Quantum Mechanics

Sample exam 4

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Question 1 – Perturbation theory

In Lorenz gauge, the potential of a charge Ze moving with constant velocity $\boldsymbol{v}$ can be written as (in Gaussian units)

$$\phi(\boldsymbol{r}) = \frac{Ze}{\sqrt{r^2 - \left(\frac{\boldsymbol{r}\times\boldsymbol{v}}{c}\right)^2}},$$

where $\boldsymbol{r}$ is the instantaneous position vector of the moving charge. This potential takes into account that the Coulomb interaction propagates with the speed of light, resulting in a retardation effect for moving charges, which is essentially a relativistic effect.

(a) Expand the expression for $V(\boldsymbol{r}) = -e\phi(\boldsymbol{r})$ and find the lowest-order correction to the potential energy of static charges.

(b) Consider this *retardation correction* as a perturbation to the hydrogen-like atom and find the first-order correction to the energy levels. You do not need to calculate the radial matrix element explicitly.

Question 2 – Scattering theory

Consider a particle of mass m that scatters at the repulsive potential

$$V(r) = \frac{\lambda}{r^2}, \qquad \lambda > 0.$$

(a) Calculate the differential cross section in the first Born approximation.

(b) Solve the scattering problem exactly and use the asymptotic form

$$j_\alpha(kr) \propto \frac{e^{-ikr}}{r} - e^{-i\pi\alpha}\frac{e^{ikr}}{r},$$

to find the phase shift δ_l. Make sure you obtain the free result $\lim_{\lambda\to 0}\delta_l = 0$.

(c) Expand the phase shift to first order in $m\lambda/\hbar^2$.

(d) Use this result to calculate the scattering amplitude to first order and show that you obtain the same result as the first Born approximation.

Question 3 – Relativistic quantum mechanics

Show that the energy spectrum of a relativistic spinless boson with charge q in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is given by

$$E_n^2(k_z) = m^2c^4 + \hbar^2 k_z^2 c^2 + 2mc^2\hbar\omega\left(n + \frac{1}{2}\right),$$

with $\omega = \frac{|qB|}{mc}$ and $n = 0, 1, \ldots$. Reduce the problem to a well-known non-relativistic problem to find the solution. Use minimal coupling $\boldsymbol{p} \to \boldsymbol{p} - q\boldsymbol{A}/c$ with the Landau gauge $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$ to introduce the magnetic field.

Hints

- The eigenkets of the hydrogen-like atom can be written as

$$|nlm\rangle = |nl\rangle_r \, |lm\rangle_\Omega \,.$$

- The differential equation

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\alpha(\alpha+1)}{r^2} - k^2\right] f(r) = 0,$$

  has one solution for $\alpha \geq 0$ that is regular at the origin:

$$f(r) \propto j_\alpha(kr),$$

  where j_α is the spherical Bessel function of the first kind.

- Partial-wave expansion of the scattering amplitude for spherically-symmetric potentials:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\sin\delta_l \, P_l\left(\cos\theta\right).$$

- Generating function for Legendre polynomials:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l.$$