# Bosonization: Worksheet 1

Basic concepts of condensed-matter field theory

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# 1 Fourier transform of the Yukawa potential

Calculate the three-dimensional Fourier transform of the Yukawa potential

$$V(\boldsymbol{q}) = \int d^3r \, V(r) \, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}, \qquad V(r) = \frac{e^2}{r} \, e^{-\mu r},$$

and take the limit  $\mu \to 0$  to obtain the regularized Fourier transform of the Coulomb potential. Hint: Use spherical coordinates with the z-axis pointing along q to perform the integration.

#### 2 Free electrons

The free electron Hamiltonian can be written as

$$\hat{H}_0 = \sum_{\sigma = \uparrow, \downarrow} \int_V d^d r \, \hat{\psi}^{\dagger}_{\sigma}(\boldsymbol{r}) \left[ \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{r}) \right] \hat{\psi}_{\sigma}(\boldsymbol{r}),$$

where  $\hat{\boldsymbol{p}} = -i\hbar\nabla$ . The field operators obey the anti-commutation relations  $\{\hat{\psi}_{\sigma}(\boldsymbol{r}), \hat{\psi}_{\sigma'}^{\dagger}(\boldsymbol{r'})\} = \delta_{\sigma\sigma'}\delta(\boldsymbol{r} - \boldsymbol{r'})$  and  $\{\hat{\psi}_{\sigma}(\boldsymbol{r}), \hat{\psi}_{\sigma'}(\boldsymbol{r'})\} = \{\hat{\psi}_{\sigma}^{\dagger}(\boldsymbol{r}), \hat{\psi}_{\sigma'}^{\dagger}(\boldsymbol{r'})\} = 0$ .

(1) Consider the Fourier transform

$$\hat{\psi}_{\sigma}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{c}_{\mathbf{k}\sigma},$$

where  $\hat{c}_{\boldsymbol{k}\sigma}$  and  $\hat{c}_{\boldsymbol{k}\sigma}^{\dagger}$  are field operators in momentum space. Assume the volume  $V=L^d$  is a d-dimensional cube with sides L and use periodic boundary conditions  $\hat{\psi}_{\sigma}(\boldsymbol{r}+L\boldsymbol{e}_i)=\hat{\psi}_{\sigma}(\boldsymbol{r})$   $(i=1,\ldots,d)$  to find the allowed momenta. Also find an expression for the anti-commutation relations  $\{\hat{c}_{\boldsymbol{k}\sigma},\hat{c}_{\boldsymbol{k}'\sigma'}^{\dagger}\}$ ,  $\{\hat{c}_{\boldsymbol{k}\sigma},\hat{c}_{\boldsymbol{k}'\sigma'}\}$ , and  $\{\hat{c}_{\boldsymbol{k}\sigma}^{\dagger},\hat{c}_{\boldsymbol{k}'\sigma'}^{\dagger}\}$ .

- (2) For the remainder of this exercise we put  $V(\mathbf{r}) = 0$ . Use the above result to obtain a representation of  $\hat{H}_0$  in terms of  $c_{\mathbf{k}\sigma}$  and  $\hat{c}_{\mathbf{k}\sigma}^{\dagger}$ .
- (3) Calculate the non-interacting ground state energy

$$E_0 = \langle \Omega | \hat{H}_0 | \Omega \rangle$$
,

where  $|\Omega\rangle = \prod_{|\boldsymbol{k}| \leq k_F, \sigma} \hat{c}^{\dagger}_{\boldsymbol{k}\sigma} |0\rangle$  is the Fermi sea with  $k_F$  the Fermi wave number. Perform the calculation for d=1 and d=3. Hint:  $\sum_{\boldsymbol{k}} \stackrel{L \to \infty}{\longrightarrow} \left(\frac{L}{2\pi}\right)^d \int d^d k$ .

