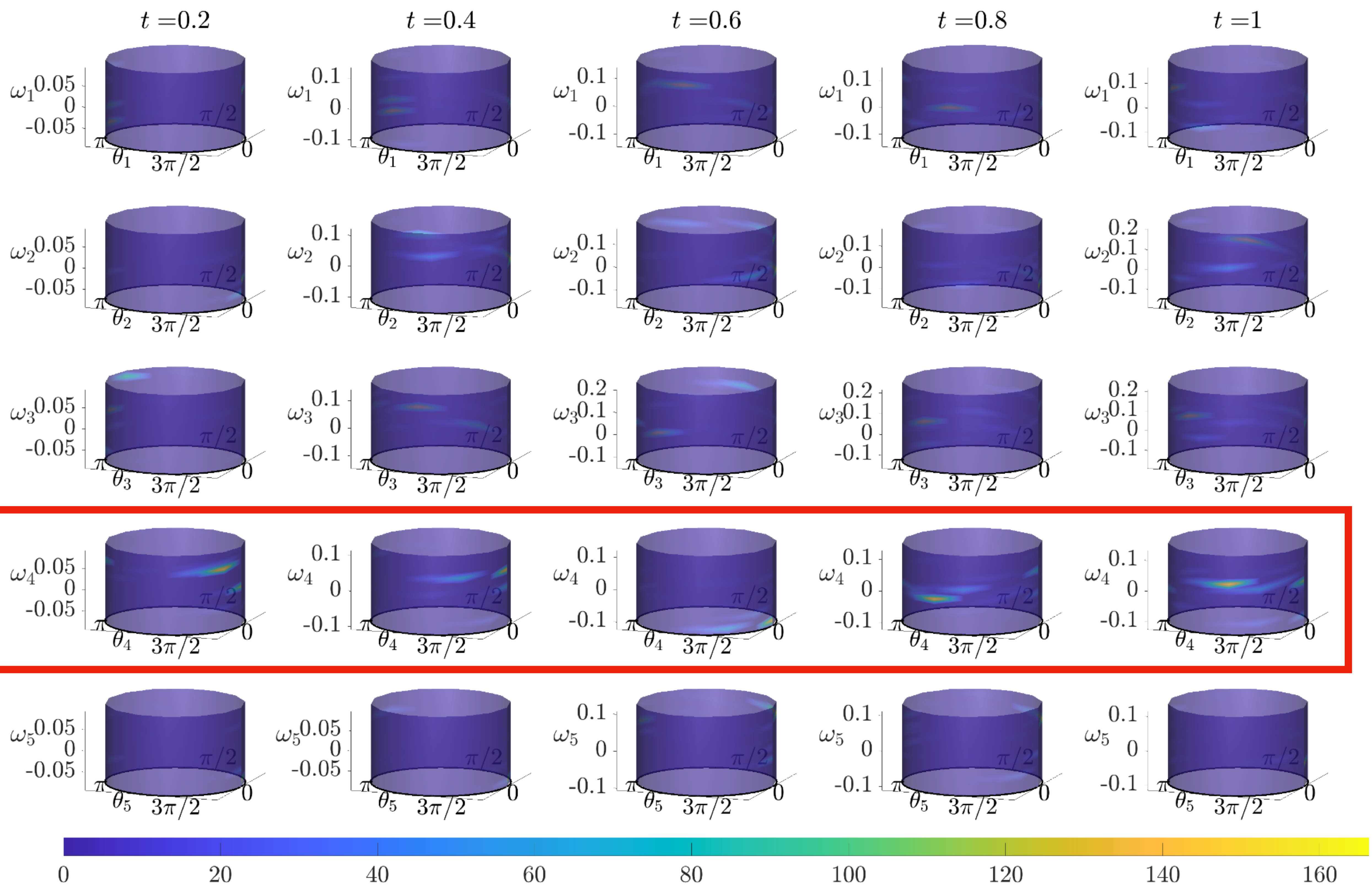
IEEE 14 bus case study: bivariate (θ, ω) marginals

