

### Topological Systems: Worksheet 3

*Majorana bound states from helical edge states*

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The minimal model for the helical edge state of a two-dimensional  $\mathbb{Z}_2$  topological insulator can be written as

$$H_0 = \sum_k c_k^\dagger \hbar v k s_z c_k,$$

where  $v$  is the Fermi velocity,  $s_z$  is a spin-1/2 Pauli matrix, and  $c_k = (c_{k\uparrow}, c_{k\downarrow})^t$ . The helical edge state is protected by time-reversal and charge-conservation symmetry. Magnetic impurities along the edge break time-reversal symmetry and lead to single-particle backscattering between the Kramers partners, gapping out the helical edge. Such a process can be included as,

$$H_M = \sum_k c_k^\dagger M s_x c_k.$$

Another way to open gap without breaking time-reversal symmetry, is by placing a superconductor on the edge, which breaks charge-conservation symmetry. An incoming spin-up electron can be reflected at the superconductor as an outgoing spin-down hole. In this process, called Andreev reflection, a Cooper pair of charge  $2e$  is transmitted to the superconductor. The Hamiltonian that describes both processes can be written as

$$H_{\text{BdG}} = \frac{1}{2} \sum_k \Psi_k^\dagger \mathcal{H}_{\text{BdG}}(k) \Psi_k,$$

$$\mathcal{H}_{\text{BdG}}(k) = \tau_z (\hbar v k s_z - \mu s_0) + M \tau_0 s_x + \Delta_0 (\cos \phi \tau_x + \sin \phi \tau_y) s_0,$$

in the Nambu basis  $\Psi_k = (c_{k\uparrow}, c_{k\downarrow}, -c_{-k\downarrow}^\dagger, c_{-k\uparrow}^\dagger)^t$  and where  $\tau_i s_j \equiv \tau_i \otimes s_j$ . Here  $\Delta = \Delta_0 e^{i\phi}$  is the superconducting pairing potential and  $\tau_i$  are Pauli matrices in the electron-hole subspace.

- (1) Find the excitation spectrum  $\varepsilon(k)$  of  $\mathcal{H}_{\text{BdG}}(k)$  and make a sketch of the energy bands for  $M = \Delta = 0$ ,  $M = 0$ , and  $\Delta = 0$ .
- (2) Consider a system with  $M(x) = M_0 \theta(-x)$  and  $\Delta(x) = \Delta_0 \theta(x)$  with  $\Delta_0 > 0$ , where  $\theta(x)$  is the step function, as illustrated in Fig. 1, and find the eigenstates in the Nambu basis in region I ( $x < 0$ ) and region II ( $x > 0$ ) for  $\varepsilon = 0$ . Hint: use the *ansatz*  $\psi(x) = e^{\lambda x} \psi_0$  and note that the electron and hole equations decouple for  $\Delta = 0$ .
- (3) The general solution at  $\varepsilon = 0$  can be written as (setting  $\hbar v = 1$ )

$$\psi_I(x) = a_1 \begin{pmatrix} M_0 \\ \mu + i\lambda_I \\ 0 \\ 0 \end{pmatrix} e^{\lambda_I x} + a_2 \begin{pmatrix} M_0 \\ \mu - i\lambda_I \\ 0 \\ 0 \end{pmatrix} e^{-\lambda_I x} + b_1 \begin{pmatrix} 0 \\ 0 \\ -M_0 \\ \mu + i\lambda_I \end{pmatrix} e^{\lambda_I x} + b_2 \begin{pmatrix} 0 \\ 0 \\ -M_0 \\ \mu - i\lambda_I \end{pmatrix} e^{-\lambda_I x}$$

for region I, where  $\lambda_I = \sqrt{M_0^2 - \mu^2}$ , and for region II

$$\psi_{II}(x) = c_1 \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} e^{\lambda_{II+}x} + c_2 \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix} e^{-\lambda_{II+}^*x} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix} e^{\lambda_{II-}x} + c_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix} e^{-\lambda_{II-}^*x},$$

where  $\lambda_{II\pm} = \Delta_0 \pm i\mu$ . Now assume the Fermi energy lies in the magnetic gap  $M_0^2 > \mu^2$  and that  $\Delta_0 > 0$ . Since there are no propagating modes in the energy gap, the eigenmodes at  $\varepsilon = 0$  are evanescent modes in both regions. Select the correct modes in both regions by noting that the wave function has to be normalizable.

