Quantum Mechanics: Worksheet 6

Klein-Gordon equation and the Klein paradox

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1 Klein-Gordon equation (theory)

The Klein-Gordon equation is given by

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0,$$

where $\phi(x)$ is a scalar function of the space-time coordinates $x = (ct, \boldsymbol{r})$. Here ∂^2 is the d'Alembertian,

$$\partial^2 \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

Since $\phi(x)$ is a scalar it represents a spinless boson, which transforms according to the trivial representation of the Lorentz transformation. With $\phi(x) = e^{-\frac{i}{\hbar}Et}\varphi(\boldsymbol{r})$, the time-independent Klein-Gordon equation becomes

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{E^2 - m^2c^4}{2mc^2}\right)\varphi(\boldsymbol{r}) = 0.$$

Static external electromagnetic fields are included in the same way as for the Schrödinger equation, with minimal coupling: $p_\mu \to p_\mu - \frac{q}{c}A_\mu$ (in Gaussian units).

2 Scattering of a charged spinless boson at a potential step

Consider a charged spinless boson (e.g. a charged pion) that scatters at a potential step:

$$V(x) = V_0\,\theta(x),$$

with $V_0 \geq 0$. This potential can be introduced in the time-independent Klein-Gordon equation by substituting $E \to E - V(x)$.

(a) Write down the general solution $\varphi^<(x)$ for $x < 0$ and $\varphi^>(x)$ for $x > 0$.

(b) In which energy range is the wave vector inside the potential real, and in which case is it imaginary?

(c) Use the continuity of the wave function and its derivative to find an expression for the coefficients of the wave functions.

(d) Calculate the current for both $x < 0$ and $x > 0$, separating the incident (j_i), the reflected (j_r), and the transmitted wave (j_t). The current is in general given by

$$j = \frac{\hbar}{2mi}\left(\varphi^*\frac{d\varphi}{dx} - \phi\frac{d\varphi^*}{dx}\right) = \frac{\hbar}{m}\mathrm{Im}\left[\varphi^*\frac{d\varphi}{dx}\right].$$

(e) Find the probability for reflection R and transmission T, defined as

$$R = -\frac{j_r}{j_i}, \qquad T = \frac{j_t}{j_i}.$$

The transmission probability is shown in Figure 1 as a function of the potential height.

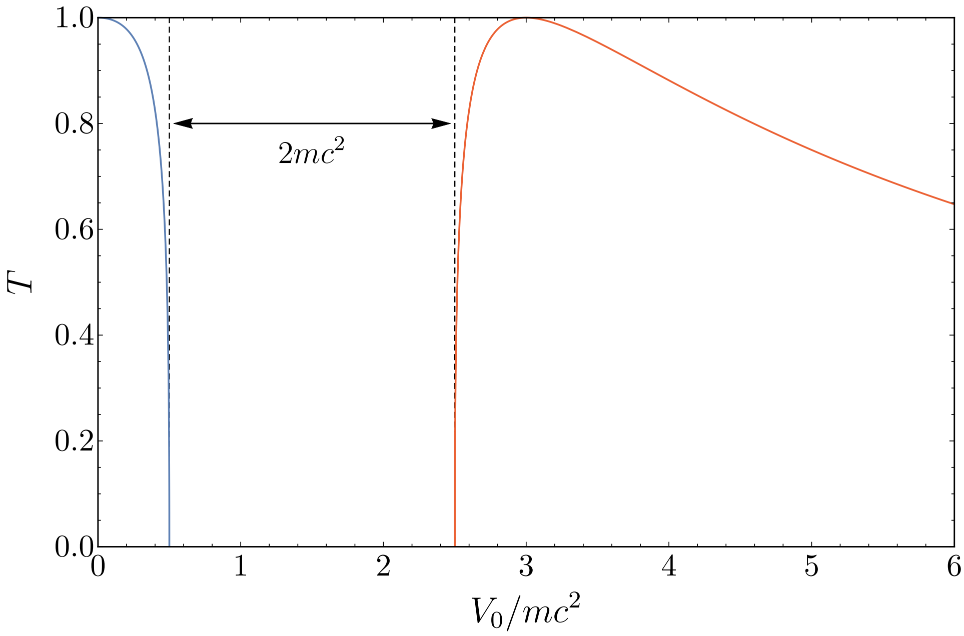

Figure 1: Transmission probability T for an incident particle with energy $E = \frac{3}{2}mc^2$.

(f) Show that $R + T = 1$, for real and imaginary wave vector inside the potential. In which cases is there perfect transmission? Check with Figure 1.

(g) For which values of the potential does the Klein paradox occur? What is the difference with the Schrödinger solution for a potential step?

(h) Calculate the asymptotic behavior for the transmission probability for $V_0 \gg mc^2$.

3 Compton wavelength

Consider a particle with mass m confined to a region of size L.

(a) Use the uncertainty relation $L\Delta p \geq \hbar/2$ to find the length scale at which the uncertainty on the energy becomes of the order of the rest energy.

(b) The Compton wavelength of a particle with mass m is given by $\lambda = h/mc$ (about 2×10^{-12} meters for an electron). What happens if we confine particles to regions smaller than λ.

(c) How much smaller is the reduced Compton wavelength of an electron to the atomic length scale, i.e. the Bohr radius (size of the hydrogen atom)? This reflects the separation of scales between the relativistic and non-relativistic atomic structure.

The uncertainty in the momentum is $\Delta p \geq \hbar/L$. In relativity momentum and energy are on equal footing so that the uncertainty on the (kinetic) energy $\Delta E \geq \hbar c/L$. Find the typical length scale at which relativistic effects become important. What would you expect to find at distances shorter than this length, called the *Compton wavelength*? What do you conclude for massless particles? The *de Broglie wavelength* is the distance at which the wave nature of matter becomes important; the Compton wavelength is the distance at which quantum field theory becomes important and the concept of a single point like particle breaks down.