Bosonization: Worksheet 4

Green's function for free fermions

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1 Thermal average of boson operators

Consider a free boson Hamiltonian,

$$H = \sum_i \hbar\omega_i \left(b_i^\dagger b_i + \frac{1}{2} \right),$$

where we put $\hbar = 1$, $\omega_i > 0$, b_i and $b_i^\dagger$ are boson operators with $[b_i, b_j^\dagger] = \delta_{ij}$. The goal of this exercise is to prove that the thermal average of e^ϕ, where $\phi = \sum_i(\lambda_i b_i + \tilde{\lambda}_i b_i^\dagger)$ is linear in the boson operators and $\lambda_i, \tilde{\lambda}_i \in \mathbb{C}$, obeys the following relation:

$$\langle e^\phi \rangle = e^{\frac{1}{2}\langle\phi^2\rangle}, \qquad \langle O \rangle \equiv \frac{\mathrm{Tr}(e^{-\beta H} O)}{Z},$$

where Z is the partition function, $Z = \mathrm{Tr}(e^{-\beta H})$ and β is the inverse temperature.

(1) Show that the partition function can be written as $Z = \prod_i Z_i$, where $Z_i = \mathrm{Tr}_i(e^{-\beta H})$ and the trace Tr_i is over the subspace spanned by the ith boson. Furthermore, show that

$$Z_i = \frac{x_i^{1/2}}{1 - x_i}, \qquad x_i \equiv e^{-\beta\omega_i} < 1.$$

Hint: Use the complete basis of eigenstates of H to evaluate the trace. The geometric series $\sum_{n=0}^\infty z^n = \frac{1}{1-z}$ with $z \in \mathbb{C}$ converges only for $|z| < 1$.

(2) Show that $\langle b_i^\dagger b_i \rangle = (x_i^{-1} - 1)^{-1}$ (Bose distribution). Use this result to show that

$$\langle \phi^2 \rangle = \sum_i \lambda_i \tilde{\lambda}_i \coth \frac{\beta\omega_i}{2}.$$

(3) Use the following theorem that we proved in the first exercise sheet,

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \qquad [A, B] \text{ commutes with } A \text{ and } B,$$

to show that

$$\langle e^\phi \rangle = f(\lambda_i, \tilde{\lambda}_i) e^{-\frac{1}{2}\sum_i \lambda_i \tilde{\lambda}_i},$$

where $f(\lambda_i, \tilde{\lambda}_i) = \langle e^{\sum_i \lambda_i b_i} e^{\sum_i \tilde{\lambda}_i b_i^\dagger} \rangle$. Use the same theorem with $A \leftrightarrow B$ to show that

$$f(\lambda_i, \tilde{\lambda}_i) = \langle e^{\sum_i \tilde{\lambda}_i b_i^\dagger} e^{\sum_i \lambda_i b_i} \rangle e^{\sum_i \lambda_i \tilde{\lambda}_i}.$$

(4) Use the fact that the trace is invariant under cyclic permutation and (prove it)

$$e^{\beta H} e^A e^{-\beta H} = e^{e^{\beta H} A e^{-\beta H}} \qquad \text{for any operator } A,$$

together with the Baker-Hausdorff theorem to evaluate $e^{\beta H} A e^{-\beta H}$, to show that $f(\lambda_i, \tilde{\lambda}_i) = f(\lambda_i x_i, \tilde{\lambda}_i) e^{\sum_i \lambda_i \tilde{\lambda}_i}$. Now apply this relation $n + 1$ times to show

$$f(\lambda_i, \tilde{\lambda}_i) = f(\lambda_i x_i^{n+1}, \tilde{\lambda}_i) e^{\sum_i \lambda_i \tilde{\lambda}_i (1 + x_i + \cdots + x_i^n)}.$$

(5) Take the limit $n \to \infty$ and remember that $\omega_i > 0$, to finally show that

$$\langle e^\phi \rangle = f(\lambda_i, \tilde{\lambda}_i) e^{-\frac{1}{2}\sum_i \lambda_i \tilde{\lambda}_i} = e^{\frac{1}{2}\sum_i \lambda_i \tilde{\lambda}_i \coth \frac{\beta\hbar\omega_i}{2}} = e^{\frac{1}{2}\langle\phi^2\rangle}.$$

(6) Use the result from (5) together with the theorem in (3) to show that

$$\langle e^{-i\phi(x)} e^{i\theta(x')} \rangle = e^{\frac{1}{2}[\phi(x),\theta(x')]} e^{-\frac{1}{2}\langle (\phi(x)-\theta(x'))^2 \rangle},$$

where $\phi(x)$ and $\theta(x)$ are operators that are linear in the boson operators with coefficients that are functions of x.

