Statistical **I**nference and **M**achine **L**earning in **E**arth **S**cience
SIMLES

Module 1
Probability Models

Lecture 3
Moments
Sampling Distributions

Cristi Proistosescu

Probability Models

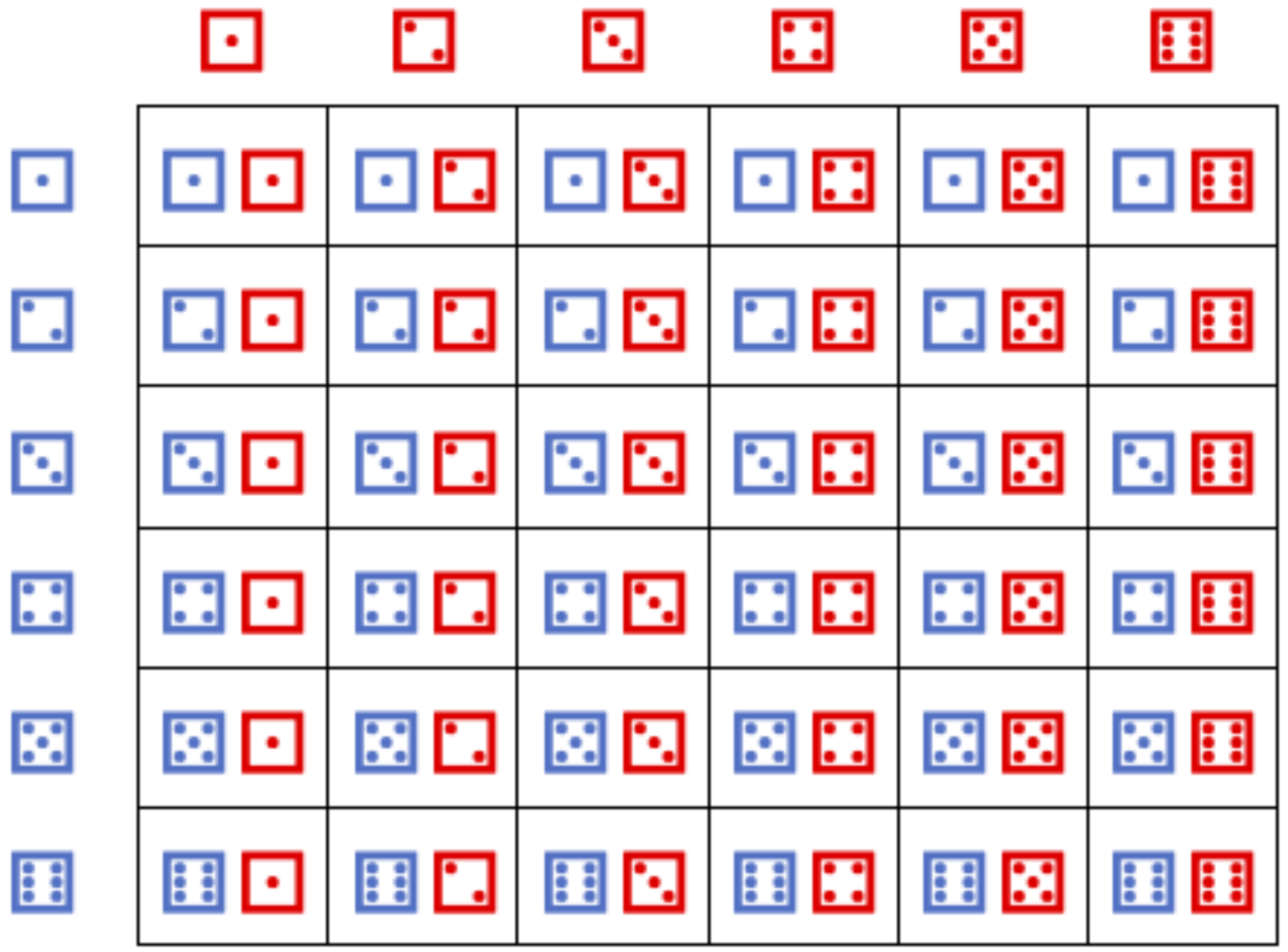
Probability models: allow us to model a random process

Probability Models: Random Variable

Probability models: allow us to model a random process

Random variables allow us to simulate the outcome of random processes.

- dice roll



```
X=stats.randint.rvs(low=1, high=7, size=1)
Y=stats.randint.rvs(low=1, high=7, size=1)
S=X+Y
print(S)
```

```
[6]
```

Probability Models: Random Variable

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- dice roll
- coin flip

0 1

$X \sim \text{Bern}(p)$

```
# Rerun this cell to generate Bernoulli random variables with $p=0.9$
X=stats.bernoulli.rvs(p=0.9, size=50)
print(X)
```

```
[1 1 1 1 0 1 1 1 1 0 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 0
 1 1 1 1 1 1 1 1 1 1 0 1 1]
```

Probability Models: Random Variable

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- dice roll
- coin flip

0 1

$X \sim \text{Bern}(p)$

```
# rerun this cell to simulate more realizations of a bernoulli random variable
X=stats.randint.rvs(low=1, high=7, size=1)
if X>3:
    Y=1
else:
    Y=0
print(Y)
```

```
1
```

Probability Models: Random Variable

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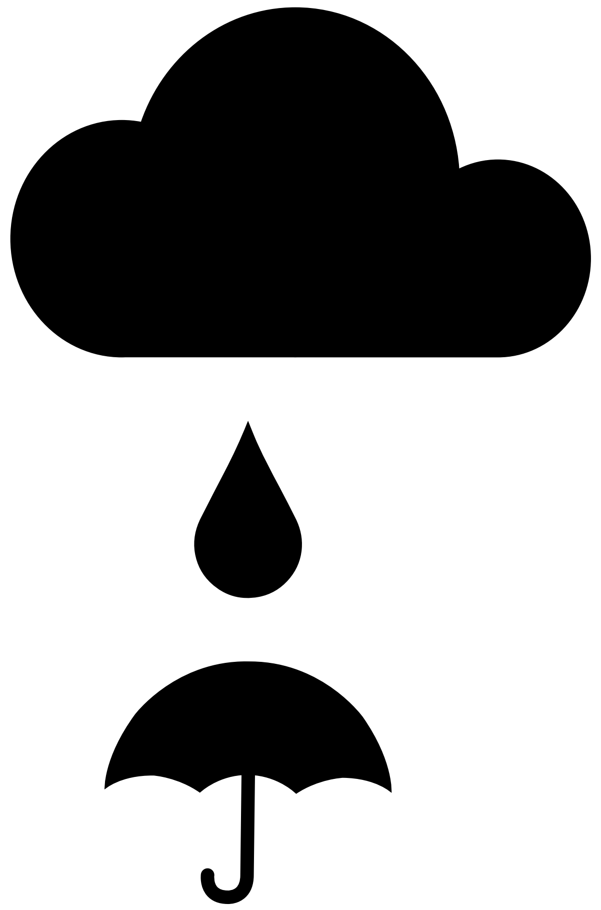
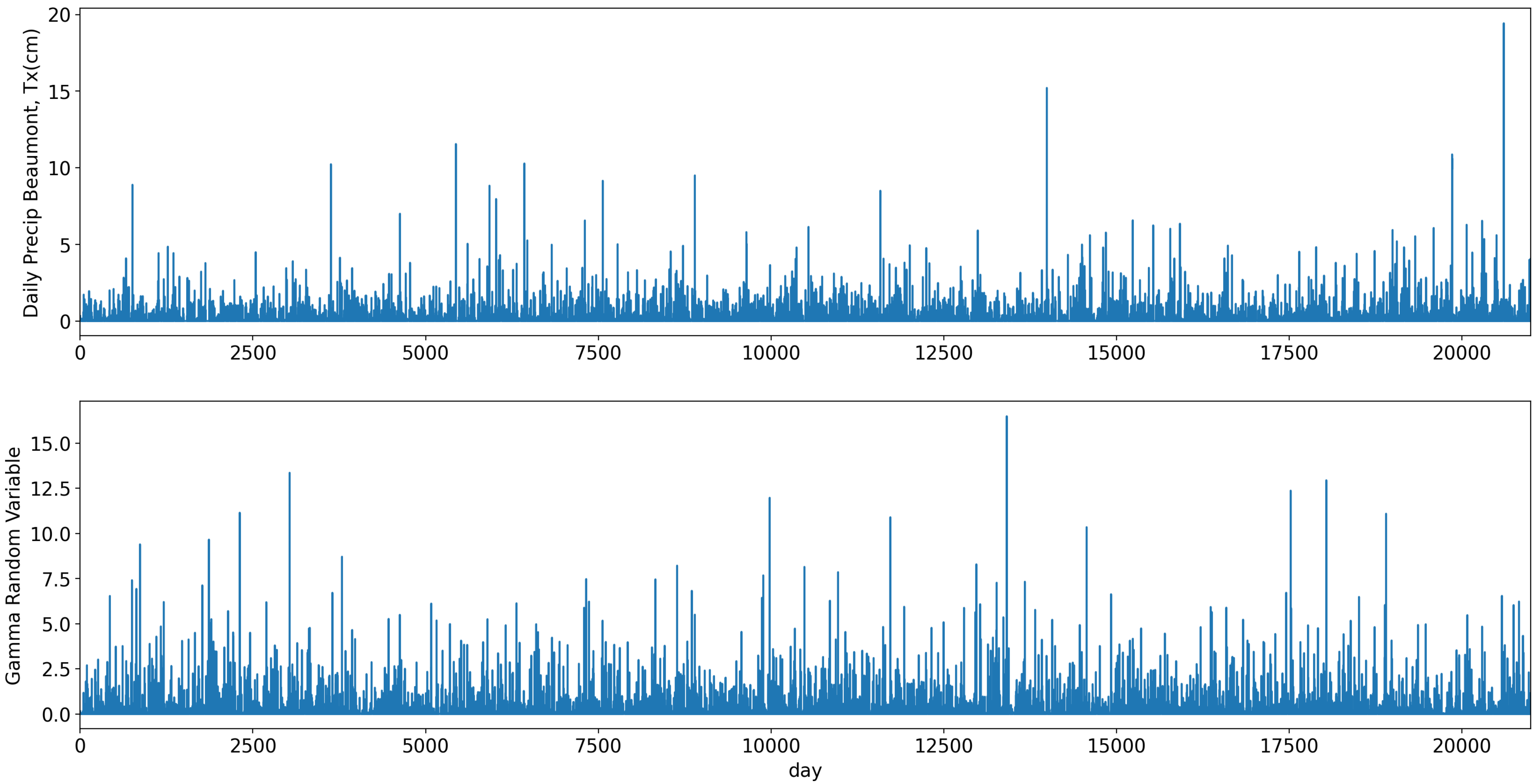
- dice roll
- coin flip
- precipitation



$$X \sim \Gamma(\alpha, \beta)$$



Probability Models: Random Variable

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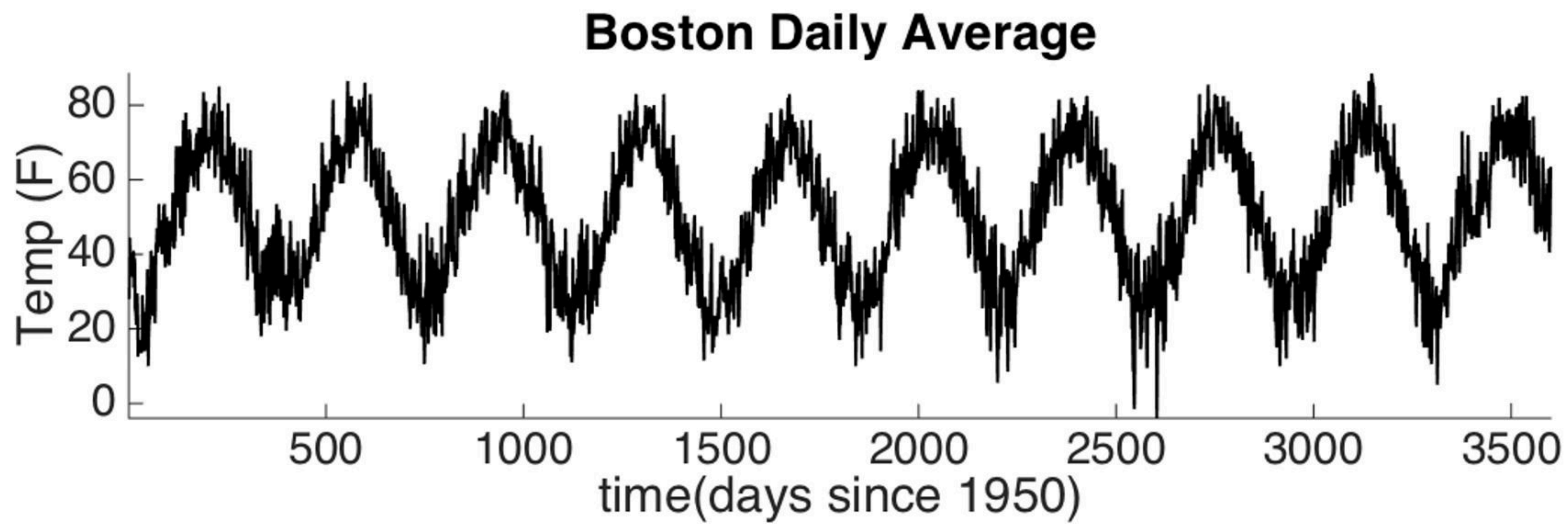
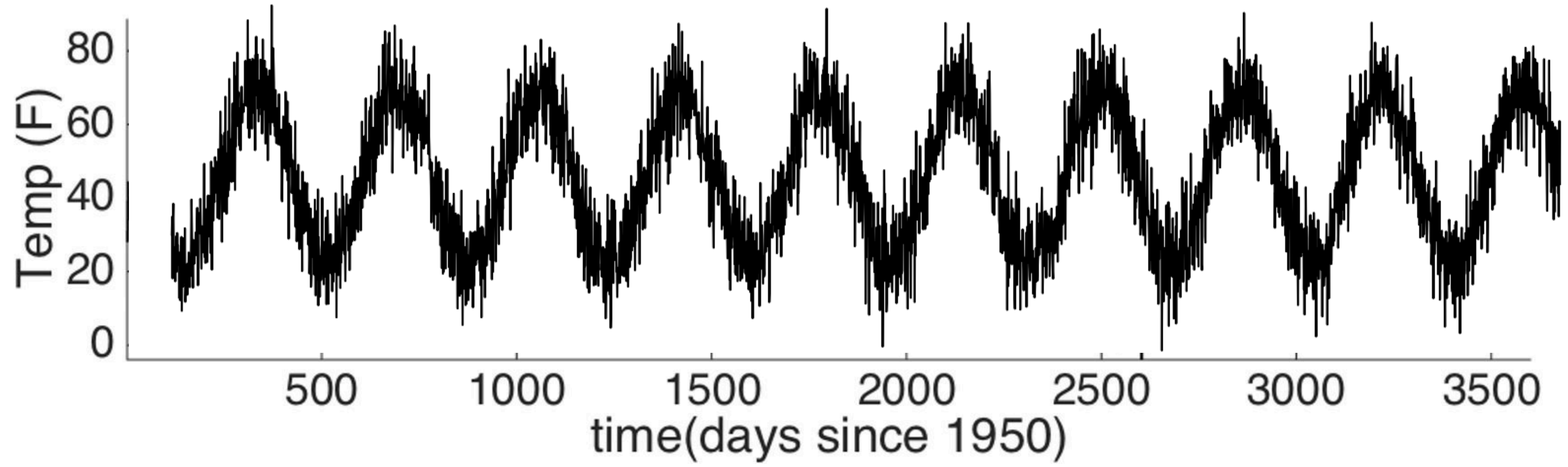
- dice roll
- coin flip
- precipitation
- temperature



$$T \sim A \sin\left(\frac{2\pi}{T}t + \phi\right) + X$$

$$X \sim \mathcal{N}(0,\sigma)$$

Probabilistic model



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- precipitation
- temperature

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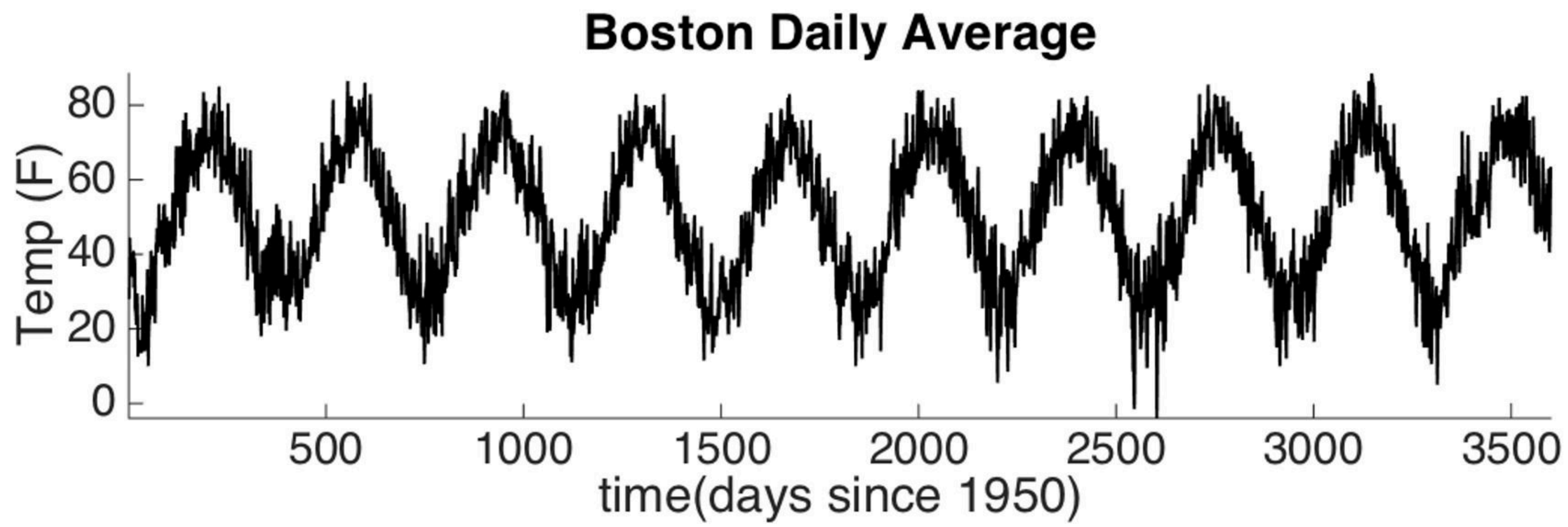
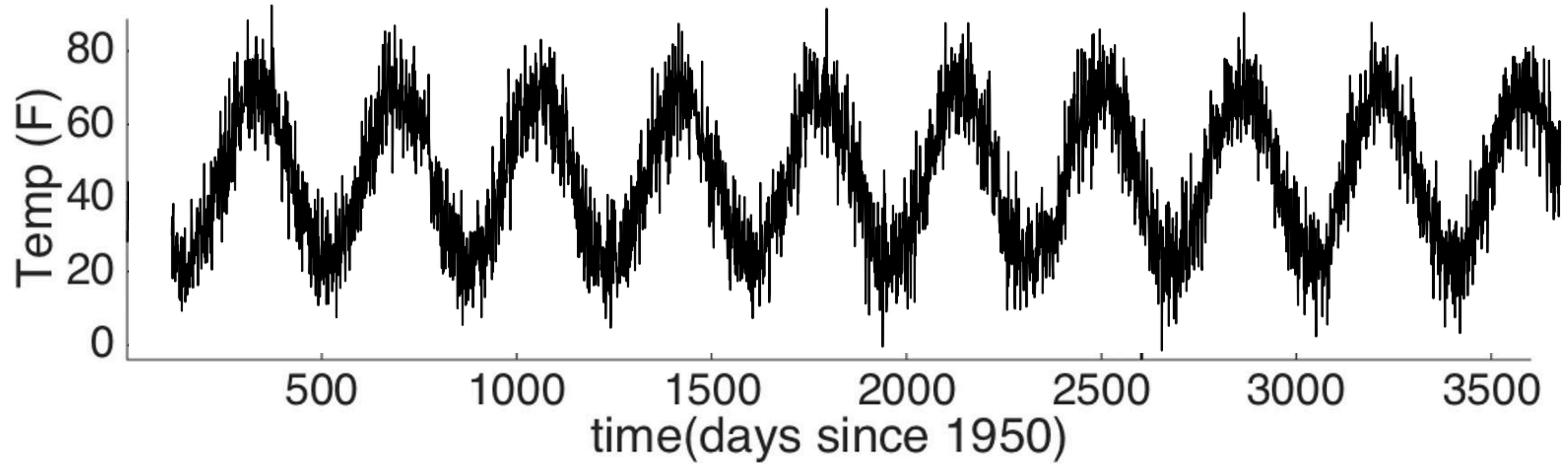
$$T \sim \mathcal{N}(0,\sigma)$$

```
Ndays=3600        #number of days to simulat
T = 365.25        #period of the annual cycle (days)
A = 30            #amplitude of annual cycle
phase = 130       # phase shift (to the sine to start at the right poin
mu     = 50       # mean temperature
sig= 30           # spread of weather

T=A*np.sin(2*np.pi*(t-phase)/T) + stats.norm.rvs(loc=mu,scale=sig,size
```



Probabilistic model



Probability Models: Distributions

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Distributions: summarize the outcome of a random process.