Nominal case:

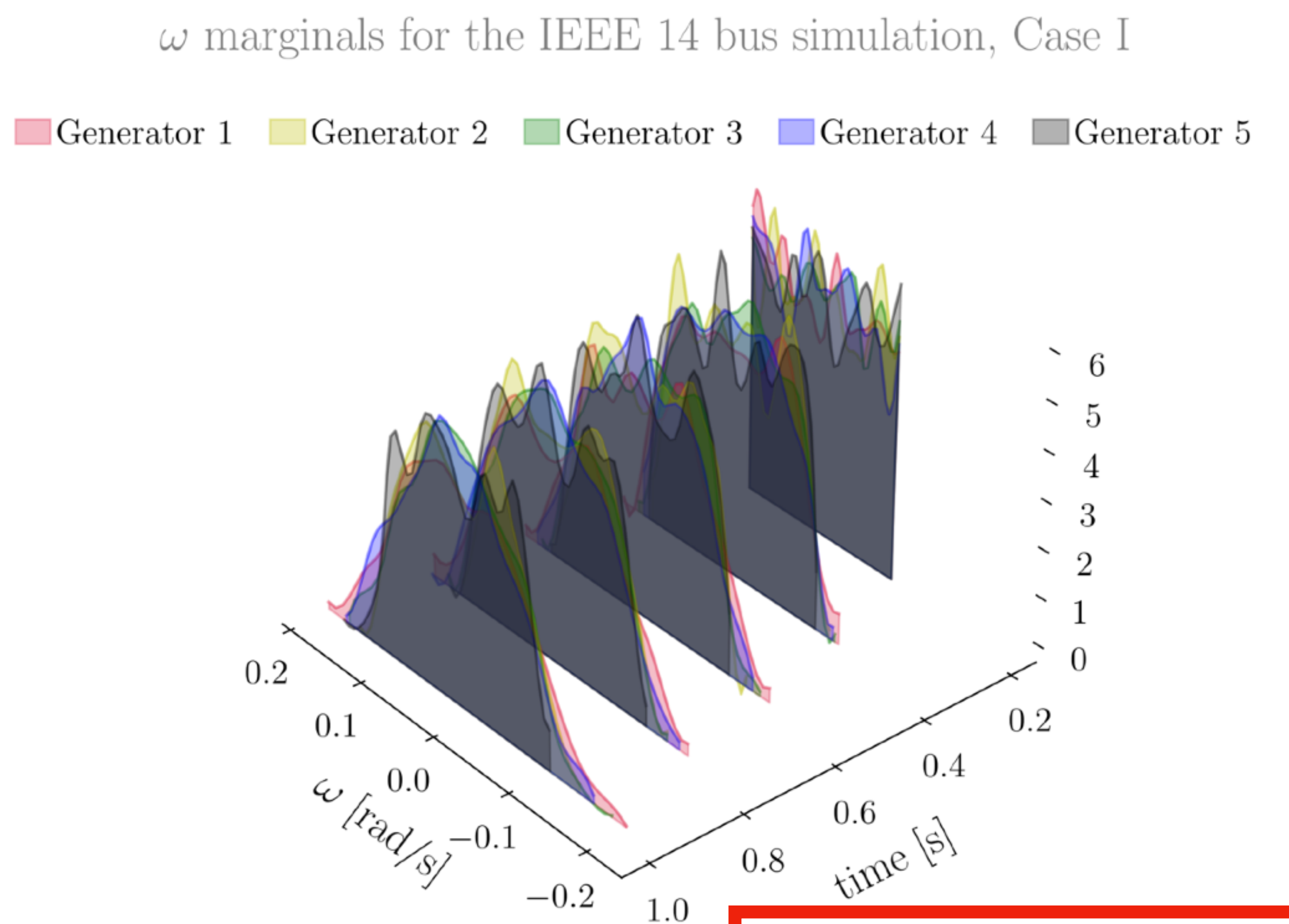


