Energy-time uncertainty relation

Somewhat rigorous

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Take a state $|\Psi\rangle$ and an arbitrary hermitian operator A with expectation value

$$\langle A\rangle = \langle\Psi|\, A \,|\Psi\rangle \,,$$

The standard deviation of the observable A in Ψ is given by

$$\Delta A = \sqrt{\langle\Psi|\, (A - \langle A\rangle)^2 \,|\Psi\rangle}.$$

Now take $r \in \mathbb{R}$ and define the operator,

$$B = r\,(A - \langle A\rangle) + i\,(H - \langle H\rangle)\,.$$

Since

$$\langle\Psi|\, B^\dagger B \,|\Psi\rangle = |B\,|\Psi\rangle|^2 \geq 0,$$

we find

$$(\Delta A)^2\, r^2 + i\,\langle[H, A]\rangle\, r + (\Delta E)^2 \geq 0.$$

So there exists one real root if the equality holds and no real roots otherwise. This implies that the discriminant is equal to, or lesser than zero,

$$\langle[H, A]\rangle^2 + 4\,(\Delta E)^2\,(\Delta A)^2 \geq 0.$$

From the Heisenberg equation we have

$$[H, A(t)] = -i\hbar\frac{d}{dt}A(t) \Rightarrow [H, A] = -i\hbar\,\left.\frac{d}{dt}A(t)\right|_{t=0}.$$

where we assumed that A itself is not time dependent. Putting everything together we obtain:

$$\Delta E\frac{\Delta A}{\left|\left.\frac{d}{dt}A(t)\right|_{t=0}\right|} \geq \frac{\hbar}{2}.$$

The second term on the left is the average time before the expectation value of A changes by an amount ΔA, i.e. the lifetime of the state Ψ with respect to the observable A. This is a reasonable definition for the uncertainty on time, as it is the smallest time scale for which Ψ changes noticeably by measuring $\langle A\rangle$. The operator A serves as a kind of clock.