Analytical Mechanics: Worksheet 2

Lagrangian mechanics 1

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The Lagrangian of a mechanical system is defined as

$$L = T - V,$$

with T and V the kinetic and potential energy (for conservative forces). The equations of motion are obtained from the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \qquad (k = 1, \ldots, N),$$

with N the number of degrees of freedom. Constraints reduce the number of independent degrees of freedom. By definition, for a holonomic system, the constraints can be written

$$f_j(q_1, \ldots, q_N, t) = 0, \qquad (j = 1, \ldots, n),$$

giving $N - n$ independent coordinates q_k. When the constraints do not depend explicitly on time, they are called scleronomic, otherwise they are called rheonomic.

1 Obtain the Lagrangian

First sketch the system and choose an inertial frame. Then determine proper generalized coordinates q_k and write down the position vector $\vec{r}_i$ of each point mass in terms of the q_k.

(a) A point mass sliding frictionless down a slope with angle α.

(b) A point mass suspended on a spring with spring constant k and equilibrium length l, and which can only move vertically.

(c) A pendulum in a vertical plane where the pivot can freely move.

(d) A pendulum in a vertical plane where the pivot is attached to a point mass, which in turn is attached to a spring that can only move horizontally in the same plane.

(e) Three point masses on an horizontal line connected through four identical springs. The ends of the first and last spring are connected to two anchors separated by a distance $4l$.

2 Bead moving on a rotating horizontal line

Consider a bead that moves along a horizontal line that is rotating around a vertical axis with angular velocity ω.

(a) Use the result of Worksheet 1 to find the Lagrangian and Hamiltonian of this system.

(b) Is the Hamiltonian a constant of motion?

(c) Is the total energy equal to the Hamiltonian?

(d) How do your conclusions change if $\omega = \omega(t)$?

3 Euler angles

To describe the rotation of a rigid body, we require three angles. The most commonly used angles are the *Euler angles.* These are defined in three steps. Starting from an inertial frame, we first rotate $\vec{r}_I$ to $\vec{r}'$, then to $\vec{r}''$, and finally to $\vec{r}_B$. In particular:

1. Rotate the axes about the z_I axis by an angle ψ;
2. Rotate the axes about the x' axis by an angle θ;
3. Rotate the axes about the $z'' = z_B$ axis by an angle ϕ.

Calculate the total rotation matrix $R = R''_\phi R'_\theta R^I_\psi$ between the inertial frame and the body frame (you can use Wolfram Mathematica or similar software). Use the standard definition where a positive angle gives a counterclockwise rotation.

4 Symmetric top

Consider a symmetric top with a pivot that is fixed at the origin.

(a) Sketch the inertial frame and the body frame, and determine the degrees of freedom.

(b) From the definition of Euler angles it follows that the angular velocity equals

$$\vec{\omega} = \dot{\psi}\hat{z}_I + \dot{\theta}\hat{x}' + \dot{\phi}\hat{z}_L.$$

Determine $\vec{\omega}$ in the body frame using the rotation matrices that you calculated above.

(c) The kinetic energy is entirely rotational and takes on a simple form in the body frame:

$$T = T_r = \frac{1}{2}\sum_i I_i \omega_i^2,$$

where $I_i > 0$ is the moment of inertia along the i-th direction and the ω_i are the components of $\vec{w}$. Make use of the symmetry of the top, and write this expression in terms of the Euler angles and their time derivatives.

(d) Obtain the Lagrangian and find the cyclic coordinates q defined as $\partial L/\partial q = 0$. Find an expression for $\dot{q}$ using the canonical momentum $p_q = \partial L/\partial \dot{q}$. Remember that p_q is a constant of motion for a cyclic coordinate (first integral of motion) which follows from the Euler-Lagrange equations. What do the canonical momenta represent physically? How can we understand this from symmetry?

(e) Because this is a scleronomic system, the total energy is conserved. We can write

$$E = \frac{1}{2}I_1\dot{\theta}^2 + V(\theta) = \text{constant}, \tag{1}$$

where $V(\theta)$ is called the effective potential. Find an expression for $V(\theta)$ using your previous result.

(f) Equation (1) effectively describes a particle with mass I_1 moving on a line in a potential $V(\theta)$. Assume that θ_0 is the equilibrium position, defined as $dV/d\theta|_{\theta=\theta_0} = 0$. Formally expand $V(\theta)$ around θ_0 up to second order. This is called the *harmonic approximation.* Assume that the equilibrium is stable (when is this the case?) and use your physical intuition to write down a general solution: this harmonic motion is called *nutation.*

(g) Expand $\cos\theta$ and $\sin\theta$ around $\theta = \theta_0$ in lowest order of the nutation amplitude ϵ. Here we have to assume that ϵ is small such that the lowest-order approximation for $V(\theta)$ near equilibrium holds. Use this to obtain expressions for $\dot{\phi}$ and $\dot{\psi}$. These motions are, respectively, called *rotation* (or *spin*) [$\phi(t)$] and *precession* [$\psi(t)$].