Quantum Mechanics: Worksheet 3

Time-dependent perturbation theory

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1 Switching on a delta-function barrier

Consider a particle in a one-dimensional infinite well with $V = 0$ if $0 \leq x \leq a$. Assume that for $t < 0$, the particle is in the ground state. At $t = 0$, we turn on the following perturbation:

$$V(x,t) = \frac{aV_0}{2}\,\delta(x - a/2)\,\theta(t).$$

Calculate the probability for a transition to an excited state in lowest order.

2 Kicked harmonic oscillator

The Hamiltonian of a particle in an harmonic potential is given by

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Now consider a time-dependent uniform electric field as a perturbation:

$$H = H_0 + V(x,t), \qquad V(x,t) = -qx\mathcal{E}(t),$$

where q is the charge of the particle in the harmonic well.

(a) Consider the electric field

$$\mathcal{E}(t) = \frac{P}{q\tau\sqrt{\pi}} e^{-t^2/\tau^2},$$

where $\tau > 0$ and P is a constant. Calculate the transition probability (in lowest order) that a particle, initially in the ground state at $t_0 \to -\infty$, ends up in an excited state at $t \to \infty$. Discuss your result for $\omega\tau \gg 1$ (adiabatic) and $\omega\tau \ll 1$ (sudden). What is the effect of the adiabatic perturbation on the ground state?

(b) Now, the unperturbed system is in the ground state for $t < 0$. For $t > 0$, we turn on the electric field

$$\mathcal{E}(t) = \mathcal{E}_0 \cos \Omega t \;\theta(t).$$

Obtain an expression for the expectation value $\langle x(t) \rangle$ in *lowest* order. First, find the coefficients $c_n(t)$ in lowest order and write down $|\psi(t)\rangle$ formally. Expand your result for $\langle x(t) \rangle$ near resonance $\Omega \approx \omega$ and discuss your findings.

3 Harmonic perturbation: ionisation of the hydrogen atom

A uniform periodic electric field

$$\boldsymbol{\mathcal{E}}(t) = \boldsymbol{\mathcal{E}}_0 \sin \omega t = \frac{\boldsymbol{\mathcal{E}}_0}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right),$$

acts upon a hydrogen atom which is in the ground state at time $t = 0$. That is

$$\langle \boldsymbol{r} | 0 \rangle = \frac{1}{\sqrt{\pi}} a_0^{-3/2} e^{-r/a_0},$$

with energy $E_0 = -\frac{e^2}{2a_0}$, where $a_0 = \frac{\hbar^2}{m_e e^2}$ is the Bohr radius (in Gaussian units). In this exercise, we consider ionization, i.e. transitions to the continuum of the Coulomb potential. To simplify the calculation, we take vacuum as an approximation. The final state is therefore given by a plane wave

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

(a) What does it mean to ionise an atom? What is the minimum frequency ω_0 of the electric field necessary to ionise a hydrogen atom in the ground state?

(b) Introduce the field with a scalar potential, and calculate the matrix element that appears in Fermi's golden rule for a harmonic perturbation (absorption),

$$w_{0\to\boldsymbol{k}} = \frac{2\pi}{\hbar} |V^{\dagger}_{\boldsymbol{k}0}|^2 \delta(E_{\boldsymbol{k}} - E_0 - \hbar\omega).$$

The integrals might seem daunting, but they become relatively easy if you choose a suitable coordinate system to perform the integration.

(c) Integrate over all $\boldsymbol{k}$ to find the ionisation rate $W_{0\to i}$ and discuss its behaviour as a function of ω/ω_0.

Time-dependent perturbation theory If the system is initially in an eigenstate $|i\rangle$ at time t_0, the zeroth- and first-order coefficients are given by

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= -\frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{i\omega_{ni}t'} V_{ni}(t'), \end{aligned}$$

where

$$\omega_{ni} = \frac{E_n - E_i}{\hbar}, \qquad V_{ni}(t) = \langle n|\, V(t)\, |i\rangle\,.$$