Generalized Linear Targeting and Guidance

This memo describes the mathematical background and practical implementation of an explicit generalized linear targeting and guidance (GLTG) algorithm. The GLTG algorithm uses a combination of modern control theory and linear mathematical techniques to formulate and solve a variety of orbital transfer problems. The targeting and guidance solutions are formulated using generalized mission constraints which relate the classical orbital elements to the state vector predicted by the GLTG scheme. In the closed-loop guidance mode, this iterative algorithm resolves the orbital boundary value problem during the actual finite-burn maneuvers. A key assumption for this algorithm is that each propulsive maneuver is treated as a single impulsive delta-velocity during targeting and as a series of fixed-attitude, finite-burn maneuvers during closed-loop guidance.

The robustness and computational speed of the generalized linear scheme make it ideal for on-board targeting and guidance flight software. The proper combination of numerical methods makes it possible to carry a three-degree-of-freedom (3 DOF) simulation on-board and solve the trajectory targeting and guidance problems as the vehicle performs the mission.

In the following discussion, bold letters and symbols indicate vectors and matrices.

Equations of motion

The first-order, nonlinear equations of orbital motion are given by

$$\dot{\mathbf{x}} = f\left(\mathbf{x}, \boldsymbol{\lambda}\right)$$

where $\mathbf{x}$ is the state vector and $\boldsymbol{\lambda}$ is the control vector.

The inertial, Cartesian equations of motion of a spacecraft subject to gravitational and thrust forces can be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g} + a\left(t\right)\mathbf{u}_T \end{bmatrix}$$

where $\mathbf{r}$ is the position vector, $\mathbf{v}$ the velocity vector, $\mathbf{g}$ is the acceleration due to gravity, $a\left(t\right)$ is the magnitude of the thrust acceleration and $\mathbf{u}_T$ is the unit thrust vector given by

$$\mathbf{u}_T = \begin{bmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ \sin\beta \end{bmatrix}$$

In this equation, α is the in-plane steering angle and β is the out-of-plane steering angle.

The control variables used to solve the trajectory problem consist of the impulse time and three components of each delta-v maneuver. The steering angles provided to the control system are determined from the components of the delta-v vector.

The inertial components of a $J_2 - \textit{perturbed}$ Earth gravity model are as follows;

$$g_x = -\mu \frac{r_x}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(1 - \frac{5r_z^2}{r^2}\right)\right\}$$

$$g_y = -\mu \frac{r_y}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(1 - \frac{5r_z^2}{r^2}\right)\right\}$$

$$g_z = -\mu \frac{r_z}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(3 - \frac{5r_z^2}{r^2}\right)\right\}$$

where

$r_x, r_y, r_z =$ inertial position vector components of the spacecraft

$r = \sqrt{r_x^2 + r_y^2 + r_z^2} =$ geocentric distance of the spacecraft

$\mu =$ gravitational constant of the Earth

$r_{eq} =$ equatorial radius of the Earth

$J_2 =$ oblateness gravity coefficient for the Earth

However, the equations of motion used to predict the spacecraft trajectory can include a higher order gravity model as well as other perturbations such as aerodynamic drag, solar radiation pressure, third-body effects and so forth.

Linearization

The equations of motion can be linearized about a reference trajectory according to

$$\delta \mathbf{x} = \mathbf{F}(t)\delta \mathbf{x} + \mathbf{G}(t)\delta \boldsymbol{\lambda}$$

where

$$\mathbf{F} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} \qquad \mathbf{G} = \frac{\partial \dot{\mathbf{x}}}{\partial \boldsymbol{\lambda}}$$

Furthermore,

$$\mathbf{F} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{F}' & 0 \end{bmatrix}$$

and

$$\mathbf{F}' = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} = -\mu \frac{\mathbf{I}}{r^3} + 3\mu \frac{\mathbf{r}\mathbf{r}^T}{r^5}$$

where $\mathbf{F}'$ is the variation of the gravity vector due to position and can be formulated by ignoring second order terms in the gravity vector. $\mathbf{I}$ is the 3 by 3 identity matrix.

Note that the **F** and **G** partial derivative matrices are functions of time and are evaluated on the reference trajectory.

