# Topological Systems: Worksheet 3

*Majorana bound states from helical edge states*

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The minimal model for the helical edge state of a two-dimensional $\mathbb{Z}_2$ topological insulator can be written as

$$H_0 = \sum_k c_k^\dagger \, \hbar v k s_z \, c_k,$$

where $v$ is the Fermi velocity, $s_z$ is a spin-1/2 Pauli matrix, and $c_k = (c_{k\uparrow}, c_{k\downarrow})^t$. The helical edge state is protected by time-reversal and charge-conservation symmetry. Magnetic impurities along the edge break time-reversal symmetry and lead to single-particle backscattering between the Kramers partners, gapping out the helical edge. Such a process can be included as,

$$H_M = \sum_k c_k^\dagger \, M s_x \, c_k.$$

Another way to open gap without breaking time-reversal symmetry, is by placing a superconductor on the edge, which breaks charge-conservation symmetry. An incoming spin-up electron can be reflected at the superconductor as an outgoing spin-down hole. In this process, called Andreev reflection, a Cooper pair of charge $2e$ is transmitted to the superconductor. The Hamiltonian that describes both processes can be written as

$$H_{\rm BdG} = \frac{1}{2} \sum_k \Psi_k^\dagger \, \mathcal{H}_{\rm BdG}(k) \, \Psi_k,$$

$$\mathcal{H}_{\rm BdG}(k) = \tau_z \left( \hbar v k s_z - \mu s_0 \right) + M \tau_0 s_x + \Delta_0 \left( \cos\phi \, \tau_x + \sin\phi \, \tau_y \right) s_0,$$

in the Nambu basis $\Psi_k = (c_{k\uparrow}, c_{k\downarrow}, -c_{-k\downarrow}^\dagger, c_{-k\uparrow}^\dagger)^t$ and where $\tau_i s_j \equiv \tau_i \otimes s_j$. Here $\Delta = \Delta_0 e^{i\phi}$ is the superconducting pairing potential and $\tau_i$ are Pauli matrices in the electron-hole subspace.

(1) Find the excitation spectrum $\varepsilon(k)$ of $\mathcal{H}_{\rm BdG}(k)$ and make a sketch of the energy bands for $M = \Delta = 0$, $M = 0$, and $\Delta = 0$.

(2) Consider a system with $M(x) = M_0 \theta(-x)$ and $\Delta(x) = \Delta_0 \theta(x)$ with $\Delta_0 > 0$, where $\theta(x)$ is the step function, as illustrated in Fig. 1, and find the eigenstates in the Nambu basis in region I ($x < 0$) and region II ($x > 0$) for $\varepsilon = 0$. Hint: use the *ansatz* $\psi(x) = e^{\lambda x} \psi_0$ and note that the electron and hole equations decouple for $\Delta = 0$.

(3) The general solution at $\varepsilon = 0$ can be written as (setting $\hbar v = 1$)

$$\psi_I(x) = a_1 \begin{pmatrix} M_0 \\ \mu + i\lambda_I \\ 0 \\ 0 \end{pmatrix} e^{\lambda_I x} + a_2 \begin{pmatrix} M_0 \\ \mu - i\lambda_I \\ 0 \\ 0 \end{pmatrix} e^{-\lambda_I x} + b_1 \begin{pmatrix} 0 \\ 0 \\ -M_0 \\ \mu + i\lambda_I \end{pmatrix} e^{\lambda_I x} + b_2 \begin{pmatrix} 0 \\ 0 \\ -M_0 \\ \mu - i\lambda_I \end{pmatrix} e^{-\lambda_I x}$$

for region I, where $\lambda_I = \sqrt{M_0^2 - \mu^2}$, and for region II

$$\psi_{II}(x) = c_1 \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} e^{\lambda_{II+}x} + c_2 \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix} e^{-\lambda^*_{II+}x} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix} e^{\lambda_{II-}x} + c_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix} e^{-\lambda^*_{II-}x},$$

where $\lambda_{II\pm} = \Delta_0 \pm i\mu$. Now assume the Fermi energy lies in the magnetic gap $M_0^2 > \mu^2$ and that $\Delta_0 > 0$. Since there are no propagating modes in the energy gap, the eigenmodes at $\varepsilon = 0$ are evanescent modes in both regions. Select the correct modes in both regions by noting that the wave function has to be normalizable.

