1 Discrete Markov process

What	$P(X_{k+1} = x_{k+1} \| X_0 = x_0, \ldots, X_k = x_k) = P(X_{k+1} = x_{k+1} \| X_k = x_k)$
Transition probability	$P_{ij}(k) = Pr(X_{k+1} = j \| X_k = i)$
Stationarity	$P_{ij}(k) = P_{ij}$
State vector	$s[k] = \begin{bmatrix} s_1[k] & s_2[k] & \ldots & s_N[k] \end{bmatrix}$
	$s_i[k] = Pr\left[X_k = i\right]$
	$s[k+n] = s[k]\, P^n$
Reachability	Irreducible vs reducible
Periodicity	$d_i = 1$ vs $d_i > 1$
Hitting time	$T_j = \min_{k \geq 0} \{X_k = j\}$
Reaching and returning	$r_{ij} = Pr(T_j < \infty \| X_0 = i)$
Transience and Recurrence	$r_{jj} < 1$ vs $r_{jj} = 1$
Expected number of visits	$\mathbb{E}\left[N_i(j)\right] = \frac{r_{ij}}{1-r_{jj}} = \sum_{n=1}^{\infty} (P^n)_{ij}$
Steady state	$\pi = \pi P$
	Fundamental Theorem of Markov Chains

2 Continuous Markov process

What	$Pr\left[X(t+\tau) = j \| X(\tau) = i, X(u) = x(u),\, 0 \leq u < \tau\right] = Pr\left[X(t+\tau) = j \| X(\tau) = i\right]$
Infinitesimal generator	$Q := \lim_{h \to 0} \frac{P(h) - \mathbb{I}}{h} = P'(0)$
Stationarity	$P_{ij}(t) = Pr\left[X(t+\tau) = j \| X(\tau) = i\right] = Pr\left[X(t) = j \| X(0) = i\right],\, \forall t > 0$
State vector	$s(t) = \begin{bmatrix} s_1(t) & s_2(t) & \ldots & s_N(t) \end{bmatrix}$
	$s_i(t) = Pr\left[X(t) = i\right]$
	$s(t+\tau) = s(\tau)\, P(t)$
Chapman-Kolmogorov	$P(t+u) = P(t)\, P(u) = P(u)\, P(t)$
Memoryless property and transition times	$\tau_{ij} \sim \mathrm{Exp}(\tau; q_{ij}) = q_{ij}\, e^{-q_{ij}\tau}$
Sojourn time	$\tau_j \sim \mathrm{Exp}(\tau; q_j) = q_j\, e^{-q_j \tau}$
P vs Q	$P'(t) = P(t)\, Q = Q\, P(t)$
	$P(t) = e^{Q\, t}$
s vs Q	$s'(t) = s(t)\, Q$
	$s(t) = s(0)\, e^{Q\, t}$
Steady state	$\pi Q = 0$
Embedded Markov chain	$V_{ij} = \frac{q_{ij}}{q_i}$
	$\Delta t \leq \frac{1}{\max_j q_j}$

3 Poisson process

What	Counting process
	Independent increments
	$Pr\left[X(t+s)-X(s)=k\right]=\frac{e^{-\lambda t}\,(\lambda\,t)^k}{k!}$
When	$P(\tau_n=t_n-t_{n-1};\lambda)=\mathrm{Exp}(\tau;\lambda)$
Equivalence what and when	$\{\tau_n>s\}\equiv\{X(t_{n-1}+s)-X(t_{n-1})=0\}$
Memoryless property	$Pr\left[\tau_n>s+t\vert\tau_n>s\right]=Pr\left[\tau_n>t\right]$
Arrival time n-th event	$\Pr\left[\sum_{k=1}^n\tau_k=t\right]=\mathrm{Erlang}(t;n,\lambda)$
One event per small interval	$Pr\left[X(t)=1\right]=\lambda\,h+o(h)$
	$Pr\left[X(h)>1\right]=o(h)$
Uniform time 1 event	$Pr\left[\tau_1<s\vert X(t)=1\right]=\frac{s}{t}$
Non-homogenous	$Pr\left[X(t)-X(s)=k\right]=\mathrm{Poisson}(k;\Lambda(t)-\Lambda(s))$
	$\Lambda(t)=\int_0^t\lambda(u)\,du$

4 Queuing theory

M/M/1	$\pi_i=(1-\rho)\,\rho^i$
Traffic intensity	$\rho=\frac{\lambda}{\mu}$
Individuals in the system	$\mathbb{E}\left[N_S(t)\right]=\mathbb{E}\left[\mathrm{geom}(\rho)\right]=\frac{\rho}{1-\rho}$
Little's law	$\mathbb{E}\left[N_S\right]=\lambda\,\mathbb{E}\left[T\right]$
M/M/∞	$q_{i,i-1}=i$

5 Epidemics

Compartmental model	$\frac{dS(t)}{dt}=-\beta\,I(t)\,S(t)+\delta\,I$
	$\frac{dI(t)}{dt}=\beta\,I(t)\,S(t)-\delta\,I(t)$
Steady state	$\pi_1\,(\delta-\beta\,\pi_0)=0$
	$\pi_0+\pi_1=1$
SIS continuous-time (network)	$\frac{d\mathbb{E}[X_i(t)]}{dt}=\epsilon-(\delta+\epsilon)\,\mathbb{E}\left[X_i(t)\right]+\beta\sum_k a_{ik}\mathbb{E}\left[X_k(t)\right]-\beta\sum_k a_{ik}\mathbb{E}\left[X_i(t)\,X_k(t)\right]$
Epidemic threshold	$\frac{\beta_c}{\delta_c}=\tau_c\geq\tau_c^{(1)}=\frac{1}{\lambda_1}$
NIMFA approximation	$\frac{dv_i(t)}{dt}=-\delta\,v_i(t)+(1-v_i(t))\left[\beta\sum_k a_{ik}v_k(t)+\epsilon\right]$
Steady state	$v_{i\infty}=1-\frac{1}{1+\tau\sum_k a_{ik}v_{k\infty}}$
Exact vs NIMFA	$\frac{d\mathbb{E}[X_i(t)]}{dt}\leq\frac{dv_i(t)}{dt}$