

# Quantum Mechanics

## Sample exam 2

Christophe De Beule (christophe.debeule@gmail.com)

---

### Question 1 – Perturbation theory

Consider a hydrogen atom with a finite-sized nucleus. Model the proton as a uniformly charged ball of radius  $b$  with total charge  $e$ . The corresponding potential is given by

$$V(r) = \begin{cases} -\frac{e^2}{2b} \left( 3 - \frac{r^2}{b^2} \right) & r \leq b \\ -\frac{e^2}{r} & r > b. \end{cases}$$

- (a) Find the corresponding perturbation to the hydrogen atom with point nucleus.
- (b) Calculate the first-order correction to the ground-state energy and show that the leading order is proportional to  $(b/a)^2$ .
- (c) Does the perturbation affect higher angular-momentum states more or less? Give a simple argument without performing any calculation.
- (d) Now consider the first-excited state. Is the original basis of the 4-dimensional degenerate subspace  $\{2s, 2p_x, 2p_y, 2p_z\}$  diagonal in the perturbation? Why?

The normalized ground-state wave function of the (unperturbed) hydrogen atom is given by ( $a$  is the Bohr radius)

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.$$

### Question 2 – Born approximation

An electron scatters at an electric dipole composed of charges  $-Ze$  and  $Ze$  separated by the vector  $\mathbf{d}$ .

- (a) Make a drawing and find the scattering potential.
- (b) Calculate the differential cross section in the first Born approximation. You can use the result for Rutherford scattering  $\left(\frac{d\sigma}{d\Omega}\right)_R$ .

### Question 3 – Partial waves

Consider  $p$ -wave scattering from a hard sphere of radius  $a$ .

- (a) Find the phase shift  $\delta_1(k)$  for  $p$ -wave scattering and evaluate it in the low-energy limit up to lowest order.
- (b) Compare this result with the  $s$ -wave phase shift. Which channel dominates in the low-energy limit?

The radial wave function of the  $l$ -th partial wave outside the scattering region is

$$R_l^>(r) = e^{i\delta_l} h_l^{(1)}(kr) + e^{-i\delta_l} h_l^{(2)}(kr),$$

and the spherical Hankel functions are given by (for  $x$  real)

$$h_l^{(1)}(x) = -ix^l \left( -\frac{1}{x} \frac{d}{dx} \right)^l \frac{e^{ix}}{x}, \quad h_l^{(2)}(x) = h_l^{(1)}(x)^*.$$