(4) The second boundary condition is the continuity of the Nambu spinor at $x = 0$. Show that the zero mode is a solution and find its wave function, which can be written as

$$\psi_0(x) = \frac{a}{2} \begin{pmatrix} e^{i\mu\theta(x)x} \\ e^{i\chi}e^{-i\mu\theta(x)x} \\ -ie^{i\mu\theta(x)x} \\ ie^{i\chi}e^{-i\mu\theta(x)x} \end{pmatrix} \exp\left[\left(\sqrt{M_0^2 - \mu^2}\,\theta(-x) - \Delta_0\theta(x)\right) x\right],$$

where $e^{i\chi} = (\mu + i\sqrt{M_0^2 - \mu^2}\,)/M_0$ and $|a|^2 = 2\Delta_0\sqrt{M_0^2 - \mu^2}/(\sqrt{M_0^2 - \mu^2} + \Delta_0)$ is the normalization constant.

(5) Write down the quasiparticle operator $\gamma_0 = \int dx\, \psi_0^\dagger(x)\Psi(x)$ of the Majorana mode. Show that $\bar{\gamma}_0^\dagger = \bar{\gamma}_0$ and $\bar{\gamma}_0^2 = 1$ with $\bar{\gamma}_0 = \sqrt{2}e^{i\pi/4}e^{i\chi/2}\gamma_0$. Hint: use the anticommutator relations $\{\hat{\psi}_s(x), \hat{\psi}_{s'}(x')\} = 0$ and $\{\hat{\psi}_s(x), \hat{\psi}^\dagger_{s'}(x')\} = \delta_{ss'}\delta(x - x')$.

(6) Now perform the same calculation for $M = 0$ and

$$\Delta(x) = \begin{cases} \Delta_0 & x < 0 \\ \Delta_0 e^{i\phi} & x > 0 \end{cases},$$

and show that there are two zero modes for $\phi = \pi$. However, these solutions are not independent. Show that their quasiparticle operators can be written as $f$ and $f^\dagger$ and construct two Majorana operators: $\gamma_1 = f + f^\dagger$ and $\gamma_2 = -i(f - f^\dagger)$. These Majoranas are localized in the same place but are protected by time-reversal symmetry, which is preserved for $\phi = 0$ or $\pi$.

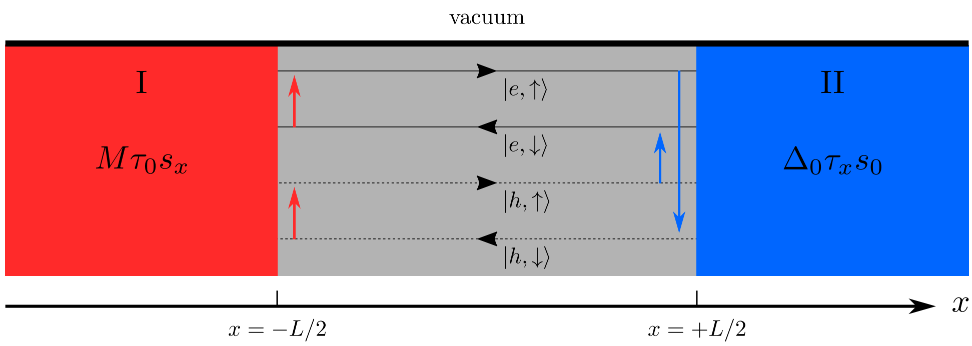


Figure 1: Hybrid junction in a helical edge state of length $L$. Here, we only consider the short-junction limit $L \to 0$. The spin-flip and Andreev reflection processes at the magnetic and superconducting region are denoted by the vertical arrows, respectively.