2 Green's function for free fermions

Consider the Hamiltonian

$$H_0 = \sum_{k,s} E_{ks} :c_{ks}^\dagger c_{ks}:,$$

where $E_{ks} = v_F \epsilon_s k$, c_{ks} and $c_{ks}^\dagger$ are fermions operators with $\{c_{ks}, c_{k's'}^\dagger\} = \delta_{ss'}\delta_{kk'}$, and :: indicates normal ordering with respect to the vacuum $|0\rangle$, defined as the state where all single-particle states with $\epsilon_s k \le 0$ are occupied and those with $\epsilon_s k > 0$ are unoccupied. Use *anti-periodic boundary conditions* so that $k = \frac{2\pi}{L}\left(n + \frac{1}{2}\right)$ with $n \in \mathbb{Z}$.

(1) Calculate the fermion partition function $Z = \mathrm{Tr}(e^{-\beta H_0})$. The result is given by

$$Z = \prod_s \prod_{\epsilon_s k > 0} Z_{ks}^2, \qquad Z_{ks} = 1 + e^{-\beta E_{ks}}.$$

Note that Z only converges because of the normal ordering in H_0. Hint: The state $|k,s\rangle$ can either be empty or occupied.

(2) Show that (Fermi distribution)

$$\langle c_{ks}^\dagger c_{k's'} \rangle = \frac{\delta_{ss'}\delta_{kk'}}{e^{\beta E_{ks}} + 1},$$

and take the zero-temperature limit $\beta \to \infty$ to show that $\langle c_{ks}^\dagger c_{ks} \rangle = \Theta(-\epsilon_s k)$, where $\Theta(x)$ is the Heaviside step function.

(3) Use the Baker-Hausdorff theorem to calculate $c_{ks}(t) = e^{iH_0 t} c_{ks} e^{-iH_0 t}$.

The time-ordered fermion Green's function $G_{ss'}(x,t) \equiv \langle \mathcal{T} \psi_s(x,t) \psi_{s'}^\dagger(0,0) \rangle$,

$$G_{ss'}(x,t) = \Theta(t) G_{ss'}^{>}(x,t) + \Theta(-t) G_{ss'}^{<}(x,t),$$

defined for $t \neq 0$ and where

$$G_{ss'}^{>}(x,t) \equiv \langle \psi_s(x,t) \psi_{s'}^\dagger(0,0) \rangle,$$
$$G_{ss'}^{<}(x,t) \equiv -\langle \psi_{s'}^\dagger(0,0) \psi_s(x,t) \rangle,$$

are only defined for $t > 0$ and $t < 0$, respectively, and are called the advanced and retarded fermion Green's functions with

$$\psi_s(x,t) = \frac{1}{\sqrt{L}} \sum_k e^{ikx - a|k|/2} c_{ks}(t),$$

where $a > 0$ is an infinitesimal parameter needed to regularize ultraviolet ($|k| \to \infty$) divergent momentum sums that arise in certain non-normal-ordered expressions and commutators. It can be interpreted as a kind of effective bandwidth.

(4) Calculate the time-ordered fermion Green's function at zero temperature. With anti-periodic boundary conditions, the final result should be $[\sigma = \mathrm{sgn}(t)]$

$$G_{ss'}(x,t)|_{T=0} = \frac{1}{2\pi i} \frac{\delta_{ss'}}{\frac{L}{\pi} \sin\left[\frac{\pi}{L}(v_F t - \epsilon_s x - i\sigma a)\right]} \xrightarrow{L\to\infty} \frac{1}{2\pi i} \frac{\delta_{ss'}}{v_F t - \epsilon_s x - i\sigma a}.$$