- **pmfs, pdfs** describe the underlying *process*.
- **Histograms** summarize a finite sample.

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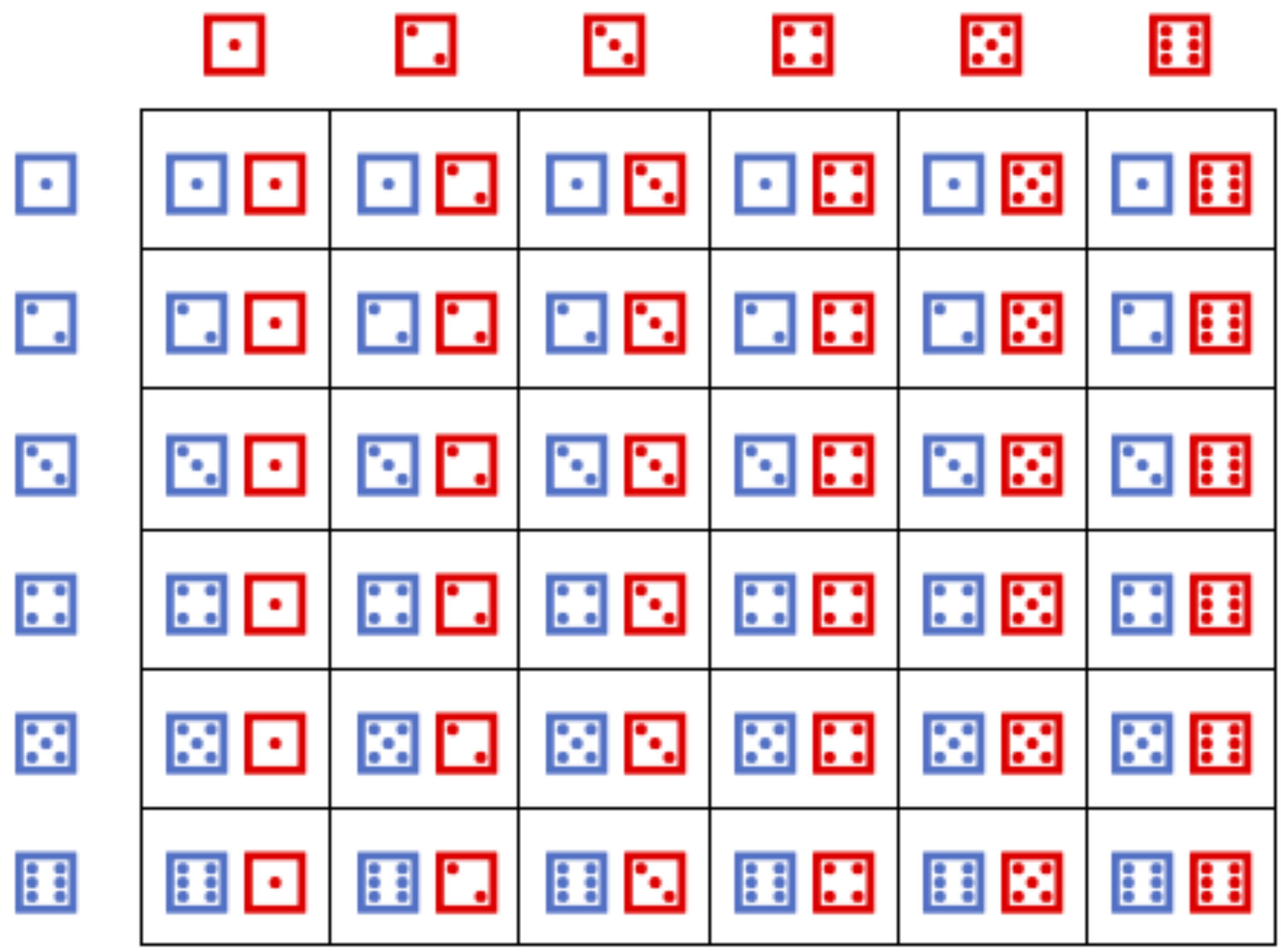

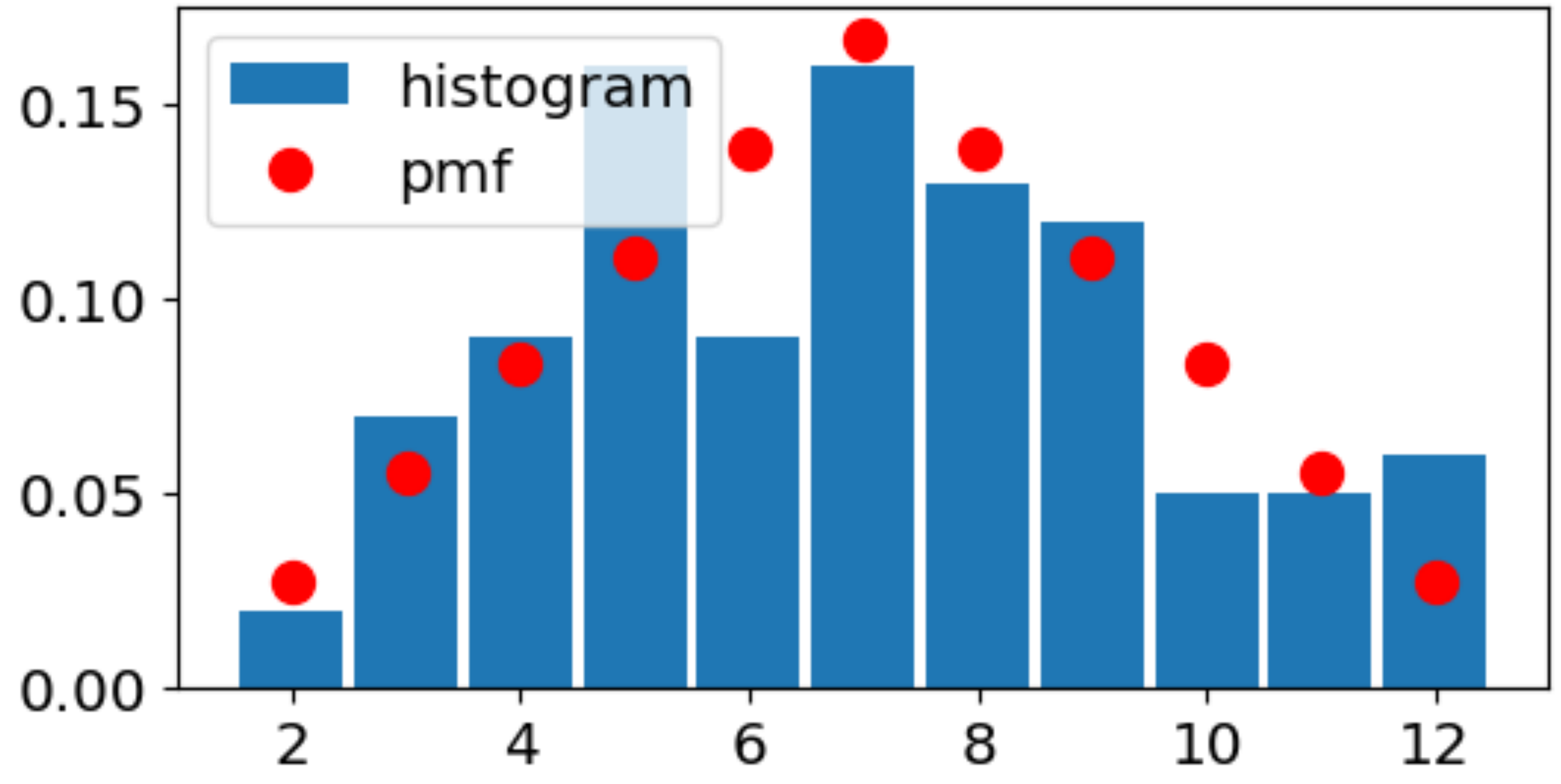

$$N_{draws} = 100$$

Probability Models: Moments

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Moments: another useful summary of the outcome of a random process.

Moments: Expected Value

Expected value

$$E(X) = \sum_{i=1}^{N} x \cdot P(X = x_i)$$

$$E(X) = \int_{\mathbb{R}} x \cdot p(x) dx$$

The expected value of a random variable is the average value we would expect to get if we could sample a random variable an infinite number of times.

It is a average of all the possible outcomes, weighted by how probably they are.

Moments: mean

Key property: linearity

$$E(aX + bY) = aE(x) + bE(y)$$

Moments: mean

Key property: linearity

$$E(aX + bY) = aE(x) + bE(y)$$

We can also define the expected value, or mean, of any function of a random variable:

$$E(g(X)) = \begin{cases} \sum_{i=1}^{N} g(x_i)P(X = x_i) & \text{if } X \text{ is discrete} \\ \int_{\mathbb{R}} g(x)f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Moments: variance

A closely related notion to the second order moment is the **variance** or centered second moment, defined as:

$$V(X) = E([X - E(x)]^2) = E([X - \mu]^2) = \int_{\mathbb{R}} (x - \mu)^2 f(x) dx$$

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Expanding the square and using the linearity:

$$V(X) = E(X^2) - (E(X))^2 = E(X^2) - \mu^2$$

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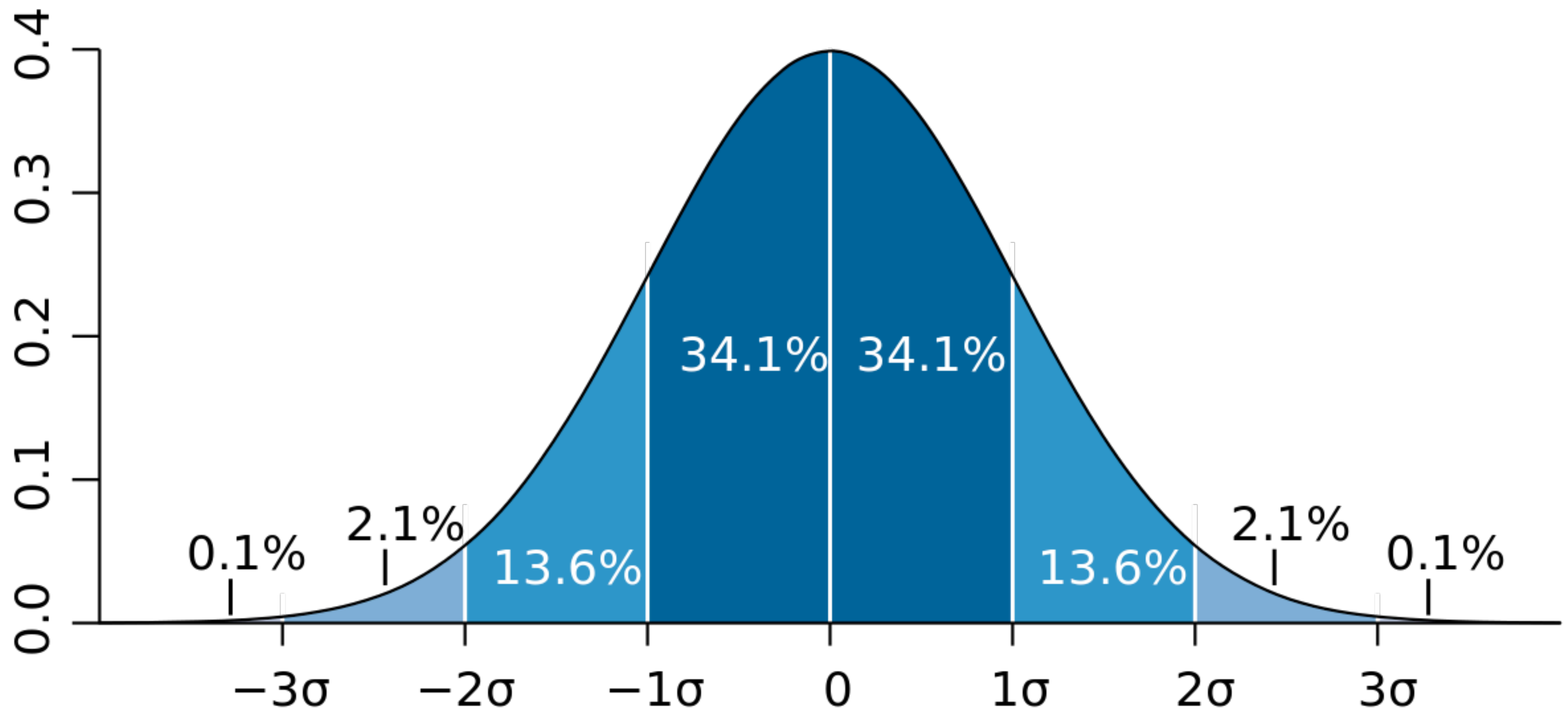
Another closely related measure is standard deviation,

$$\text{std} = \sqrt{V(X)} = \sqrt{E([X - \mu]^2)}$$

Normal distribution

X= normal with location parameter μ and scale parameter σ

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$



$$E(X) = \mu$$
$$V(X) = \sigma^2$$
$$\text{std} = \sigma$$

Normal distribution

DEMO: mean and variance of normal distribution

Moments

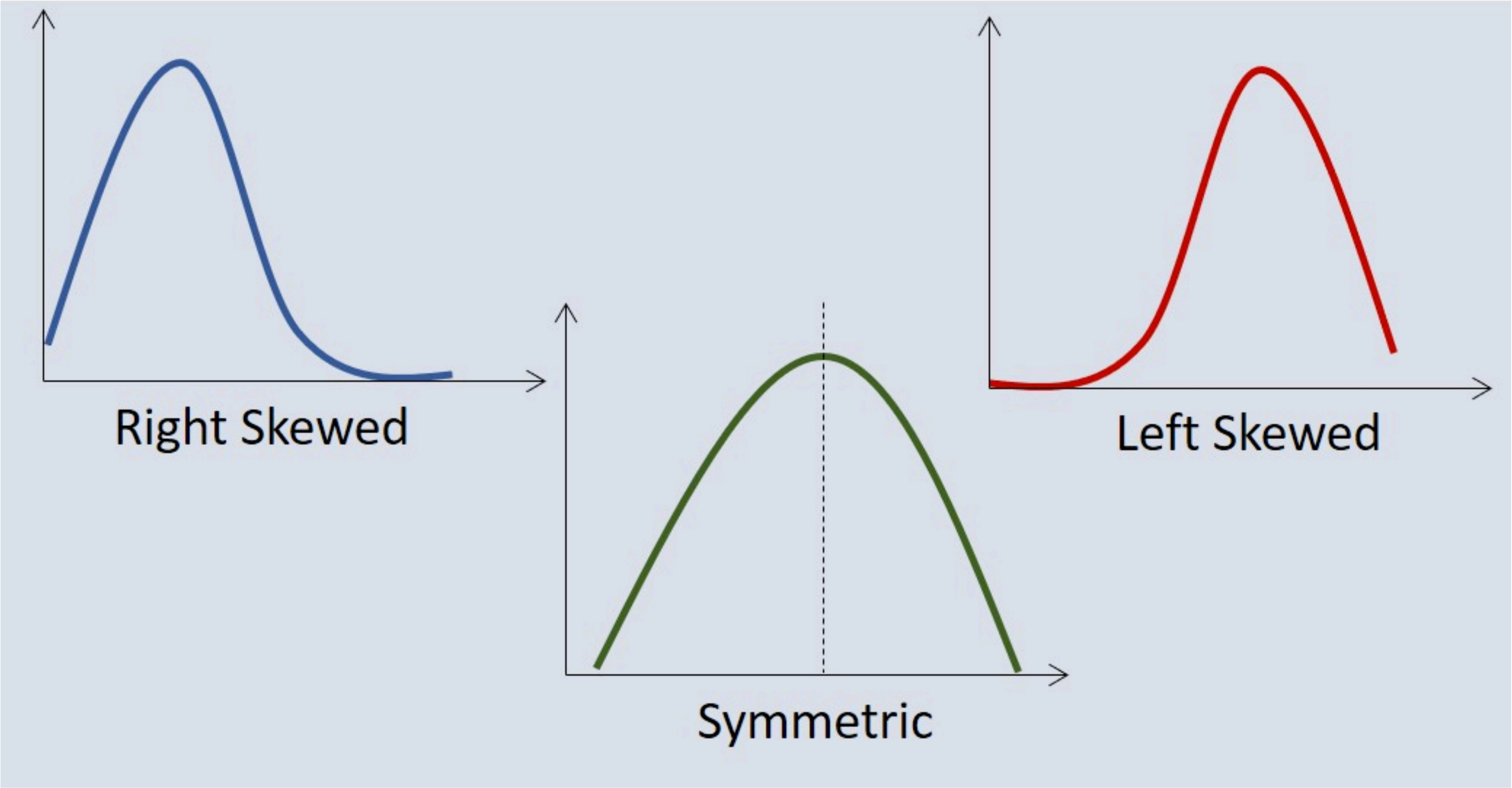
Moments of a distribution

$$m(X,n) = E(X^n) = \sum_{i=1}^{N} x_i^n \cdot P(X = x_i)$$

$$m(X,n) = E(X^n) = \int_{\mathbb{R}} x^n \cdot p(x)dx$$

Higher order moments: Skewness

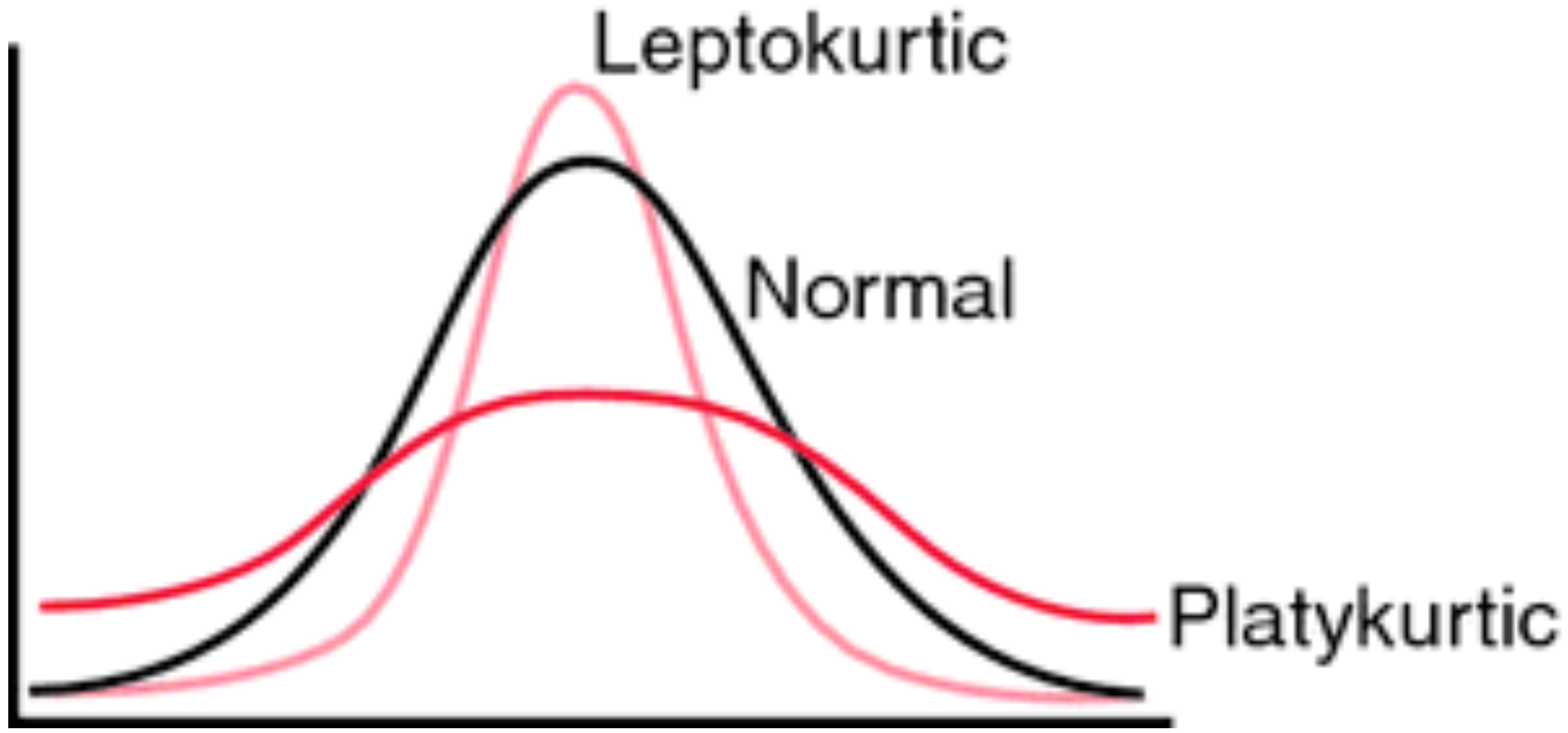
$$S = E\left(\left[\frac{X-\mu}{\sigma}\right]^3\right)$$



Higher order moments: Kurtosis

$$K = E\left(\left[\frac{X-\mu}{\sigma}\right]^4\right)$$

$$K_{\text{normal}} = 3$$



Bivariate distributions

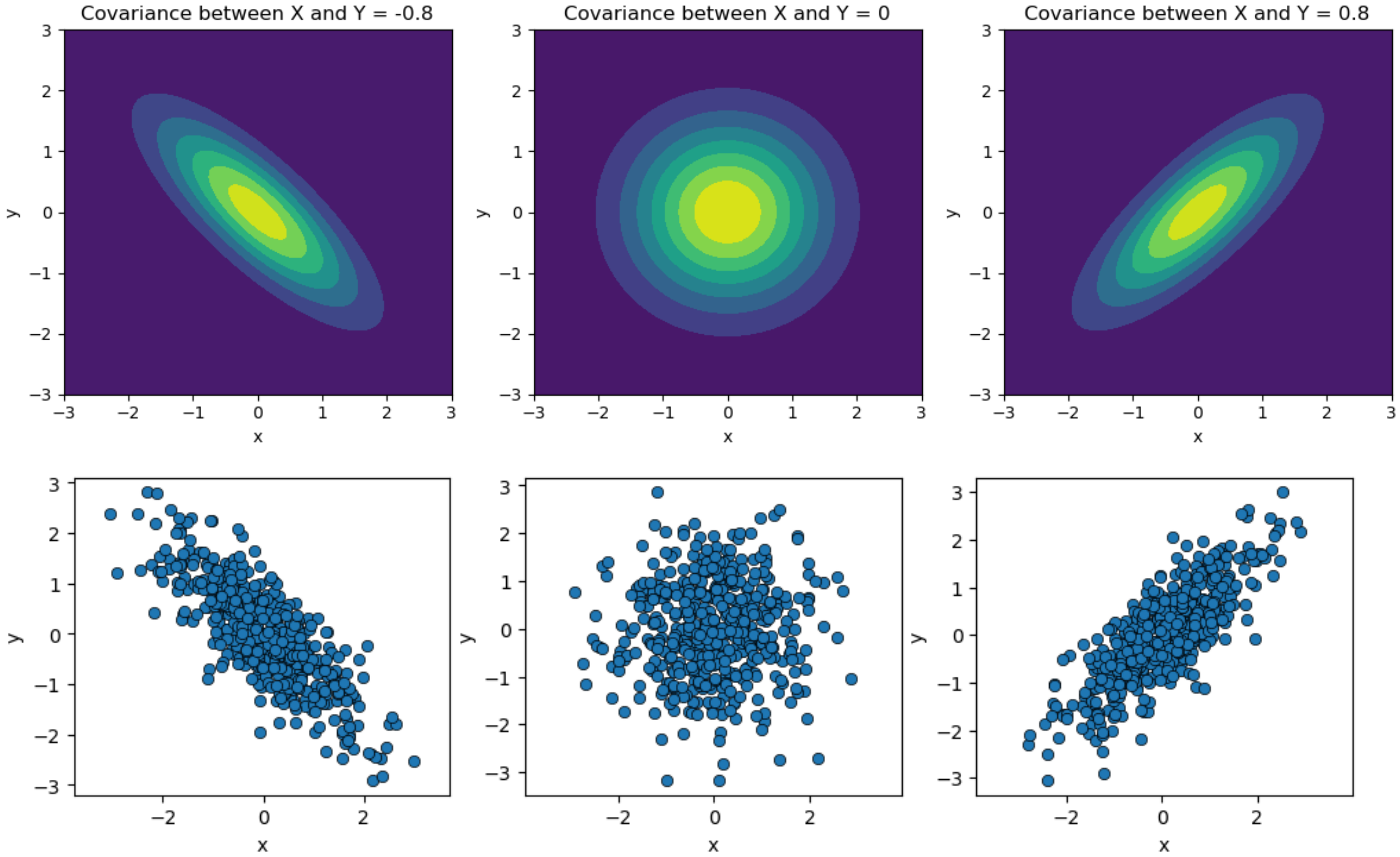
Joint Distributions

$$p(x, y) = P(X \in [x, x + dx] \text{ AND } Y \in [y, y + dy])$$

Bivariate pdf

Bivariate histogram

$N = 500$



Covariance & Correlation

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \left\langle \left(X - \langle X \rangle\right) \left(Y - \langle Y \rangle\right) \right\rangle$$

Covariance & Correlation

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \Big\langle \big(X - \langle X \rangle\big)\,\big(Y - \langle Y \rangle\big) \Big\rangle$$

$$Corr(X, Y) = \rho_{XY} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

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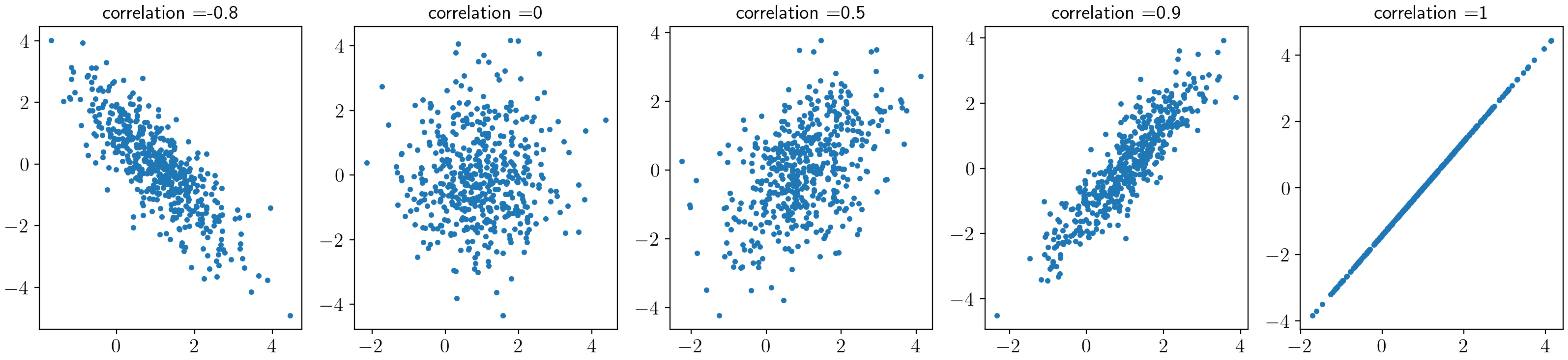

Covariance & Correlation

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$$Corr(X,Y) = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Important

Key Properties of Covariance

$$Cov(X,Y) = Cov(Y,x)$$

$$Cov(aX,Y) = a \cdot Cov(X,y)$$

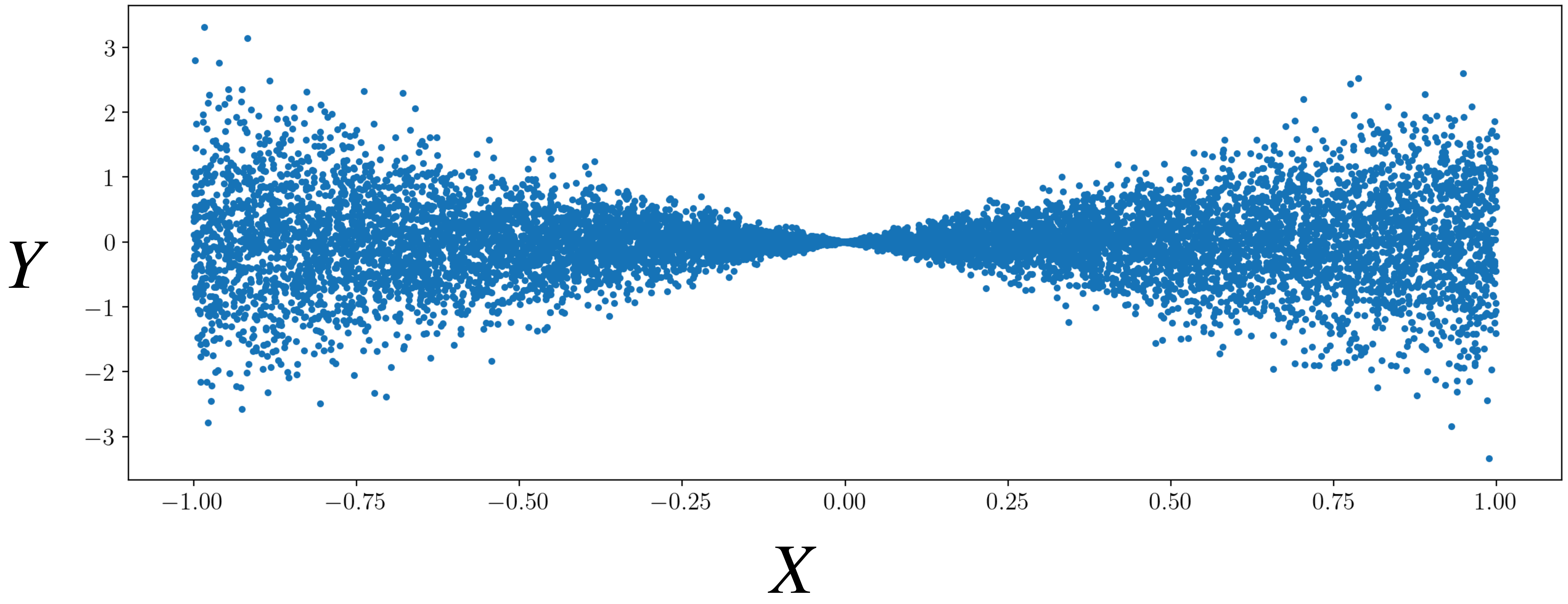
$$Cov(X,Y+Z) = Cov(X,Y) + COV(Y,Z)$$

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Correlation vs independence

$$X \sim U[-1,1]$$

$$Y|X \sim \mathcal{N}\left(0, X^2\right)$$



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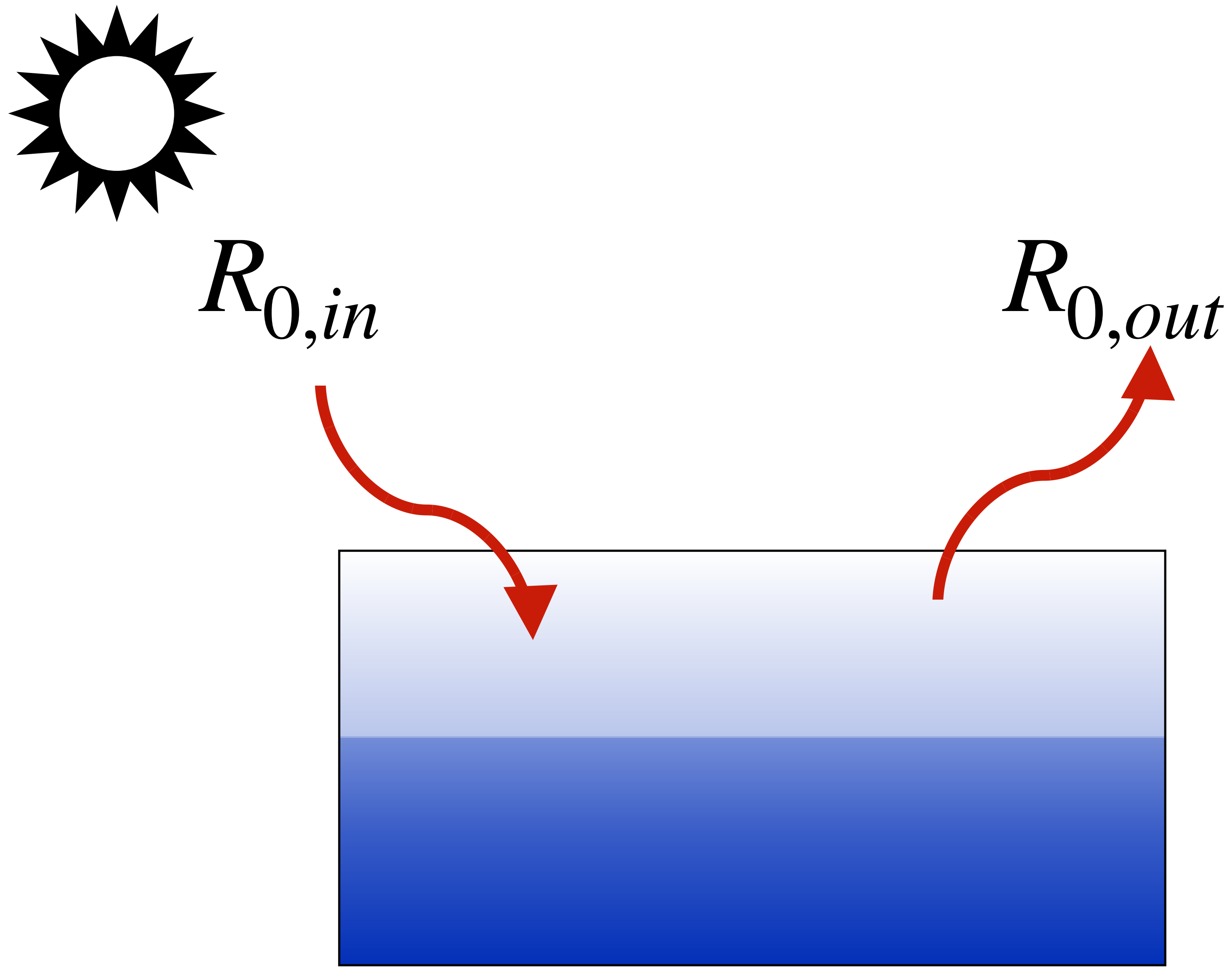
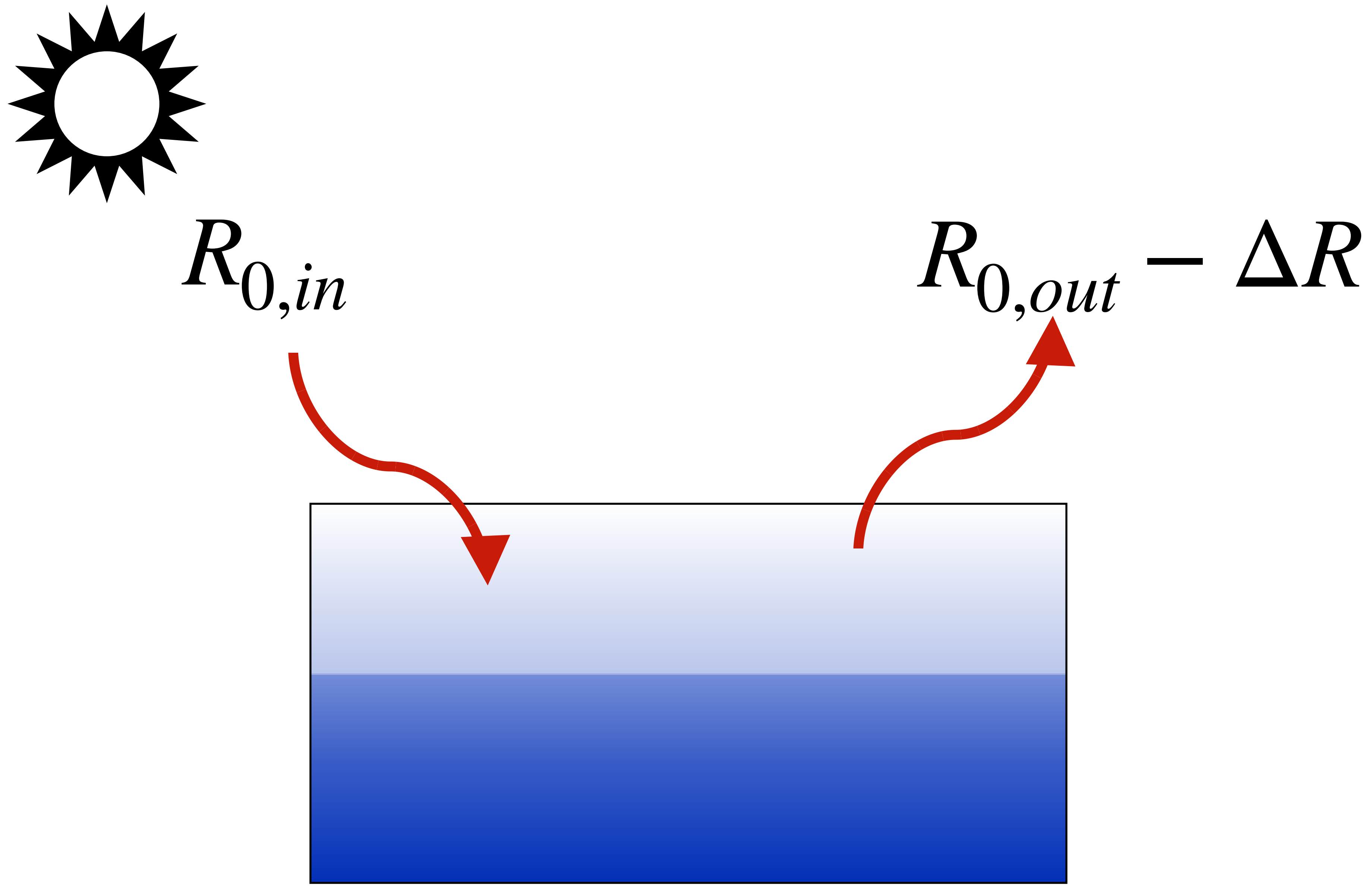
Earth's Energy Budget



Constant temperature (equilibrium)
Requires balanced energy budget

$$R_{in} + R_{out} = 0$$

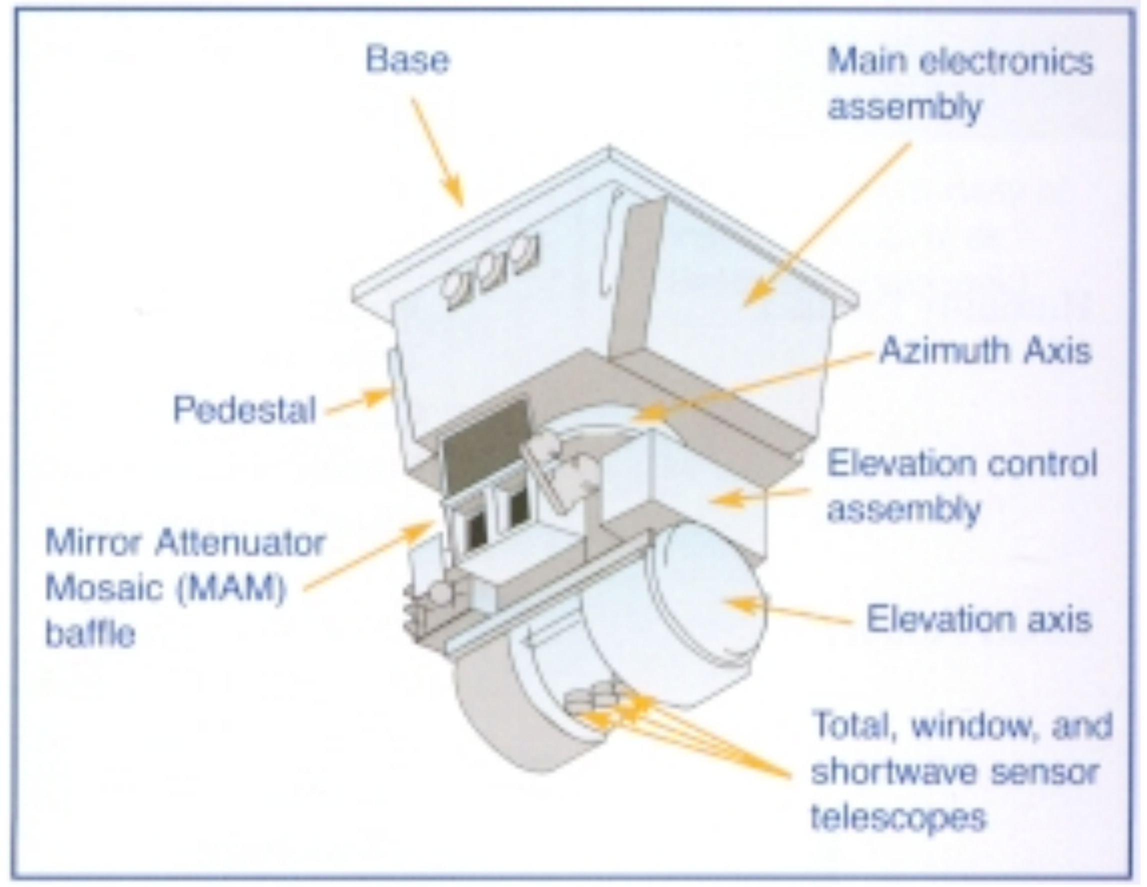
Earth's Energy Budget



Global warming:

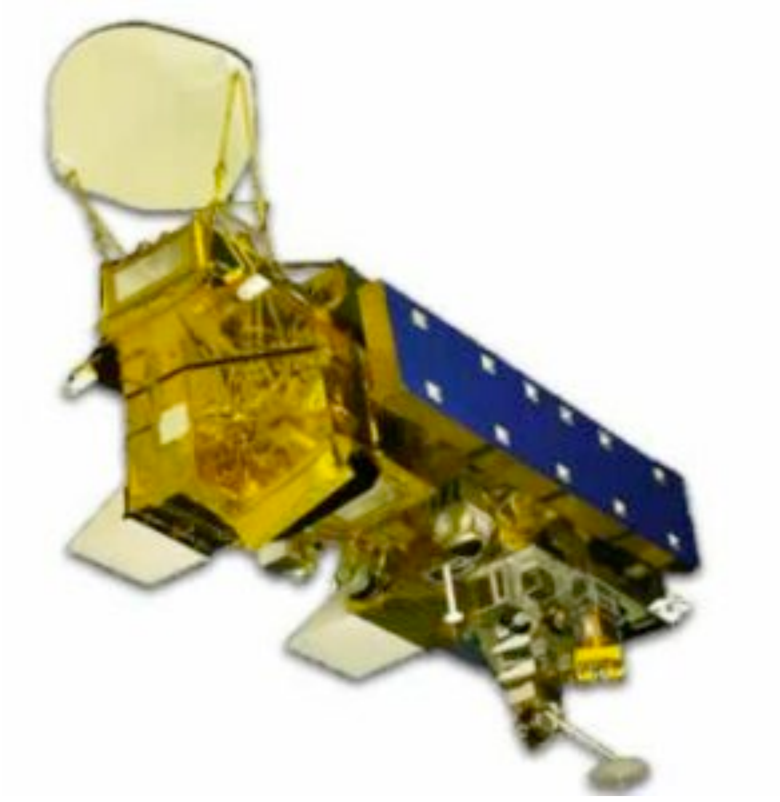
$$R_{in} + R_{out} = \Delta R > 0$$

Earth's Energy Imbalance



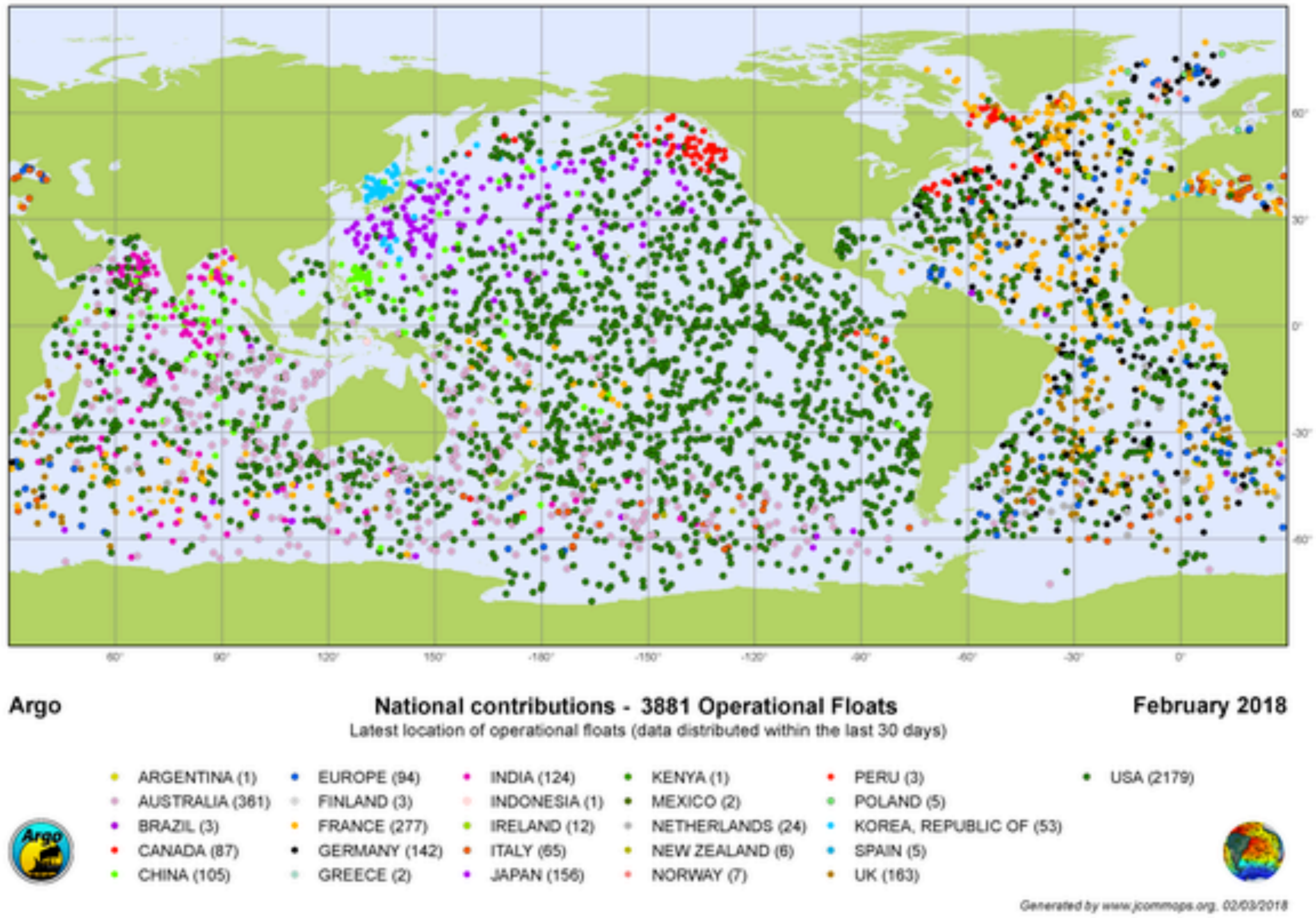
CERES instrument

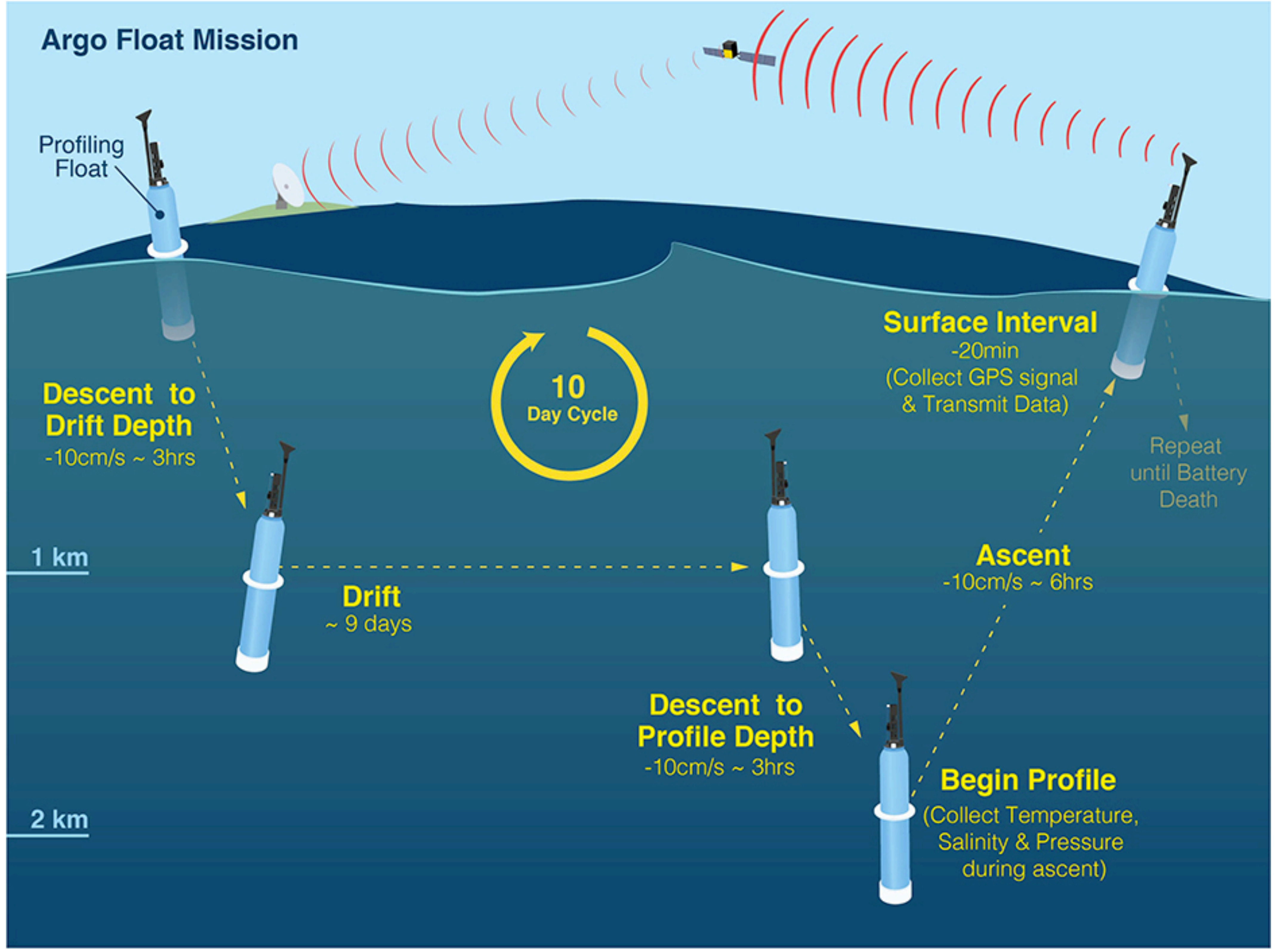
Terra satellite



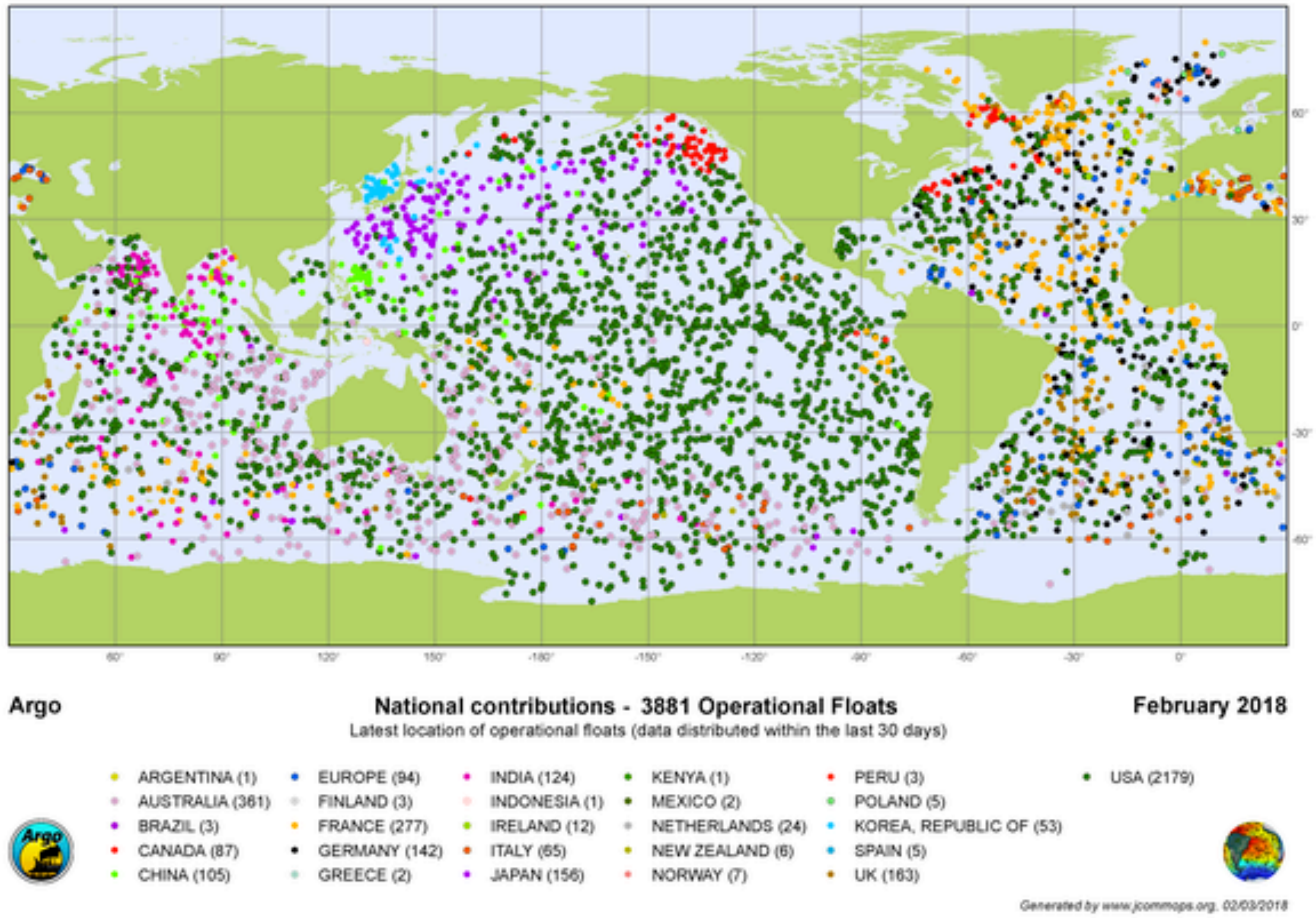
Aqua satellite

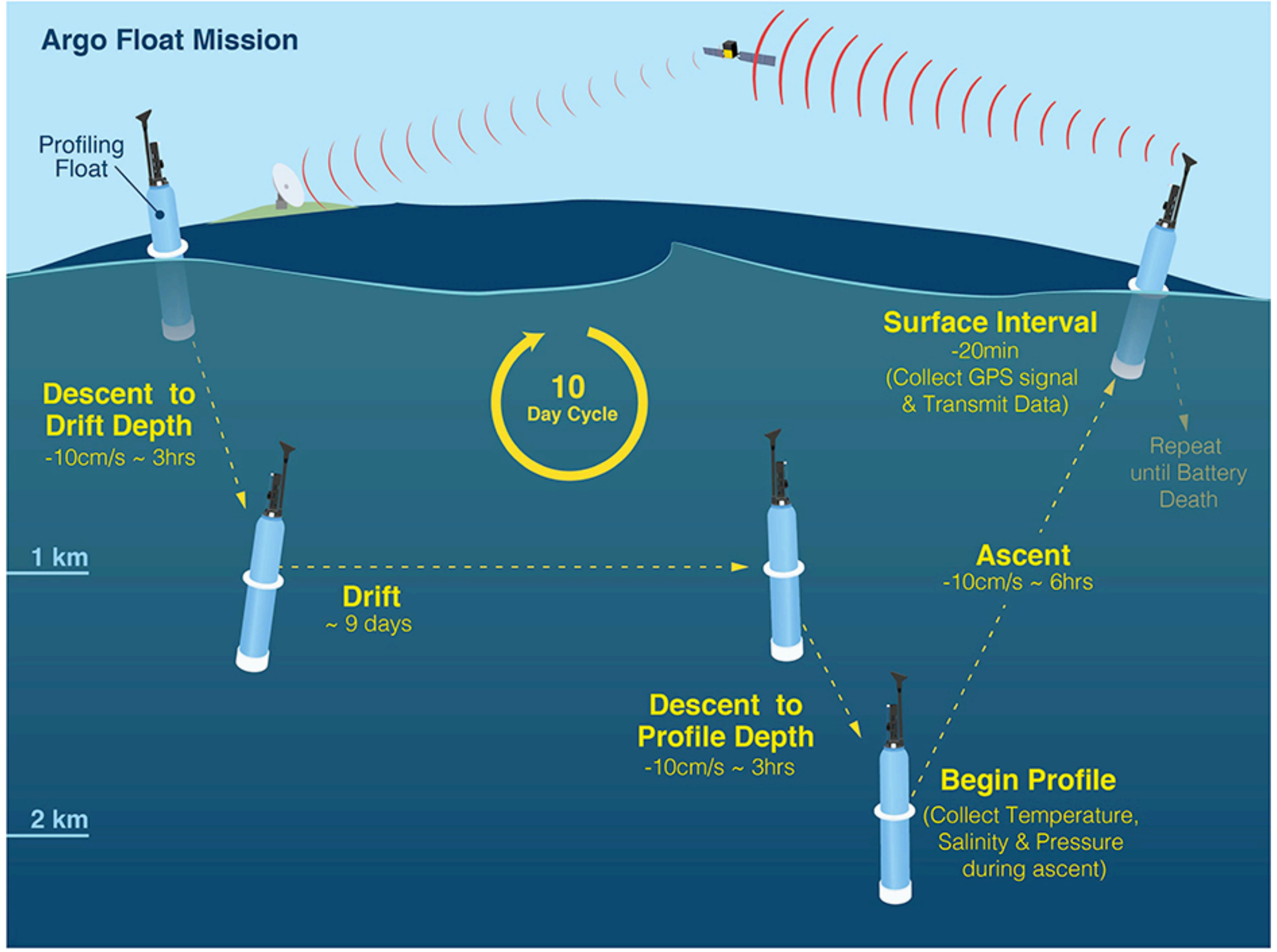
Earth's Energy Imbalance





Earth's Energy Imbalance





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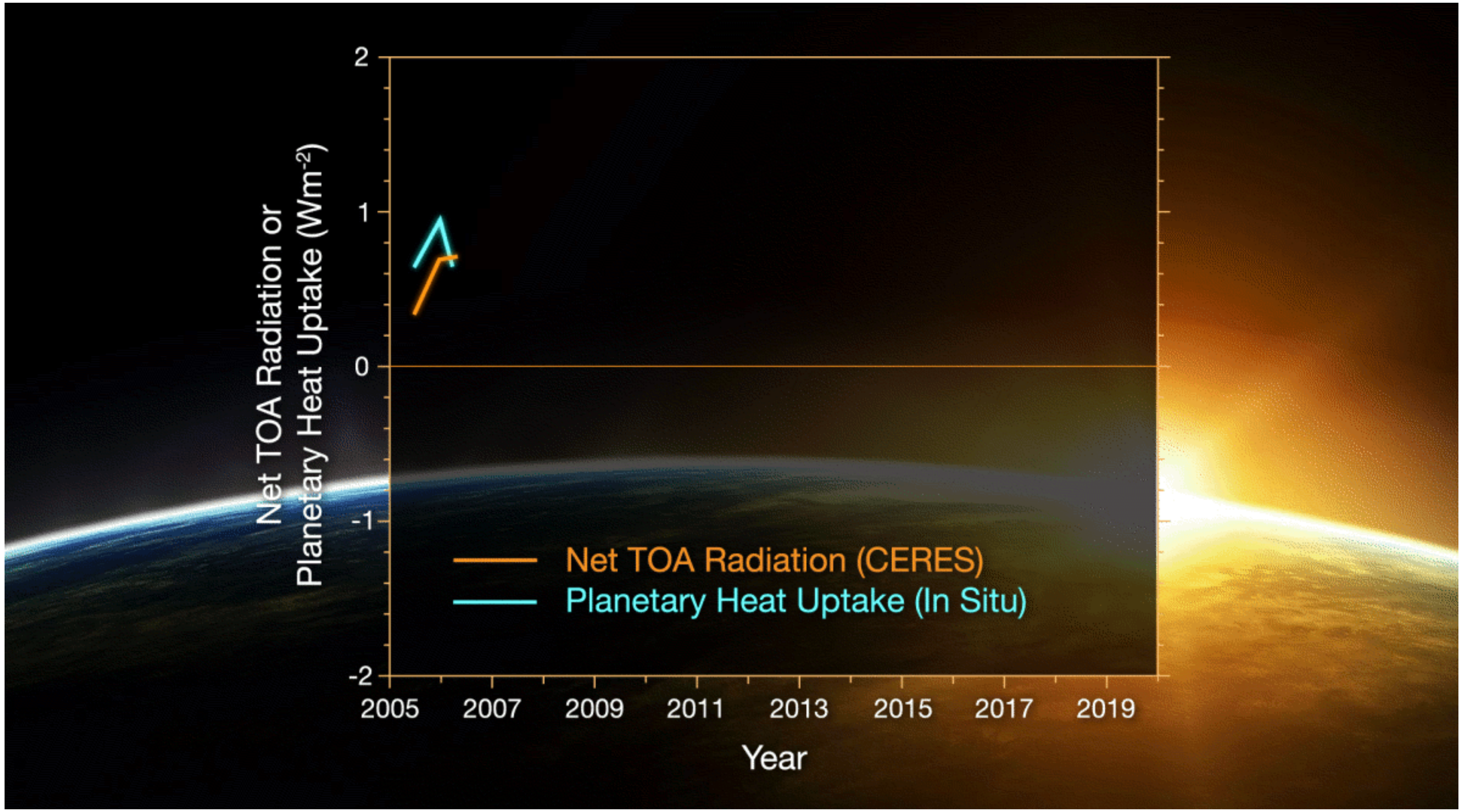

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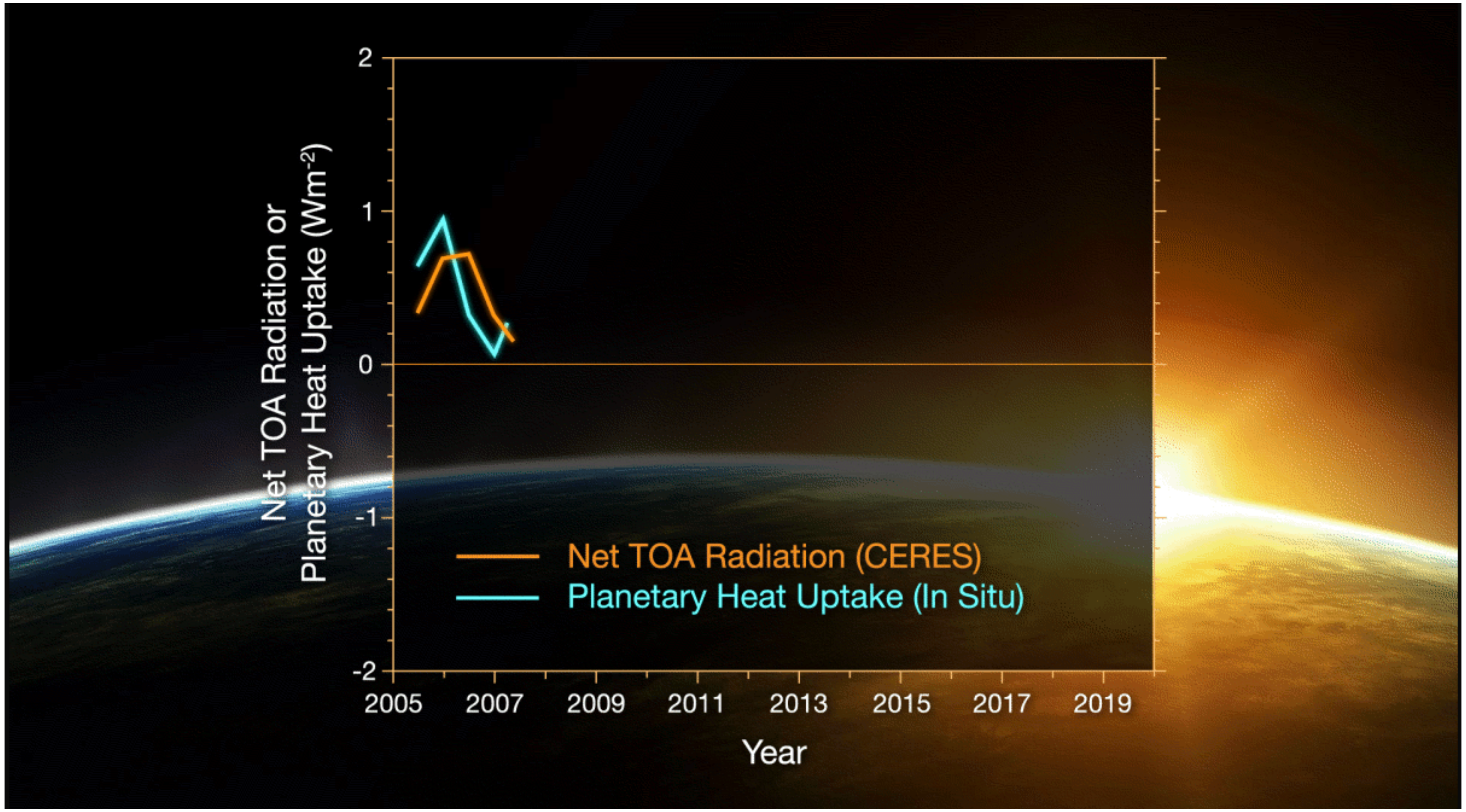

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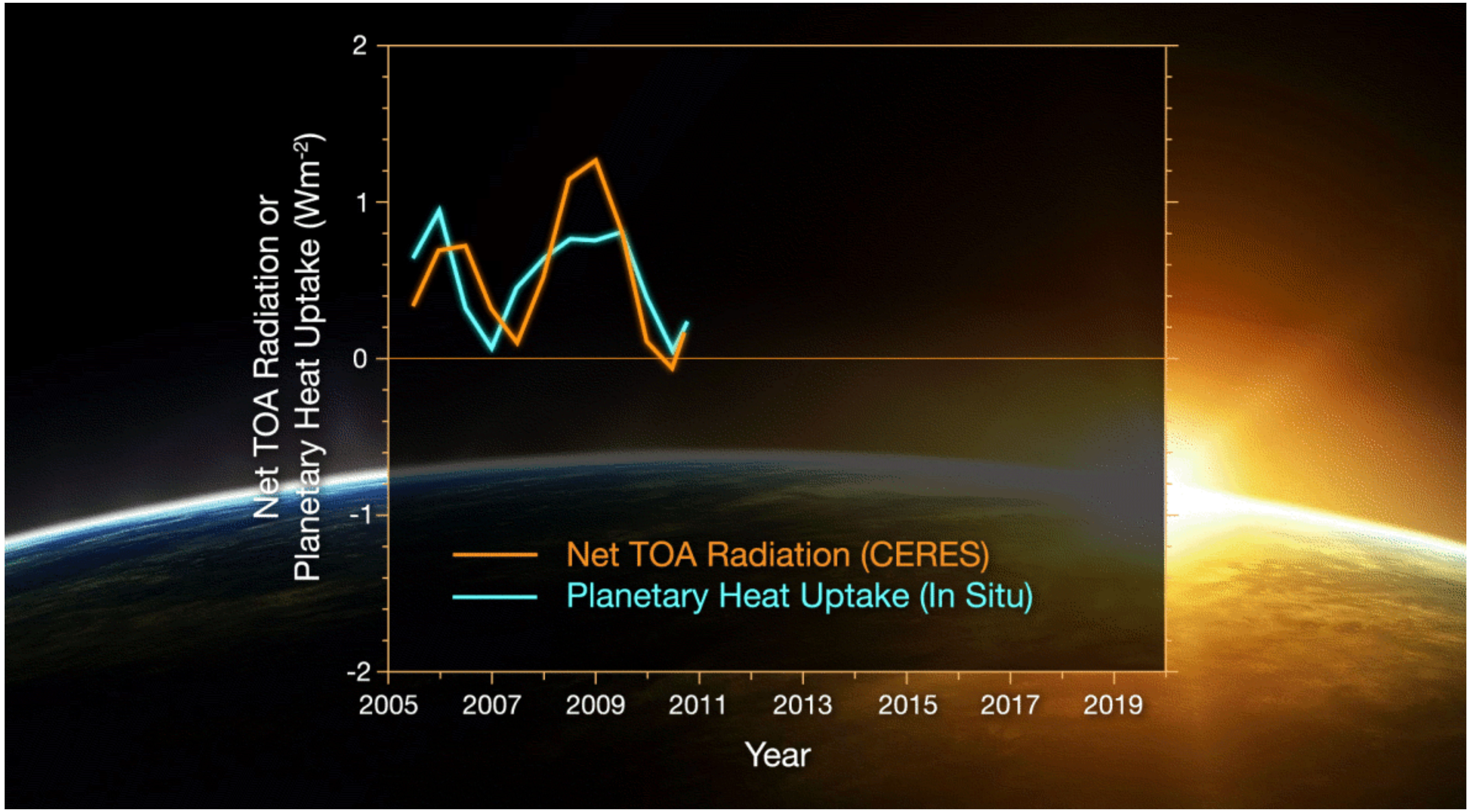

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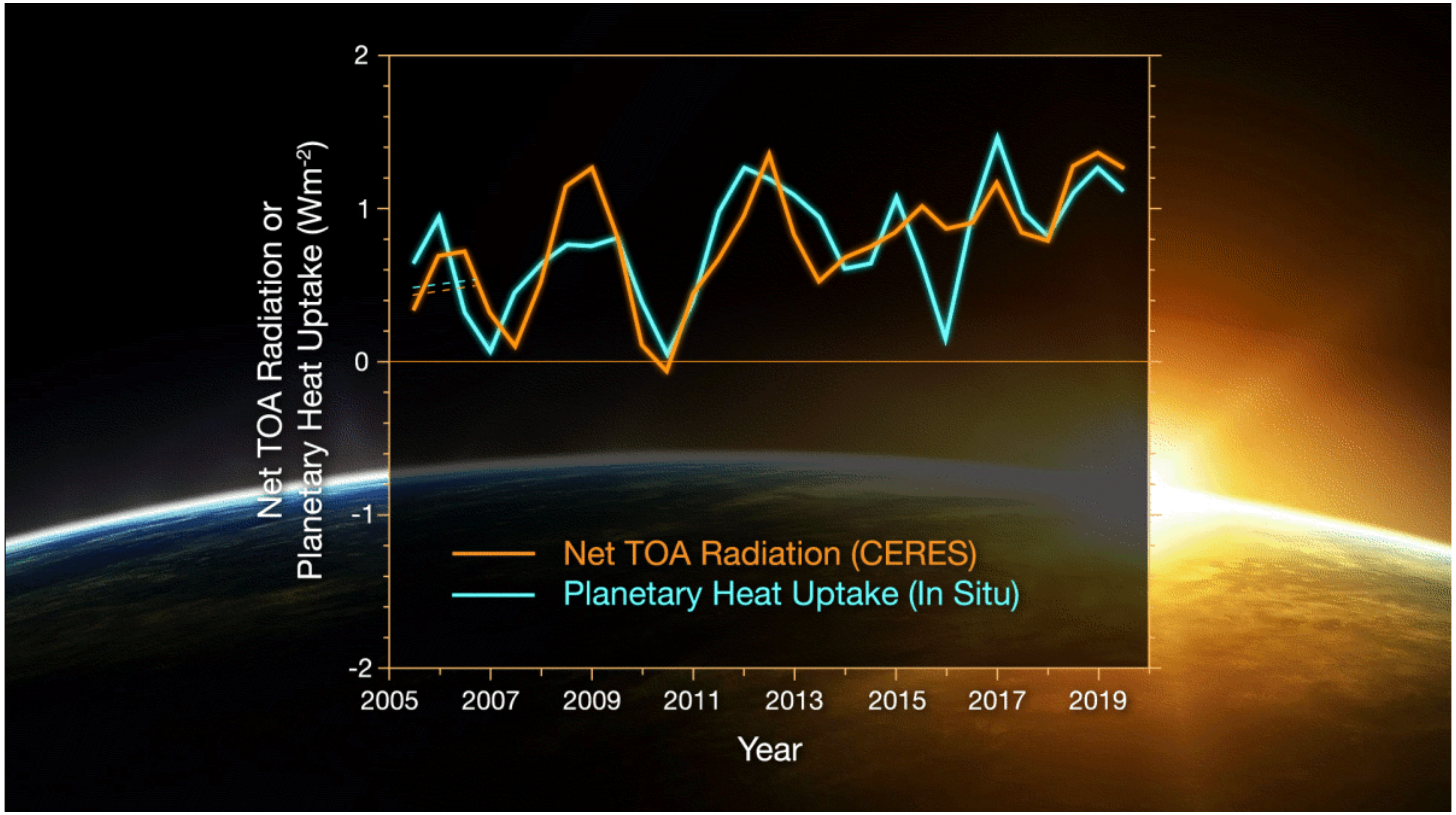

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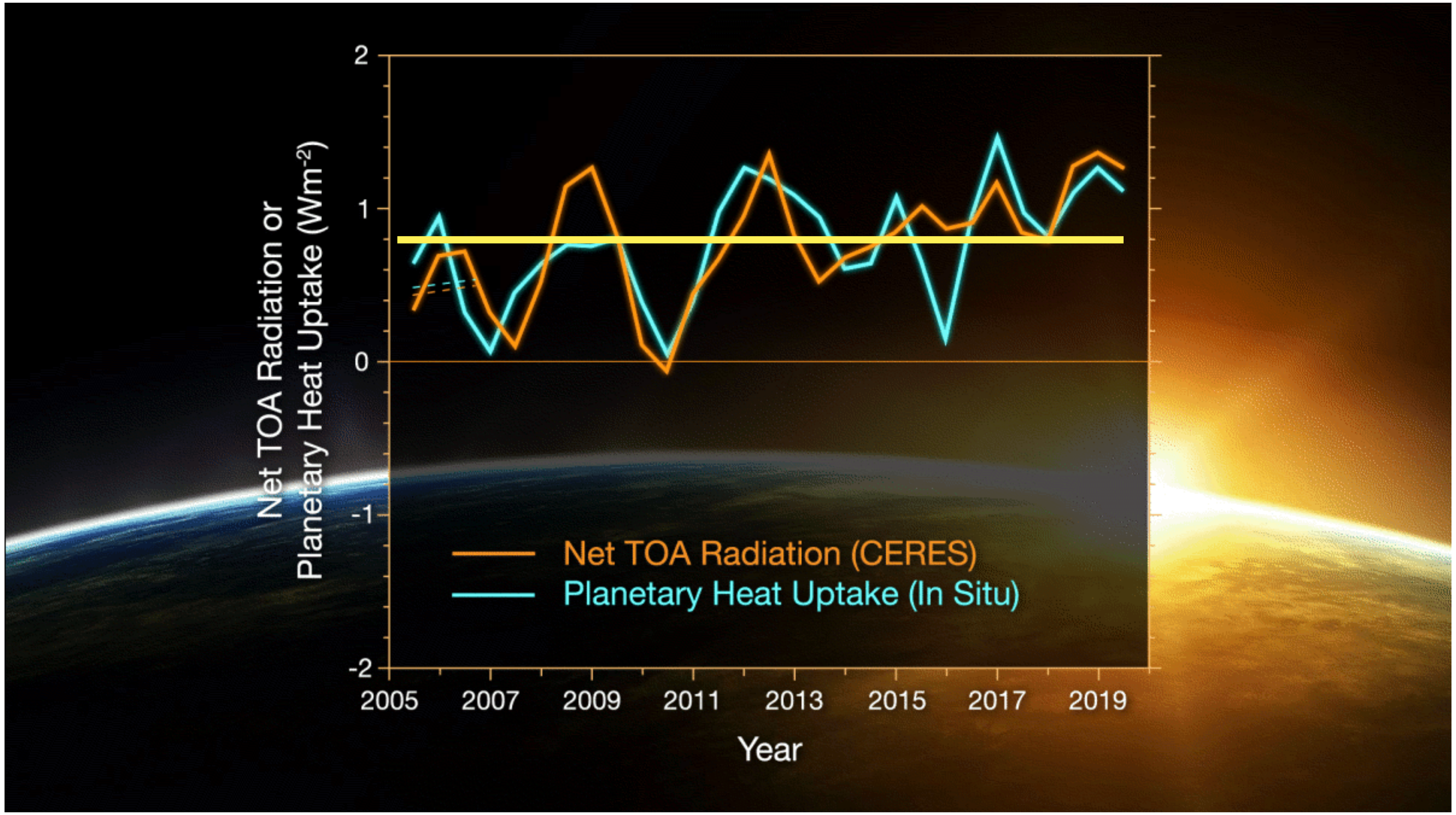
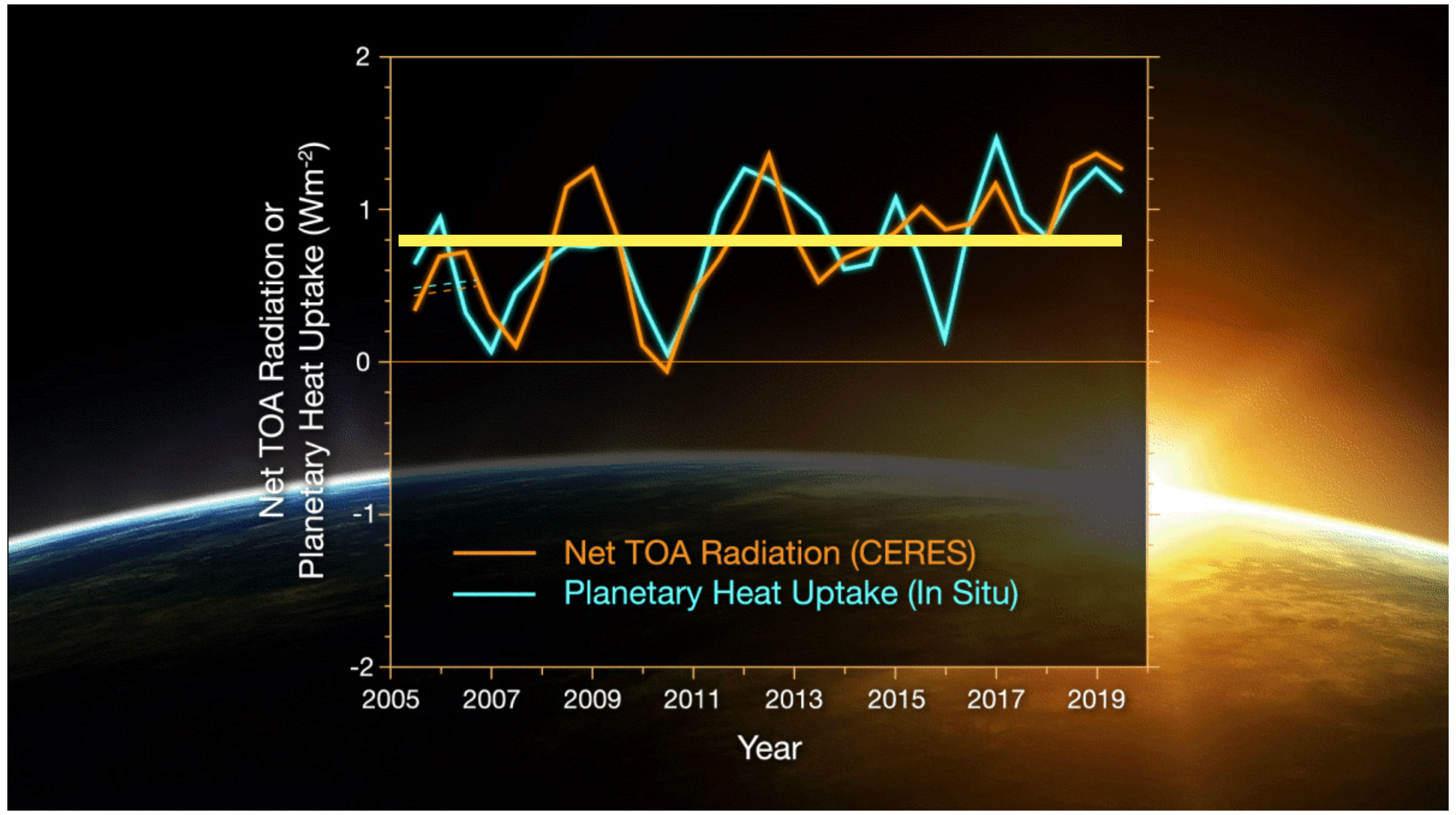


Model the data!



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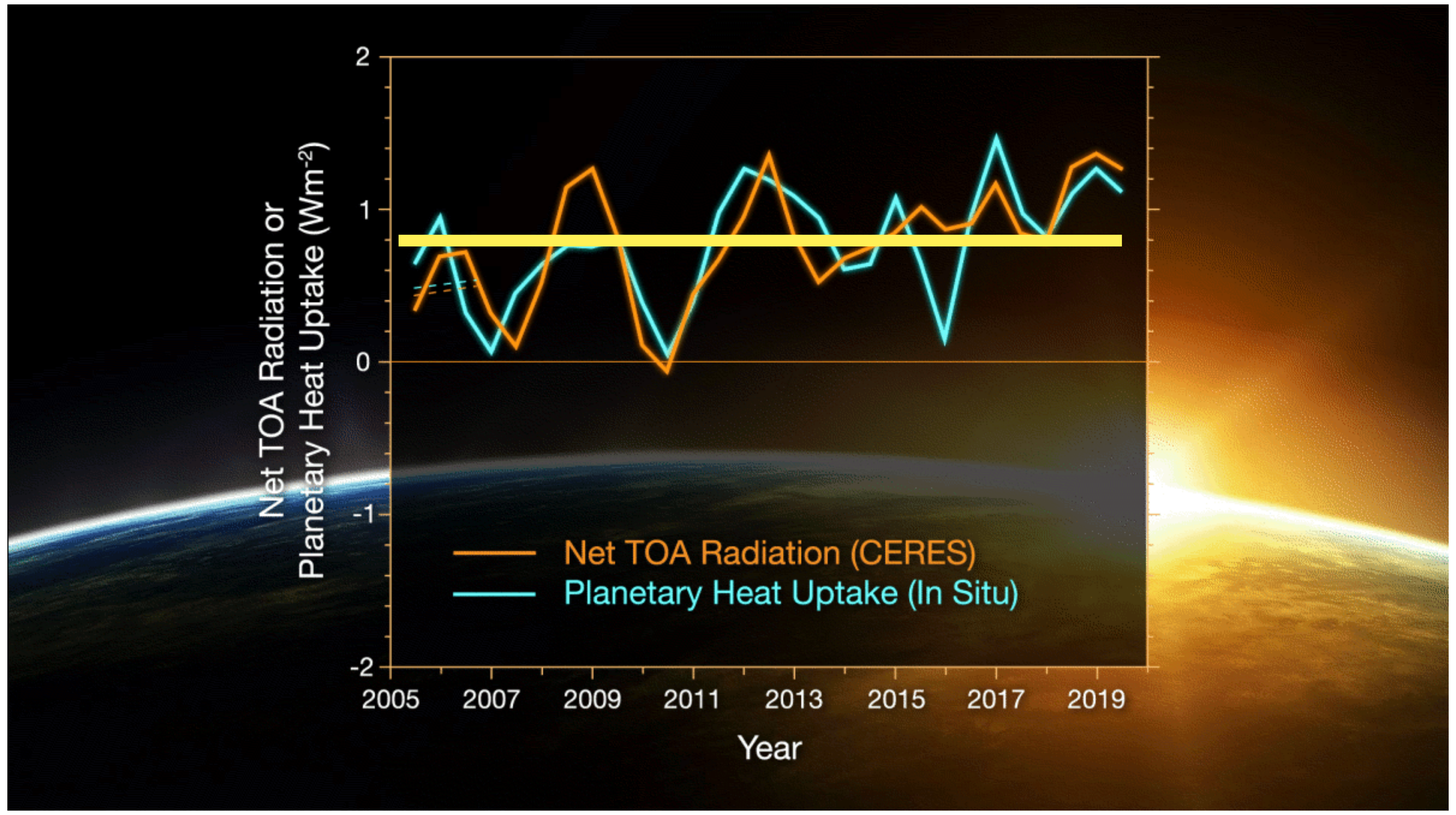


A probabilistic model for the measurements:

$$X = \mu + \varepsilon$$

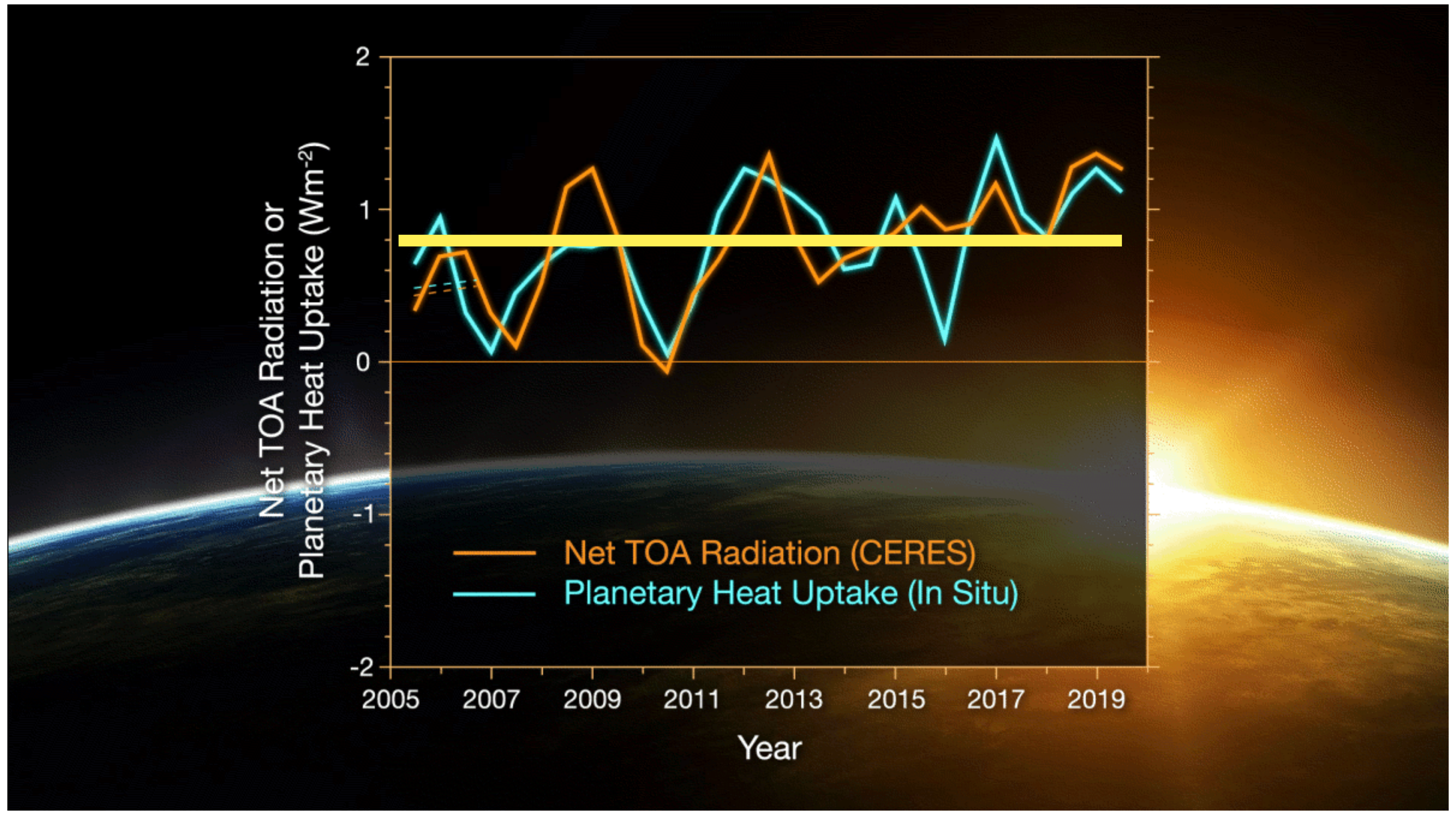
Quantity of interest (climate change component)

$$\mu$$

Random errors (weather, instrument error):

$$\varepsilon \sim \mathcal{N}(0, \sigma^2)$$

Estimate model parameters



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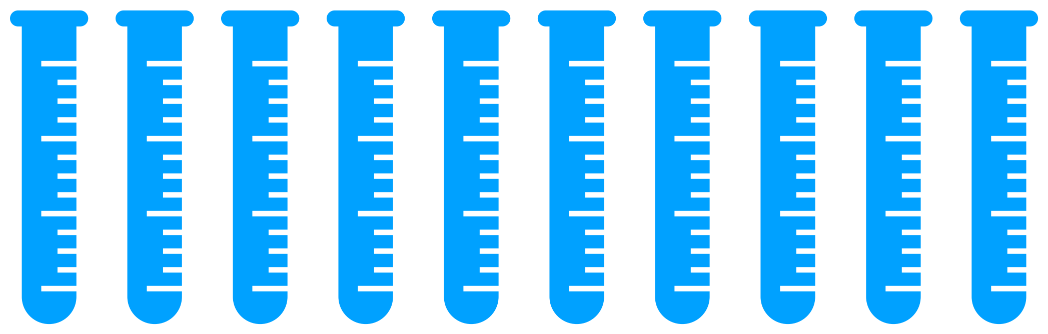
Estimator: Sample mean

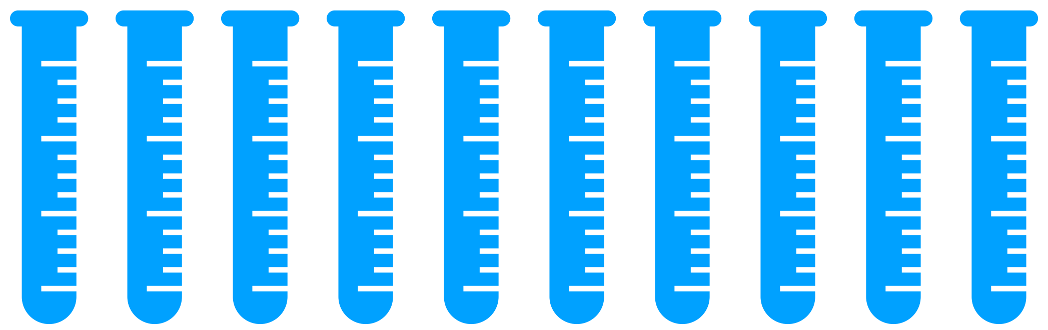
$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

A sample



N measurements

A sample



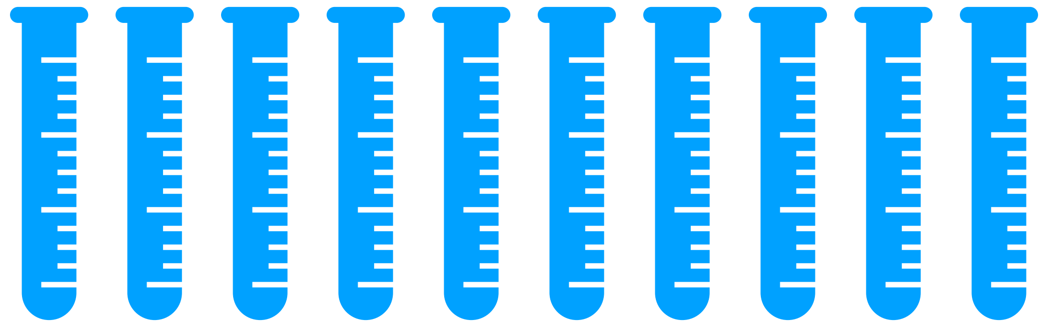
$x_1 \quad x_2 \qquad\qquad\qquad x_n$

N measurements

Sample mean

$$z = \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Many samples

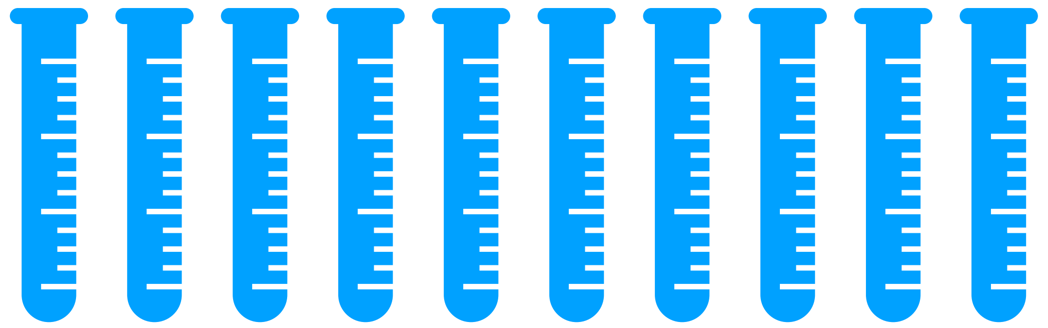


N measurements

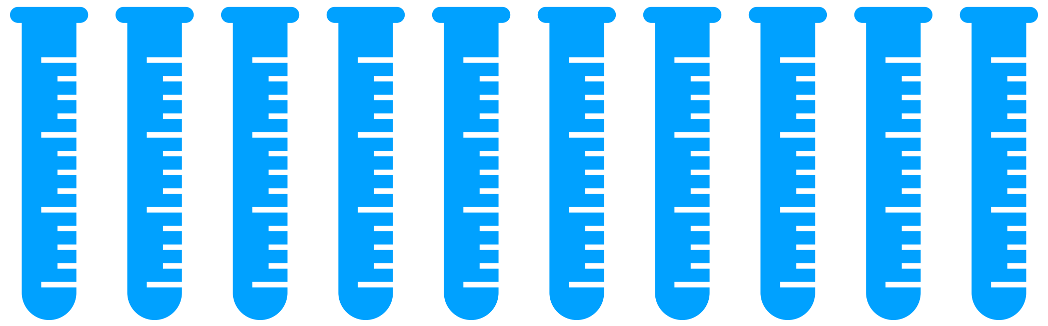
x_{11} x_{21} x_{n1}

Sample mean

$$z_1 = \bar{x}_{n1} = \frac{1}{n}\sum_{i=1}^{n} x_{i1}$$

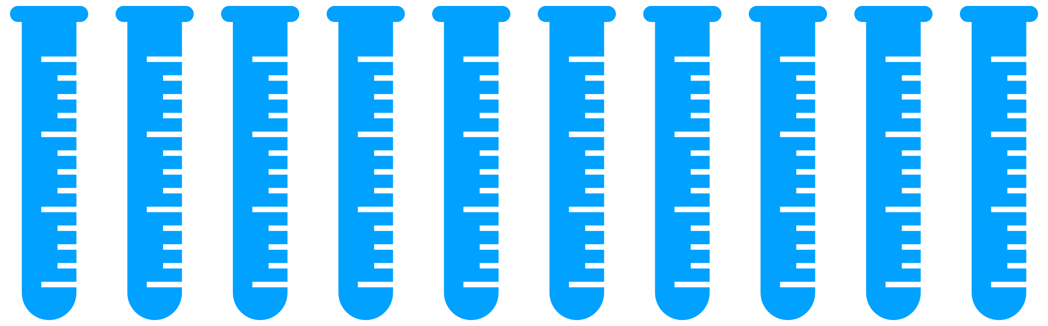


$$z_2 = \bar{x}_{n2} = \frac{1}{n}\sum_{i=1}^{n} x_{i2}$$



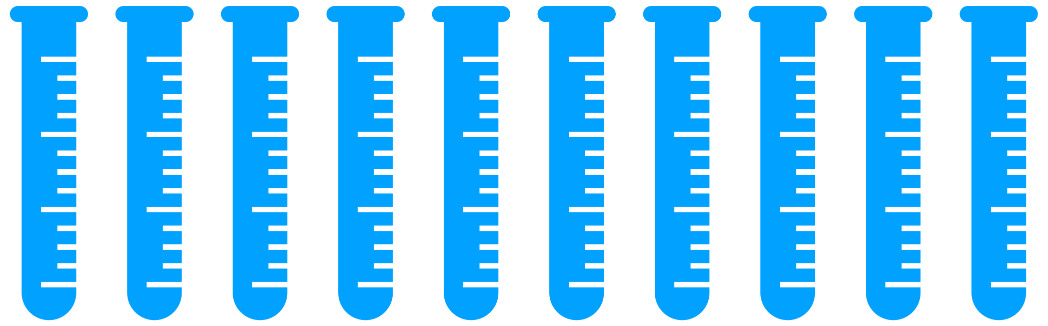
$$z_3 = \bar{x}_{n3} = \frac{1}{n}\sum_{i=1}^{n} x_{i3}$$

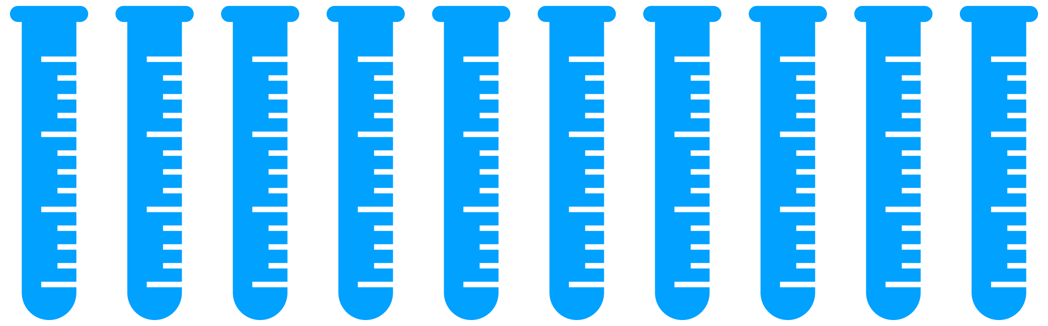
Sample mean is a random variable



X_1 X_2 X_n

N measurements





Sample mean is a random variable

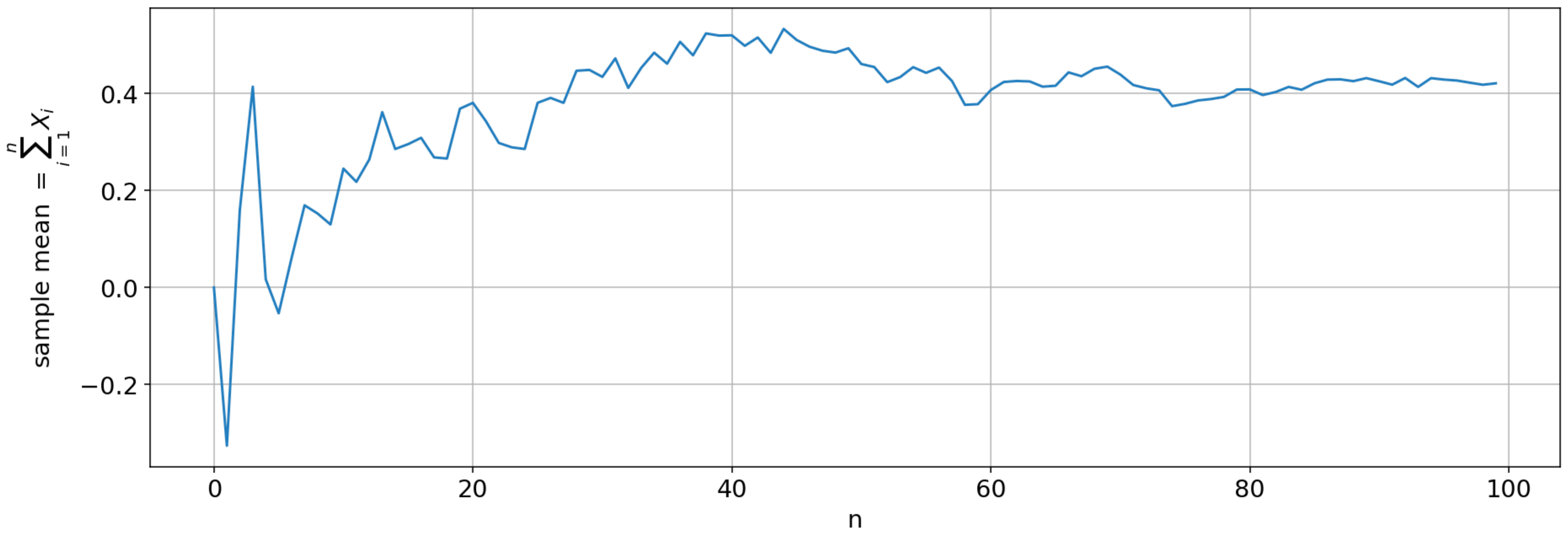
$$Z = \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_n$$

$$z_j = \overline{x}_{nj} = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$$

Law of large numbers

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$X = \mu + \varepsilon \sim \mathcal{N}(\mu, \sigma^2)$$



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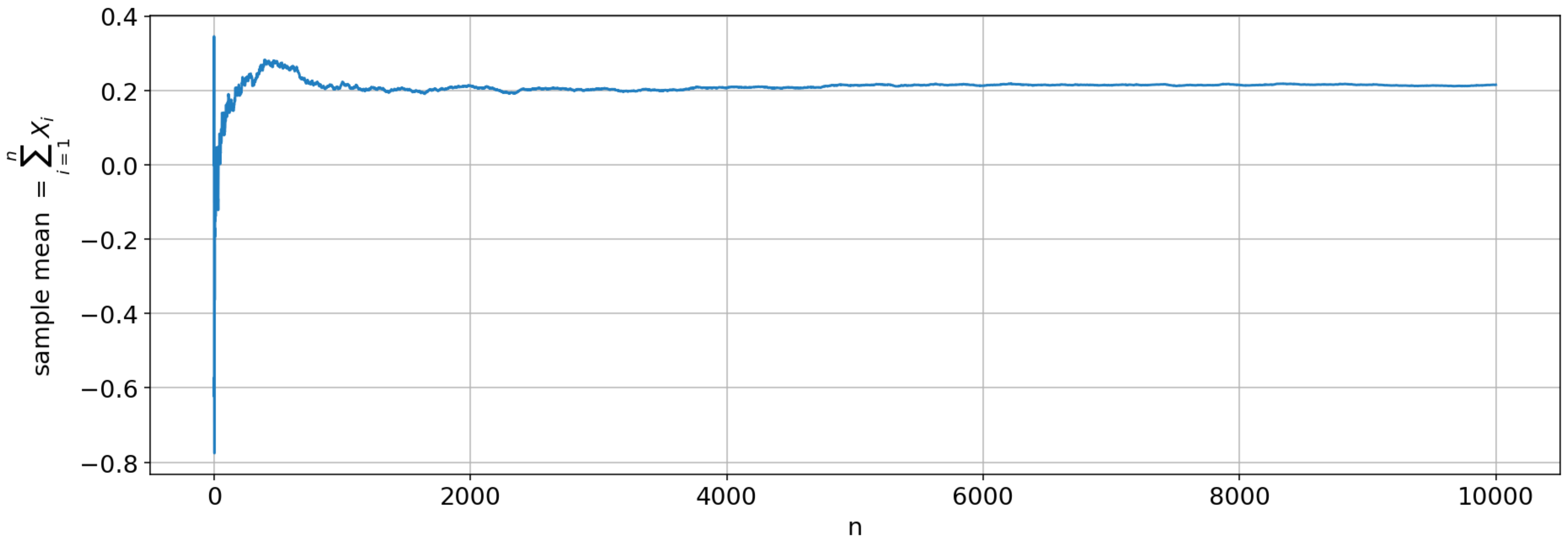

Central Limit Theorem

What is the distribution of $\overline{X}$?

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$X = \mu + \varepsilon \sim \mathcal{N}(\mu, \sigma^2)$$

Central Limit Theorem

$$\overline{X}_n \sim \mathcal{N}(\mu, \sigma^2/n)$$

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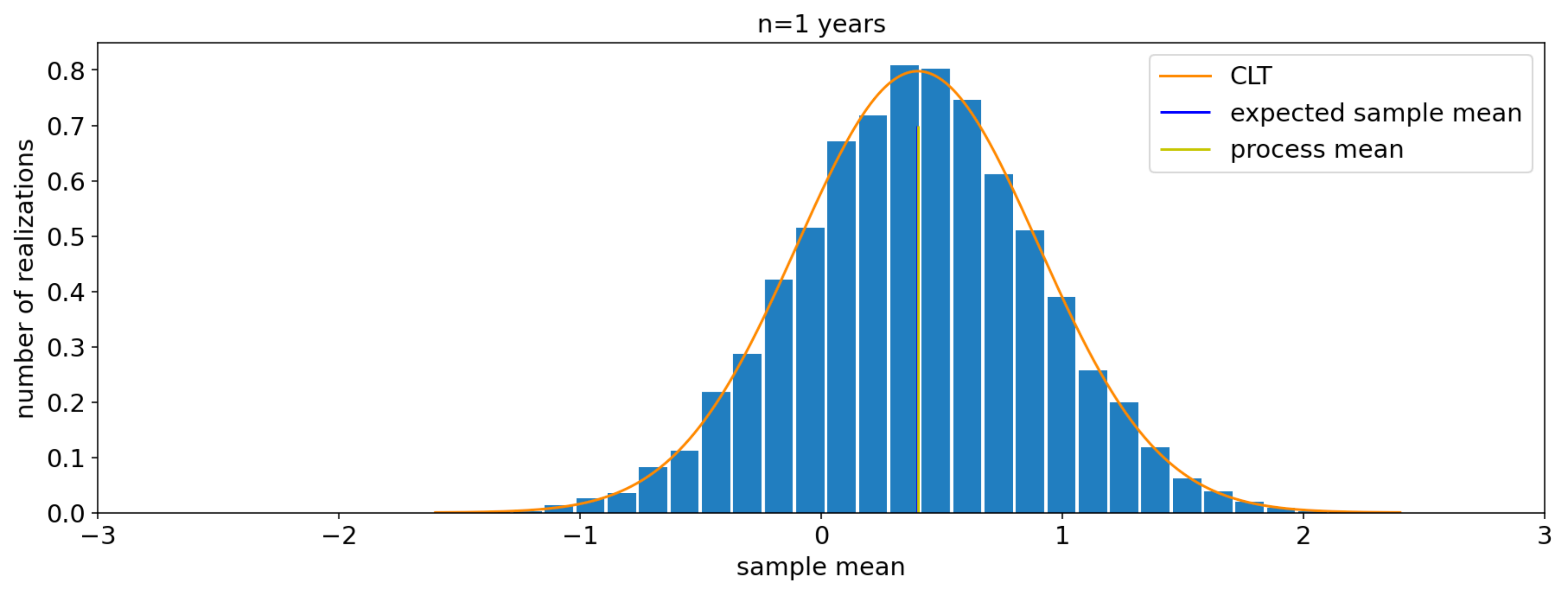

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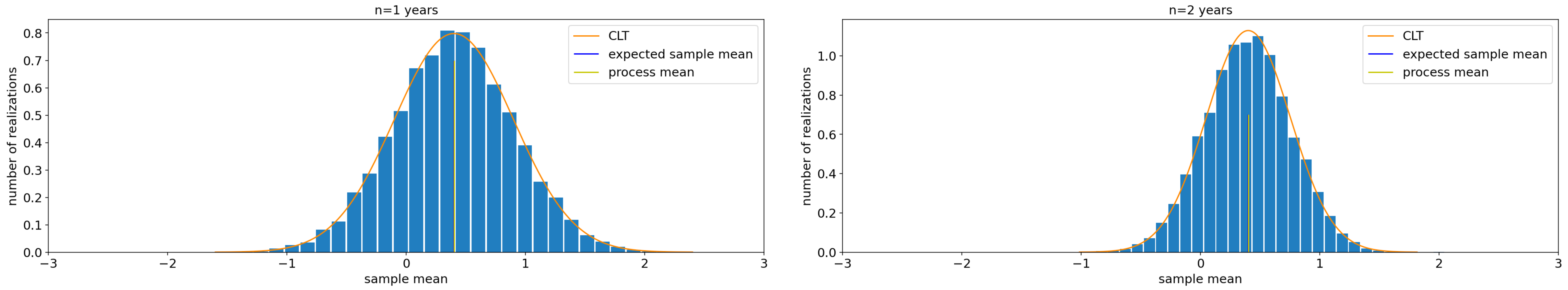
Central Limit Theorem

$$\overline{X}_n \sim \mathcal{N}(\mu, \sigma^2/n)$$



Central Limit Theorem

What is the distribution of $\bar{X}$?

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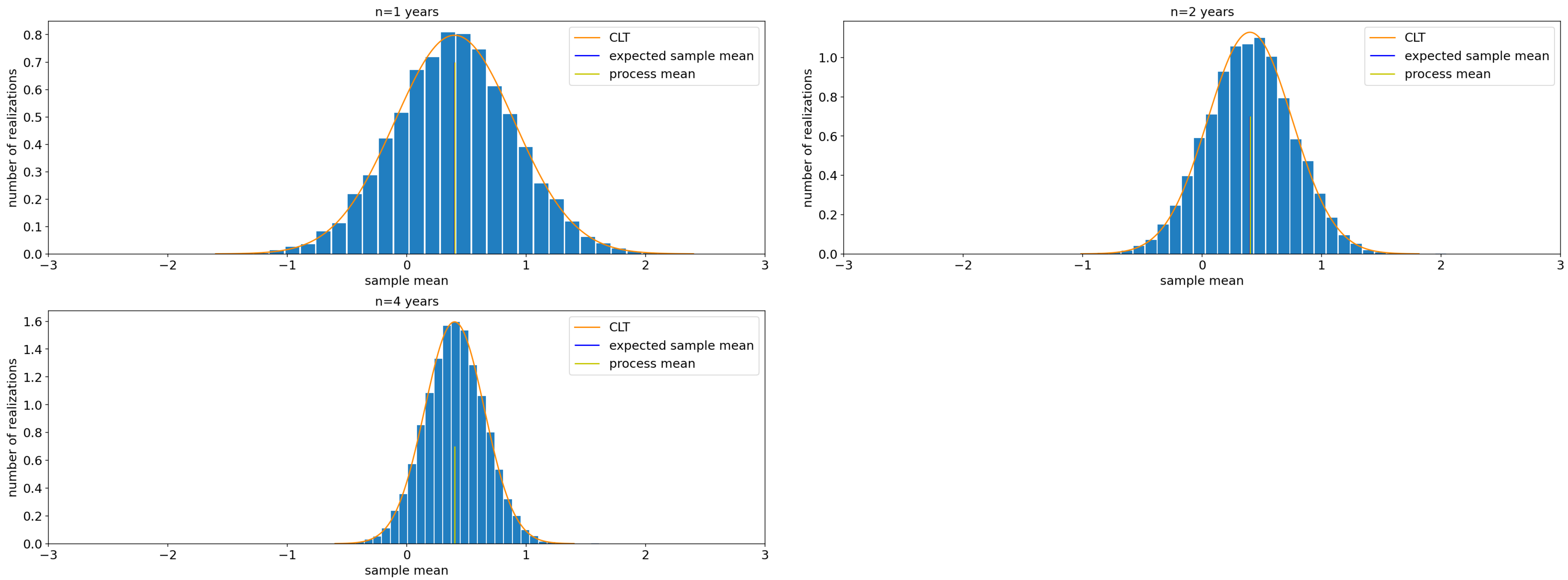

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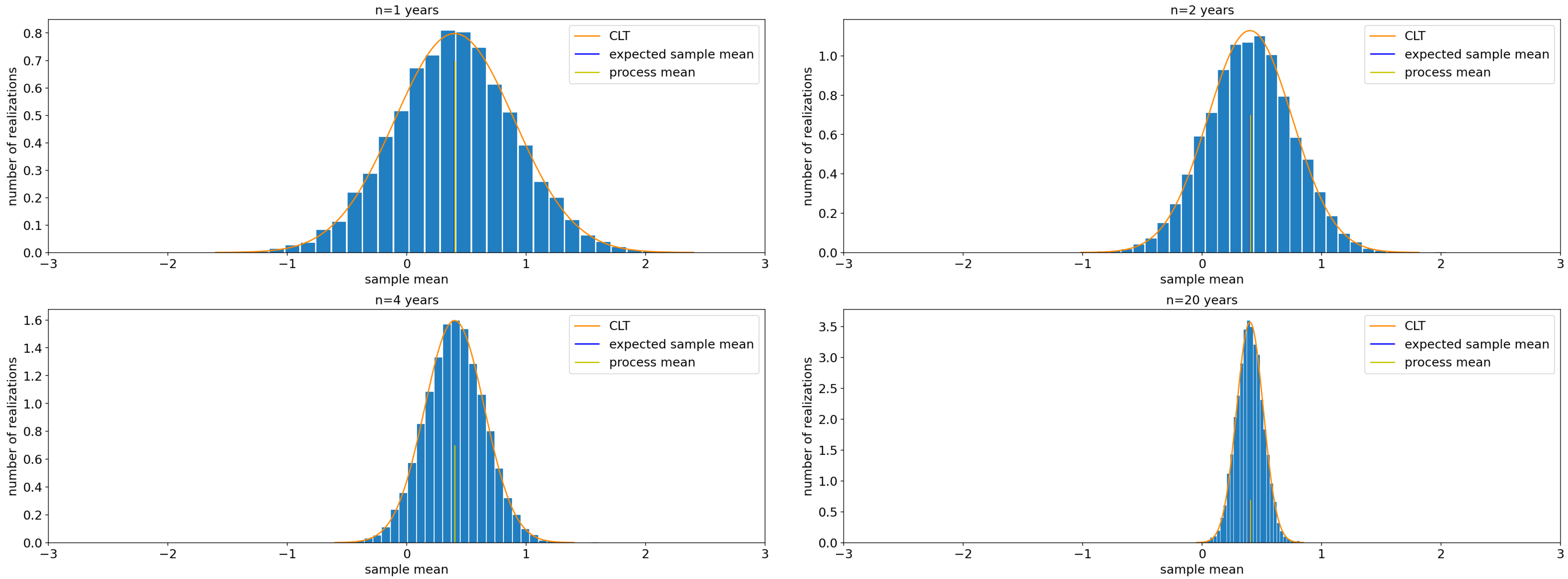

