

Bosonization: Worksheet 7

Green's function for free bosons

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1 Green's function for free bosons

In the previous exercise sheet we showed that

$$\psi_s(x) = (2\pi a)^{-1/2} U_s e^{i\frac{2\pi}{L}(N_s - \frac{1}{2})x} e^{i\sqrt{\pi}[\tilde{\varphi}(x) + \epsilon_s \tilde{\vartheta}(x)]},$$

with $a = 0^+$ which is needed to regularize certain sums (because the right-hand side is not normal ordered) and

$$\tilde{\varphi}(x) = i \sum_{q \neq 0} \frac{\text{sgn}(q) e^{-iqx}}{\sqrt{2L|q|}} (b_{-q}^\dagger + b_q) e^{-a|q|/2}, \quad \tilde{\vartheta}(x) = i \sum_{q \neq 0} \frac{e^{-iqx}}{\sqrt{2L|q|}} (b_{-q}^\dagger - b_q) e^{-a|q|/2},$$

where (for $q \neq 0$)

$$b_q = \left(\frac{2\pi}{|q|L} \right)^{1/2} (\theta(q) \rho_{qL} + \theta(-q) \rho_{qR}),$$

which is different from the result obtained in worksheet 3, because we changed the definition of ρ_{qs} in worksheet 5 to match the conventions of the theory course. These fields are related to the fields $\varphi(x)$ and $\vartheta(x)$ defined in worksheet 5 as follows:

$$\varphi(x) = \frac{\sqrt{\pi}}{L} (N_R + N_L) + \tilde{\varphi}(x), \quad \vartheta(x) = \frac{\sqrt{\pi}}{L} (N_R - N_L) + \tilde{\vartheta}(x).$$

Note that $\tilde{\varphi}(x)$ and $\tilde{\vartheta}(x)$ have the same commutation relations as $\varphi(x)$ and $\vartheta(x)$ and become equal in the limit $L \rightarrow \infty$.

- (1) Use the Baker-Hausdorf theorem to show that $b_q(t) \equiv e^{-iH_0 t} b_q e^{iH_0 t} = b_q e^{-iv_F |q|t}$, where the free Hamiltonian can be written as

$$H_0 = \sum_{q \neq 0} v_F |q| b_q^\dagger b_q + \frac{\pi v_F}{L} (N_R^2 + N_L^2),$$

where we have taken anti-periodic boundary conditions.

- (2) Calculate the boson Green's functions $\mathcal{G}_{\varphi\varphi}(x, t) = \langle \tilde{\varphi}(x, t) \tilde{\varphi}(0, 0) \rangle$, $\mathcal{G}_{\vartheta\vartheta}(x, t) = \langle \tilde{\vartheta}(x, t) \tilde{\vartheta}(0, 0) \rangle$, $\mathcal{G}_{\varphi\vartheta}(x, t) = \langle \tilde{\varphi}(x, t) \tilde{\vartheta}(0, 0) \rangle$, and $\mathcal{G}_{\vartheta\varphi}(x, t) = \langle \tilde{\vartheta}(x, t) \tilde{\varphi}(0, 0) \rangle$ at zero temperature. Hint: $\sum_{n=1}^{\infty} z^n / n = -\ln(1 - z)$ for $|z| < 1$.
- (3) In the limit $L \rightarrow \infty$, the *zero-mode* contributions N_s/L can be neglected. Show that in this case, the fermion Green's function can be obtained from the boson Green's functions using the expression for $\psi_s(x)$ in terms of $\varphi(x)$ and $\vartheta(x)$. Hint: $\langle e^{\lambda_1 B_1} e^{\lambda_2 B_2} \rangle = e^{\langle \lambda_1 B_1 \lambda_2 B_2 + \frac{1}{2} (\lambda_1^2 B_1^2 + \lambda_2^2 B_2^2) \rangle}$, where $B_{1,2}$ are linear in boson operators.

2 Useful relation for boson operators

Show that for A and B linear in boson operators, we have

$$\langle e^A B \rangle = e^{\frac{1}{2} \langle A^2 \rangle} \langle AB \rangle.$$

This is a generalization of a theorem which was proven in Worksheet 4. To proceed, first consider the quantity $\langle e^A e^{\epsilon B} \rangle$ and write it as the expectation value of a single exponent using one of the theorems from Worksheet 1. Then expand both sides up to first order in ϵ and identify like terms to obtain the desired result. Hint: Make use of an important theorem for operators linear in boson operators and note that $[A, B] \in \mathbb{C}$ for A and B linear in boson operators.