Analytical Mechanics: Worksheet 7

Lorentz transformations

Christophe De Beule (christophe.debeule@gmail.com)

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1 Time dilation for muons

Muons are created in the upper atmosphere through scattering of cosmic radiation with air molecules. Every minute, about 10000 muons per square meter reach the surface of the Earth.

What is the speed of a muon that decays on the surface of the Earth if it was produced at a height of 6 km. The mean life time of a muon in its rest frame is about 2.2 μs. How far would the muon travel if we do not consider time dilation.

2 Relativistic addition law for velocity

Consider a rocket with four velocity $U'^{\mu} = \gamma' \left(c, u'^1, u'^2\right)$ in the system of a space ship that is moving with a speed V along the X axis relative to Earth. Use a Lorentz boost to compute the four velocity U^{μ} in the Earth frame and determine the velocity $u^i = cU^i/U^0$.

3 Resolution of the twin paradox

A space traveler, called A, departs in a space ship, while their identical twin B remains on Earth. We place the X axis along the path of the space ship with (t, x) the ship coordinates measured by B and τ the time of A. The latter is the proper time of the trajectory of the ship. The goal of this exercise is to find the coordinates x and t as a function of τ.

(a) Find the four acceleration $A^{\mu} \equiv \frac{dU^{\mu}}{d\tau}$ where U^{μ} is the four velocity, and determine the four acceleration in the instantaneous inertial frame of the accelerated space traveler.

(b) Use an instantaneous Lorentz boost $\Lambda(\tau)$ to find the four acceleration in the inertial frame of B for each τ.

(c) Compare your previous result with the general form of the four acceleration and obtain a differential equation for $\beta(\tau)$. Solve this equation for the initial condition $\beta(0) = 0$ and find the solution for $\beta(\tau)$ and $\gamma(\tau)$ in the frame of B. Hints: $\int \frac{ds}{1-s^2} = \operatorname{artanh}(s)$ for $|s| < 1$ and $\frac{1}{c}\int_0^{\tau} ds\, g(s) \equiv \varphi(\tau)$ the *rapidity*, with $g(\tau)$ the acceleration measured by A.

(d) Use the four velocity in the frame of B to write down the solutions for $t(\tau)$ and $x(\tau)$ in terms of integrals. Can you resolve the twin paradox using these expressions and the properties of the cosh function?

(e) Compute $\varphi(\tau)$ for a constant acceleration g and find an explicit expression for x and t.

(f) Assume A is traveling to Proxima Centauri at 4.25 light-years from Earth[1]. The space ship is constructed to undergo constant acceleration $g = 9.8$ m/s^2 (for a comfortable journey) during the first half of the outbound trip. Halfway to the destination, the spaceship is turned around, decelerating, such that it comes at rest relative to the Earth at the destination. The journey back proceeds analogously.

[1] This question was inspired by Episode 8: Journeys in Space & Time of Carl Sagan's magnum opus Cosmos. In particular, Sagan's discussion of the theoretically proposed Bussard ramjet that scoops up hydrogen from the interstellar medium and accelerates it into a fusion engine.

How many years do the twins differ in age when A comes back from their journey? What is the maximal speed of the space ship according to B? Make a Minkowski diagram of the trajectory and plot $\beta(\tau)$.

(g) Extra: Perform a similar calculation for the North Star (446.5 ly), the center of the Milky Way Galaxy (26000 ly), the Andromeda Galaxy (152000 ly), and the circumnavigation of the observable universe (292×10^6 ly).

4 Generators of the Lorentz transformation

Show that the six 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha\right),$$

(here μ and ν label the matrices) satisfy

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\right).$$

Therefore, they form a matrix representation of the generators of Lorentz transformations (rotations and boosts). Now consider the infinitesimal transformation

$$V^\alpha \to \left(\delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu} \left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\right) V^\beta,$$

where V is a four vector and $\omega_{\mu\nu}$ are the components of an asymmetric tensor that correspond to the infinitesimal angles. What is the physical interpretation of:

(a) $\omega_{12} = -\omega_{21} = \theta$ and the rest zero;

(b) $\omega_{10} = -\omega_{01} = \varphi$ and the rest zero.