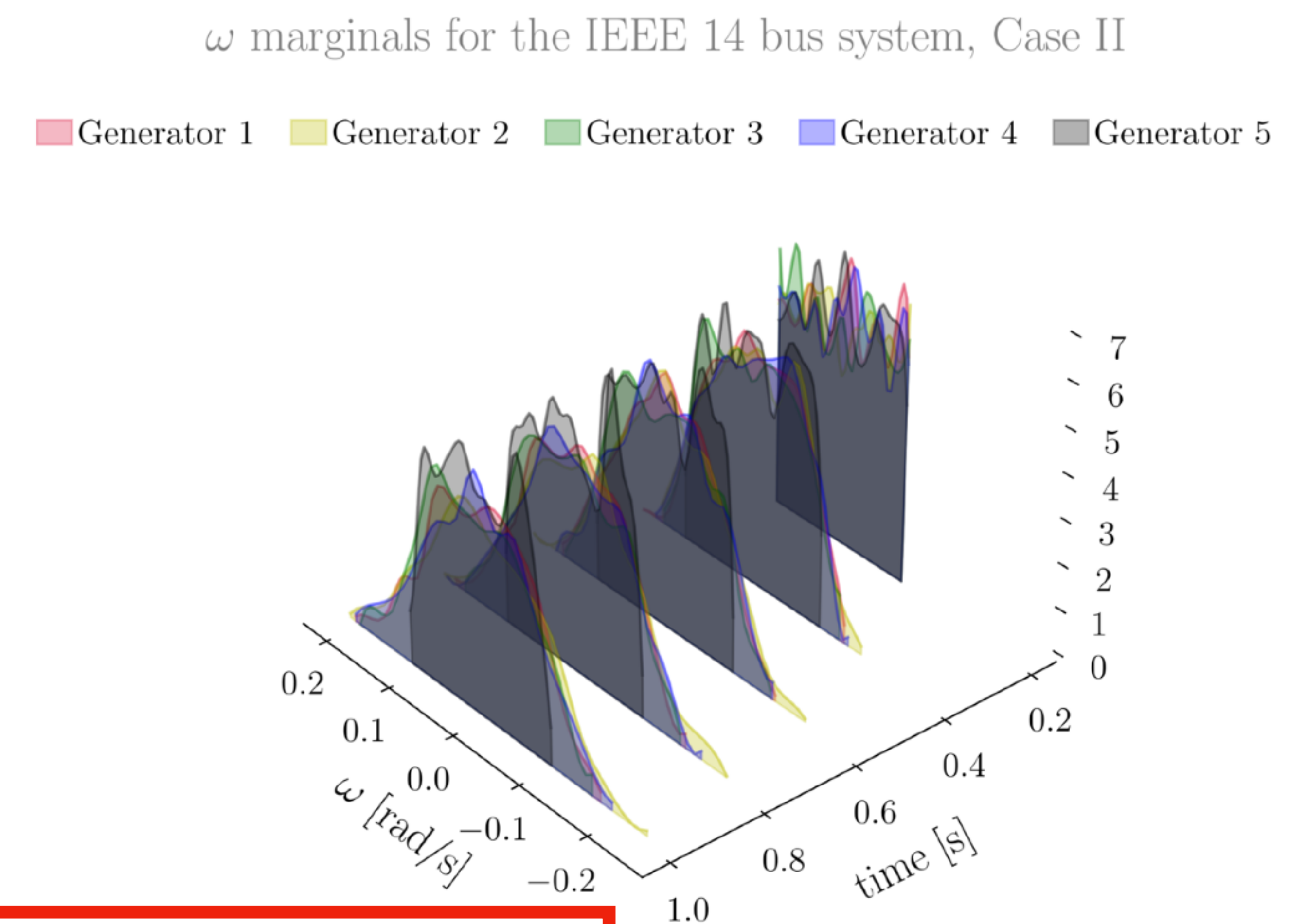
IEEE 14 bus case study: bivariate (θ, ω) marginals

