Quantum Mechanics

Sample exam 2

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Question 1 – Perturbation theory

Consider a hydrogen atom with a finite-sized nucleus. Model the proton as a uniformly charged ball of radius b with total charge e. The corresponding potential is given by

$$V(r) = \begin{cases} -\frac{e^2}{2b}\left(3 - \frac{r^2}{b^2}\right) & r \leq b \\ -\frac{e^2}{r} & r > b. \end{cases}$$

(a) Find the corresponding perturbation to the hydrogen atom with point nucleus.

(b) Calculate the first-order correction to the ground-state energy and show that the leading order is proportional to $(b/a)^2$.

(c) Does the perturbation affect higher angular-momentum states more or less? Give a simple argument without performing any calculation.

(d) Now consider the first-excited state. Is the original basis of the 4-dimensional degenerate subspace $\{2s, 2p_x, 2p_y, 2p_z\}$ diagonal in the perturbation? Why?

The normalized ground-state wave function of the (unperturbed) hydrogen atom is given by (a is the Bohr radius)

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.$$

Question 2 – Born approximation

An electron scatters at an electric dipole composed of charges $-Ze$ and Ze separated by the vector $\boldsymbol{d}$.

(a) Make a drawing and find the scattering potential.

(b) Calculate the differential cross section in the first Born approximation. You can use the result for Rutherford scattering $\left(\frac{d\sigma}{d\Omega}\right)_R$.

Question 3 – Partial waves

Consider p-wave scattering from a hard sphere of radius a.

(a) Find the phase shift $\delta_1(k)$ for p-wave scattering and evaluate it in the low-energy limit up to lowest order.

(b) Compare this result with the s-wave phase shift. Which channel dominates in the low-energy limit?

The radial wave function of the l-th partial wave outside the scattering region is

$$R_l^{>}(r) = e^{i\delta_l} h_l^{(1)}(kr) + e^{-i\delta_l} h_l^{(2)}(kr),$$

and the spherical Hankel functions are given by (for x real)

$$h_l^{(1)}(x) = -ix^l \left(-\frac{1}{x}\frac{d}{dx}\right)^l \frac{e^{ix}}{x}, \qquad h_l^{(2)}(x) = h_l^{(1)}(x)^*.$$