

# Quantum Mechanics

## Sample exam 4

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### Question 1 – Perturbation theory

The potential energy between a static charge  $Ze$  located in the origin and a charge  $-e$  moving with constant velocity  $\mathbf{v}$  can be written as (in Gaussian units)

$$V(\mathbf{r}) = -\frac{Ze^2}{\sqrt{r^2 - \left(\frac{\mathbf{r} \times \mathbf{v}}{c}\right)^2}},$$

where  $\mathbf{r}$  is the position vector of the moving charge. This potential takes into account that the Coulomb interaction propagates with the speed of light, resulting in a retardation effect for moving charges, which is essentially a relativistic effect.

- (a) Expand the expression for  $V(\mathbf{r})$  and find the lowest-order correction to the potential energy of static charges.
- (b) Consider this *retardation* correction to the potential as a perturbation to the hydrogen-like atom and find the first-order correction to the energy levels. You do not need to calculate the radial matrix element explicitly.

### Question 2 – Scattering

Consider a particle of mass  $m$  that scatters at the repulsive potential

$$V(r) = \frac{\lambda}{r^2}, \quad \lambda > 0.$$

- (a) Calculate the differential cross section in the first Born approximation.
- (b) Solve the scattering problem exactly and use the asymptotic form

$$j_\alpha(kr) \propto \frac{e^{-ikr}}{r} - e^{-i\pi\alpha} \frac{e^{ikr}}{r},$$

to find the phase shift  $\delta_l$ . Make sure you obtain the free result  $\lim_{\lambda \rightarrow 0} \delta_l = 0$ .

- (c) Expand the phase shift to first order in  $m\lambda/\hbar^2$ .
- (d) Use this result to calculate the scattering amplitude to first order and show that you obtain the same result as the first Born approximation.

### Question 3 – Relativistic quantum mechanics

Show that the energy spectrum of a relativistic spinless boson with charge  $q$  in a uniform magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$  is given by

$$E_n^2(k_z) = m^2c^4 + \hbar^2k_z^2c^2 + 2mc^2\hbar\omega \left(n + \frac{1}{2}\right),$$

with  $\omega = \frac{|qB|\hbar}{mc}$  and  $n = 0, 1, \dots$ . Reduce the problem to a well-known non-relativistic problem to find the solution. Use minimal coupling  $\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}/c$  with the Landau gauge  $\mathbf{A} = Bx\hat{\mathbf{y}}$  to introduce the magnetic field.

## Hints

- The eigenkets of the hydrogen-like atom can be written as

$$|nlm\rangle = |nl\rangle_r |lm\rangle_\Omega.$$

- The differential equation

$$\left[ -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{\alpha(\alpha+1)}{r^2} - k^2 \right] f(r) = 0,$$

has one solution for  $\alpha \geq 0$  that is regular at the origin:

$$f(r) \propto j_\alpha(kr),$$

where  $j_\alpha$  is the spherical Bessel function of the first kind.

- Partial-wave expansion of the scattering amplitude for spherically-symmetric potentials:

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta).$$

- Generating function for Legendre polynomials:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l.$$