Analytical Mechanics: Worksheet 5

Non-holonomic constraints

Christophe De Beule (christophe.debeule@gmail.com)

---

1 Theory

Consider a mechanical system with Lagrangian $L = L(q_i, \dot{q}_i, t)$ subject to a *non-holonomic* constraint:

$$c_1 \delta q_1 + \cdots + c_n \delta q_n = 0,$$

where $c_1, \ldots, c_n$ are functions of the generalized coordinates q_i. It is impossible to reduce the number of degrees of freedom with this constraint. Hence, non-holonomic constraints *necessitate* the method of Lagrange multipliers. Similar as before, we obtain

$$\frac{\delta L}{\delta q_i} \equiv \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\lambda c_i.$$

This is equivalent to a free variational problem with

$$\delta \overline{L} = \delta L + \lambda \left(c_1 \delta q_1 + \cdots + c_n \delta q_n \right).$$

In this worksheet, we will consider the problem of rolling without slipping. This is a typical example of a non-holonomic constraint, called a *non-integrable constraint.* Other non-holonomic constraints are inequalities, e.g. a mass that slides frictionless of a sphere with $r \geq R$, and systems with friction, e.g. a damped harmonic oscillator.

2 Rolling coin

Consider an upright coin that rolls without slipping down a slope, see Figure 1. The configuration of the coin is determined by the coordinates of the contact x and y and the rolling without slipping constraint.

As shown in the figure, θ is the angle between the y axis and the velocity $\vec{v}$ of the contact point, and ϕ is the angle about the central axis perpendicular to the coin. Due to the rolling constraint, we cannot express x and y as functions of θ and ϕ. This is because x and y generally do not return to themselves after a closed path in (θ, ϕ) space. Instead, they also depend on the *history of the system.* It is therefore impossible to describe the coin with only two coordinates, even though there are only two degrees of freedom in this problem. This is a general feature of non-holonomic systems.

(a) Determine the non-holonomic constrains of this system by expressing δx and δy in terms of the angles θ and ϕ.

(b) Calculate the kinetic energy. It contains both translational (x and y) and rotational contributions (θ and ϕ). For the latter, you need to calculate the moment of inertia around an axis $\hat{x}_i$:

$$I_i = \int d^3 r \, \sigma(\vec{r}) \left(r^2 - x_i^2 \right).$$

Assume that the coin has no thickness and a uniform mass density σ. Determine a relation between $I_\perp$ and $I_\parallel$.

(c) Calculate the potential energy due to gravity and determine the Lagrangian. Show that we require three coordinates: θ, ϕ, and y.

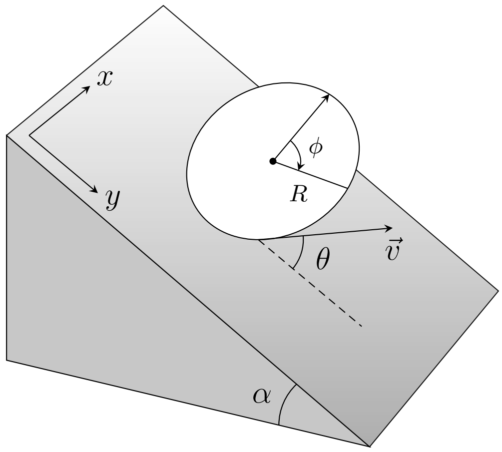

Figure 1: A coin rolls from a slope α without slipping. We also assume that the coin stays perpendicular to the surface of the slope as it rolls. As the coin rolls, it rotates through an angle ϕ about its own axis. In addition, the coin can turn by an angle θ about the axis through the contact and perpendicular to the slope.

(d) Use a Lagrange multiplier λ to include the rolling constraint on y to the variation of the action δS. Determine the functions c_θ, c_ϕ, and c_y.

(e) Find an expression for λ and solve the equations of motion for $\theta(t)$ and $\phi(t)$.

(f) Substitute $\theta(t)$ and $\phi(t)$ into the expression for $\dot{x}$ and $\dot{y}$. You can obtain the latter from the rolling constraint, taking a variation that coincides with the change over a time dt.

(g) Integrate and find $x(t)$ and $y(t)$.