Two-Body Partial Derivatives

This document is a compilation of partial derivatives of the astrodynamic two-body problem. Partial derivatives are summarized for circular, elliptical and hyperbolic two-body motion.

Downstream error propagation derivatives for a circular orbit

$$\frac{\partial v_f}{\partial v} = 2\cos\theta_f - 1$$

$$\frac{\partial r_f}{\partial v} = \frac{2r_i}{v_i}\left(1-\cos\theta_f\right)$$

$$\frac{\partial \gamma_f}{\partial v} = \frac{2}{v_i}\sin\theta_f$$

$$\frac{\partial \theta_f}{\partial v} = \frac{1}{v_i}\left(4\sin\theta_f - 3\theta_f\right)$$

$$\frac{\partial v_f}{\partial \gamma} = -v_i\sin\theta_f$$

$$\frac{\partial r_f}{\partial \gamma} = r_i\sin\theta_f$$

$$\frac{\partial \gamma_f}{\partial \gamma} = \cos\theta_f$$

$$\frac{\partial \theta_f}{\partial \gamma} = -2\left(1-\cos\theta_f\right)$$

where

$r_i =$ initial position magnitude

$r_f =$ final position magnitude

$v_i =$ initial velocity magnitude $= \sqrt{\mu/r_i}$

$v_f =$ final velocity magnitude

$\theta_f =$ final true anomaly

$\gamma_f =$ final flight path angle

$\mu =$ gravitational constant

Elliptical orbits

apoapsis radius with respect to r, v, γ

$$\frac{\partial r_a}{\partial r} = \frac{(1+e\cos\theta)(1+e)}{(1-e^2)^2}\left\{2+e-e^2+\cos\theta+e\cos\theta\right\}$$

$$\frac{\partial r_a}{\partial v} = \frac{2a^{\frac{3}{2}}(1+2e+e^2)(1+\cos\theta)}{\sqrt{\mu}(1-e^2)^{\frac{1}{2}}(1+2e\cos\theta+e^2)^{\frac{1}{2}}}$$

$$\frac{\partial r_a}{\partial \gamma} = \frac{a(1-e^2)\sin\theta}{1+e\cos\theta}$$

periapsis radius with respect to r, v, γ

$$\frac{\partial r_p}{\partial r} = \frac{(1+e\cos\theta)(1+e)}{(1-e^2)^2}\left\{2-e-e^2-\cos\theta-e\cos\theta\right\}$$

$$\frac{\partial r_p}{\partial v} = \frac{2a^{\frac{3}{2}}(1-2e+e^2)(1-\cos\theta)}{\sqrt{\mu}(1-e^2)^{\frac{1}{2}}(1+2e\cos\theta+e^2)^{\frac{1}{2}}}$$

$$\frac{\partial r_p}{\partial \gamma} = \frac{-a(1-e^2)\sin\theta}{1+e\cos\theta}$$

orbital period with respect to r, v, γ

$$\frac{\partial \tau}{\partial r} = \frac{3\tau a}{r^2} \qquad \frac{\partial \tau}{\partial v} = \frac{3\tau(2a-r)}{rv} = \frac{3av\tau}{\mu} \qquad \frac{\partial \tau}{\partial \gamma} = 0$$

true anomaly with respect to r, v, γ

$$\frac{\partial \theta}{\partial r} = -\frac{\sin\theta(1+e\cos\theta)}{ae(1-e^2)}$$

$$\frac{\partial \theta}{\partial v} = -\frac{2\sin\theta}{e}\left\{\frac{a(1-e^2)}{\mu(1+2e\cos\theta+e^2)}\right\}^{\frac{1}{2}}$$

$$\frac{\partial \theta}{\partial \gamma} = 2+\frac{(1-e^2)}{e}\left\{\frac{\cos\theta}{1+e\cos\theta}\right\}$$

orbital eccentricity with respect to r, v, γ

$$\frac{\partial e}{\partial r} = \frac{v^2 \cos^2 \gamma}{\mu e a}(a - r) = \frac{e + \cos\theta}{r}$$

$$\frac{\partial e}{\partial v} = \frac{2rv\cos^2 \gamma}{\mu e a}(a - r)$$

$$\frac{\partial e}{\partial \gamma} = \frac{r^2 v^2 \sin 2\gamma}{2\mu e a}$$

semimajor axis with respect to r, v, γ

$$\frac{\partial a}{\partial r} = \frac{2a^2}{r^2} \qquad \frac{\partial a}{\partial v} = \frac{2a^2 v}{\mu} \qquad \frac{\partial a}{\partial \gamma} = 0$$

semimajor axis, eccentricity, orbital period and apogee radius with respect to periapsis radius

$$\frac{\partial a}{\partial r_p} = 2\left\{\frac{r_a}{r_p(1+e)}\right\}^2 = 2\left(\frac{a}{r_p}\right)^2 = \frac{2(1 + e\cos\theta)^2}{(1 - e^2)^2}$$

$$\frac{\partial e}{\partial r_p} = \frac{e + \cos\theta}{r_p}$$

$$\frac{\partial \tau}{\partial r_p} = \frac{3\tau a}{r_p^2}$$

$$\frac{\partial r_a}{\partial r_p} = \frac{(1 + e\cos\theta)(2 + \cos\theta - e)}{(1 - e)^2}$$

semimajor axis, eccentricity, orbital period and apogee radius with respect to periapsis velocity