The solution to the linearized system of equations is given by

$$\delta\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\,\delta\mathbf{x}(t_0) + \mathbf{\Gamma}_x(t,t_0)\,\delta\boldsymbol{\lambda}$$

where $\mathbf{\Phi}$ is the two-body state transition matrix and $\mathbf{\Gamma}_x$ is the control matrix which satisfies

$$\dot{\mathbf{\Gamma}}_x = \mathbf{F}\mathbf{\Gamma}_x + \mathbf{G}$$

and $\mathbf{\Gamma}_x(t_0) = \mathbf{0}$. Note that the use of velocity impulses in the GLTG algorithm permits a closed-form solution for the $\mathbf{\Gamma}_x$ matrix.

The state transition matrix $\mathbf{\Phi}$ consists of the 36 partial derivatives of the final state vector defined by the six components $x, y, z, \dot{x}, \dot{y}$ and $\dot{z}$ with respect to the six components of the initial state vector defined by $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0$ and $\dot{z}_0$. This matrix can be written as

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}\right] = \begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{bmatrix}$$

$$\Phi_{12} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{v}_0}\right] = \begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix}$$

$$\Phi_{21} = \left[\frac{\partial \mathbf{v}}{\partial \mathbf{r}_0}\right] = \begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix}$$

$$\Phi_{22} = \left[\frac{\partial \mathbf{v}}{\partial \mathbf{v}_0}\right] = \begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix}$$

The state transition matrix satisfies

$$\frac{\partial}{\partial t}\mathbf{\Phi}(t,t_0) = \mathbf{F}(t)\,\mathbf{\Phi}(t,t_0)$$

and $\mathbf{\Phi}(t,t_0) = \mathbf{I}$.

Since a velocity impulse $\Delta\mathbf{v}$ is a discontinuity in velocity only, the state vector immediately following an impulse time t_I is simply

$$\mathbf{x}(t_I) = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} + \Delta\mathbf{v} \end{bmatrix}$$

and the partial derivatives of the state vector with respect to this impulse are

$$\frac{\partial \mathbf{x}}{\partial \Delta\mathbf{v}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

The partial derivatives of the state vector with respect to the impulse time are found from the first order variation according to

$$\delta\mathbf{x} = \int_{t_I}^{t_I+\delta t_I} \begin{bmatrix} \mathbf{v} \\ \mathbf{g} \end{bmatrix} dt - \int_{t_I}^{t_I+\delta t_I} \begin{bmatrix} \mathbf{v} + \Delta\mathbf{v} \\ \mathbf{g} \end{bmatrix} dt = \begin{bmatrix} -\Delta\mathbf{v}\,\delta t_I \\ \mathbf{0} \end{bmatrix}$$

Therefore, for the i^{th} delta-v and impulse time

$$\frac{\partial \mathbf{x}}{\partial t_{I_i}} = \begin{bmatrix} -\Delta\mathbf{v}_i \\ \mathbf{0} \end{bmatrix}$$

Generalized mission constraint equations

Define the mission constraint vector **y** at the final time t_F to be

$$\mathbf{y} = f(\mathbf{x}_F) = \begin{bmatrix} \mathbf{h} \\ \mathbf{e} \end{bmatrix}_F = \begin{bmatrix} \mathbf{r} \times \mathbf{v} \\ \mathbf{v} \times \mathbf{h}/\mu - \hat{\mathbf{r}} \end{bmatrix}_F$$

Linearizing the constraint vector produces

$$\delta\mathbf{y} = \mathbf{H}\delta\mathbf{x}$$

where **H** is the constraint partial derivative matrix given by

$$\mathbf{H} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$$

The equations for the elements of this matrix can be found in Appendix A.

The solution to the linearized constraint vector can be written as

$$\delta\mathbf{y} = \mathbf{H}\delta\mathbf{x}_F = \mathbf{H}\boldsymbol{\Phi}\delta\mathbf{x}_0 + \mathbf{H}\boldsymbol{\Gamma}_x\delta\boldsymbol{\lambda}$$

We can define a fundamental constraint-control matrix as

$$\mathbf{\Gamma} = \mathbf{H}\,\mathbf{\Gamma}_x$$

The relationship between the mission constraints and control variables is

$$\delta\mathbf{y} = \mathbf{\Gamma}\,\delta\boldsymbol{\lambda}$$

with

$$\delta\mathbf{x}_0 = \mathbf{0}$$

The condition that $\delta\mathbf{x}_0$ is zero implies that a reference trajectory can be created at each targeting or guidance update. The $\mathbf{\Gamma}_x$ matrix is propagated during coast arcs using the state transition matrix.