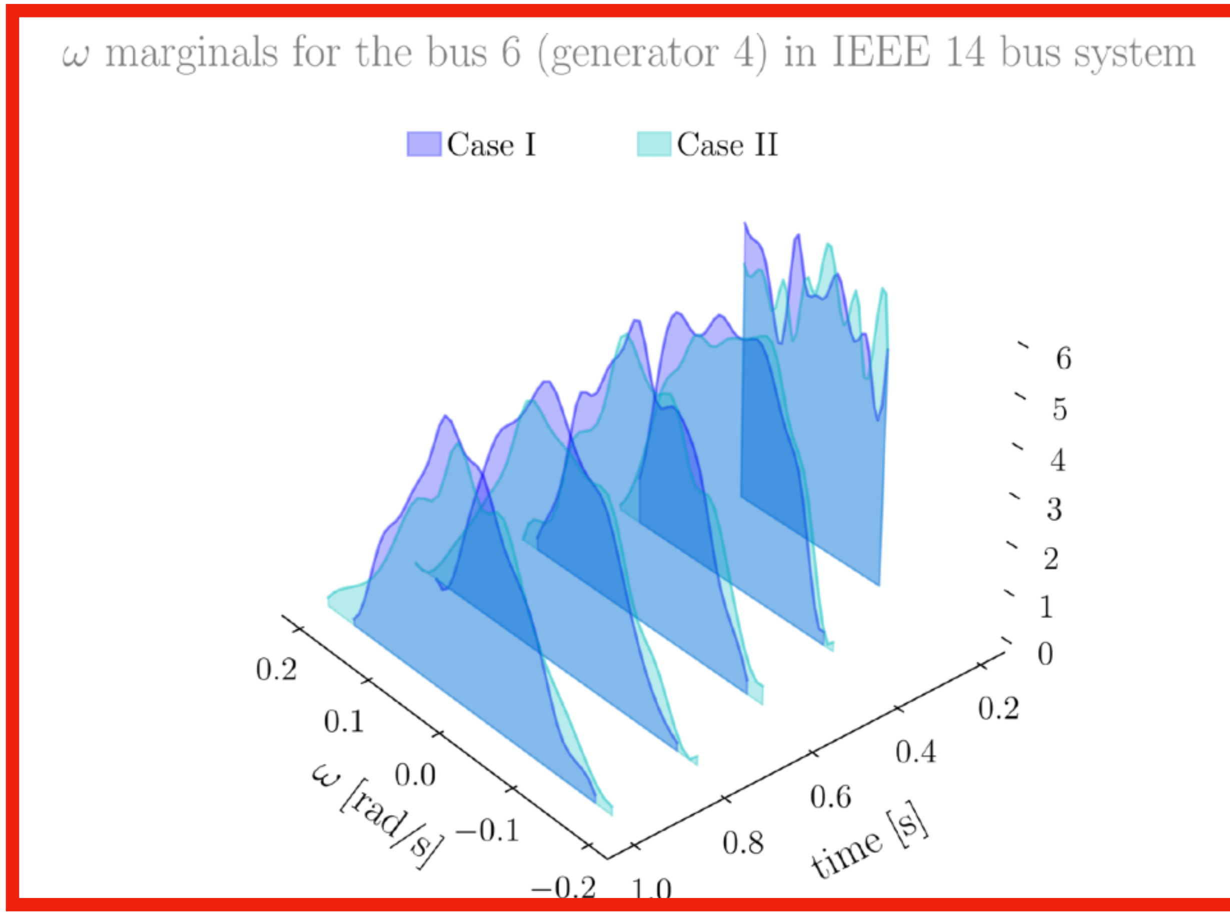
Post-contingency case:



