2 Random Variables

Expectation of a r.v.

$$E\left[X\right] = \sum_x x \Pr\left[X = x\right]$$

Expectation of a function of a r.v.

$$E\left[g\left(X\right)\right] = \sum_x g(x) \Pr\left[X = x\right]$$

Probability generating function (discrete random variable)

$$\varphi_X\left(z\right) = E\left[z^X\right] = \sum_{k=0}^{\infty} \Pr\left[X = k\right] z^k$$

Probability generating function (continuous random variable)

$$\varphi_X\left(z\right) = E\left[e^{-zX}\right] = \int_{-\infty}^{+\infty} e^{-zt} f_X(t) dt$$

Law of total probability

$$\Pr\left[A\right] = \sum_k \Pr\left[A|B_k\right] \Pr\left[B_k\right]$$

3 Basic Distributions

Binomial probability density function

$$\Pr\left[X = k\right] = \binom{n}{k} p^k q^{n-k}$$

Poisson probability density function

$$\Pr\left[X = k\right] = \frac{\lambda^k e^{-\lambda}}{k!}$$

Exponential distribution

$$\varphi_X\left(z\right) = \alpha \int_0^{\infty} e^{-\alpha t} e^{-zt} dt = \frac{\alpha}{z + \alpha}$$

Minimum of m i.i.d. random variables

$$\Pr\left[\min_{1 \le k \le m} X_k \le x\right] = 1 - \prod_{k=1}^{m} \Pr\left[X_k > x\right]$$

Maximum of m i.i.d. random variables

$$\Pr\left[\max_{1 \le k \le m} X_k \le x\right] = \prod_{k=1}^{m} \Pr\left[X_k \le x\right]$$

7 The Poisson Process

Poisson distribution

$$\Pr[X(t+s) - X(s) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Mean

$$E[X(t)] = \lambda t$$

Reliability function

$$R(t) = 1 - F_X(t) = \Pr[X > t]$$

Mean time to failure

$$E[X] = \int_0^{\infty} R(t) dt$$

Reliability function of systems in series and parallel

$$R_{ser}(t) = \prod_{j=1}^{n} R_j(t), \quad R_{par}(t) = 1 - \prod_{j=1}^{n} \left(1 - R_j(t)\right)$$

Memoryless property

$$\Pr[\tau_n > s + t | \tau_n > t] = \Pr[\tau_n > s]$$

9 Discrete-Time Markov Chains

Markov chain (iterated)

$$\begin{aligned} &\Pr[X_0 = x_0, .., X_k = x_k] = \\ &\qquad = \prod_{j=1}^{k} \Pr[X_j = x_j | X_{j-1} = x_{j-1}] \Pr[X_0 = x_0] \end{aligned}$$

Probability that starting in state i will ever reach j

$$r_{ij} = \Pr\left[T_j < \infty | X_0 = i\right]$$

Mean return time to state j

$$m_j = E\left[T_j | X_0 = j\right]$$

Average numb. times the state is in j if started in i

$$\begin{aligned} E\left[N\left(j\right)|X_0 = i\right] &= \sum_{n=1}^{\infty} \Pr\left[X_n = j | X_0 = i\right] \\ &= \sum_{n=1}^{\infty} P_{ij}^n \end{aligned}$$

Steady-state vector: $\pi = \pi P$

Steady state vector (componentwise)

$$\pi_j = \sum_{k=1}^{N} P_{kj} \pi_k = \lim_{k \to \infty} P_{ij}^k$$

Fraction of time the chain is in state j during $[1, n]$

$$\begin{aligned} \pi_j &= \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P_{ij}^m \\ &= \lim_{n \to \infty} \frac{E\left[N_n\left(j\right)|X_0 = i\right]}{n} = \frac{r_{ij}}{m_j} \end{aligned}$$

2-state transition probability (from time k to $k+1$)

$$\begin{aligned} \Pr\left[X_{k+1} = 0\right] &= q \Pr\left[X_k = 1\right] + (1-p) \Pr\left[X_k = 0\right] \\ &= q\left(1 - \Pr\left[X_k = 0\right]\right) + (1-p) \Pr\left[X_k = 0\right] \\ &= (1 - p - q) \Pr\left[X_k = 0\right] + q \end{aligned}$$

2-state transition probability (at time k given X_0)

$$\begin{aligned} \Pr\left[X_k = 0\right] &= (1-p-q)^k \Pr\left[X_0 = 0\right] + q \sum_{j=0}^{k-1} (1-p-q)^j \\ &= \frac{q}{p+q} + (1-p-q)^k \left(\Pr\left[X_0 = 0\right] - \frac{q}{p+q}\right) \end{aligned}$$

2-state transition probability: If $|1-p-q| < 1$, then the steady-state vector equals

$$\pi = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$$

10 Continuous-Time Markov Chains

Transition probability matrix

$$\begin{aligned} P_{ij}\left(t\right) &= \Pr\left[X\left(t+\tau\right) = j | X\left(\tau\right) = i\right] \\ &= \Pr\left[X\left(t\right) = j | X\left(0\right) = i\right], \\ P\left(t+u\right) &= P\left(u\right) P\left(t\right) = P\left(t\right) P\left(u\right), \\ P\left(0\right) &= I \end{aligned}$$

Infinitesimal generator (matrix)

$$Q = \begin{bmatrix} -q_1 & q_{12} & \cdots & q_{1N} \\ q_{21} & -q_2 & \ddots & q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \ddots & -q_N \end{bmatrix},$$

$$\begin{aligned} Q &= \lim_{h \to 0} \frac{P\left(h\right) - I}{h} = P'\left(0\right), \\ P'(t) &= P(t) Q \end{aligned}$$

