# Quantum Mechanics

*Sample exam 1*

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## Question 1 – Perturbation theory

Consider a two-dimensional harmonic oscillator with the perturbation $V = x^2 y^2$.

(a) Find the energy correction to the second-excited state up to first order, together with the corresponding eigenstates. Draw an energy diagram.

(b) Now consider the perturbation $V = x^2 f(y)$, where $f(y)$ is an *analytic* (operator-valued) function. Show that there is no first-order correction for the second-excited state if $f(y)$ is an odd function, i.e. for $f(-y) = -f(y)$.

Recall that the position operator can be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left( a + a^\dagger \right),$$

where $a$ and $a^\dagger$ are the annihilation and creation operators, for which

$$\begin{aligned} a \left| n \right\rangle &= \sqrt{n} \left| n - 1 \right\rangle \\ a^\dagger \left| n \right\rangle &= \sqrt{n+1} \left| n + 1 \right\rangle . \end{aligned}$$

## Question 2 – Born approximation

The spherical potential well is given by

$$V(r) = \begin{cases} V_0 & r \leq a \\ 0 & r > a. \end{cases}$$

(a) Calculate the scattering amplitude in the (first) Born approximation.

(b) Expand your result for small $qa$ up to second order.

(c) Write down the differential cross section and use your approximate result to find the total cross section in the low-energy limit as a function of $ka$ with $k = \sqrt{2mE}/\hbar$.

## Question 3 – Partial waves

Scattering resonances of the delta-shell potential,

$$V(r) = \lambda \delta(r - a).$$

(a) Consider $s$-waves ($l = 0$) and find $u(r) = rR_0(r)$ inside and outside the shell. Assume that $u(r)$ is continuous and find the other boundary condition. Use these to determine the $s$-wave phase shift $\delta_0$. Show that you recover the result of the hard sphere if $\mu \equiv 2m\lambda/\hbar^2 k \gg 1 - \cot ka$, where $k$ is the length of the incident wave vector.

(b) Consider $\delta_0$ as a function of $a$ while keeping $k$ fixed for different values of $\mu$ and find the condition for resonances ($\delta_0 = \pi/2$).

(c) Determine the $s$-wave bound states of the infinite spherical potential *well* of radius $a$ and discuss their relationship with the delta-shell resonances in case $\mu \gg 1$.

## Formulas

Series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7),$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6).$$

Trigonometric identities:

$$\cos^2 x - \sin^2 x = \cos 2x,$$
$$2\cos x \sin x = \sin 2x.$$

For a spherically-symmetric potential, the radial Schrödinger equation for angular momentum quantum number $l$, is given by

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}\left(E - V(r)\right)\right] R_l(r) = 0.$$