Analytical Mechanics: Worksheet 3

Lagrangian mechanics 2

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1 Spherical pendulum in a magnetic field

The Lagrangian of a spherical pendulum can be written as

$$L = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) + mgl \cos\theta,$$

where l is the distance from the pivot to the point mass m. In addition, we now consider a mass with charge e in a uniform magnetic field $\vec{B} = -B\hat{z}$. We can account for a static magnetic field with a velocity-dependent generalized potential:

$$U = -e\,\vec{A} \cdot \vec{v}, \qquad \text{with} \qquad \vec{B} = \nabla \times \vec{A}.$$

(a) Choose a proper gauge for $\vec{A}$ and write the magnetic coupling as $U = -\vec{\mu} \cdot \vec{B}$ where $\vec{\mu} = \gamma \vec{L} = \gamma m \vec{r} \times \vec{v}$ is the orbital magnetic moment, and find the gyromagnetic ratio γ.

(b) Write down the total Lagrangian and substitute the Larmor frequency $\omega = eB/(2m)$.

(c) Determine the cyclic coordinate and corresponding canonical momentum. Relate this to the symmetry of the system.

(d) This system has another conserved quantity. Find it and explain why it is conserved. Use it to find an expression for the angle θ of the form:

$$\frac{1}{2} m l^2 \dot{\theta}^2 + V(\theta) = \text{constant}.$$

(e) Find the solution $\theta(t)$ near the minimum of the effective potential $V(\theta)$ in the harmonic approximation.

(f) Find an expression for $\phi(t)$ to lowest order in the deviation amplitude of $\theta(t)$.

2 Virial theorem

The virial (from Latin *vis* "force") theorem for a one-dimensional scleronomic system states

$$2\,\langle T \rangle = -\langle F \cdot q \rangle,$$

for bound motion. Here, T is the kinetic energy and $F = -dV/dq$ the force from the potential $V = V(q)$. The angle brackets $\langle . \rangle$ indicate a long-time average,

$$2\,\langle T \rangle = \frac{1}{\tau} \int_0^\tau m \dot{q}^2 \, dt,$$

where τ is much longer than any time scale of the system. Prove the virial theorem with the principal of stationary action.

(a) Calculate the variation of the action δS due to a variation of the path $q(t) \to (1+\epsilon)\, q(t)$ from an initial time $t = 0$ to $t = \tau$. Then write $\delta S/\tau$ as a time average.

(b) Use the equations of motion to write δS as a boundary term. Argue that these are finite for bounded motion. Use this to prove the virial theorem in the limit $\tau \to \infty$.

(c) Consider $V = aq^n$ and use the virial theorem to write the total energy with $\langle V \rangle$ or $\langle T \rangle$ only. Note that $n = -1$ for gravity or electrostatics and $n = 2$ for a harmonic oscillator.

Interestingly, the virial theorem also holds in quantum mechanics for a time-independent Hamiltonian where the average is now defined as the expectation value with respect to an eigenstate.

3 Variational principal for the Schrödinger equation

The one-dimensional Schrödinger equation is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x,t)\psi. \tag{$\clubsuit$}$$

Find a variational principal that gives this equation in terms of a (generalized) Euler-Lagrange equation, using the complex fields $\psi(x,t)$ en $\psi^*(x,t)$ which depend on independent variables x and t. Hence, you have to find a Lagrangian density $\mathcal{L}$ that yields the Schrödinger equation for

$$\int\int \delta\mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial x}, \frac{\partial \psi^*}{\partial t}, x, t\right) dx\, dt = 0.$$

We start by reviewing the variational method for a functional that depends on two variables.

(a) Calculate the variation $\delta\mathcal{F} = \overline{\mathcal{F}} - \mathcal{F}$ of the function $\mathcal{F}\left(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, x, t\right)$ by varying $y = y(x,t)$ while keeping x and t constant. Here we defined $\overline{\mathcal{F}} = \mathcal{F}\left(\overline{y}, \frac{\partial \overline{y}}{\partial x}, \frac{\partial \overline{y}}{\partial t}, x, t\right)$ and $\overline{y} = y + \delta y$.

(b) Calculate the double integral over x and t of $\delta\mathcal{F}$. Make use of the chain rule and Green's theorem:

$$\int\int_D (\nabla \cdot \vec{u})\, dA = \oint_{\partial D} (\vec{u} \cdot \hat{n})\, ds.$$

Here $\vec{u} = (u_x, u_t)$ is a vector field and $\hat{n} = \hat{n}(s)$ is a unit vector that lies in the plane and is perpendicular to the boundary ∂D of the 2D surface D on the left-hand side. How can we get rid of the boundary integral?

Our goal is to find an equation of motion for $y(x,t)$ together with boundary conditions on ∂D that extremize the functional. One can either use fixed boundary fields (Dirichlet) which restricts the space of allowed variations. Alternatively, natural boundary conditions (Neumann) are obtained by demanding that the boundary term vanishes for any δy. One can even have mixed versions with different boundary conditions on disjoint parts of the boundary. All these statements hold in higher dimensions (making use of the general divergence theorem) or with multiple fields. It is also instructive to check that for a single degree of freedom, the natural boundary condition gives $\dot{q}(s) = 0$ with $\partial D = \{t_1, t_2\}$.

(c) The variational principal requires that $\int dx \int dt\, \delta\mathcal{F} = 0$. We saw that vanishing of the boundary term is related to specifying the boundary conditions. Now consider the bulk term and find a partial differential equation for $y(x,t)$.

(d) Apply this method and find the Schrödinger Lagrangian density $\mathcal{L}$ that reproduces Eq. ($\clubsuit$). Assume that $\mathcal{L}$ is real such that it only contains terms such as $\psi\psi^*$ and $\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x}$.