Topological Systems: Worksheet 5

Abelian toric code

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Consider a square lattice with qubits (i.e. two-level systems) or spin-1/2 particles that are fixed at the center of the lattice links, as shown in Fig. 1(a). The qubits or spins interact with each other via the vertex and plaquette operators:

$$A(v) = \sigma^x_{v1}\sigma^x_{v2}\sigma^x_{v3}\sigma^x_{v4}, \qquad B(p) = \sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4},$$

where v and p label vertices and plaquettes which both consist of four spins, with the labeling defined in Fig. 1(a). For each spin in the bulk, there exists two vertices and two plaquettes which contain it. The total Hamiltonian is then given by

$$H = -\sum_v A(v) - \sum_p B(p).$$

(1) Show that the $A(v)$ and $B(p)$ operators commute with each other and the Hamiltonian. Find their eigenvalues and show that the ground state energy is given by

$$E_0 = -N_V - N_P,$$

where N_V and N_P are the total number of vertices and plaquettes, respectively.

(2) Show that the ground state is given by

$$|\xi\rangle = \prod_v \frac{1}{\sqrt{2}} (1 + A(v)) |00\cdots 0\rangle .$$

where $|00\cdots 0\rangle$ is the state where all spins are in the $|0\rangle$ state with $\sigma^z |0\rangle = |0\rangle$. Hint: you have to show that $A(v)|\xi\rangle = |\xi\rangle$ and $B(p)|\xi\rangle = |\xi\rangle$ for any v and p.

(3) Show that the lowest-energy excitations correspond to the application of σ^z or σ^x to one of the spins. Hint: show that there exist two v such that $\sigma^z A(v) = -A(v)\sigma^z$ and similar for σ^x and the $B(p)$ operators.

Hence, the σ^z and σ^x excitations can be interpreted as two quasiparticles, where each corresponds to an eigenvalue -1 of one of the $A(v)$ and $B(p)$ operators, respectively, see Fig. 1(b). We call these particles e and m anyons since we do not yet know their statistics:

$$|e, e\rangle = \sigma^z |\xi\rangle , \qquad |m, m\rangle = \sigma^x |\xi\rangle .$$

Since the e (m) anyons correspond to a certain vertex (plaquette) operator, we can think of them as being positioned at the corresponding vertex (plaquette center). Furthermore, because the anyons are always created in pairs, we draw a straight line between them, which we call a string.

(4) Show that the combination of e and m anyons creates the composite quasiparticle ϵ with $|\epsilon, \epsilon\rangle = \sigma^z\sigma^x |\xi\rangle$. Make a drawing of such an excitation on the lattice similar to Fig. 1(b).

(5) As we have discussed above, the presence of an e or m anyon is detected by the eigenvalues of the corresponding $A(v)$ or $B(p)$ operators. What happens if the same Pauli rotation is applied to another spin of the same vertex or plaquette? Show that it results in four anyons where two of them are overlapping. Show that the eigenvalues of the vertex or plaquette corresponding to the overlapping pair is given by $+1$.

Because overlapping anyons result in a $+1$ eigenvalue of the corresponding vertex or plaquette operator, the anyons are annihilated, giving rise to the fusion rules

$$e \times e = m \times m = \epsilon \times \epsilon = 1, \quad e \times m = \epsilon, \quad \epsilon \times e = m, \quad \epsilon \times m = e,$$

where 1 corresponds to the vacuum state with no anyon at the vertex or plaquette center.

(6) Show that the anyons can be moved around by applying successive Pauli rotations on neighboring spins. When two anyons annihilate, their strings are glued together such that the endpoints of a string always correspond to anyons.

(7) Consider a lattice with two e anyons and exchange their positions by applying successive σ^z operations. Show that the final state is the same up to the application of a $B(p)$ operator and show that the statistics of e anyons is bosonic. Similarly, show that m anyons are also bosons.

(8) Consider a lattice with one $\epsilon = e \times m$ composite anyon and rotate it around by 2π by moving the constituent e anyon around the m anyon with successive σ^z operations. Show that this gives a phase factor -1, revealing that ϵ has spin $1/2$. Hence, the ϵ particles are fermions, which can also be checked explicitly by exchange two ϵ anyons without rotating them.

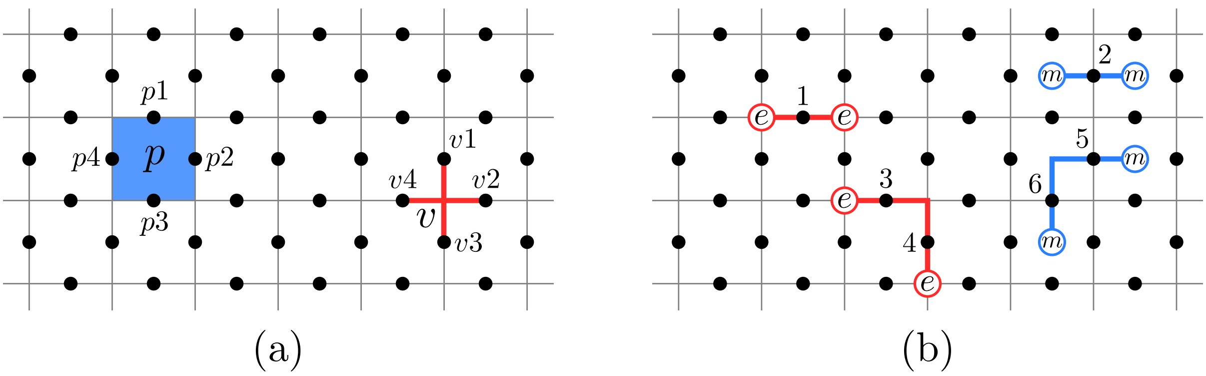

Figure 1: (a) Toric code on a square lattice with spin-1/2 particles (dots) at the center of the links. The spins interact with each other through the vertex operators $A(v) = \sigma^x_{v1}\sigma^x_{v2}\sigma^x_{v3}\sigma^x_{v4}$ and the plaquette operators $B(p) = \sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}$. (b) The action of σ^z_1 on the ground state $|\xi\rangle$ creates two e anyons located at the two vertices that contain the spin labeled as 1. Similarly, σ^x_2 creates two m anyons located at the center of the plaquettes that contain spin 2. Also shown is the result of applying $\sigma^z_4\sigma^z_3$ and $\sigma^x_6\sigma^x_5$, which illustrates the annihilation of e and m anyons, respectively.