Updated control variables

The control variable vector at successive targeting or guidance updates can be determined from

$$\boldsymbol{\lambda}_{i+1} = \boldsymbol{\lambda}_i + \delta\boldsymbol{\lambda}$$

where

$$\delta\boldsymbol{\lambda} = \mathbf{\Gamma}^{-1}\delta\mathbf{y}$$

The linear system defined by this last equation is solved using the Householder Orthogonal Decomposition method. A key characteristic of this numerical method is that it can be used to solve overdetermined, exact or underdetermined linear systems which may be encountered during different flight phases of a complex space mission.

Mission constraints and classical orbital elements

The following table illustrates the relationship between classical orbital elements and the corresponding mission constraint subsets. In this table, *a* is semimajor axis, *e* is orbital eccentricity, *i* is orbital inclination, ω is argument of periapsis, and Ω is the right ascension of the ascending node (RAAN).

Table 1. Orbital Elements and Mission Constraint Subsets

orbital element	**constraint subset**						
	$\mathbf{h}, r, \gamma$	$\mathbf{h}, r, \gamma$	h, h_z, r, γ	h, h_z, C_3	h, h_z, C_3, e_z	$\mathbf{h}, C_3$	$\mathbf{h}, \mathbf{e}$
a	constrained	constrained	constrained	constrained	constrained	constrained	constrained
e	$= 0$	$= 0$	$= 0$	> 0	> 0	> 0	> 0
i	$= 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	constrained	constrained
ω	undefined	undefined	undefined	free	constrained	free	constrained
Ω	undefined	constrained	free	free	free	constrained*	constrained*

*undefined if $i = 0$

where

$\mathbf{h} =$ angular momentum vector

$\mathbf{e} =$ orbital eccentricity vector

$r =$ geocentric radius

$\gamma =$ flight path angle

$C_3 =$ specific orbital energy

$h =$ angular momentum magnitude

$h_z =$ z-component of angular momentum vector

$e_z =$ z-component of eccentricity vector

This next table illustrates example constraint subsets for each classical orbital element. Constraints for special cases such as circular or equatorial orbits are also noted.

Table 2. Example Mission Constraints

constraint elements	**equivalent constraint**
C_3 or r and v (circular orbit)	a
h and C_3 or r, v and γ (circular orbit)	e
h and h_z or $\mathbf{h}$ $(i = 0)$	i
h, h_z, e_z and C_3	ω
h_x and h_y	Ω

Targeting to an Outgoing Hyperbola

For lunar and interplanetary missions, the "target specs" are usually specified by the specific orbital energy C_3, and the right ascension (RLA) and declination (DLA) of the outgoing asymptote of the launch or departure hyperbola. This section describes the computation of an "energy-scaled" mission constraint useful for these types of missions.

A unit vector in the direction of the departure asymptote is given by

$$\hat{\mathbf{s}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where

$\alpha_\infty =$ right ascension of departure asymptote

$\delta_\infty =$ declination of departure asymptote

The specific orbital energy is computed from the following expression

$$C_3 = v^2 - \frac{2\mu}{r}$$

The outgoing unit asymptote vector is given by

$$\hat{\mathbf{s}} = \frac{1}{1 + C_3 \dfrac{h^2}{\mu^2}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\} = \frac{1}{1 + C_3 \dfrac{p}{\mu}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\}$$

Finally, the targeting or constraint vector consists of the "energy-scaled" unit asymptote vector given by

$$\tilde{\mathbf{s}} = C_3 \, \hat{\mathbf{s}}$$

which is usually evaluated at the end of the final finite-burn orbit transfer maneuver or perhaps at a trajectory interface point (TIP) some time after final stage burnout.

Calculating the mission constraints

This section summarizes the equations used to compute the complete set of mission constraint components from the inertial state vector. In the equations that follow, a "hat" symbol indicates a unit vector.