$$\frac{\partial a}{\partial v_p} = \frac{2a(2a - r_p)}{r_p v_p}$$

$$\frac{\partial e}{\partial v_p} = \frac{2(e + \cos\theta)}{v_p}$$

$$\frac{\partial \tau}{\partial v_p} = \frac{3\tau(2a - r_p)}{r_p v_p}$$

$$\frac{\partial r_a}{\partial v_p} = \frac{2a(1 + \cos\theta)(1 + e)}{v_p(1 - e)}$$

where

$$r = \text{ position magnitude}$$

$$v = \text{ velocity magnitude}$$

$$\gamma = \text{ flight path angle}$$

$$\theta = \text{ true anomaly}$$

$$e = \text{ orbital eccentricity}$$

$$r_p = \text{ periapsis radius}$$

$$r_a = \text{ apoapsis radius}$$

$$a = \text{ semimajor axis}$$

$$\tau = \text{ orbital period}$$

In the following series of equations, **x** represents the Cartesian state vector consisting of position vector **r**, and velocity velocity **v**.

position vector

$$\frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \left[\mathbf{r}^T / r \quad \mathbf{0} \right]$$

velocity vector

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \left[\mathbf{0} \quad \mathbf{v}^T / v \right]$$

angular momentum vector

$$\frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \left[\mathbf{I} \times \mathbf{v} \quad \mathbf{r} \times \mathbf{I} \right]$$

angular momentum magnitude

$$\frac{\partial h}{\partial \mathbf{x}} = \left[\mathbf{h}^T \left(\mathbf{I} \times \mathbf{v} \right) / h \quad \mathbf{h}^T \left(\mathbf{r} \times \mathbf{I} \right) / h \right]$$

eccentricity vector

$$\frac{\partial \mathbf{e}}{\partial \mathbf{x}} = \left[\left(v^2 / \mu - 1/r \right) \mathbf{I} + \mathbf{r}\,\mathbf{r}^T / r^3 - \mathbf{v}\,\mathbf{v}^T / \mu \quad 2\mathbf{r}\,\mathbf{v}^T / \mu - \left(\mathbf{v} \bullet \mathbf{r} \right) \mathbf{I} / \mu - \mathbf{v}\,\mathbf{r}^T / \mu \right]$$

specific orbital energy

$$\frac{\partial C_3}{\partial \mathbf{x}} = \left[2\mu \mathbf{r}^T / r^3 \quad 2\mathbf{v}^T \right]$$

sine of flight path angle

$$\frac{\partial \sin\gamma}{\partial \mathbf{x}} = \left[\hat{\mathbf{v}}^T \left(\mathbf{I}/r - \mathbf{r}\mathbf{r}^T/r^3 \right) \quad \hat{\mathbf{r}}^T \left(\mathbf{I}/v - \mathbf{v}\mathbf{v}^T/v^3 \right) \right]$$

semimajor axis

$$\frac{\partial a}{\partial \mathbf{r}} = \frac{2a^2\mathbf{r}}{r^3} \qquad \frac{\partial a}{\partial \mathbf{v}} = \frac{2a^2\mathbf{v}}{\mu}$$

where

$\mathbf{r} = \left[r_x, r_y, r_z \right]^T =$ position vector

$r = |\mathbf{r}| =$ position magnitude

$\mathbf{v} = \left[v_x, v_y, v_z \right]^T =$ velocity vector

$v = |\mathbf{v}| =$ velocity magnitude

$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \left[h_x, h_y, h_z \right]^T =$ angular momentum vector

$h = |\mathbf{r} \times \mathbf{v}| =$ angular momentum magnitude

$\mathbf{e} =$ orbital eccentricity vector

$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$ identity matrix

Hyperbolic orbits

In the following series of equations, $\mathbf{x}$ represents the state vector consisting of position vector $\mathbf{r}$, and velocity velocity $\mathbf{v}$. C_3 is equal to twice the specific (per unit mass) orbital energy.

scaled unit asymptote vector

$$\frac{\partial \tilde{\mathbf{s}}}{\partial \mathbf{x}} = C_3 \frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{x}} + \hat{\mathbf{s}} \frac{\partial C_3}{\partial \mathbf{x}}$$

$$= \frac{\sqrt{C_3^3}}{\mu e^2} \left\{ \left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \times \mathbf{e} \right) + \left(\mathbf{h} \times \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \right) \right\} - \frac{C_3}{e^2} \frac{\partial \mathbf{e}}{\partial \mathbf{x}} + \left\{ \frac{3}{2C_3} \frac{\partial C_3}{\partial \mathbf{x}} - \frac{2}{e^2} \left(\mathbf{e} \bullet \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \right) \right\} \tilde{\mathbf{s}} + \frac{1}{2e^2} \frac{\partial C_3}{\partial \mathbf{x}} \mathbf{e}$$

The asymptote unit vector of a hyperbolic orbit is given by

$$\hat{\mathbf{s}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

In this expression, α_∞ is the right ascension of the asymptote (RLA), and δ_∞ is the declination of the asymptote (DLA).

The asymptote unit vector at any trajectory time can also be computed from

$$\hat{\mathbf{s}} = \frac{1}{1 + C_3 \dfrac{h^2}{\mu^2}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\} = \frac{1}{1 + C_3 \dfrac{p}{\mu}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\}$$

where p is the semi-parameter of the orbit.