Quantum Mechanics: Worksheet 5

Scattering theory: partial waves

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1 Partial-wave expansion of a plane wave

A plane wave with wave vector $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ can be expanded in partial waves as

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} A_l j_l(kr) P_l(\cos\theta).$$

Find an expression for the coefficient A_l using the orthogonality of the Legendre polynomials:

$$\int_{-1}^{1} du\, P_l(u) P_{l'}(u) = \frac{2}{2l+1}\delta_{ll'}.$$

Use any source to find a suitable integral representation of $j_l(kr)$ and solve for A_l.

2 Method of partial waves

The method of partial waves consists of a convenient basis expansion which separates the total scattering problem in independent smaller problems.

In a scattering problem, we consider an incoming plane wave that scatters into outgoing channels. In one dimension, we typically have two outgoing channels (reflected and transmitted). In higher dimensions, there are an infinite amount of channels. However, one can often expand the incoming and outgoing wave in a suitable basis to simplify the problem. For a spherically-symmetric potential in particular, we can decompose a plane wave in contributions with definite l and m quantum numbers since $\boldsymbol{L}^2$ and L_z are conserved. These are called *partial waves* because each represents a part of the plane wave. Partial waves are free-particle solutions in spherical coordinates and are ideal for spherically-symmetric potentials because mixing of different partial waves is forbidden by symmetry. Hence, outgoing partial waves do not lose any amplitude and only pick up a phase because of current conservation (see Figure 1).

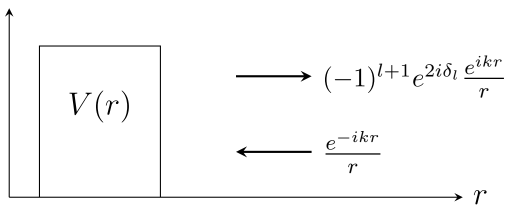

Figure 1: Scattering of partial waves at a spherically-symmetric potential $V(r)$. Only the asymptotic form of the radial wave function is shown (exact for $l = 0$).

The radial Schrödinger equation for a spherically-symmetric potential is given by

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_l}{dr}\right) + \left[\frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}\left(V(r) - E\right)\right] R_l = 0, \tag{1}$$

where $R_l(r)$ is the radial wave function. For a constant potential, the solutions of (1) are the spherical Bessel functions:

$$R_l(r) = a_l j_l(qr) + b_l n_l(qr), \qquad q = \sqrt{2m(E-V)/\hbar^2},$$

where a_l and b_l are constants. The solution can also be expressed with spherical Hankel functions $h_l^{(1)} = j_l + in_l$ and $h_l^{(2)} = j_l - in_l$, which are useful if q becomes imaginary.

3 Spherical potential well

Consider a spherical potential well:

$$V(r) = -V_0\,\theta(a-r), \qquad V_0 > 0.$$

Before we investigate scattering, we consider bound states localized inside the well that have discrete energies in the range $-V_0 < E < 0$.

(a) Find an equation for the s-wave ($l = 0$) bound-state spectrum. For which values of V_0 does a bound state exist? Compare this to the one-dimensional potential well. Hint: substitute $u(r) = rR_0(r)$ to solve the radial equation.

(b) Now consider s-wave scattering ($E > 0$). Outside the well, the scattering state is

$$u(r) = \sin(kr + \delta_0) \propto \underbrace{e^{-ikr}}_{\text{in}} - \underbrace{e^{2i\delta_0}e^{ikr}}_{\text{out}},$$

where $\delta_0(k)$ is the phase shift of the outgoing s wave. The phase shift δ_0 contains all the information on scattering through the s-wave channel.

(c) Find an expression for $\tan\delta_0$ and calculate the s-wave cross section σ_0.

(d) Consider low-energy scattering and evaluate $\tan\delta_0$ to order k^2. In general, the phase shift is small for low-energy scattering. When is this not the case and why?

4 Delta-shell scattering

Consider scattering by a δ-shell potential:

$$V(r) = \gamma\delta(r-a).$$

(a) Integrate the radial equation over a tiny spherical shell around $r = a$ of thickness 2ϵ in the limit $\epsilon \to 0$ to find the boundary condition on the derivative.

(b) Consider s-wave scattering and find an expression for $S_0(k) = e^{2i\delta_0}$.

(c) Look for the poles (zeros of the denominator) of S_0. Substitute $k = \sqrt{2mE}/\hbar = i\kappa$ to get an expression for the bound states. For what values of γ does a bound state exist?

(d) Evaluate the poles of S_0 up to $\left(\frac{\hbar^2}{ma\gamma}\right)^2$ and solve the resulting quadratic equation for ka. Show that the quasibound states evolve to confined states inside the shell for $\left|\frac{\hbar^2}{ma\gamma}\right| \ll 1$.

(e) Why do the poles have to lie in the lower half of the complex k-plane?

(f) Expand the *physical* solution of the quadratic equation up to order $\left(\frac{\hbar^2}{ma\gamma}\right)^4$ and find the expressions for the resonance energy and width in lowest order.

Resonances Resonances correspond to sharp peaks in the cross section due to quasibound states arising from reflections inside the scattering region. This effectively traps the incident wave for a finite time, while it gradually decays to the continuum outside the scattering region, as it is slowly transmitted. Quasibound states have complex energies so they cannot be normalizable, as they are eigenstates of the Hamiltonian. Nevertheless, they can contribute to observable quantities by their proximity to the real axis. At resonance, a quasibound state is almost degenerate with the continuum.