

Analytical Mechanics: Worksheet 4

Lagrange multipliers

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1 Theory

Consider a mechanical system with Lagrangian $L = L(q_i, \dot{q}_i, t)$ and a holonomic constraint,

$$f(q_1, \dots, q_n, t) = 0. \quad (\hookrightarrow)$$

Instead of using this constraint to directly reduce the number of independent coordinates by one, we can account for it using the method of *Lagrange multipliers*. To this end, first take the variation of the constraint (\hookrightarrow),

$$\delta f = \frac{\partial f}{\partial q_1} \delta q_1 + \dots + \frac{\partial f}{\partial q_n} \delta q_n = \delta 0 = 0,$$

which holds for all times t . If we multiply this equation with an unknown function $\lambda = \lambda(t)$ and integrate over time, the right-hand side still vanishes. Adding this result to the variation of the action:

$$\delta S \rightarrow \delta S + \int dt \lambda \delta f = \int dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \lambda \frac{\partial f}{\partial q_i} \right) \delta q_i = 0,$$

yields n equations

$$\frac{\delta L}{\delta q_i} \equiv \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\lambda \frac{\partial f}{\partial q_i},$$

where the right-hand side is a constraint force (e.g. normal forces or tension).

Hence, together with (\hookrightarrow) we have a total of $n+1$ equations for the $n+1$ functions $\{q_1, \dots, q_n, \lambda\}$ where λ is called a *Lagrange multiplier*. The method of Lagrange multipliers transforms a constrained problem to a free variational problem with an extra degree of freedom λ :

$$\bar{L} = L + \lambda f, \quad \frac{\delta \bar{L}}{\delta q_i} = \frac{\delta \bar{L}}{\delta \lambda} = 0,$$

where the variation with respect to λ gives the constraint (\hookrightarrow).

2 Applications

2.1 Planar pendulum

The Lagrangian of a point mass moving in a vertical plane is given by

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy.$$

Let us apply an additional constraint such that the distance of the point mass to the origin remains fixed:

$$f(x, y) = \sqrt{x^2 + y^2} - l = 0.$$

- Use a Lagrange multiplier λ and find the equations of motion for x and y .
- Find a relation between the constraint force (tension) that keeps the point mass along a circle, the constraint, and the Lagrange multiplier λ .
- Define an effective potential energy that includes the contribution from the constraint.

2.2 Catenary

The method of Lagrange multipliers can also be used to enforce constraints in a static equilibrium problem. As an example, consider a rope or chain with a constant mass density ρ and length l . Our goal is to find the shape $y(x)$ that minimizes the potential energy in a uniform gravitational field. This is called the catenary problem.

The potential energy of the rope is given by

$$V = \rho g \int ds y(s),$$

where ds is the line element along the rope. A convenient parameterization of the curve is given by $(x, y(x))$ with

$$ds = dx \sqrt{1 + y'(x)^2}.$$

The constraint of fixed length can be expressed as

$$f = \int ds - l = \int dx \sqrt{1 + y'(x)^2} - l = 0,$$

which is called an *isoperimetric* constraint. Hence, using the method of Lagrange multipliers, we have to minimize the functional $V + \lambda f$ with respect to $y(x)$.

- (a) Define an effective Lagrangian and write down the Euler-Lagrange equation for $y(x)$.
- (b) Instead of directly solving the equation of motion, note that the Lagrangian does not depend explicitly on x . Hence, the “Hamiltonian”

$$H \equiv L - y' \frac{\partial L}{\partial y'} = \text{constant},$$

because $dH/dx = -\partial L/\partial x$. Prove this and calculate the Hamiltonian. In this context, this is also called the *Beltrami identity*.

- (c) Define constants a and b such that $H = \rho g a$ and $\lambda = \rho g b$ and obtain a solution by solving the Beltrami identity for $y(x)$. Hint: $\int \frac{du}{\sqrt{u^2 - 1}} = \text{arcosh } u$.
- (d) Consider a symmetric solution $y(-x) = y(x)$ and set $b = 0$. Now assume that the rope is anchored between two points at $x = \pm w/2$. Find an equation for l in terms of a and w .