Bosonization: Worksheet 7

Green's function for free bosons

Christophe De Beule (christophe.debeule@gmail.com)

1 Green's function for free bosons

In the previous exercise sheet we showed that

$$\psi_s(x) = (2\pi a)^{-1/2}\, U_s\, e^{i\frac{2\pi}{L}\left(N_s - \frac{1}{2}\right)x}\, e^{i\sqrt{\pi}\left[\tilde{\varphi}(x) + \epsilon_s \tilde{\vartheta}(x)\right]},$$

with $a = 0^+$ which is needed to regularize certain sums (because the right side is not normal ordered) and

$$\tilde{\varphi}(x) = i \sum_{q \neq 0} \frac{\mathrm{sgn}(q) e^{-iqx}}{\sqrt{2L|q|}} (b_{-q}^\dagger + b_q) e^{-a|q|/2}, \quad \tilde{\vartheta}(x) = i \sum_{q \neq 0} \frac{e^{-iqx}}{\sqrt{2L|q|}} (b_{-q}^\dagger - b_q) e^{-a|q|/2},$$

where (for $q \neq 0$)

$$b_q = \left(\frac{2\pi}{|q|L}\right)^{1/2} \left(\theta(q)\rho_{qL} + \theta(-q)\rho_{qR}\right),$$

which is different from the result obtained in worksheet 3, because we changed the definition of ρ_{qs} in worksheet 5 to match the conventions of the theory course. These fields are related to the fields $\varphi(x)$ and $\vartheta(x)$ defined in worksheet 5 as follows:

$$\varphi(x) = \frac{\sqrt{\pi}}{L}\left(N_R + N_L\right) + \tilde{\varphi}(x), \quad \vartheta(x) = \frac{\sqrt{\pi}}{L}\left(N_R - N_L\right) + \tilde{\vartheta}(x).$$

Note that $\tilde{\varphi}(x)$ and $\tilde{\vartheta}(x)$ have the same commutation relations as $\varphi(x)$ and $\vartheta(x)$ and become equal in the limit $L \to \infty$.

(1) Use the Baker-Hausdorf theorem to show that $b_q(t) \equiv e^{-iH_0 t} b_q e^{iH_0 t} = b_q\, e^{-iv_F|q|t}$, where the free Hamiltonian can be written as

$$H_0 = \sum_{q \neq 0} v_F |q| b_q^\dagger b_q + \frac{\pi v_F}{L}\left(N_R^2 + N_L^2\right),$$

where we have taken anti-periodic boundary conditions.

(2) Calculate the boson Green's functions $\mathcal{G}_{\varphi\varphi}(x,t) = \langle \tilde{\varphi}(x,t)\tilde{\varphi}(0,0)\rangle$, $\mathcal{G}_{\vartheta\vartheta}(x,t) = \langle \tilde{\vartheta}(x,t)\tilde{\vartheta}(0,0)\rangle$, $\mathcal{G}_{\varphi\vartheta}(x,t) = \langle \tilde{\varphi}(x,t)\tilde{\vartheta}(0,0)\rangle$, and $\mathcal{G}_{\vartheta\varphi}(x,t) = \langle \tilde{\vartheta}(x,t)\tilde{\varphi}(0,0)\rangle$ at zero temperature. Hint: $\sum_{n=1}^\infty z^n/n = -\ln(1-z)$ for $|z| < 1$.

(3) In the limit $L \to \infty$, the *zero-mode* contributions N_s/L can be neglected. Show that in this case, the fermion Green's function can be obtained from the boson Green's functions using the expression for $\psi_s(x)$ in terms of $\varphi(x)$ and $\vartheta(x)$. Hint: $\left\langle e^{\lambda_1 B_1} e^{\lambda_2 B_2}\right\rangle = e^{\left\langle \lambda_1 B_1 \lambda_2 B_2 + \frac{1}{2}\left(\lambda_1^2 B_1^2 + \lambda_2^2 B_2^2\right)\right\rangle}$, where $B_{1,2}$ are linear in boson operators.

2 Useful relation for boson operators

Show that for A and B linear in boson operators, we have

$$\left\langle e^A B \right\rangle = e^{\frac{1}{2}\langle A^2\rangle} \langle AB\rangle.$$

This is a generalization of a theorem which was proven in Worksheet 4. To proceed, first consider the quantity $\left\langle e^{A}e^{\epsilon B}\right\rangle$ and write it as the expectation value of a single exponent using one of the theorems from Worksheet 1. Then expand both sides up to first order in ϵ and identify like terms to obtain the desired result. Hint: Make use of an important theorem for operators linear in boson operators and note that $[A, B] \in \mathbb{C}$ for A and B linear in boson operators.