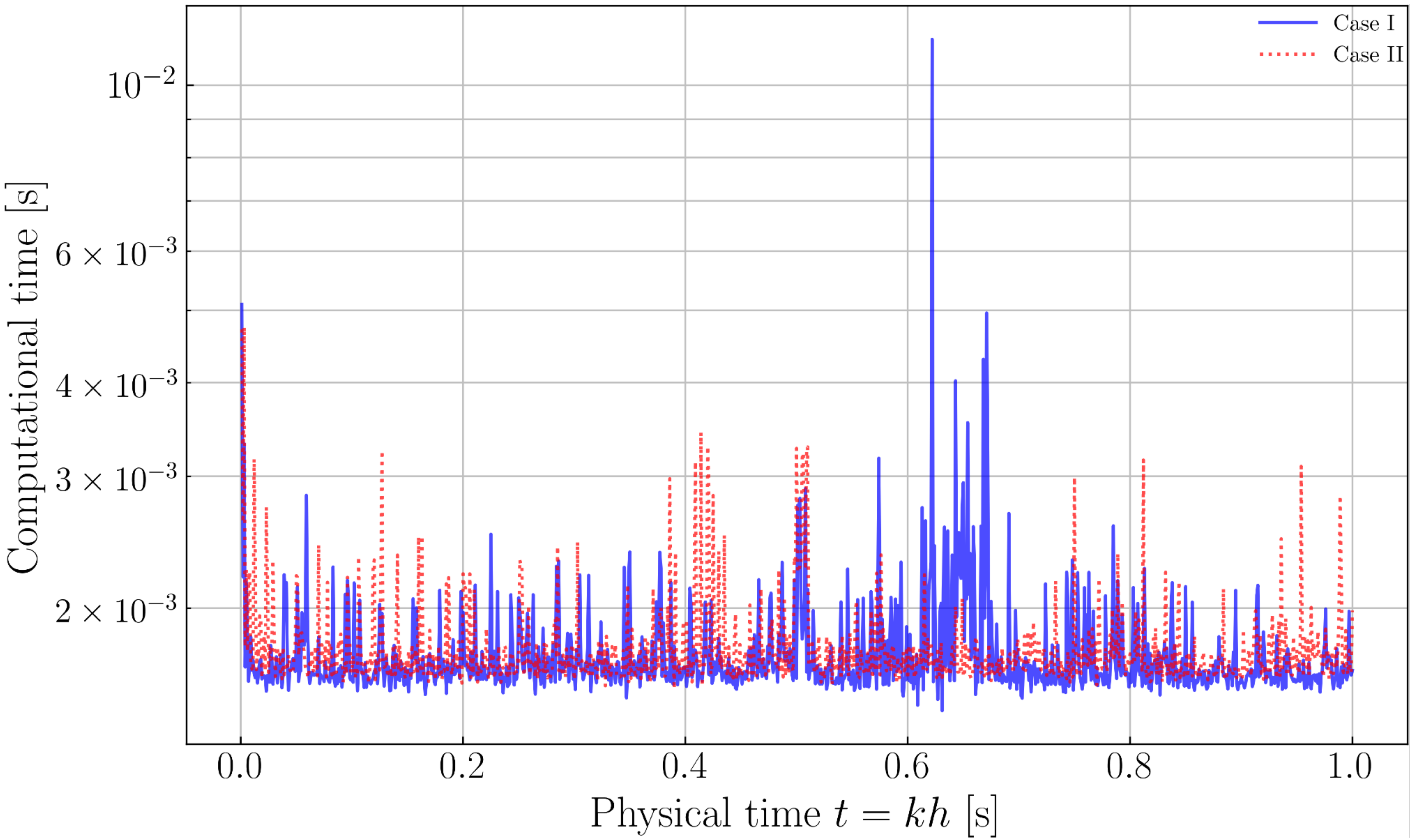
IEEE 14 bus case study: univariate ω marginals



