Quantum Mechanics: Worksheet 6

Klein-Gordon equation and the Klein paradox

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1 Klein-Gordon equation

The Klein-Gordon equation is given by

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0,$$

where $\phi(x)$ is a scalar function of the space-time coordinates $x^\mu = (ct, \boldsymbol{r})$. Here ∂^2 is the d'Alembertian,

$$\partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

with $\partial^\mu = \eta^{\mu\nu}\partial_\nu = (c^{-1}\partial_t, -\nabla)$. For stationary solutions $\phi(x) = e^{-\frac{i}{\hbar}Et}\varphi(\boldsymbol{r})$, the time-independent Klein-Gordon equation becomes

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{E^2 - m^2c^4}{2mc^2}\right)\varphi(\boldsymbol{r}) = 0.$$

External electromagnetic fields are included in the same way as for the Schrödinger equation, with minimal coupling: $p_\mu \to p_\mu - \frac{q}{c}A_\mu$ (in Gaussian units).

The Klein-Gordon equation has a conserved U(1) charge with a conserved four-current

$$j^\mu = \frac{i\hbar}{2m}\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) = \left(c\rho, \boldsymbol{j}\right),$$

where the prefactor is chosen such that it matches the Schrödinger current for positive energy states. Show that $\partial_\mu j^\mu = 0$ and $j^\mu = (\hbar/m)\,k^\mu$ for a plane wave. Also calculate j^μ for a general stationary solution. Also show that

$$\frac{dQ}{dt} = -\oint_{\partial\mathcal{V}} \boldsymbol{j}\cdot d\boldsymbol{a}, \qquad Q = \int_{\mathcal{V}} \rho d\tau,$$

with Q the total "charge" in a volume $\mathcal{V}$ with boundary $\partial\mathcal{V}$.

Why does the Klein-Gordon equation not admit a consistent interpretation in terms of a relativistic single-particle wave equation?

2 Klein paradox

The Klein paradox illustrates that, unlike the Dirac equation, the Klein-Gordon equation cannot be consistently interpreted as a relativistic single-particle equation. To this end, consider scattering at a potential step:

$$V(x) = V_0\,\theta(x),$$

with $V_0 \geq 0$. This potential can be introduced in the time-independent Klein-Gordon equation by substituting $E \to E - V(x)$. This problem has translational symmetry in the yz plane. For simplicity, we consider the case where $k_y = k_z = 0$ (normal incidence).

(a) Write down the general solution $\varphi^<(x)$ for $x < 0$ and $\varphi^>(x)$ for $x > 0$.

(b) Now consider a solution for an incoming plane wave for $x < 0$ with positive energy $E > mc^2$ and wave vector $k > 0$. Call the amplitude of the reflected mode r and of the transmitted mode t. To ensure causality, the transmitted mode always needs to have positive group velocity. Hint: make a drawing of the energy dispersion in both regions and consider three cases: $V_0 < E - mc^2$, $E - mc^2 < V_0 < E + mc^2$, and $V_0 > E + mc^2$.

(c) Calculate the Klein-Gordon current for $x < 0$ and $x > 0$. Show that $\partial_x j = 0$. Now separate the current in incoming (j_i), reflected (j_r), and transmitted (j_t) parts.

(d) From the previous question, we find that

$$j_i + j_r = j_t.$$

Use the continuity of ϕ and $\partial_x \phi$ to find an expression for the coefficients r and t. Check that your solution satisfies current conservation.

(e) Show that $j_t = 0$ for $E - mc^2 < V_0 < E + mc^2$? Hint: write down the wave vector q for $x > 0$ and show that q is imaginary in this case.

(f) Consider the case $V_0 > E + mc^2$. Show that $j_t < 0$ and thus $j_r < -j_i$. This is the Klein paradox. Argue why there is no paradox if we interpret j as a charge current. Namely, explain when $j_t < 0$ is consistent with a right-moving wave with density ρ.