Analytical Mechanics: Worksheet 4

Langrange multipliers

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1 Theory

Consider a mechanical system with Lagrangian $L = L(q_i, \dot{q}_i, t)$ and a holonomic constraint,

$$f\left(q_1, \ldots, q_n, t\right) = 0.$$

Instead of using this constraint to directly reduce the number of independent coordinates by one, we can account for it using the method of *Lagrange multipliers*. To this end, first take the variation of the constraint,

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f}{\partial q_n}\delta q_n = \delta 0 = 0,$$

which holds for all times t. If we multiply this equation with an unknown function $\lambda = \lambda(t)$ and integrate over time, the right-hand side still vanishes. Adding this result to the variation of the action:

$$\delta S \to \delta S + \int dt\, \lambda\, \delta f = \int dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \lambda \frac{\partial f}{\partial q_i} \right) \delta q_i = 0,$$

yields n equations

$$\frac{\delta L}{\delta q_i} \equiv \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\lambda \frac{\partial f}{\partial q_i},$$

where the right-hand side is a constraint force (e.g. normal forces or tension).

Hence, together with the constraint we have a total of $n+1$ equations for the $n+1$ functions $\{q_1, \ldots, q_n, \lambda\}$ where λ is called a *Lagrange multiplier*. The method of Lagrange multipliers transforms a constrained problem to a free variational problem with an extra degree of freedom λ:

$$\overline{L} = L + \lambda f, \qquad \frac{\delta \overline{L}}{\delta q_i} = \frac{\delta \overline{L}}{\delta \lambda} = 0,$$

where the variation with respect to λ gives the constraint.

2 Applications

2.1 Planar pendulum

The Lagrangian of a point mass moving in a vertical plane is given by

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy.$$

Let us apply an additional constraint such that the distance of the point mass to the origin remains fixed:

$$f(x, y) = \sqrt{x^2 + y^2} - l = 0.$$

(a) Use a Lagrange multiplier λ and find the equations of motion for x and y.

(b) Find a relation between the constraint force (tension) that keeps the point mass along a circle, the constraint, and the Lagrange multiplier λ.

(c) Define an effective potential energy that includes the contribution from the constraint.

2.2 Catenary

The method of Lagrange multipliers can also be used to enforce constraints in a static equilibrium problem. As an example, consider a rope or chain with a constant mass density ρ and length l. Our goal is to find the shape $y(x)$ that minimizes the potential energy in a uniform gravitational field. This is called the catenary problem.

The potential energy of the rope is given by

$$V = \rho g \int ds\, y(s),$$

where ds is the line element along the rope. A convenient parameterization of the curve is given by $(x, y(x))$ with

$$ds = dx\sqrt{1 + y'(x)^2},$$

where $y' = dy/dx$. The constraint of fixed length can be expressed as

$$f = \int ds - l = \int dx\sqrt{1 + y'(x)^2} - l = 0,$$

which is called an *isoperimetric* constraint. Hence, using the method of Lagrange multipliers, we have to minimize the functional $V + \lambda f$ with respect to $y(x)$.

(a) Define an effective Lagrangian and write down the Euler-Lagrange equation for $y(x)$.

(b) Instead of directly solving the equation of motion, note that the Lagrangian does not depend explicitly on x. Hence, the "Hamiltonian"

$$H \equiv L - y' \frac{\partial L}{\partial y'} = \text{constant},$$

because $dH/dx = -\partial L/\partial x$. Prove this and calculate the Hamiltonian. In this context, this is also called the *Beltrami identity*.

(c) Define constants a and b such that $H = \rho g a$ and $\lambda = \rho g b$ and obtain a solution by solving the Beltrami identity for $y(x)$. Hint: $\int \frac{du}{\sqrt{u^2-1}} = \operatorname{arcosh} u$.

(d) Consider a symmetric solution $y(-x) = y(x)$ and set $b = 0$. Now assume that the rope is anchored between two points at $x = \pm w/2$. Find an equation for l in terms of a and w.