

Analytical Mechanics: Worksheet 8

Relativistic kinematics

Christophe De Beule (christophe.debeule@gmail.com)

1 Relativistic aberration

A light source in frame S emits a conical beam of light with opening angle θ such that at the edge $u^1 = c \cos \frac{\theta}{2}$ and $u^2 = c \sin \frac{\theta}{2}$. Find the opening angle θ' in a frame S' that is moving along the X -axis with speed V relative to S . Show that $\theta' \rightarrow \theta$ for $V/c \rightarrow 0$.

2 Spontaneous decay of a particle

A particle at rest with mass M and four momentum k decays spontaneously into two particles with mass m_1 and m_2 with four momenta p_1 and p_2 . Write down the four momentum of each particle in the rest frame of the original particle. Then use conservation of four momentum to find the energy of both particles E_1 and E_2 in terms of M , m_1 , and m_2 .

3 Compton scattering

Compton scattering is the inelastic scattering of a photon with a free electron. Denote the four momentum of the incoming and scattering photon as k and k' , and the four momentum of the electron before and after the collision as p and p' .

- (a) Calculate the energy of the scattered photon in the rest frame of the electron before the collision, in function of the scattering angle θ .
- (b) Show that the maximal value for the energy of a back-scattered photon ($\theta = \pi$) in this frame is given by $\frac{1}{2}mc^2$.

4 Relativistic collision

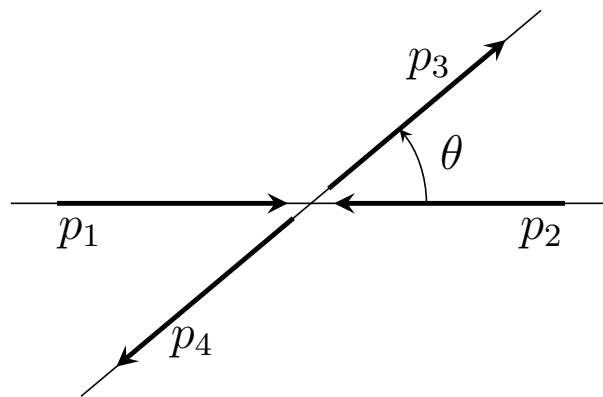


Figure 1: Scattering in the center-of-mass frame.

A particle with four momentum p_1 collides with a second particle at rest and with four momentum p_2 in the lab frame. Hence, in the lab frame

$$p_1^\mu = \left(\frac{E_1}{c}, p, 0 \right), \quad p_2^\mu = (m_2 c, 0, 0).$$

In the center-of-mass frame, by definition, $\vec{P} = 0$ with $P = p_1 + p_2$ the total four momentum.

- (a) Use the Lorentz invariance of P^2 to find the total energy in the center-of-mass frame.
- (b) Calculate the speed β and Lorentz factor γ corresponding to the Lorentz boost to the center-of-mass frame, and determine p_1 in this frame.
- (c) Suppose that scattering is elastic such that the kinetic energy of each particle is conserved in the center-of-mass frame. Determine p_3 of one of the outgoing particle.
- (d) Use the inverse Lorentz boost to find p_3 in the lab frame. Determine the scattering angle ψ in the lab frame.
- (e) Write down the energy of the scattered particle p_3 in the lab frame as a function of β , γ , θ , and the initial energy. Discuss your results for $\theta = 0$.
- (f) Use your results to compute the ratio of the kinetic energy $T_i = E_i - m_i c^2$ in the lab frame before and after the collision. Define $r = m_2/m_1$ and $\mathcal{E} = T_1/(m_1 c^2)$.
- (g) Which scattering angle θ minimizes this ratio? Evaluate this case in the non-relativistic and ultra-relativistic limit.