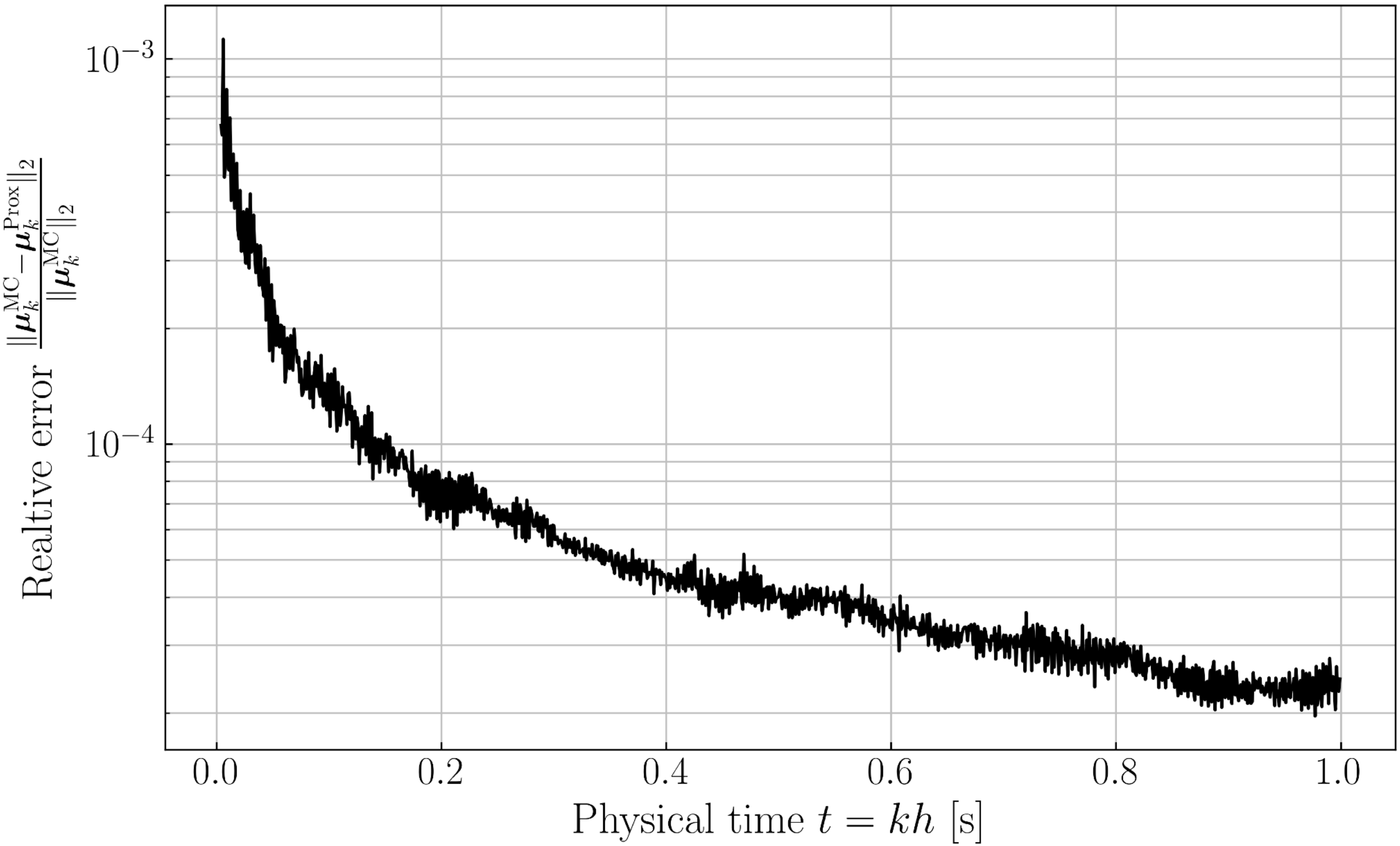




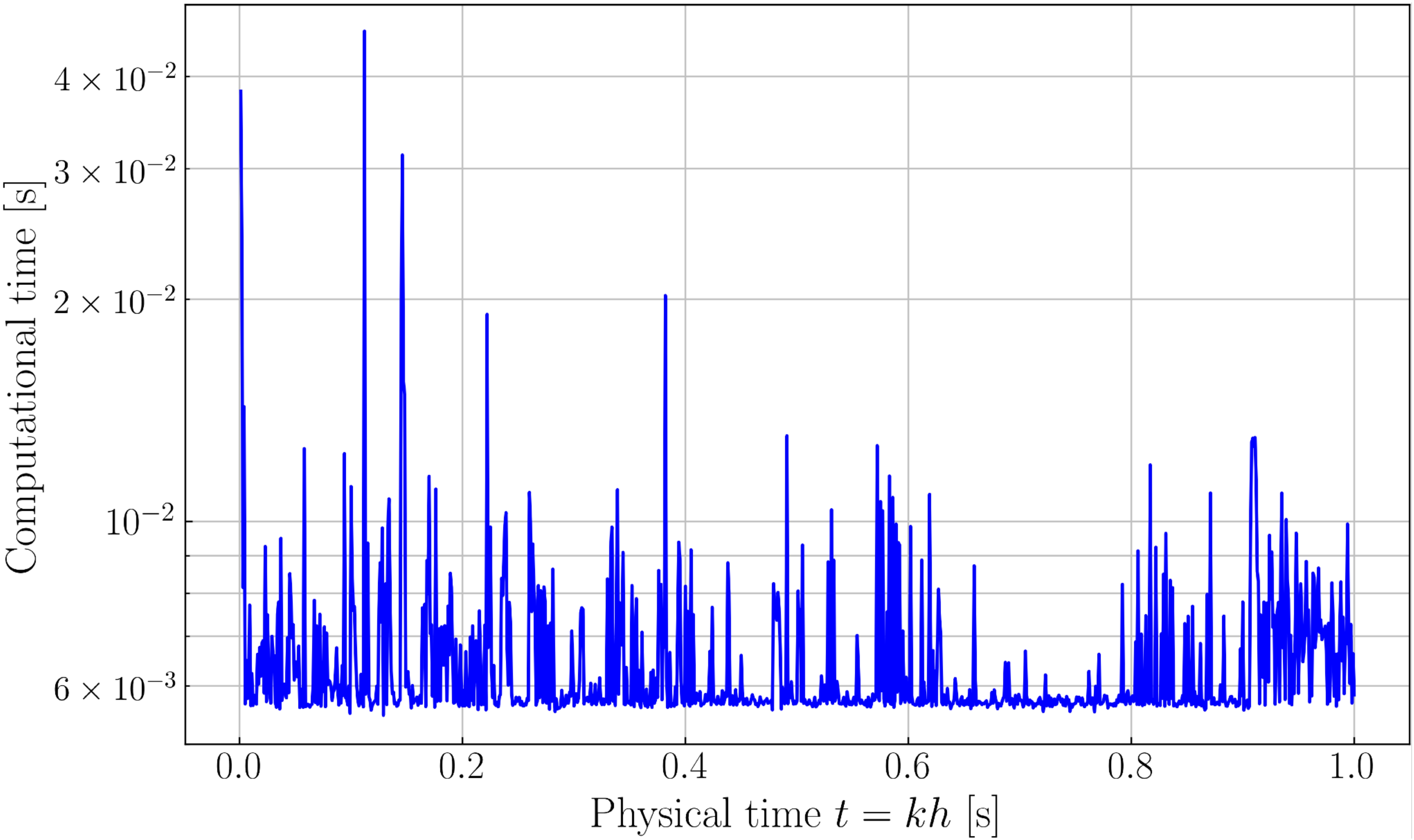
IEEE 14 bus case study: computational time



Synthetic system: $n = 50$: accuracy



Synthetic system: $n = 50$: computational time



Summary

Fast proximal recursions for PDF propagation in power systems

Ongoing work

Large scale implementation: ~1000 generators in ~seconds

Address Stochastic Differential Algebraic Equations (SDAEs)

More complex models

Control of distributional uncertainties

Thank You

Support: