Statistical **I**nference and **M**achine **L**earning in **E**arth **S**cience
SIMLES

Module 1

Basics

Lecture 1

Discrete Probabilities and RVs

Cristi Proistosescu

Probability: Fair Die

Sample space: (Set of all possible outcomes/rolls)	$\mathcal{S} = \{$	⚀	⚁	⚂	⚃	⚄	⚅ $\}$
Probability:	$P:$	1/6	1/6	1/6	1/6	1/6	1/6

Probability: Fair Die

Sample space: (Set of all possible outcomes/rolls)	$\mathcal{S} = \{$  $\}$
Probability:	$P:$ 1/6 1/6 1/6 1/6 1/6 1/6

Event: a subset of $\mathcal{S}$ corresponding to that (single) event being true. Examples:

E_1: ‘roll a 1’. True if roll in $\{$  $\}$ $P(E_1) = 1/6$

E_2: 'roll a 2’. True if roll in $\{$  $\}$ $P(E_2) = 1/6$

E_3: ‘roll even’. True if roll in $\{$  $\}$ $P(E_3) = 1/2$

E_4: 'roll <3’. True if roll in $\{$  $\}$ $P(E_4) = 1/3$

Probability: Fair Die

Sample space: (Set of all possible outcomes/rolls)	$\mathcal{S} = \{$	⚀	⚁	⚂	⚃	⚄	⚅ $\}$
Probability:	P :	1/6	1/6	1/6	1/6	1/6	1/6

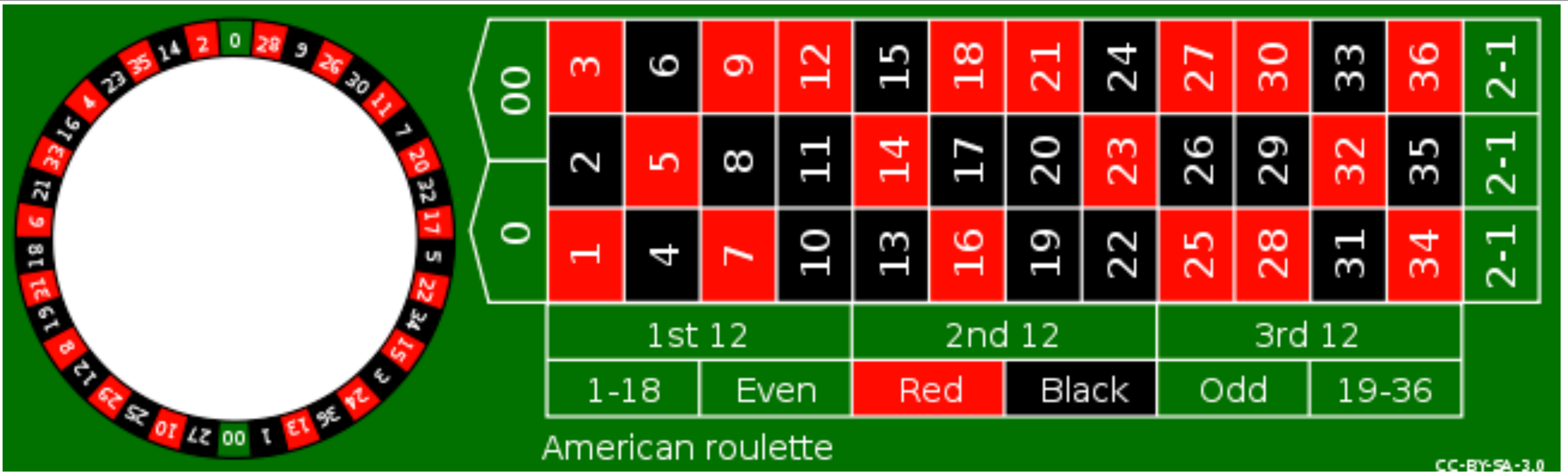
Event: a subset of $\mathcal{S}$ corresponding to that (single) event being true. Examples:

E_1: 'roll a 1'.	True if roll in	{ ⚀ }	$P(E_1) = 1/6$
E_2: 'roll a 2'.	True if roll in	{ ⚁ }	$P(E_2) = 1/6$
E_3: 'roll even'.	True if roll in	{ ⚁ ⚃ ⚅ }	$P(E_3) = 1/2$
E_4: 'roll <3'.	True if roll in	{ ⚀ ⚁ }	$P(E_4) = 1/3$

$\mathcal{E}$: the set of all possible events. You can think of it as all the possible ways you can bet on the outcome of a roll of the die.

$\mathcal{E}$ = {{ ⚀ }, { ⚁ }, …, { ⚅ }, { ⚀ ⚁ }, { ⚀ ⚁ }, …, { ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ }}

Probability: Roulette

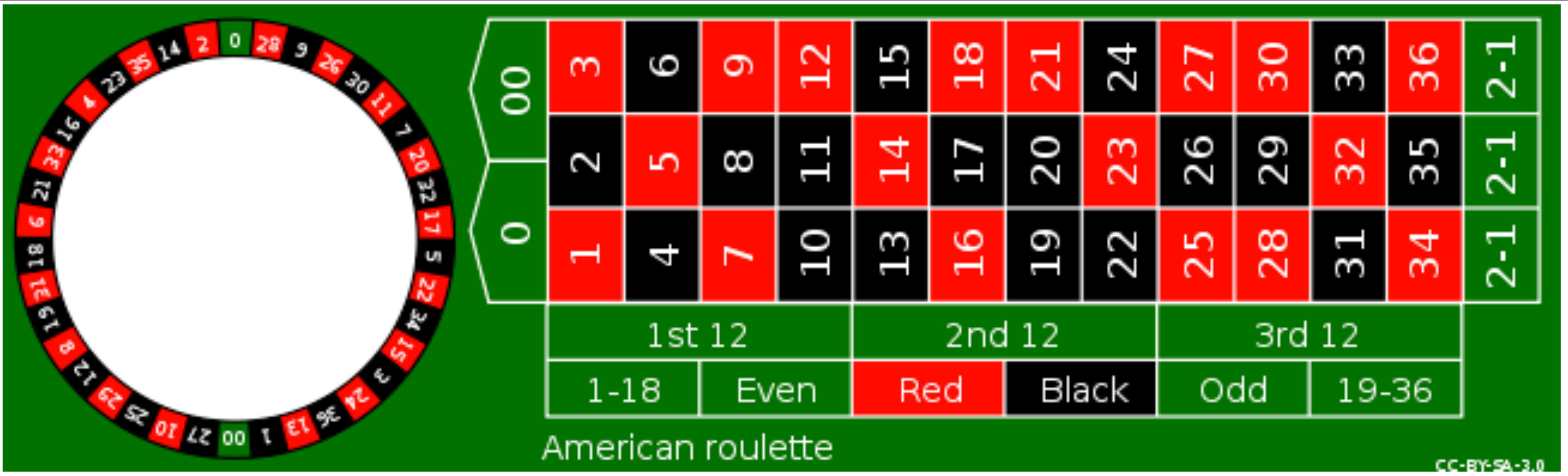


Sample space: $\mathcal{S} = \{00, 0, 1,2,3,4,\ldots,36\}$
(set of all possible outcomes)

Probability: P=1/38 for every outcome in $\mathcal{S}$

Event space: $\mathscr{E} = \big\{\{00\}, \{0,\}\{1\}, \{2\}, \{3\}\ldots, \{1,2\}, \ldots, \{1, \ldots, 36\}\big\}$
All possible roulette bets

Probability: Roulette



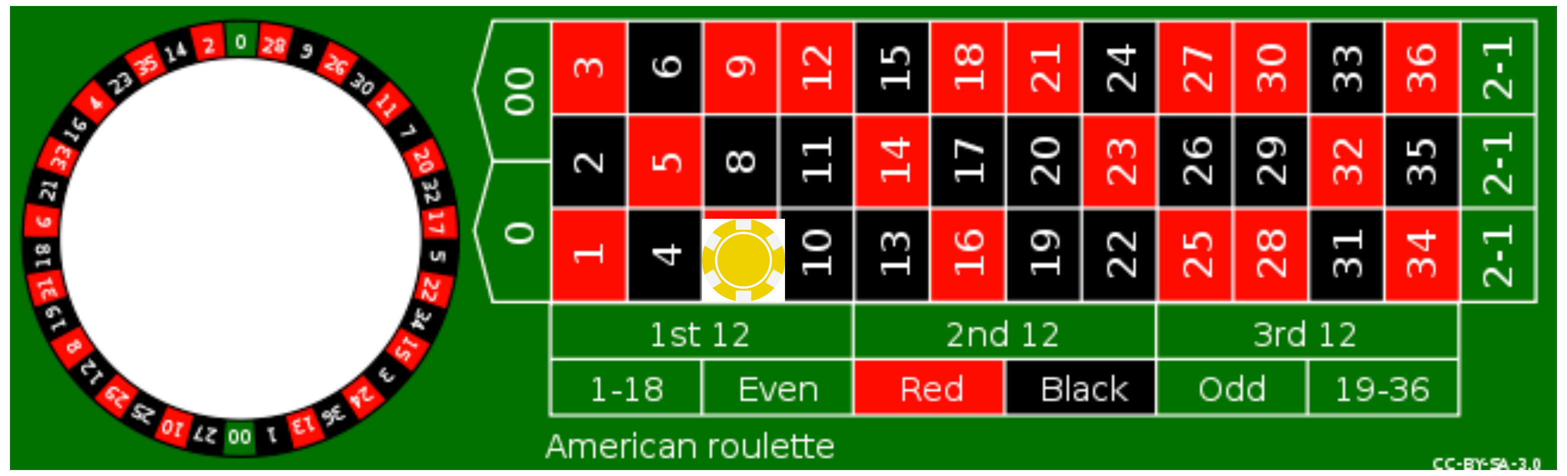
Sample space: $\mathcal{S} = \{00, 0, 1,2,3,4,\ldots, 36\}$
(set of all possible outcomes)

Probability: P=1/38 for every outcome in $\mathcal{S}$

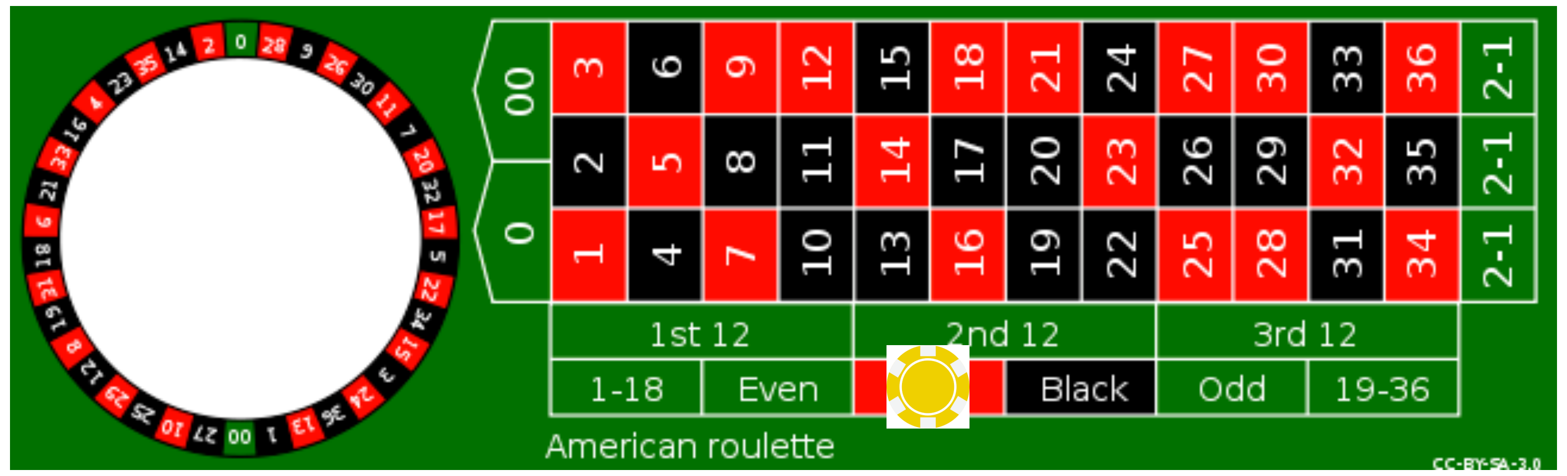
Event space: $\mathcal{E} = \{\{00\}, \{0,\}\{1\}, \{2\}, \{3\}\ldots, \{1,2\}, \ldots, \{1, \ldots, 36\}\}$
All possible roulette bets

Example of bets - i.e., events corresponding to a single spin

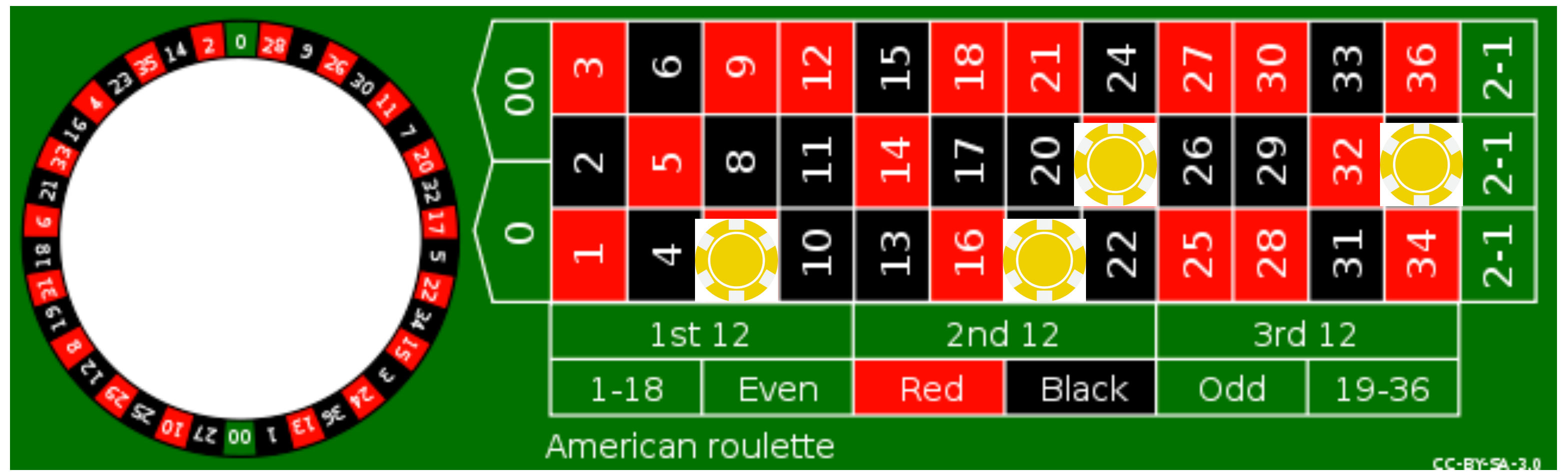
Straight up bet on 7:
Outcome in $E_1 = \{7\}$
$P(E_1) = 1/38$



Bet on red:
outcome in $E_2 = \{1,3,\ldots,34,36\}$
$P(E_2) = 18/38$

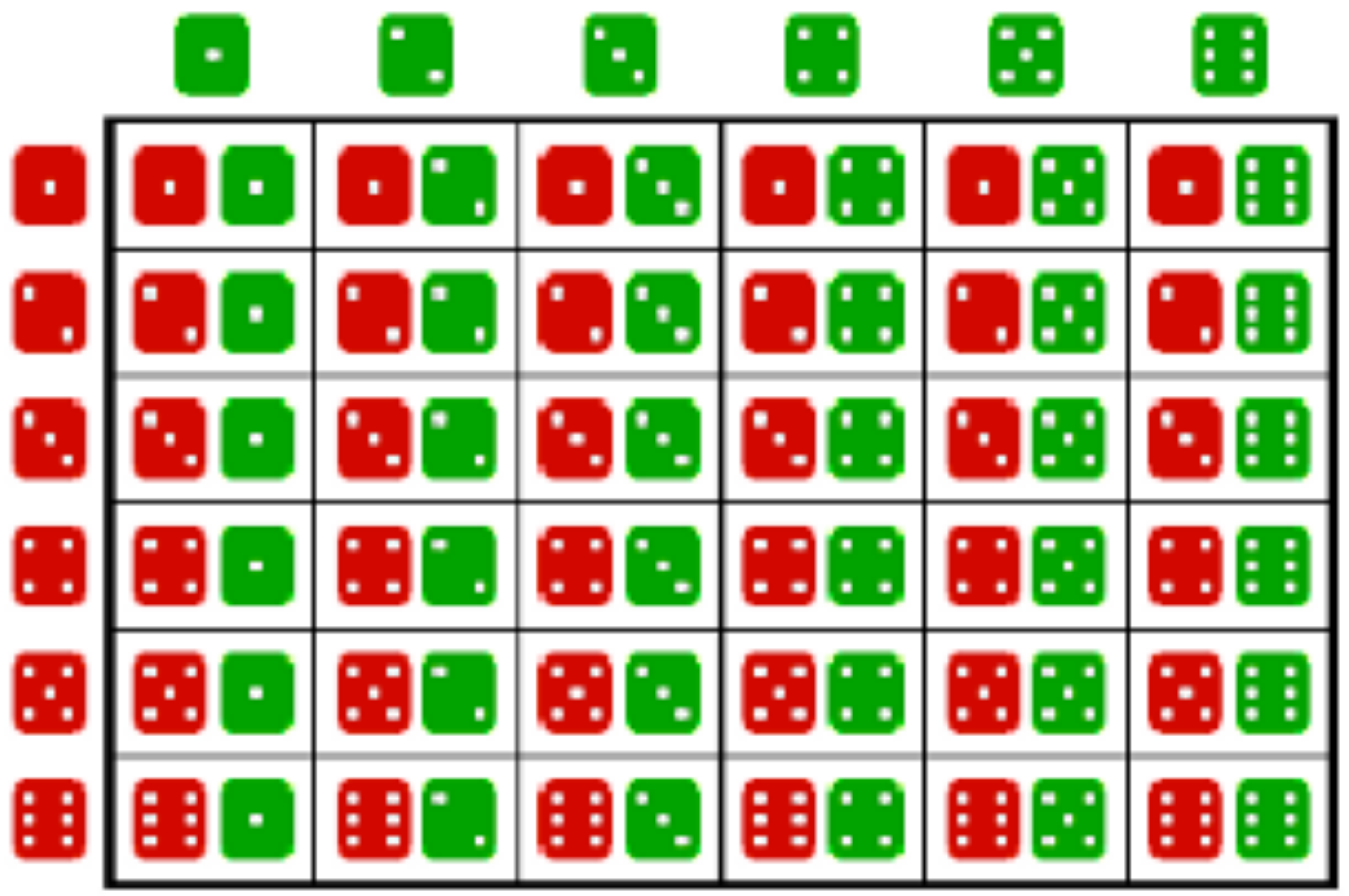


Bet on your favorite numbers
Outcome in $E_3 = \{7,23,19,35\}$
$P(E_3) = 4/38$



Probability: Two fair dice

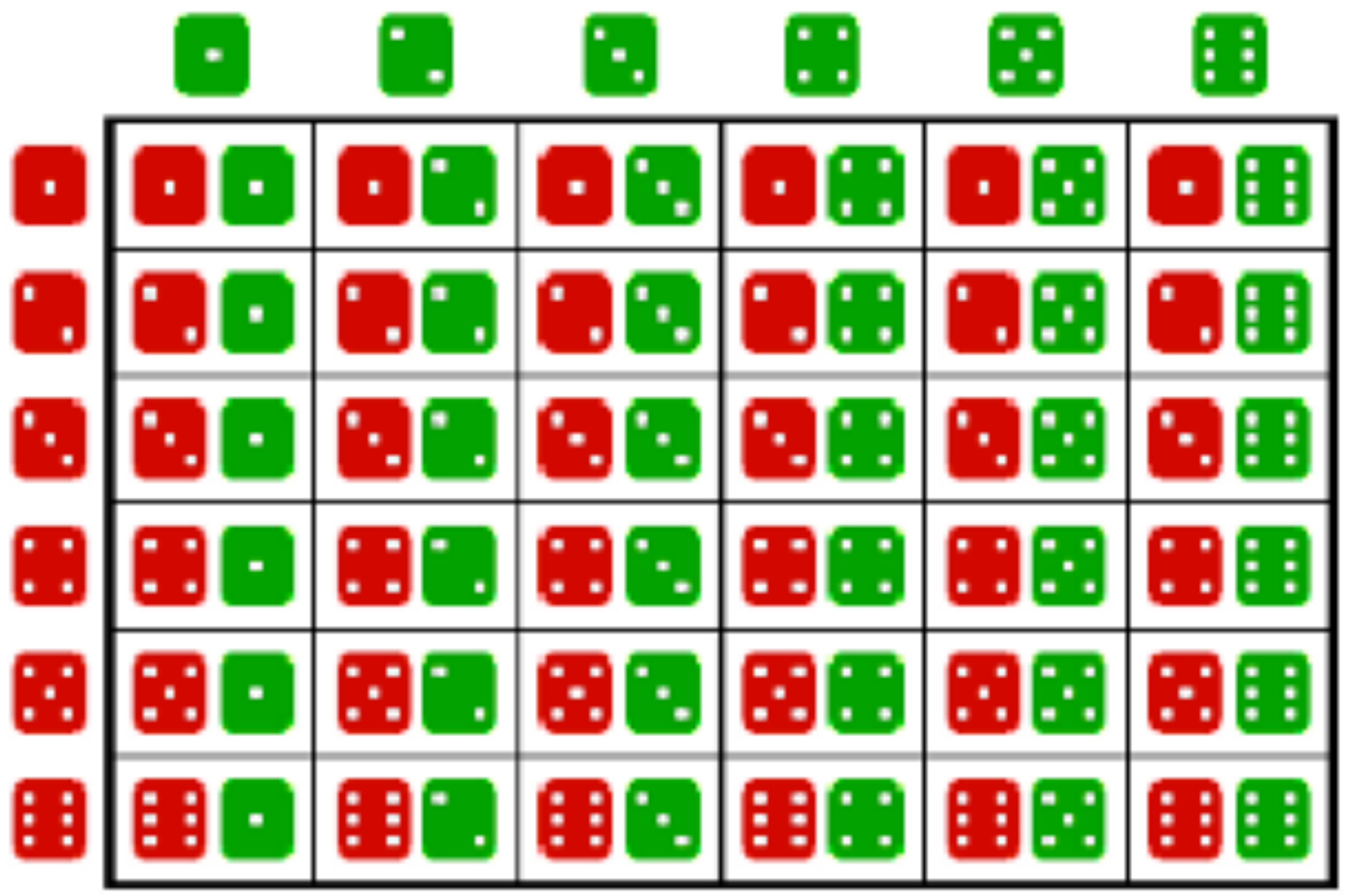
Sample space $\mathcal{S}$:



Probability $P = 1/36$ for each event in sample space

Probability: Two fair dice

Sample space $\mathcal{S}$:



Probability $P = 1/36$ for each event in sample space

Example events:

E_1: 'snake eyes'. True if outcome in: {  } $P(E_1) = 1/36$

E_2: 'roll a 4'. True if outcome in: {  } $P(E_2) = 3/36 = 1/12$

Probability: Combining probabilities

How do we compute the probability of event E_1:

E: 'roll <3'. True if roll in { }

A: 'roll a 1'. True if roll in {} $P(A) = 1/6$

B: 'roll a 2'. True if roll in {} $P(B) = 1/6$

Rolling <3 is, by definition, equivalent to rolling either  or 

$$P(E) = P(A \text{ OR } B)$$

Probability: Combining probabilities

How do we compute the probability of event E_1:

E: 'roll <3'. True if roll in {⚀ ⚁}

A: 'roll a 1'. True if roll in {⚀} $P(A) = 1/6$

B: 'roll a 2'. True if roll in {⚁} $P(B) = 1/6$

Rolling <3 is, by definition, equivalent to rolling either ⚀ or ⚁

$$P(E) = P(A \text{ OR } B)$$

Notice that:

{⚀ ⚁} = {⚀} ∪ {⚁}

{⚀} ∩ {⚁} = ϕ

$P($⚀ ⚁$) = P($⚀$) + P($⚁$)$

Property of probability: $\boxed{P(A \cup B) = P(A) + P(B) \text{ if } A \cap B = \phi}$

Probability: Combining probabilities

What if the sets are not dissjoint?

A: 'roll <3'. True if roll in {  } $P(A) = 1/3$

B: 'roll even'. True if roll in {  } $P(B) = 1/2$

Probability: Combining probabilities

What if the sets are not dissjoint?

A: 'roll <3'. True if roll in {⚀ ⚁} $P(A) = 1/3$

B: 'roll even'. True if roll in {⚁ ⚃ ⚅} $P(B) = 1/2$

Say you're playing craps and you bet $1 on rolling <3 and $1 on rolling even.
What's the probability of winning something?
Well, it's the probability of the outcome being in:

$P(A$ OR $B) = P($ {⚀ ⚁ ⚃ ⚅}$) = 4/6$

Probability: Combining probabilities

What if the sets are not dissjoint?

A: 'roll <3'. True if roll in {⚀ ⚁} $P(A) = 1/3$

B: 'roll even'. True if roll in {⚁ ⚃ ⚅} $P(B) = 1/2$

Say you're playing craps and you bet $1 on rolling <3 and $1 on rolling even.
What's the probability of winning something?
Well, it's the probability of the outcome being in:

$P(A$ OR $B) = P($ {⚀ ⚁ ⚃ ⚅}$) = 4/6$

Notice that although: {⚀ ⚁ ⚃ ⚅} = {⚀ ⚁} ∪ {⚁ ⚃ ⚅}

$P(A \text{ OR } B) = P(A \cup B)$

Probability: Combining probabilities

What if the sets are not dissjoint?

A: 'roll <3'. True if roll in {⚀ ⚁} $P(A) = 1/3$

B: 'roll even'. True if roll in {⚁ ⚃ ⚅} $P(B) = 1/2$

Say you're playing craps and you bet \$1 on rolling <3 and \$1 on rolling even.
What's the probability of winning something?
Well, it's the probability of the outcome being in:

$P(A$ OR $B) = P($ {⚀ ⚁ ⚃ ⚅}$) = 4/6$

Notice that although: {⚀ ⚁ ⚃ ⚅} = {⚀ ⚁} ∪ {⚁ ⚃ ⚅}

$P(A$ OR $B) = P(A \cup B)$

But, since $A \cap B \neq \phi$: $P(A \cup B) = 4/6 \neq P(A) + P(B) = 1/3 + 1/2 = 5/6$

Instead: $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)} = 5/6 - 1/6 = 4/6$

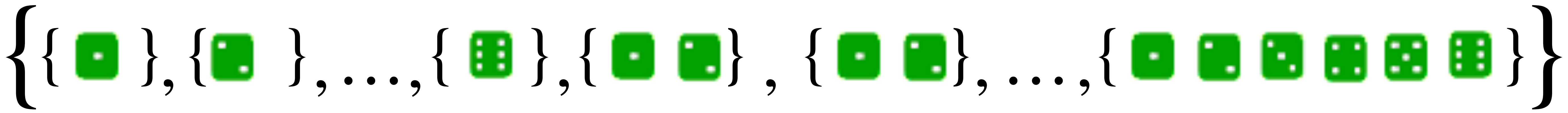
Extra (you won't need this in this class): probability space $(\mathcal{S}, \mathcal{E}, P)$

In order to define a random process and work with probabilities, we need to construct a probability space.

Sample space: (Set of all possible outcomes/rolls)

$\mathcal{S} = \{$  $\}$

Event space: (Set of all subsets of the sample space)



Valid Probability: P

- $P > 0$, $P(\phi) = 0$, $P(\mathcal{S}) = 1$
- If E_1, E_2 are mutual exclusive $(E_1 \cap E_2 = \phi)$, the probability of E_1 OR E_2 is:

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

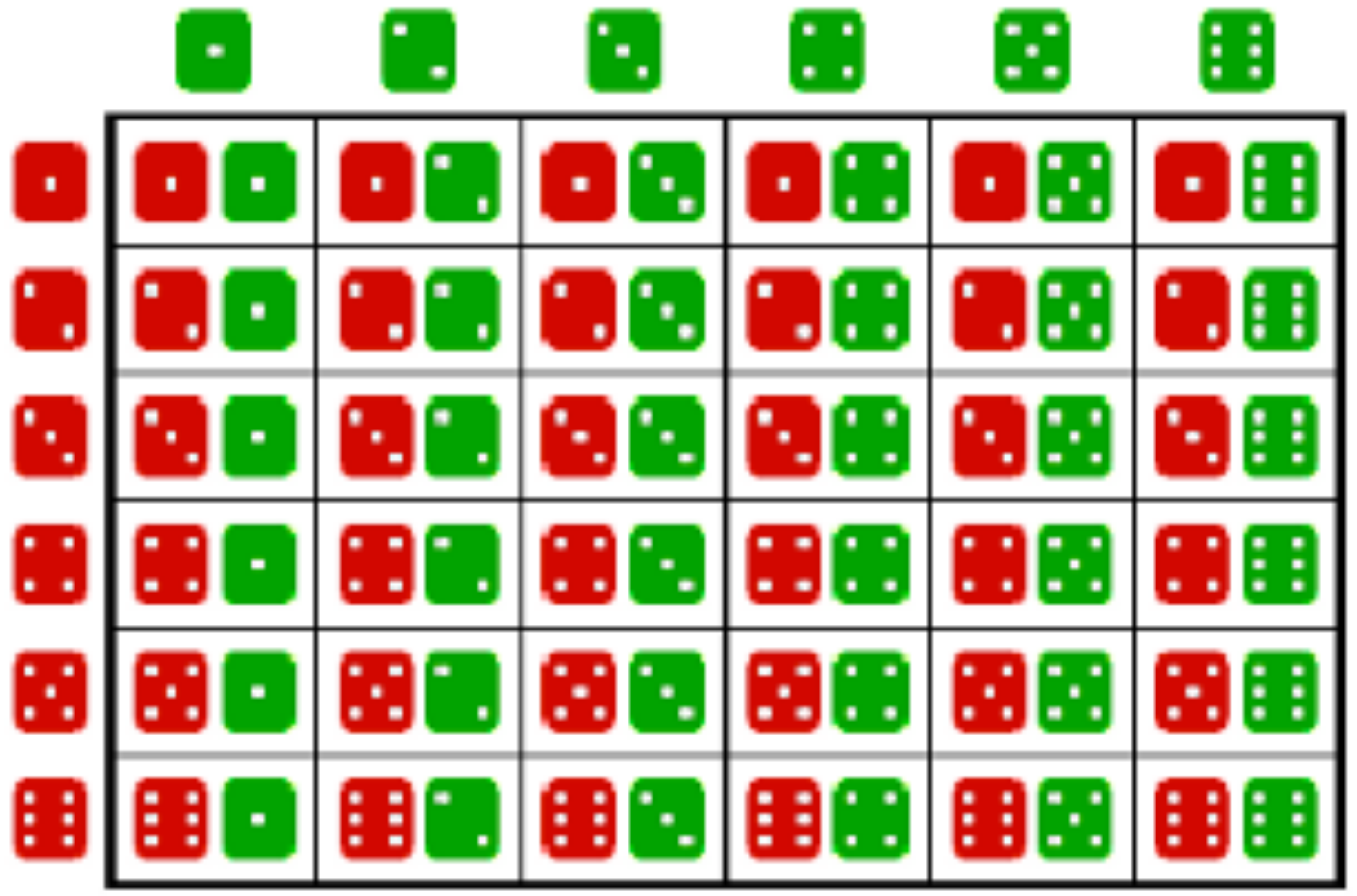
Properties of probability calculus

We’ll go through some properties of probabilities.

These are properties we will use throughout the course

Independent events

Two events are independent IF: $P(A \text{ AND } B) = P(A, B) = P(A)P(B)$



Consider the roll of two fair dice

What is the probability of rolling  ?

P() =1/36

It is also the probability of rolling both  and 
The role of one dice should not affect the role of the other. The dice should be independent.

P() =1/36=P()P()=1/6*1/6

Conditional Probabilities

Conditional probability= the probability of Event A being true **given** that event B is true

Consider the events: A :“Die rolled less than three” True if roll is in $E_9 = \{$ ⚀ ⚁ $\}$

B :“Die rolled even” True if roll is in $A = \{$ ⚁ ⚃ ⚅ $\}$

$P(A\,|\,B)$ is the probability of A, conditional on B being true. Or simply, conditional on B.

$P(A\,|\,B)$ is the probability of rolling<3, conditional (i.e. ***after***) rolling even.

It should be obvious that, for a fair dice, that probability is 1/3

How to compute it? It is the probability of both A, B, but taking into account the fact that B is already true:

$$\boxed{P(A\,|\,B) = \frac{P(A,B)}{P(B)}}$$

Conditional Probabilities: flipped

More generally, we'll use conditional probabilities to compute joint probabilities.

Consider the events:

A :"Die rolled even" True if roll is in $A = \{$  $\}$

B :"Die rolled less than three" True if roll is in $E_9 = \{$  $\}$

The probability of A and B can also be thought of as the probability of 'rolling <3', and Then also making sure the roll is even.

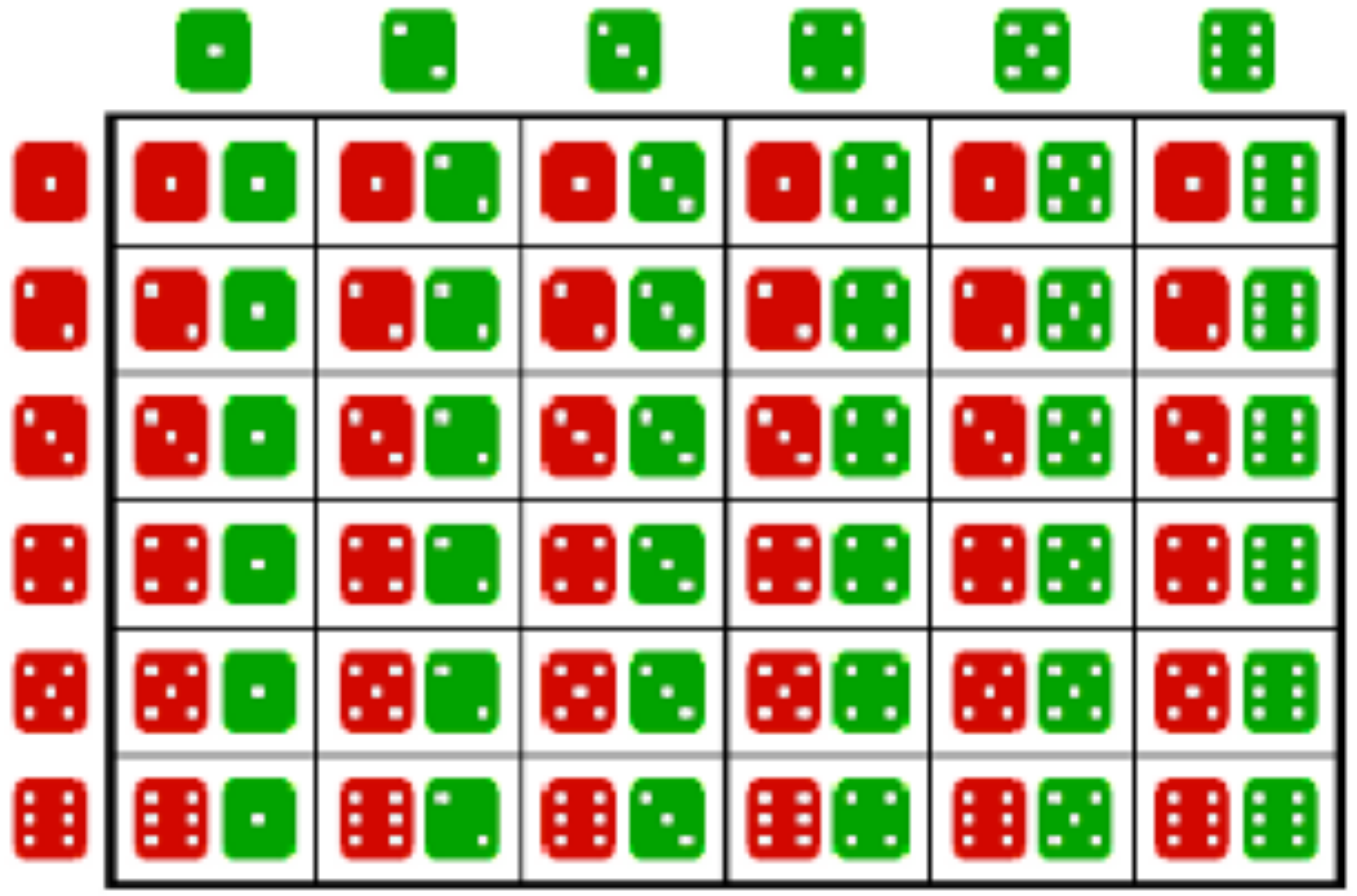
So the joint probability can also be written as:

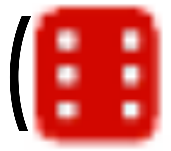
$$\boxed{P(A, B) = P(A \mid B)P(B)}$$

Conditional Probabilities and Independence

If A, B are independent what is the conditional probability $P(A\,|\,B) = \,?$

Consider the roll of two independent fair dice



What is the conditional probability P(|)?

Well, regardless of what the green dice rolls, the Probability that the red dice rolls  is the same.

This is a general property of independent events:

$\boxed{\text{If } A, B \text{ are independent } P(A\,|\,B) = P(A)}$

Bayes rule

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

$$P(A \mid B) = \frac{P(B, A)}{P(B)}$$

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Law of total probability

A :"Die rolled even"   True if roll is in $A = \{$  $\}$

$P(A) = 1/2$

$B_1 = \{$  $\}$ $B_2 = \{$  $\}$ $B_6 = \{$  $\}$

$P(A \mid B_1) = 0$ $P(A \mid B_2) = 1$ $P(A \mid B_6) = 1$

Law of total probabilities: $P(A) = \sum P(A \mid B_j) P(B_j)$

If $\bigcup_j B_j = S$, then

Recap:

Independence: $P(A, B) = P(A)P(B)$

Conditional probabilities: $P(A \mid B) = \dfrac{P(A, B)}{P(B)}$

Bayes Rule: $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$

If $P\left(\bigcup_j B_j\right) = 1,$ then

Law of total probabilities: $P(A) = \sum P(A \mid B_j)P(B_j)$

Probability space $(\mathscr{S}, \mathscr{E}, P)$

In order to define a random process and work with probabilities, we need to construct a probability space.

Sample space: (Set of all possible outcomes/rolls)

$$\mathscr{S} = \{\text{⚀ ⚁ ⚂ ⚃ ⚄ ⚅}\}$$

Event space: (Set of all subsets of the sample space)

$$\mathscr{E} = \Big\{\{\text{⚀}\}, \{\text{⚁}\}, \ldots, \{\text{⚅}\}, \{\text{⚀ ⚁}\}, \{\text{⚀ ⚁}\}, \ldots, \{\text{⚀ ⚁ ⚂ ⚃ ⚄ ⚅}\}\Big\}$$

Valid Probability: P

- $P > 0$, $P(\phi) = 0$, $P(\mathscr{S}) = 1$
- If E_1, E_2 are mutual exclusive $(E_1 \cap E_2 = \phi)$, the probability of E_1 OR E_2 is:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Random variable

A random variable X is a function defined on the sample space,
That associates a number for outcome and event:

$$X : \mathcal{S} \to \mathbb{R}$$

X: random variable

x : a specific value the random variable takes in the real numbers

Uniform discrete random variable

$$X : \mathcal{S} \rightarrow \{1,2,3,4,5,6\}$$

X: value of each die face

$\mathcal{S} = \{$  $\}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

1 2 3 4 5 6

Bernoulli Random variable

$$X : \mathcal{S} \rightarrow \{0,1\}$$

$$X = 0 \text{ if roll } \leq 3$$
$$= 1 \text{ if roll } > 3$$

$$p = P(X = 1)$$

$$p = 0.5$$

$\mathcal{S} = \{$  $\}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

0 0 0 1 1 1

Same sample space, different RV

Uniform discrete

$$X : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$$

Bernoulli

$$X : \{1,2,3,4,5,6\} \to \{0,1\}$$

Bernoulli Random variable

$$X : \mathscr{S} \rightarrow \{0,1\}$$

$X = 0$ if heads

$= 1$ if tails

$$p = P(X = 1)$$

$$p = 0.5$$

0 1

Bernoulli Random variable

$$X : \mathcal{S} \to \{0,1\}$$

X = 0 if roll ≤ 2

= 1 if roll > 2

$$p = P(X = 1)$$

$$p = 2/3$$

$\mathcal{S} = \{$  $\}$

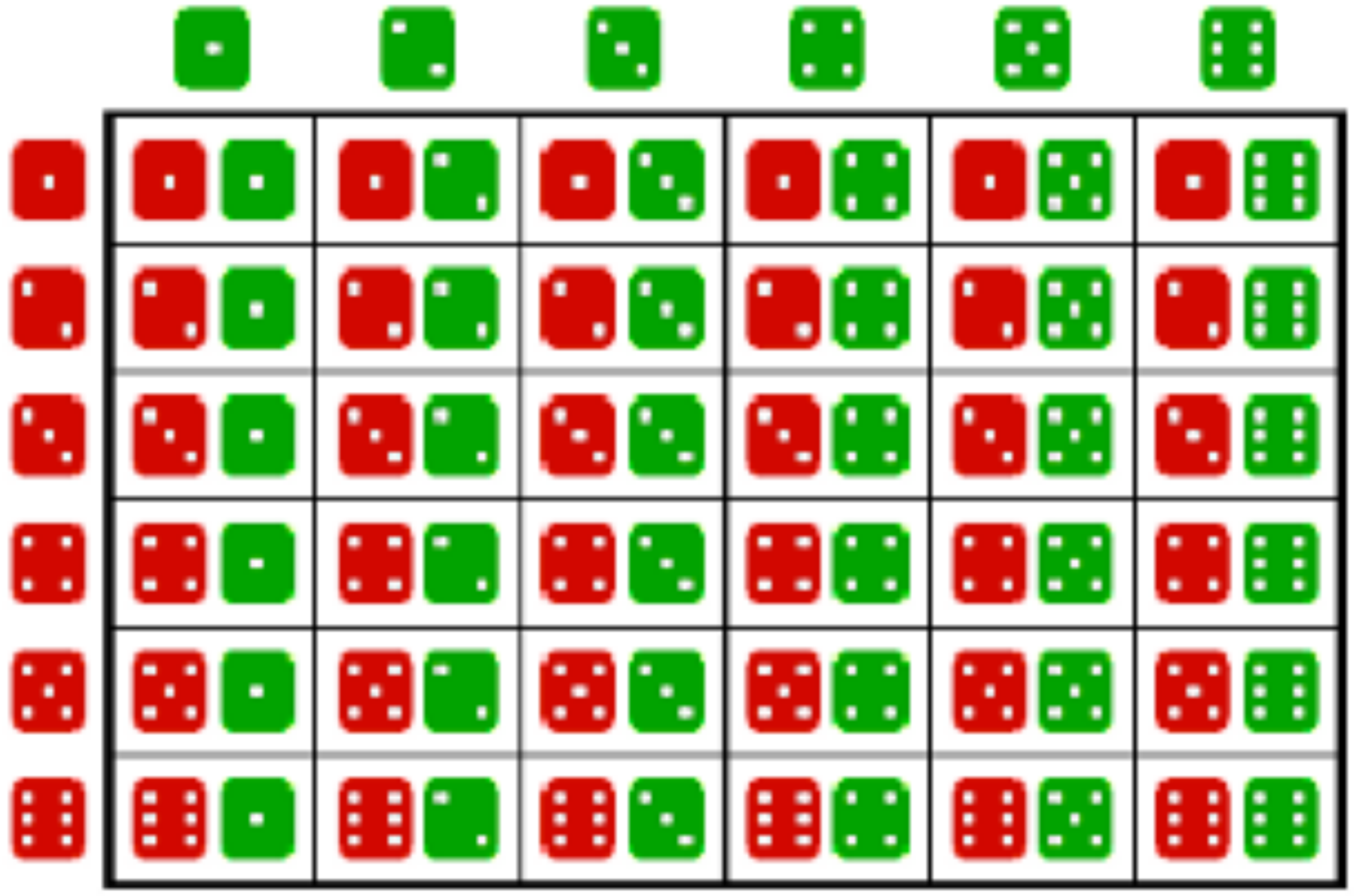
$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

0 0 1 1 1 1

Sum of two dice

$$X : \mathscr{S} \rightarrow \{2, \ldots, 12\}$$

$\mathscr{S}$:



$\rightarrow$

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

$P = 1/36$ for each event in sample space

Distributions: probability mass function

Random variables are characterized (and defined) by their distribution

$$X : \mathcal{S} \to \{x_i\}$$

Probability mass function (pmf) of X:

$$p(x_i) = P(X = x_i)$$

Distributions: probability mass function

Uniform discrete: $p(x_i) = 1/6$

Bernoulli: $p(0) = 1 - p;\ p(1) = p$

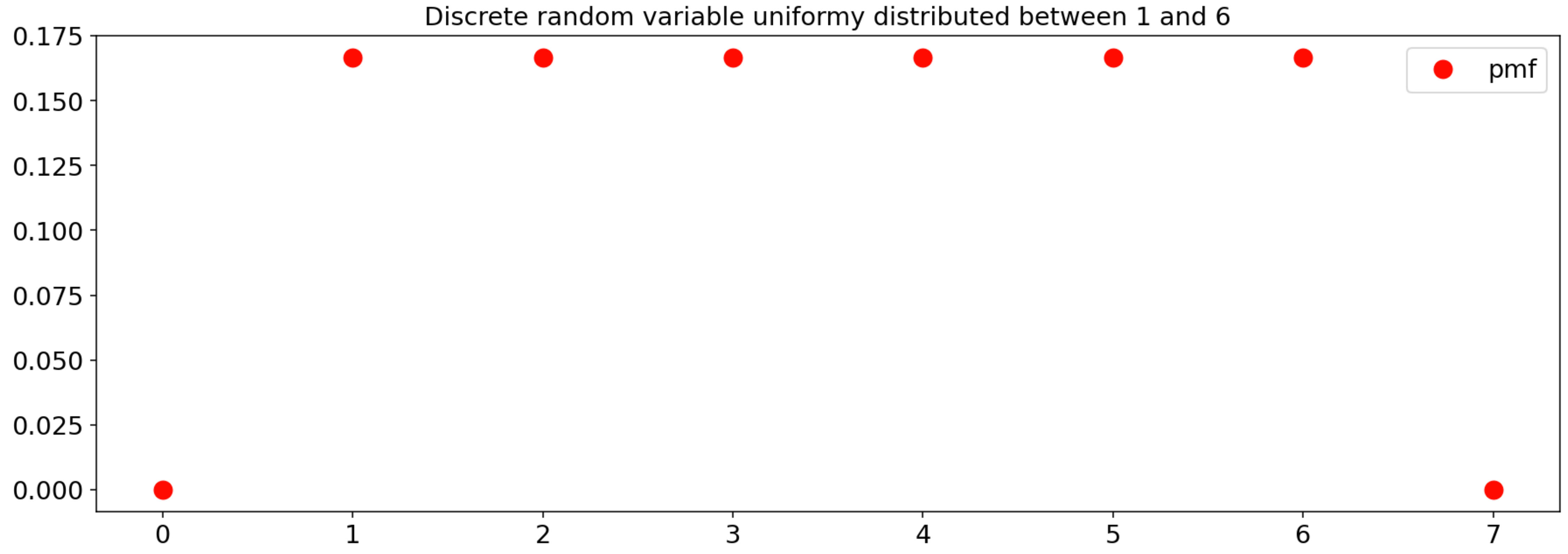
Sum of two dice:

- $p(2) = 1/36$
- $p(3) = 2/36$
- …
- $p(12) = 1/36$

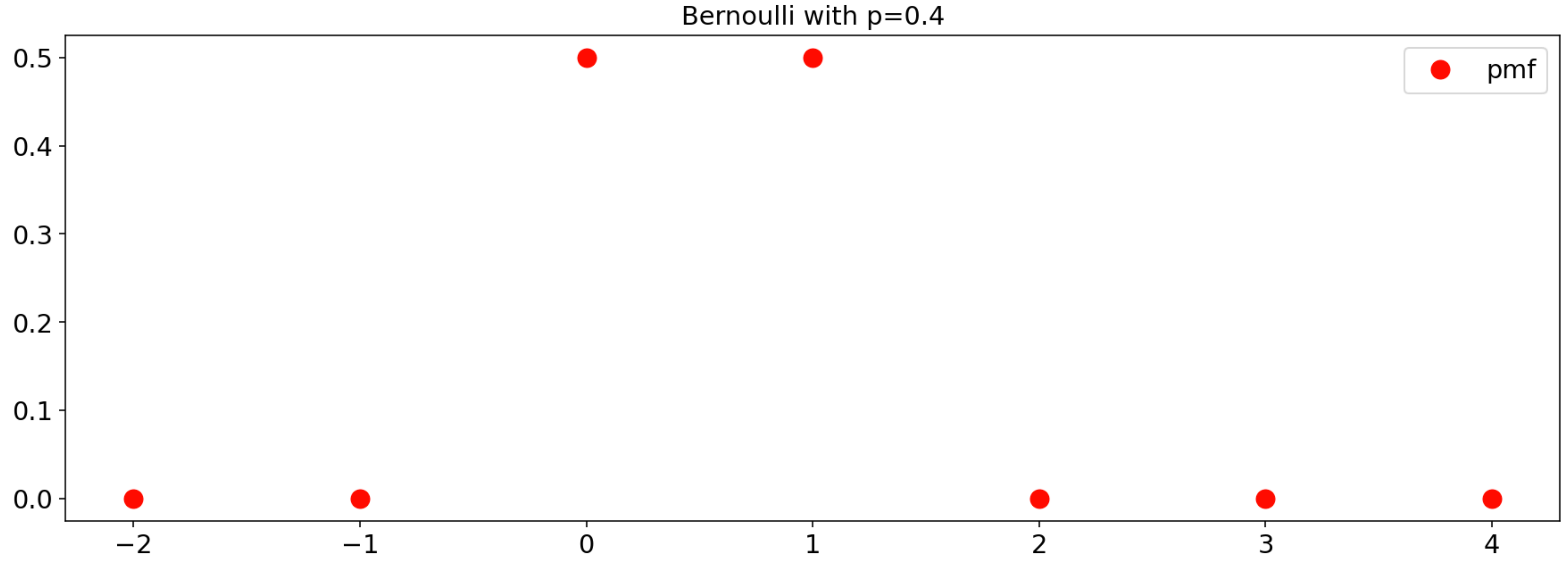
	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Distributions: probability mass function

