Analytical Mechanics: Worksheet 5

Non-holonomic constraints

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1 Theory

Consider a mechanical system with Lagrangian $L = L(q, \dot{q}, t)$ subject to a constraint

$$c_1(q,t)\delta q_1 + \cdots + c_n(q,t)\delta q_n = 0, \tag{$\heartsuit$}$$

where c_k are functions of the generalized coordinates $q = \{q_1, \ldots, q_n\}$ and time t. If these functions can be written as

$$c_i(q,t) = \frac{\partial f}{\partial q_i},$$

then the constraint is *integrable* and can be expressed as $f(q,t) = 0$ (holonomic). However, if this is not the case then the constraint is called *non-integrable*; it cannot be reduced to a constraint on coordinates alone. Equation ($\heartsuit$) is a special case of a *Pfaffian* constraint:

$$\sum_i c_i(q,t)\dot{q}_i + c_0(q,t) = 0,$$

with $c_0 = 0$. Non-integrable Pfaffian constraints are a subset of general *non-holonomic* constraints $f(q, \dot{q}, t) = 0$ that are linear in the generalized velocities. An example of a non-Pfaffian constraint is a constant-speed constraint, or inequalities, called *unilateral constraints* $f(q,t) \geq 0$, e.g. a mass that slides off a sphere with $r \geq R$.

A non-holonomic constraint of the form ($\heartsuit$) can still be dealt with using the method of Lagrange multipliers. Similar as before, we obtain

$$\frac{\delta L}{\delta q_i} \equiv \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\lambda(t) c_i(q,t),$$

which is equivalent to a free variational problem with

$$\delta \overline{L} = \delta L + \lambda \left(c_1 \delta q_1 + \cdots + c_n \delta q_n \right).$$

In this worksheet, we consider rolling without slipping. This is a typical example of a non-integrable constraint that involves terms linear in the velocities. Another example is an ice skate, where motion is constrained along the skate blade without sideways slipping.

2 Rolling disk

We consider an upright coin that rolls without slipping down a slope, see Figure 1. The configuration of the coin is determined by the coordinates of the contact point x and y and the constraint of rolling without slipping.

As shown in the figure, θ is the heading angle of the coin, defined here as the angle between the y axis and the velocity $\vec{v}$ of the contact point, and ϕ is the rotation angle of the coin around its axle. Due to the rolling constraint, we cannot express x and y as functions of θ and ϕ. We can understand this because x and y generally do not return to themselves after a closed path in the (θ, ϕ) plane. Instead, they also depend on the *history of the system*. It is therefore impossible to describe the coin with only two coordinates, even though there are only two degrees of freedom in this problem. This is a general feature of non-holonomic systems.

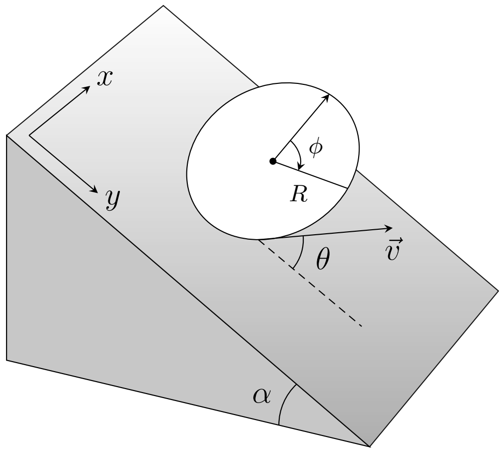

Figure 1: A coin rolls from a slope α without slipping. We also assume that the coin stays perpendicular to the surface of the slope as it rolls. As the coin rolls, it rotates through an angle ϕ about its own axis. In addition, the coin can turn by a heading angle θ about the axis through the contact and perpendicular to the slope.

(a) Determine the non-holonomic constraints of this system by expressing δx and δy in terms of the angles θ and ϕ and their variations. Determine the functions $c_i(q,t)$.

(b) Calculate the kinetic energy. It contains both translational (x and y) and rotational contributions (θ and ϕ). For the latter, you need to calculate the moment of inertia around an axis $\hat{x}_k$:

$$I_k = \int d^3r\, \sigma(\vec{r})\left(r^2 - x_k^2\right).$$

Assume that the coin has no thickness and a uniform mass density σ. Determine a relation between $I_\perp$ and $I_\parallel$.

(c) Calculate the potential energy and determine the Lagrangian. Show that we require three coordinates: θ, ϕ, and y, even though there are only two degrees of freedom.

(d) Use a Lagrange multiplier λ to include the rolling constraint on y to the variation of the action δS and obtain the equations of motion.

(e) Find an expression for λ and solve the equations of motion for $\theta(t)$ and $\phi(t)$.

(f) Substitute $\theta(t)$ and $\phi(t)$ into the expression for $\dot{x}$ and $\dot{y}$. You can obtain the latter from the rolling constraint, taking a variation that coincides with the change over a time dt.

(g) Integrate and find $x(t)$ and $y(t)$.