

Bosonization: Worksheet 2

Operators in second quantization and exchange correction

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1 Current-density operator in second quantization

The current-density operator can be written in first quantization as

$$\hat{\mathbf{J}}(\mathbf{x}) = \sum_{n=1}^N \hat{\mathbf{j}}(\mathbf{x}, \hat{\mathbf{r}}_n), \quad \hat{\mathbf{j}}(\mathbf{x}, \hat{\mathbf{r}}) = \frac{1}{2} \left\{ \frac{\hat{\mathbf{p}}}{m}, \delta(\mathbf{x} - \hat{\mathbf{r}}) \right\},$$

where $\hat{\mathbf{p}} = -i\hbar\nabla_{\mathbf{r}}$ and \mathbf{x} is a parameter indicating the position. Remember that a one-body operator $\hat{O}_1 = \sum_{n=1}^N \hat{o}_n$, is represented in second quantization as

$$\hat{O}_1 = \sum_{\mu, \nu} \langle \mu | \hat{o} | \nu \rangle \hat{c}_{\mu}^{\dagger} \hat{c}_{\nu}.$$

- (1) Show that the current-density operator in second quantization, in the representation with field operators in position space, is given by

$$\hat{\mathbf{J}}(\mathbf{x}) = \sum_{\sigma} \int d^d r \hat{\psi}_{\sigma}^{\dagger}(\mathbf{r}) \hat{\mathbf{j}}(\mathbf{x}, \mathbf{r}) \hat{\psi}_{\sigma}(\mathbf{r}),$$

and work out the integral. Hint: Use periodic boundary conditions.

- (2) Write down the current-density operator with field operators in momentum space. Hint: Use a resolution of identity and $\langle \mathbf{r} | \mathbf{k} \rangle = e^{i\mathbf{k} \cdot \mathbf{r}} / \sqrt{V}$. The final result can be written as

$$\hat{\mathbf{J}}(\mathbf{x}) = \frac{1}{V} \sum_{\sigma} \sum_{\mathbf{k}, \mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} \frac{\hbar \mathbf{k}}{m} \hat{c}_{\mathbf{k} - \frac{\mathbf{q}}{2}\sigma}^{\dagger} \hat{c}_{\mathbf{k} + \frac{\mathbf{q}}{2}\sigma}.$$

- (3) Calculate the current-density operator in momentum representation,

$$\hat{\mathbf{J}}_{\mathbf{q}} = \int d^d x e^{-i\mathbf{q} \cdot \mathbf{x}} \hat{\mathbf{J}}(\mathbf{x}).$$

- (4) Find an expression for $\mathbf{q} \cdot \hat{\mathbf{J}}_{\mathbf{q}}$ for the Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}, \sigma} E_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^{\dagger} \hat{c}_{\mathbf{k}\sigma},$$

by Fourier transforming the continuity equation, $\partial_t \hat{\rho}(\mathbf{x}, t) + \nabla \cdot \hat{\mathbf{J}}(\mathbf{x}, t) = 0$, where

$$\hat{\rho}(\mathbf{x}) = \sum_{\sigma} \hat{\psi}_{\sigma}^{\dagger}(\mathbf{x}) \hat{\psi}_{\sigma}(\mathbf{x}) = \frac{1}{V} \sum_{\sigma} \sum_{\mathbf{k}, \mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} \hat{c}_{\mathbf{k} - \mathbf{q}\sigma}^{\dagger} \hat{c}_{\mathbf{k}\sigma},$$

and calculating $\hbar \partial_t \hat{\rho}_{\mathbf{q}} = i[\hat{H}, \hat{\rho}_{\mathbf{q}}]$. Show that you recover the result of (3) when $E_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}$.

2 Spin operators in second quantization

The spin-density operator can be written in first quantization as

$$\hat{\mathbf{S}}(\mathbf{x}) = \sum_{n=1}^N \delta(\mathbf{x} - \hat{\mathbf{r}}_n) \hat{\mathbf{s}}_n,$$

where $\hat{\mathbf{s}}_n$ are the spin operators of the n -th particle. Find $\hat{\mathbf{S}}(\mathbf{x})$ and $\hat{\mathbf{S}}_{\mathbf{q}}$ in second quantization, in the representation with field operators in momentum space. Finally, also calculate the spin operator in second quantization: $\hat{\mathbf{S}} = \int d^d x \hat{\mathbf{S}}(\mathbf{x}) = \hat{\mathbf{S}}_{\mathbf{q}=\mathbf{0}}$.

3 Exchange integral

Calculate the exchange integral

$$\Sigma_{\text{ex}}(\mathbf{k}) = -\frac{1}{V} \sum_{\mathbf{q} \neq 0} V_{\text{ee}}(\mathbf{q}) n_{\mathbf{k}-\mathbf{q}},$$

where $n_{\mathbf{k}-\mathbf{q}} = \Theta(|\mathbf{k} - \mathbf{q}| \leq k_F)$ is the occupation number at zero temperature and $V_{\text{ee}}(\mathbf{q}) = 4\pi e^2/q^2$ is the Fourier transform of the Coulomb potential.

- (1) Shift the integrand by \mathbf{k} and then use spherical coordinates with the z -axis along \mathbf{k} . Also get rid of units in the integral by using momenta in units of k_F .
- (2) Solve the following integral first,

$$\int_{-1}^1 \frac{du}{a + bu},$$

where a and b are functions of k and q . This integral converges only for $a > b$. Check that this is always the case.

- (3) The final integral should be over the momentum q/k_F and is a bit tricky. Hint: Keep in mind that $\ln x^2 = 2 \ln |x|$ for $x \in \mathbb{R}$.

The final result is given by

$$\Sigma_{\text{ex}}(k) = -\frac{2e^2 k_F}{\pi} F(k/k_F),$$

with

$$F(y) = \frac{1}{2} + \frac{1-y^2}{4y} \ln \left| \frac{1+y}{1-y} \right|,$$

which is shown in the figure below.

