Topological Systems: Worksheet 4

Vortices in p-wave superconductors and non-Abelian statistics

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The continuum model of a 2D spinless p-wave superconductor is given by

$$H_{\text{BdG}} = \frac{1}{2} \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}^\dagger \begin{pmatrix} \frac{k^2}{2m} - \mu & 2i\Delta(k_x + ik_y) \\ -2i\Delta^*(k_x - ik_y) & -\frac{k^2}{2m} + \mu \end{pmatrix} \Psi_{\boldsymbol{k}},$$

where $k^2 = k_x^2 + k_y^2$ and the Nambu operator $\Psi_{\boldsymbol{k}} = (c_{\boldsymbol{k}}, c_{-\boldsymbol{k}}^\dagger)^t$. Now consider a vortex at $r = 0$,

$$\Delta(r, \theta) = \Delta_0(r) e^{i\phi(\boldsymbol{r})},$$

where we take $\Delta_0 > 0$ and the superconducting phase $\phi(\boldsymbol{r})$ winds one time around the origin. To keep Δ single-valued at the origin, we require $\Delta_0(0) = 0$.

(1) Perform a gauge transformation on the fermion operators $c(\boldsymbol{r}) \to e^{i\phi(\boldsymbol{r})/2} c(\boldsymbol{r})$. This removes the phase of Δ from the BdG equations. Moreover, in the new basis, the wave function $\psi(\boldsymbol{r})$ obeys antiperiodic boundary conditions:

$$\psi(r, \theta + 2\pi) = -\psi(r, \theta), \tag{1}$$

as the components are multiplied by $e^{\pm i\phi(\boldsymbol{r})/2}$. Hint: to find $\partial_{x,y}$ in polar coordinates, consider $\partial_{x,y} f(r, \theta)$ together with the equations $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

(2) Show that there exists a zero-energy solution: $\mathcal{H}\psi(\boldsymbol{r}) = 0$ with

$$\mathcal{H} = \begin{pmatrix} -\mu(r) & 2\Delta_0(r) e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \\ -2\Delta_0(r) e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) & \mu(r) \end{pmatrix}.$$

where $\Delta_0 > 0$ and $\mu > 0$. To this end, use the *ansatz*

$$\psi_0(r, \theta) = g(r) \begin{pmatrix} e^{i\theta/2} \\ a e^{-i\theta/2} \end{pmatrix},$$

which solves the angular part and corresponds to a state with zero angular momentum, where $j_z = -i\partial_\theta - \tau_z/2$ is the angular momentum operator. Show that $[\mathcal{H}, j_z] = 0$.

What are the allowed values of a? Hint: to solve the radial equation, first set $g(r) = h(r)/\sqrt{r}$ and solve for $h(r)$. Which value of a should we take to obtain a normalizable solution?

(3) The wave function of the zero mode can be written as,

$$\psi_0(r, \theta) = \frac{i}{\mathcal{N}\sqrt{r}} \exp\left[-\frac{1}{2} \int_0^r dr' \, \frac{\mu(r')}{\Delta_0(r')} \right] \begin{pmatrix} e^{i\theta/2} \\ -e^{-i\theta/2} \end{pmatrix} \equiv i g(r) \begin{pmatrix} e^{i\theta/2} \\ -e^{-i\theta/2} \end{pmatrix},$$

with $\mathcal{N}$ a normalization constant. Observe that the wave function obeys the anti-periodic boundary condition (1). Construct the corresponding operator, $\gamma = \sqrt{2} \int d^2 r \, \psi_0(\boldsymbol{r})^\dagger \Psi(\boldsymbol{r})$, and show that $\gamma^\dagger = \gamma$ and $\gamma^2 = 1$. Thus, we find that a vortex supports a single Majorana bound state with

$$\gamma = \sqrt{2} \int dr \int d\theta \, i r g(r) \left[e^{-i\theta/2} e^{i\phi(\boldsymbol{r})/2} c(\boldsymbol{r}) - e^{i\theta/2} e^{-i\phi(\boldsymbol{r})/2} c^\dagger(\boldsymbol{r}) \right].$$

Now consider a collection of $2N$ vortices that are sufficiently separated so that there is no overlap between Majorana wave functions. Hence, we have a zero-energy ground state with 2^N degeneracy corresponding to N non-local fermions $f_n = (\gamma_{2n-1} + i\gamma_{2n})/2$ with $\gamma_j^\dagger = \gamma_j$ and $\gamma_j^2 = 1$ $(j = 1, \ldots, 2N)$ that can be either occupied or unoccupied. The ground-state subspace can be further divided into two parity sectors as the BdG equations conserve the fermion parity $\prod_{n=1}^{N} (-1)^{f_n^\dagger f_n}$ such that we have 2^{N-1} states with even parity and 2^{N-1} states with odd parity.

(4) Consider a single pair of vortices with Majoranas γ_1 and γ_2 that are well separated. If we move the vortices adiabatically, we can exchange the Majoranas. Specifically, we consider a process where the Hamiltonian returns to itself, i.e. $H(t+T) = H(t)$, such that the wave functions only differ by a phase. Under this process, one of the Majoranas picks up a sign,

$$\gamma_1 \to \gamma_2, \qquad \gamma_2 \to -\gamma_1,$$

as illustrated in Fig. 1. Why is this the case?

(5) Consider the exchange operator T_{ij} that exchanges Majoranas γ_i and γ_j and leaves the rest invariant, such that $T_{ij}\gamma_i T_{ij}^{-1} = \gamma_j$, $T_{ij}\gamma_j T_{ij}^{-1} = -\gamma_i$, and $T_{ij}\gamma_k T_{ij}^{-1} = \gamma_k$ for $k \neq i, j$. Show that a representation is given by

$$T_{ij} = \frac{1}{\sqrt{2}} \left(1 - \gamma_i \gamma_j\right).$$

(6) Consider four vortices with Majorana operators γ_1, γ_2, γ_3, and γ_4, which can be paired into two non-local fermions $a = (\gamma_1 + i\gamma_2)/2$ and $b = (\gamma_3 + i\gamma_4)/2$. Calculate the exchange operators T_{12}, T_{23}, and T_{34} in the occupation number representation of the ground-state subspace. Show that they are block diagonal due to fermion parity conservation.

(7) Which exchange operation represents non-Abelian statistics? What is the final state under exchange of γ_2 and γ_3, given an initial state $|\psi_i\rangle = |0\rangle_a |0\rangle_b$?

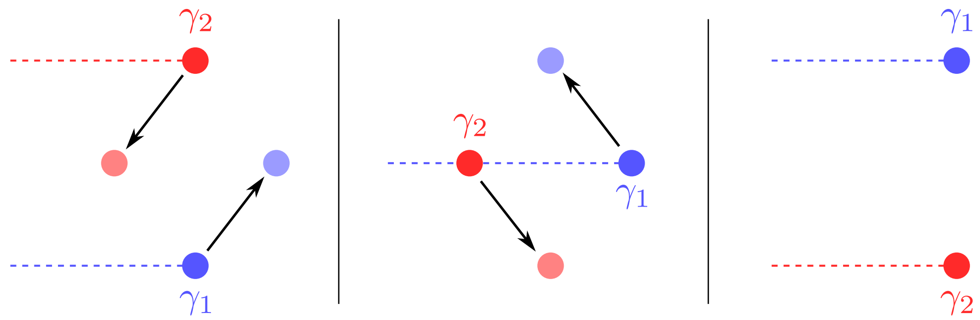

Figure 1: Exchange of two vortices in a p-wave superconductor. The dashed lines correspond to branch cuts where the superconducting phase changes by 2π.