- (4) The second boundary condition is the continuity of the Nambu spinor at  $x = 0$ . Show that the zero mode is a solution and find its wave function, which can be written as

$$\psi_0(x) = \frac{a}{2} \begin{pmatrix} e^{i\mu\theta(x)x} \\ e^{i\chi}e^{-i\mu\theta(x)x} \\ -ie^{i\mu\theta(x)x} \\ ie^{i\chi}e^{-i\mu\theta(x)x} \end{pmatrix} \exp \left[ \left( \sqrt{M_0^2 - \mu^2} \theta(-x) - \Delta_0 \theta(x) \right) x \right],$$

where  $e^{i\chi} = (\mu + i\sqrt{M_0^2 - \mu^2})/M_0$  and  $|a|^2 = 2\Delta_0\sqrt{M_0^2 - \mu^2}/(\sqrt{M_0^2 - \mu^2} + \Delta_0)$  is the normalization constant.

- (5) Write down the quasiparticle operator  $\gamma_0 = \int dx \psi_0^\dagger(x) \Psi(x)$  of the Majorana mode. Show that  $\bar{\gamma}_0^\dagger = \bar{\gamma}_0$  and  $\bar{\gamma}_0^2 = 1$  with  $\bar{\gamma}_0 = \sqrt{2}e^{i\pi/4}e^{i\chi/2}\gamma_0$ . Hint: use the anticommutator relations  $\{\hat{\psi}_s(x), \hat{\psi}_{s'}(x')\} = 0$  and  $\{\hat{\psi}_s(x), \hat{\psi}_{s'}^\dagger(x')\} = \delta_{ss'}\delta(x - x')$ .
- (6) Now perform the same calculation for  $M = 0$  and

$$\Delta(x) = \begin{cases} \Delta_0 & x < 0 \\ \Delta_0 e^{i\phi} & x > 0 \end{cases},$$

and show that there are two zero modes for  $\phi = \pi$ . However, these solutions are not independent. Show that their quasiparticle operators can be written as  $f$  and  $f^\dagger$  and construct two Majorana operators:  $\gamma_1 = f + f^\dagger$  and  $\gamma_2 = -i(f - f^\dagger)$ . These Majoranas are localized in the same place but are protected by time-reversal symmetry, which is preserved for  $\phi = 0$  or  $\pi$ .

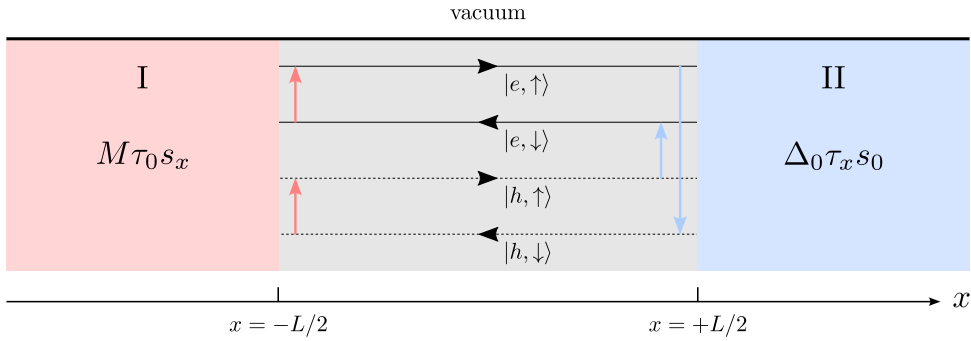


Figure 1: Hybrid junction in a helical edge state of length  $L$ . Here, we only consider the short-junction limit  $L \rightarrow 0$ . The spin-flip and Andreev reflection processes at the magnetic and superconducting region are denoted by the vertical arrows, respectively.