## 3 Lattice version of the interaction Hamiltonian

The interaction Hamiltonian

$$\hat{V}_{ee} = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^d r \int d^d r' V_{ee}(\boldsymbol{r} - \boldsymbol{r}') \hat{\psi}_{\sigma}^{\dagger}(\boldsymbol{r}) \hat{\psi}_{\sigma'}^{\dagger}(\boldsymbol{r}') \hat{\psi}_{\sigma'}(\boldsymbol{r}') \hat{\psi}_{\sigma}(\boldsymbol{r}),$$

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where  $V_{ee}(\mathbf{r} - \mathbf{r}')$  is the interaction potential, can also be written in terms of Wannier field operators  $\hat{c}_{i\sigma}$  and  $\hat{c}_{i\sigma}^{\dagger}$  through a unitary transformation

$$\hat{\psi}_{\sigma}(\mathbf{r}) = \sum_{i} \phi_{\mathbf{R}_{i}}(\mathbf{r}) \, \hat{c}_{i\sigma},$$

where the Wannier function  $\phi_{\mathbf{R}_i}(\mathbf{r})$  is localized around the Bravais lattice site  $\mathbf{R}_i$  and the set of Wannier states  $\{|\phi_{\mathbf{R}_i}\rangle\}$  form an orthonormal basis of the single-particle Hilbert space. The result is given by

$$\hat{V}_{ee} = \sum_{\sigma,\sigma'} \sum_{ii'jj'} V_{ii'jj'} \hat{c}^{\dagger}_{i\sigma} \hat{c}^{\dagger}_{i'\sigma'} \hat{c}_{j'\sigma'} \hat{c}_{j\sigma}.$$

- (1) Find an expression for  $V_{ii'jj'}$  in terms of the Wannier functions.
- (2) Direct terms  $V_{ii'ii'} \equiv V_{ii'}$  couple density fluctuations at sites  $i \neq i'$ . Determine  $V_{ii'}$  for a contact potential  $V_{ee}(\mathbf{r}) = \lambda \delta(\mathbf{r})$  and wave functions  $\phi_{\mathbf{R}_i}(\mathbf{r}) = \frac{e^{-(\mathbf{r} \mathbf{R}_i)^2/(2\Delta^2)}}{(\Delta \sqrt{\pi})^d}$ , for
  - (i) an on-site interaction i = i';
  - (ii) a nearest-neighbor interaction in a square lattice (d=2) with lattice constant a.
- (3) In the limit  $\Delta \ll a$ , the wave functions are strongly localized on the sites and the onsite interaction  $V_{ii} \equiv U/2$ , called the Hubbard interaction, is the dominant contribution. Determine the interaction Hamiltonian  $\hat{V}_{ee}$  in this approximation.

### 4 Useful identities

Consider two operators A and B. The Baker-Hausdorff theorem is given by

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty} \frac{[A,B]_n}{n!} = A + [A,B] + \frac{1}{2}[[A,B],B] + \dots,$$

where  $[A,B]_{n+1} \equiv [[A,B]_n,B]$  and  $[A,B]_0 \equiv A$ . Now define  $C \equiv [A,B]$  and assume that C commutes with A and B.

- (1) Use the Baker-Hausdorff theorem to show that  $e^{-B}Ae^{B} = A + C$ .
- (2) Consider the operator-valued function  $\mathcal{T}(s) = e^{sA}e^{sB}$  ( $s \in \mathbb{R}$ ) and use (1) to write

$$\frac{d\mathcal{T}(s)}{ds} = \mathcal{T}(s)\,\mathcal{P}(s),$$

where  $\mathcal{P}(s)$  is to be found. Solve this differential equation by inspection, with the boundary condition  $\mathcal{T}(0) = 1$ . Use your result to prove

$$e^A e^B = e^{A+B+C/2} = e^{A+B} e^{C/2}$$

where the last equality follows from [A + B, C] = 0.

(3) First use (1) to prove by induction that  $e^{-B}A^ne^B=(A+C)^n$ , and then show that

$$e^{-B} f(A)e^{B} = f(A+C).$$

(4) Use (3) to show that  $e^A e^B = e^B e^A e^C$ .

<sup>&</sup>lt;sup>1</sup>These functions are not strictly Wannier functions because they are not orthogonal. Wannier functions can be constructed from a superposition of Gaussians, but this is beyond the scope of the exercise.