The angular momentum vector is normal to the orbit plane and is given by the following right-handed cross product

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

The unit angular momentum vector is given by

$$\hat{\mathbf{h}} = \frac{\mathbf{r} \times \mathbf{v}}{|\mathbf{r} \times \mathbf{v}|}$$

The angular momentum magnitude can be computed from

$$h = |\mathbf{r} \times \mathbf{v}| = \sqrt{h_x^2 + h_y^2 + h_z^2} = \sqrt{\mu a \left(1 - e^2\right)}$$

The inertial flight path angle can be determined from

$$\gamma = \sin^{-1}\left(\frac{\mathbf{r} \bullet \mathbf{v}}{|\mathbf{r} \bullet \mathbf{v}|} \right) = \sin^{-1}\left(\hat{\mathbf{r}} \bullet \hat{\mathbf{v}} \right)$$

Flight path angle is positive above the local horizon and negative below. The sine of the flight path angle is often used as a mission constraint.

Twice the specific (per unit mass) orbital energy can be determined from

$$C_3 = v^2 - \frac{2\mu}{r} = -\frac{\mu}{a}$$

The eccentricity vector is in the direction of perigee and is given by

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \mathbf{r} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - \left(\mathbf{r} \bullet \mathbf{v}\right)\mathbf{v}\right]$$

Calculating the classical orbital elements

This section describes the equations used to compute the classical *angular* or *orientation* orbital elements from the state vector. In the equations to follow, the correct quadrant for each angular orbital element can be determined from the corresponding sine and cosine equations using a four quadrant inverse tangent evaluation.

The semimajor axis can be determined from the following equation:

$$a = \frac{r}{2 - rv^2/\mu} = -\frac{\mu h^2}{\left\{(e\mu)^2 - \mu^2\right\}} = \frac{\mu r}{\left(2\mu - r\,v^2\right)} = -\frac{\mu}{C_3}$$

The sine and cosine of orbital inclination can be determined from the following two expressions:

$$\sin i = \left|\hat{\mathbf{k}} \times \hat{\mathbf{h}}\right|$$

$$\cos i = \hat{\mathbf{h}} \bullet \hat{\mathbf{k}} = \frac{h_z}{h} = \hat{h}_z$$

where $\hat{\mathbf{k}}$ is a geocentric unit vector in the direction of the planet's spin axis and is given by $\left[0\;0\;1\right]^T$.

An orbital inclination constraint can be enforced using the z-component and magnitude of the angular momentum vector according to the second equation.

A unit vector in the direction of the ascending node is given by the following cross product

$$\hat{\mathbf{n}} = \hat{\mathbf{k}} \times \hat{\mathbf{h}}$$

The sine and cosine of the right ascension of the ascending node (RAAN) can be determined from

$$\sin\Omega = \left(\hat{\mathbf{i}} \times \hat{\mathbf{n}}\right) \bullet \hat{\mathbf{k}}$$

$$\cos\Omega = \hat{\mathbf{n}} \times \hat{\mathbf{i}}$$

where $\hat{\mathbf{i}}$ is a geocentric unit vector in the direction of the x-axis of the planet-centered-inertial (PCI) coordinate system and is given by $\left[1\ 0\ 0\right]^T$. A RAAN constraint can be enforced using the x and y components of the angular momentum vector according to $\Omega = \tan^{-1}\left(h_x, -h_y\right)$.

The sine and cosine of the argument of periapsis for elliptical and hyperbolic orbits can be determined from the next two expressions

$$\sin\omega = \left(\hat{\mathbf{n}} \times \hat{\mathbf{e}}\right) \bullet \hat{\mathbf{h}}$$

$$\cos\omega = \hat{\mathbf{h}} \bullet \hat{\mathbf{e}}$$

Finally, an expression for the scalar orbital eccentricity in terms of specific orbital energy and angular momentum is given by

$$e = \frac{h\sqrt{C_3 + \mu^2 / h^2}}{\mu}$$

and the eccentricity vector which points in the direction of perigee is

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \hat{\mathbf{r}}$$

Applications

The GLTG algorithm has been used to solve the following orbital transfer problems;

(1) Fixed one and two impulse orbital transfer

For this type of problem, the delta-v components and the impulse times are the control variables. However, the scalar magnitude of each delta-v is fixed.