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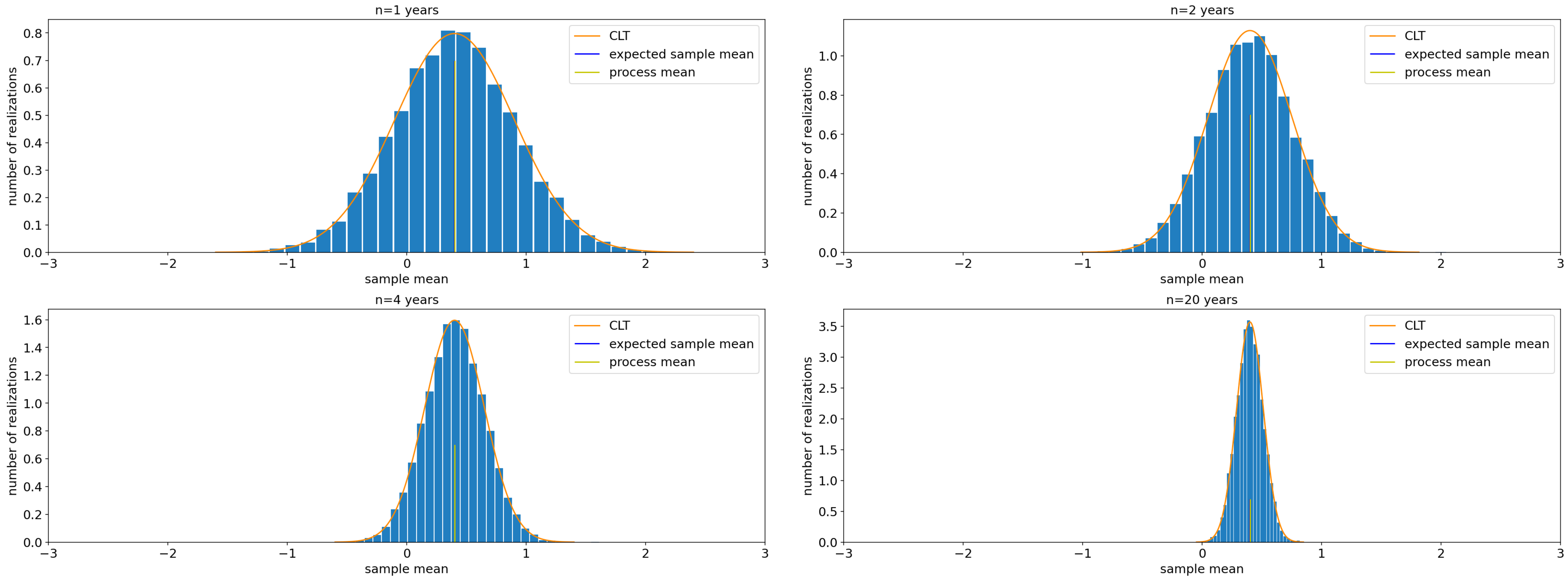

But what is σ?!

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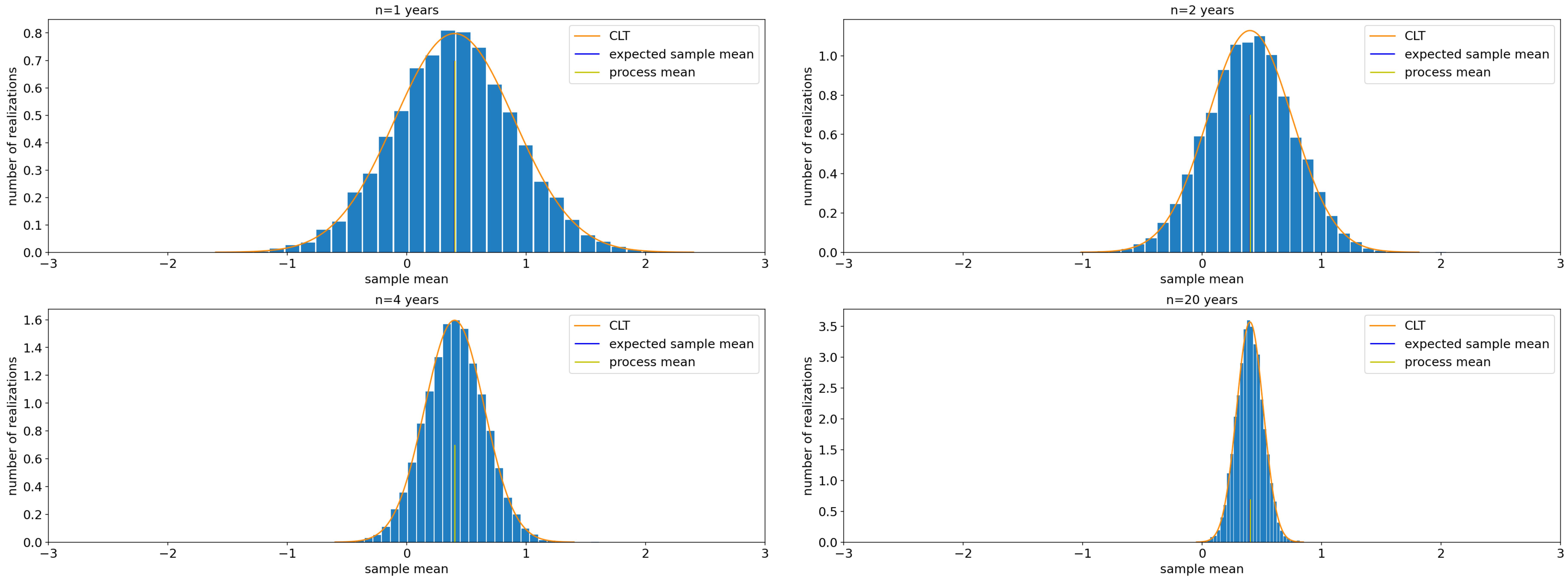

Sample Variance

$$s_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

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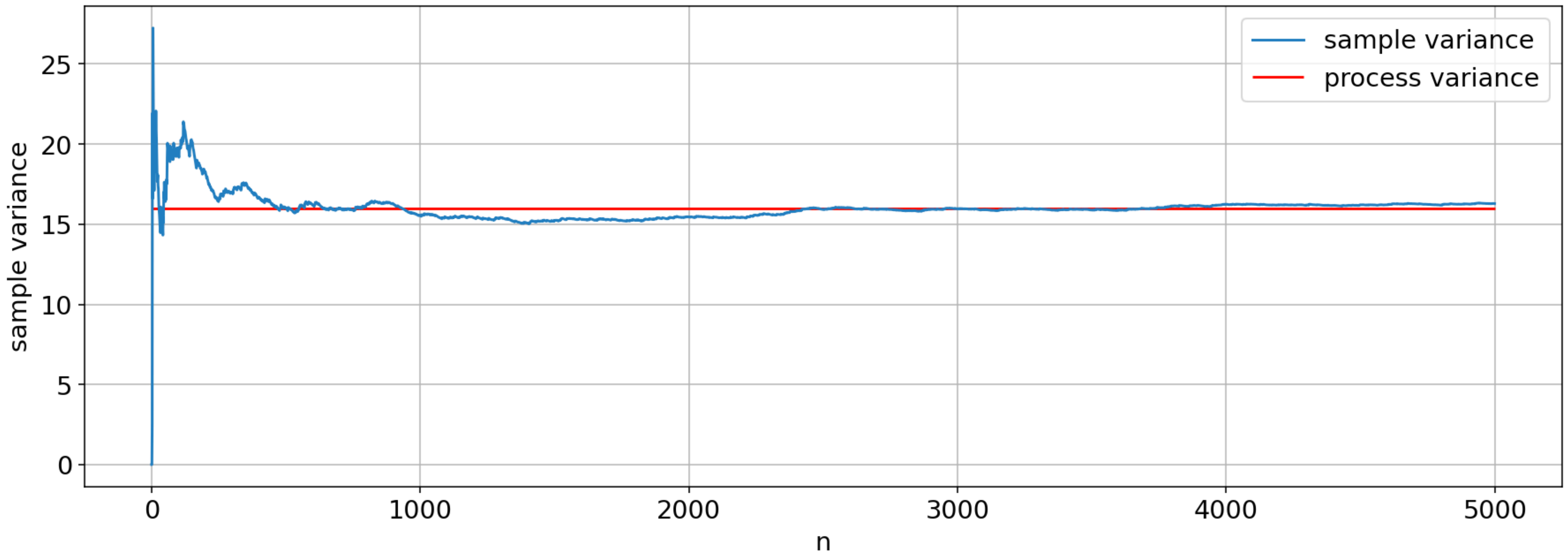

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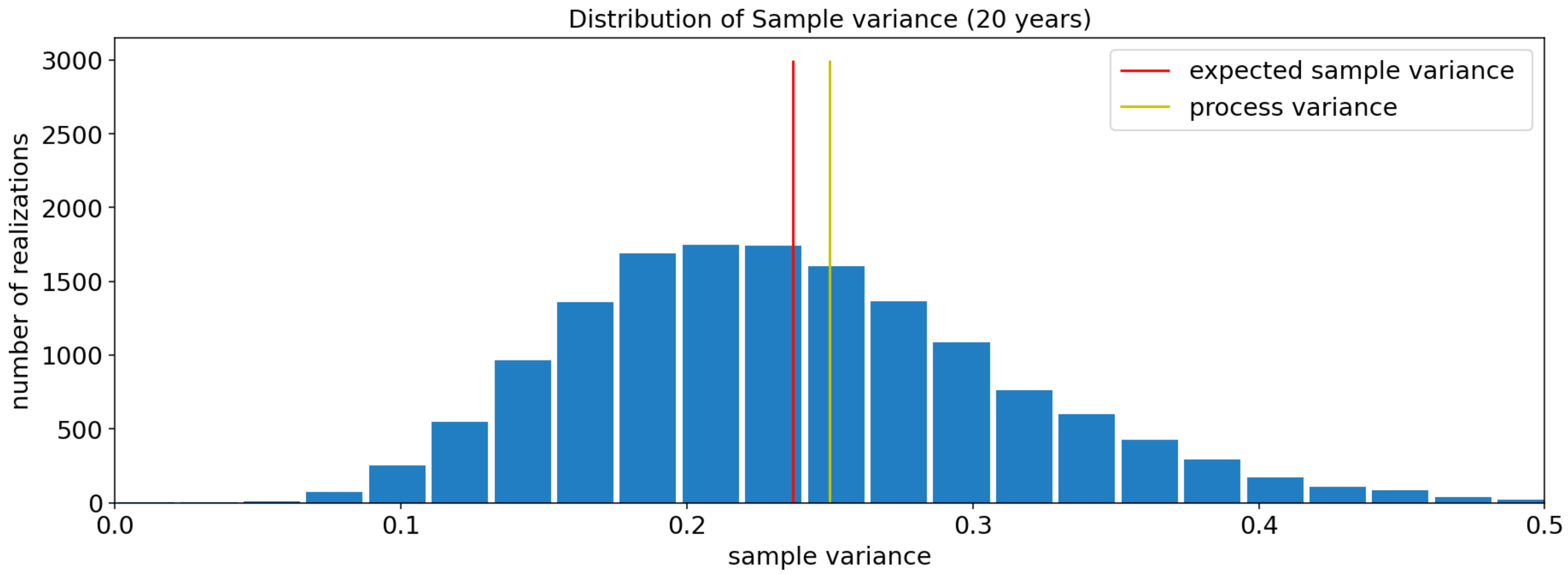
Biased



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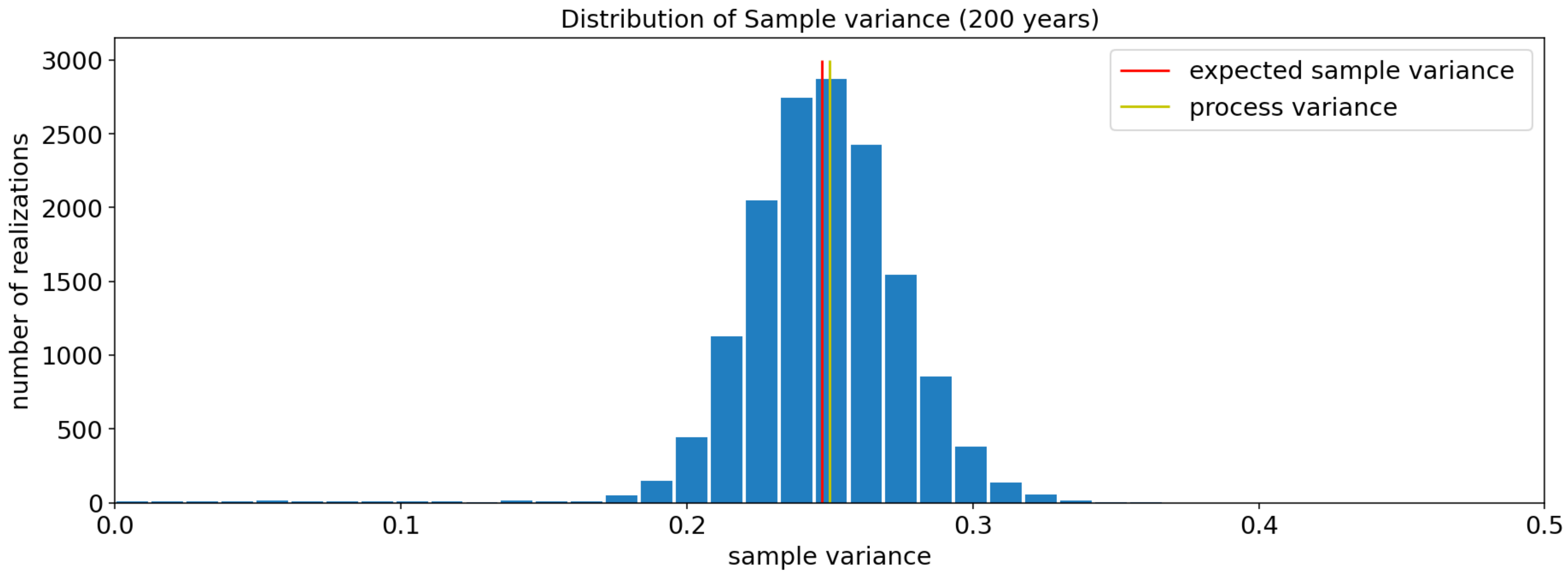
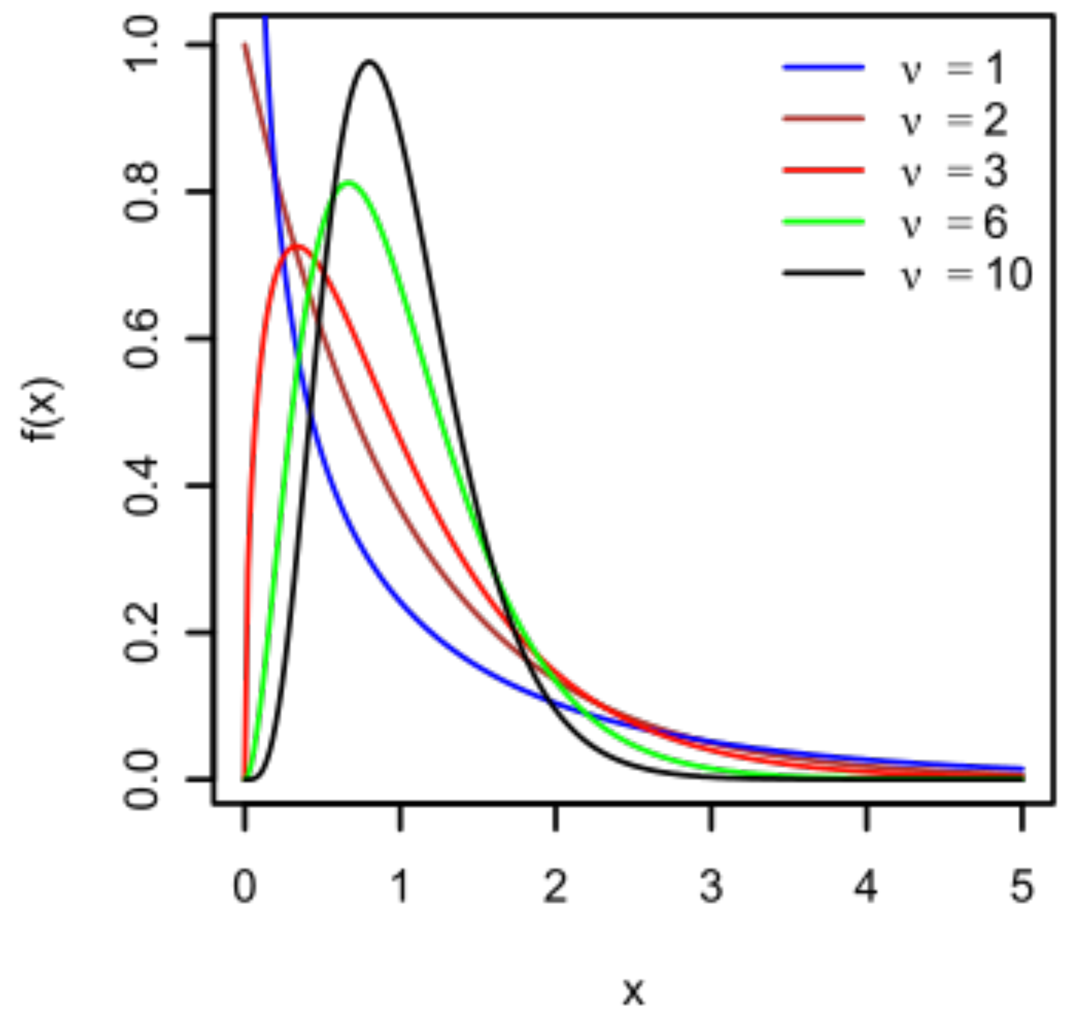


Sample Distributions:

Sample mean: $\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma_X}{n}\right)$

$$\bar{X}_n \sim \frac{\sigma_X}{\sqrt{(n)}}\mathcal{N}(\mu,1)$$

Sample variance: $s_{n-1} \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$



Sample mean with unknown variance

$$\bar{X}_n \sim \frac{s_{n-1}}{\sqrt{(n)}}\mathcal{N}(\mu,1/n)$$