Steady state (balance equations)

$$\pi Q = 0, \quad P_\infty = \lim_{t\to\infty} s(0)\, e^{Qt}$$

Transition probability embedded Markov Chain

$$V_{ij}(h) = \Pr\left[X(h) = j \middle| X(h) \neq i, X(0) = i\right]$$

Limiting process

$$V_{ij} = \lim_{h \downarrow 0} V_{ij}(h) = \frac{q_{ij}}{q_i}$$

11 Applications of Markov Chains

General random walk: steady state

$$\pi_j = p_{j-1}\pi_{j-1} + r_j \pi_j + q_{j+1}\pi_{j+1}$$

General random walk: steady state

$$\pi_j = \frac{\prod_{m=0}^{j-1} \frac{p_m}{q_{m+1}}}{1 + \sum_{k=1}^{N} \prod_{m=0}^{k-1} \frac{p_m}{q_{m+1}}}$$

General random walk: steady state (if $p_k = p$ and $q_k = q$)

$$\pi_j = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}} \quad \text{with} \quad \rho = \frac{p}{q}$$

General birth and death: steady state

$$\pi_j = \begin{cases} \dfrac{1}{1 + \sum_{k=1}^{\infty} \prod_{m=0}^{k-1} \frac{\lambda_m}{\mu_{m+1}}} & \text{if} \quad j = 0, \\ \dfrac{\prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}}{1 + \sum_{k=1}^{\infty} \prod_{m=0}^{k-1} \frac{\lambda_m}{\mu_{m+1}}} & \text{if} \quad j \geq 1. \end{cases}$$

13 General Queueing Theory

Poisson arrivals see time averages (PASTA)

$$\Pr[N_{S;A} = j] = \Pr[N_S = j]$$

Little's Law

$$E[N_S] = \lambda E[T], \quad E[N_Q] = \lambda E[w], \quad E[N_x] = \frac{\lambda}{\mu}$$

14 Queueing Models

Burke's Theorem (M/M/1)

$$\Pr[N_S = j] = (1-\rho)\,\rho^j \quad \text{with} \quad j \geq 0$$
$$E[N_S] = \varphi'_{N_S}(1) = \frac{\rho}{1-\rho}$$

Average number of packets (M/M/1)

$$E\left[N_{S;M/M/1}\right] = \frac{\rho}{1-\rho}$$

Variance of the number of packets (M/M/1)

$$\text{Var}\left[N_{S;M/M/1}\right] = \frac{\rho}{(1-\rho)^2}$$

Average system waiting time

$$E\left[T_{M/M/1}\right] = \frac{E[N_S]}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$$

Average queue waiting time

$$E\left[w_{M/M/1}\right] = \frac{1}{\mu(1-\rho)} - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

Probability of queueing (Erlang C) (M/M/m)

$$\Pr[N_S \geq m] = \frac{\Pr[N_S = 0]\,\lambda^m}{m!\left(1 - \frac{\lambda}{m\mu}\right)\mu^m}$$

Busy probability (M/M/m/m)

$$\Pr[N_S = j] = \begin{cases} \frac{\lambda^j}{j!\mu^j} \Pr[N_S = 0] & \text{if} \quad j \leq m, \\ 0 & \text{if} \quad j > m. \end{cases}$$

Call blocking probability (Erlang B) (M/M/m/m)

$$\Pr[N_S = m] = \frac{\frac{\lambda^m}{m!\mu^m}}{\sum_{j=0}^{m} \frac{\lambda^j}{j!\mu^j}}$$

Average number of packets (M/M/m/m)

$$E[N_S] = \frac{\lambda}{\mu}\left(1 - \Pr[N_S = m]\right)$$

Probability of a filled system / loss (M/M/1/K)

$$\Pr[N_S = K] = \frac{(1-\rho)\,\rho^K}{1-\rho^{K+1}}$$

Average waiting time in the queue (M/G/1)

$$E[\omega] = \frac{\lambda E\left[x^2\right]}{2(1-\rho)}$$

System content in steady state (Pollaczek-Khinchin)

$$S(z) = (1-\rho)\,\frac{(z-1)\,\varphi_x(\lambda - \lambda z)}{z - \varphi_x(\lambda - \lambda z)}$$

15 General characteristics of graphs

Average number of paths with j hops between two nodes

$$E[X_j] = \frac{(N-2)!}{(N-j-1)!}p^j$$

Pdf of the degree D_{rg} of an arbitrary (randomly chosen) node in $G_p(N)$

$$\Pr[D_{rg} = k] = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

16 The shortest path problem

Pgf of the hopcount in the URT

$$\varphi_{h_N}(z) = E\left[z^{h_N}\right] = \frac{\Gamma(N+z)}{\Gamma(N+1)\,\Gamma(z+1)}$$
$$= \frac{1}{N!}\prod_{k=1}^{N-1}(z+k)$$

Hopcount probability (Poisson)

$$\Pr[h_N = k] \sim \frac{(\log N)^k}{Nk!}$$

Average hopcount h_N in a URT

$$E[h_N] = \varphi'_{h_N}(1) = \frac{d}{dz}\log \varphi_{h_N}(z)\Big|_{z=1} = \sum_{l=2}^{N}\frac{1}{l}$$

17 Epidemics in networks

NIMFA differential equation

$$\frac{dv_i(t)}{dt} = -\delta v_i(t) + \beta(1 - v_i(t))\sum_{j=1}^{N} a_{ij} v_j(t)$$

NIMFA epidemic threshold

$$\tau_c^{(1)} = \frac{1}{\lambda_1(A)}$$