

Analytical Mechanics: Worksheet 2

Lagrangian mechanics 1

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The Lagrangian of a mechanical system is defined as

$$L = T - V,$$

with T and V the kinetic and potential energy (for conservative forces). The equations of motion are obtained from the Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0, \quad (k = 1, \dots, N),$$

with N the number of degrees of freedom. Constraints reduce the number of independent degrees of freedom. By definition, for a holonomic system, the constraints can be written as

$$f_j(q_1, \dots, q_N, t) = 0, \quad (j = 1, \dots, n),$$

giving $N - n$ independent generalized coordinates q_k . When the constraints do not depend explicitly on time, they are called scleronomic, otherwise they are called rheonomic.

1 Obtain the Lagrangian

First sketch the system and choose an inertial frame. Then determine proper generalized coordinates q_k and write down the position vector \vec{r}_i of each particle in terms of the q_k .

- (a) A point mass sliding frictionless down a slope with angle α .
- (b) A point mass attached to a spring (spring constant k , equilibrium length l) fixed at one end and constrained to move only vertically.
- (c) A pendulum in a vertical plane with a freely moving pivot.
- (d) A pendulum confined to a vertical plane, whose pivot is attached to a point mass connected to a horizontally moving spring in the same plane.
- (e) Three point masses on an horizontal line connected through four identical springs. The ends of the first and last spring are connected to two anchors separated by a distance $4l$.

2 Bead moving on a rotating horizontal line

Consider a bead that moves along a horizontal line that is rotating around a vertical axis with angular velocity ω .

- (a) Use the result of Worksheet 1 to find the Lagrangian and Hamiltonian of this system.
- (b) Is the Hamiltonian a constant of motion?
- (c) Is the total energy equal to the Hamiltonian?
- (d) How do your conclusions change if $\omega = \omega(t)$?

3 Euler angles

To describe the rotation of a rigid body, we require three angles. The most commonly used angles are the *Euler angles*. These are defined in three steps. Starting from an inertial frame, we first rotate \vec{r}_I to \vec{r}' , then to \vec{r}'' , and finally to \vec{r}_B . In particular:

1. Rotate the axes about the z_I axis by an angle ψ ;
2. Rotate the axes about the x' axis by an angle θ ;
3. Rotate the axes about the $z'' = z_B$ axis by an angle ϕ .

Calculate the total rotation matrix $R = R''_\phi R'_\theta R^I_\psi$ between the inertial frame and the body frame (you can use Wolfram Mathematica or similar software). Use the standard definition where a positive angle corresponds to a counterclockwise rotation about a given axis.

4 Symmetric top

Consider a symmetric top with a pivot that is fixed at the origin.

- (a) Sketch the inertial frame and the body frame, and determine the degrees of freedom.
- (b) From the definition of the Euler angles it follows that the angular velocity equals

$$\vec{\omega} = \dot{\psi}\hat{z}_I + \dot{\theta}\hat{x}' + \dot{\phi}\hat{z}_L.$$

Determine $\vec{\omega}$ in the body frame using the rotation matrices that you calculated above.

- (c) The kinetic energy is entirely rotational and takes on a simple form in the body frame:

$$T = T_r = \frac{1}{2} \sum_i I_i \omega_i^2,$$

where $I_i > 0$ is the moment of inertia along the i -th direction and the ω_i are the components of $\vec{\omega}$. Make use of the symmetry of the top, and write this expression in terms of the Euler angles and their time derivatives.

- (d) Obtain the Lagrangian and find the cyclic coordinates q defined as $\partial L / \partial q = 0$. Find an expression for \dot{q} using the canonical momentum $p_q = \partial L / \partial \dot{q}$. Remember that p_q is a constant of motion for a cyclic coordinate (first integral of motion). What do the canonical momenta represent physically? How can we understand this from symmetry?
- (e) Because this is a scleronomous system, the total energy is conserved. We can write

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) = \text{constant}, \quad (1)$$

where $V(\theta)$ is called the effective potential. Find an expression for $V(\theta)$ using your previous results.

- (f) Equation (1) describes a particle with mass I_1 moving on a line in a potential $V(\theta)$. Assume that θ_0 is the equilibrium position, defined as $dV/d\theta|_{\theta=\theta_0} = 0$. Formally expand $V(\theta)$ around θ_0 up to second order. This is called the *harmonic approximation*. Assume that the equilibrium is stable (when is this the case?) and use your physical intuition to write down a general solution: this harmonic motion is called *nutation*.
- (g) Expand $\cos \theta$ and $\sin \theta$ around $\theta = \theta_0$ in lowest order of the nutation amplitude ϵ . Here we have to assume that ϵ is small such that the harmonic approximation holds. Use this to obtain expressions for $\dot{\phi}$ and $\dot{\psi}$. These motions are referred to as *rotation* (or *spin*) $[\phi(t)]$ and *precession* $[\psi(t)]$.