(2) Minimum one and two impulse orbital transfer

The minimum impulse orbital transfer problems are solved by wrapping an unconstrained minimization routine around the GLTG algorithm. As in the previous example, the delta-v components and the impulse times are the control variables. For the “outside” or minimization loop, the optimization control variables are the impulse times and the objective function is the scalar magnitude of the impulsive maneuver(s).

(3) One and two impulse Lambert trajectories

For the Lambert applications, the impulse times are fixed and the mission constraints consist of the three components of the final position (one impulse) and velocity vector (two impulse).

(4) Single impulse deorbit maneuver

The mission constraints for this problem are the inertial flight path angle and geodetic altitude at the entry interface (EI). The delta-v components and the time of the deorbit impulse are the control variables.

References and Bibliography

"Two-body Linear Guidance Matrices", L. P. Abrahamson and R. G. Stern, NASA CR 70465, June 1965.

"An Analysis and Evaluation of Guidance Modes", C. G. Pfeiffer, NASA CR 113902, May 1970.

"Gamma Guidance for the Inertial Upper Stage", J. W. Hardtla, AIAA 78-1292, August 1978.

"Linear Guidance Laws for Space Missions", W. Tempelman, AIAA Journal of Guidance, Control and Dynamics, Vol. 9, No. 4, July-August 1986.

"Linear and Nonlinear Perturbation Analysis Applied to the Two-body Problem", W. Tempelman, AIAA-1988-4216, AIAA/AAS Astrodynamics Conference, August 15-17, 1988.

Appendix A

Elements of the H Matrix

This appendix summarizes the elements of the H matrix which represent the partial derivatives of the mission orbit constraint vector **y** with respect to the state vector **x**.

position vector

$$\frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \left[\mathbf{r}^T / r \quad \mathbf{0} \right]$$

velocity vector

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \left[\mathbf{0} \quad \mathbf{v}^T / v \right]$$

angular momentum vector

$$\frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \left[\mathbf{I} \times \mathbf{v} \quad \mathbf{r} \times \mathbf{I} \right]$$

angular momentum magnitude

$$\frac{\partial h}{\partial \mathbf{x}} = \left[\mathbf{h}^T \left(\mathbf{I} \times \mathbf{v} \right) / h \quad \mathbf{h}^T \left(\mathbf{r} \times \mathbf{I} \right) / h \right]$$

eccentricity vector

$$\frac{\partial \mathbf{e}}{\partial \mathbf{x}} = \left[\left(v^2 / \mu - 1/r \right) \mathbf{I} + \mathbf{r}\mathbf{r}^T / r^3 - \mathbf{v}\mathbf{v}^T / \mu \quad 2\mathbf{r}\mathbf{v}^T / \mu - \left(\mathbf{v} \bullet \mathbf{r} \right) \mathbf{I} / \mu - \mathbf{v}\mathbf{r}^T / \mu \right]$$

specific orbital energy

$$\frac{\partial C_3}{\partial \mathbf{x}} = \left[2\mu \mathbf{r}^T / r^3 \quad 2\mathbf{v}^T \right]$$

sine of flight path angle

$$\frac{\partial \sin\gamma}{\partial \mathbf{x}} = \left[\hat{\mathbf{v}}^T \left(\mathbf{I}/r - \mathbf{r}\mathbf{r}^T / r^3 \right) \quad \hat{\mathbf{r}}^T \left(\mathbf{I}/v - \mathbf{v}\mathbf{v}^T / v^3 \right) \right]$$

scaled unit asymptote vector

$$\begin{aligned} \frac{\partial \tilde{\mathbf{s}}}{\partial \mathbf{x}} &= C_3 \frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{x}} + \hat{\mathbf{s}} \frac{\partial C_3}{\partial \mathbf{x}} \\ &= \frac{\sqrt{C_3^3}}{\mu e^2} \left\{ \left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \times \mathbf{e} \right) + \left(\mathbf{h} \times \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \right) \right\} - \frac{C_3}{e^2} \frac{\partial \mathbf{e}}{\partial \mathbf{x}} + \left\{ \frac{3}{2C_3} \frac{\partial C_3}{\partial \mathbf{x}} - \frac{2}{e^2} \left(\mathbf{e} \bullet \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \right) \right\} \tilde{\mathbf{s}} + \frac{1}{2e^2} \frac{\partial C_3}{\partial \mathbf{x}} \mathbf{e} \end{aligned}$$