Uniform discrete: $p(x_i) = 1/6$

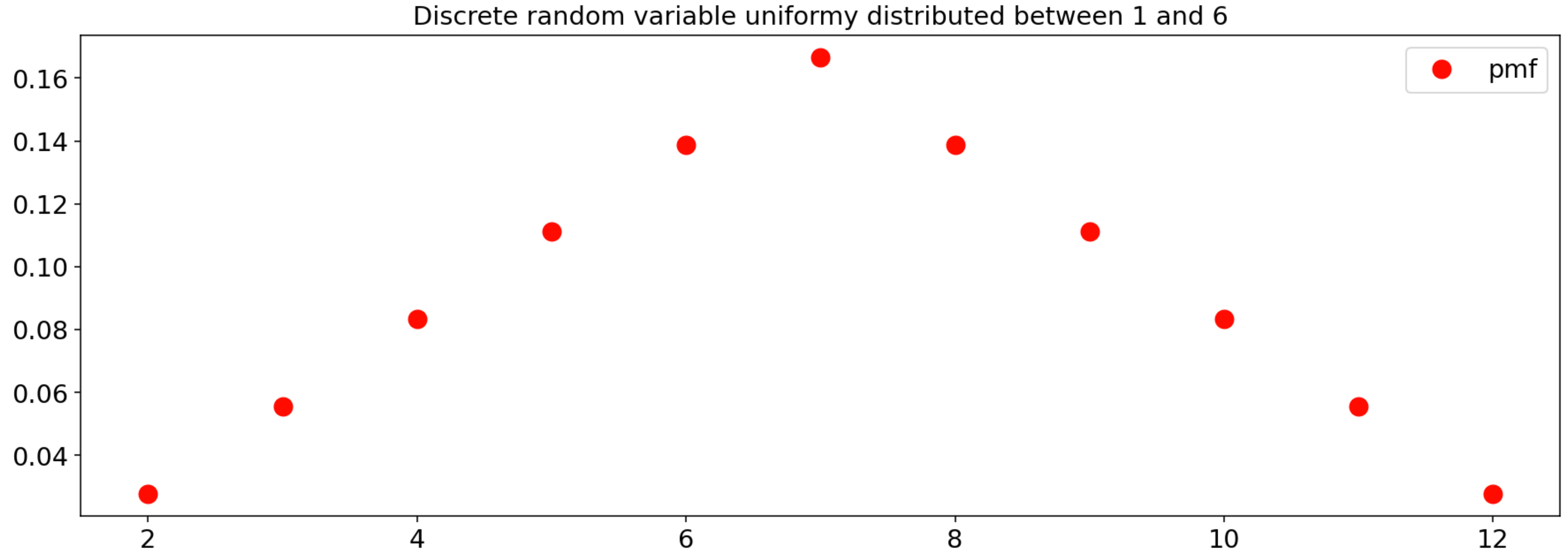


Bernoulli: $p(0) = 1 - p;\ p(1) = p$



Sum of two dice:

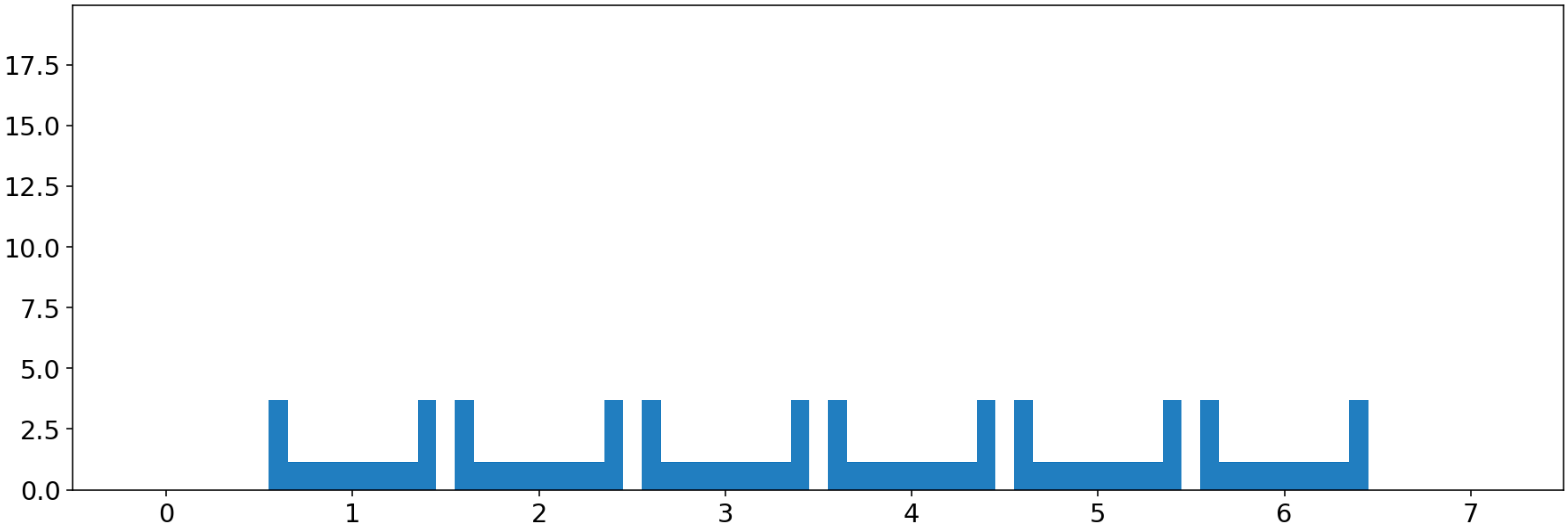
- $p(2) = 1/36$
- $p(3) = 2/36$
- $\ldots$
- $p(12) = 1/36$



Histograms

An *approximate* representation of the distribution of a random variable

Definition: the frequency of realizations occurring in certain ranges of values (bins)

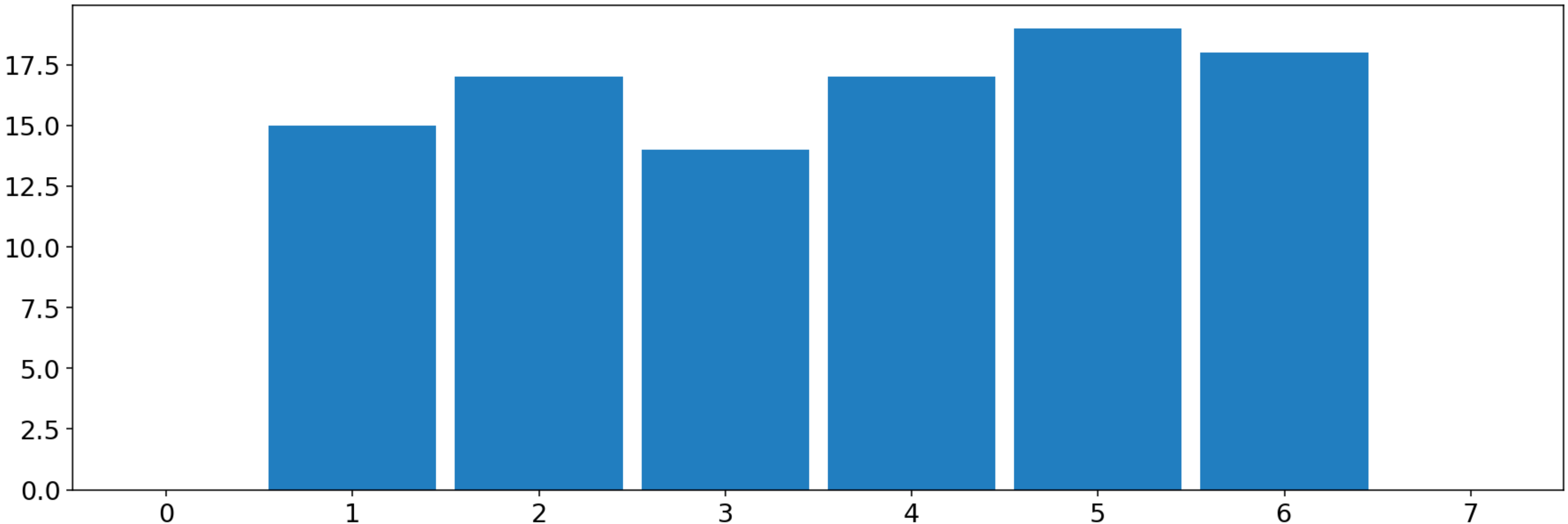


Histograms

An *approximate* representation of the distribution of a random variable

Definition: the frequency of realizations occurring in certain ranges of values (bins)

Count: the *number* of realizations occurring in each bin



Histograms

An *approximate* representation of the distribution of a random variable

Definition: the frequency of realizations occurring in certain ranges of values (bins)

Frequency: the *relative number* of realizations occurring in each bin
(Number in each bin / total number)