green's functions

A short introduction

Chris Deimert

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Department of Electrical and Computer Engineering, University of Calgary

- This is intended as a short introduction to Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them.
- For a short presentation, we can either cover a lot of material incompletely, or a small amount of material thoroughly. The second is chosen here, because I think it leads to a clearer understanding of Green's functions as a whole. Of course, the downside is that we'll miss out on a lot of interesting topics and applications.
- Suggested further reading provided at the end.

OUTLINE

1. Basic idea of Green’s functions.
2. Simplest method for solving the Green’s function equation.
3. How to use the Green’s function to solve a problem with boundary conditions. (Biggest section!)
4. Useful properties of Green’s functions for special types of problems.
5. Relationship between Green’s functions and eigenvalues/eigenfunctions.
6. Summary and suggested further reading.

BASIC IDEA

- The basic idea of Green’s functions is really simple. You’ve actually used them before!

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x')f(x')\,\mathrm{d}x}$$

(Some conditions apply.)

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.
- Beware the fine print! This formula actually only works under certain assumptions about the boundary conditions.
- We'll deal with the more general approach later. For now, this gets the key idea across.

FAMILIAR GREEN'S FUNCTIONS

Impulse response of an LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}t'$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t - t')$ from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find $h(t - t')$ using Fourier transform of the transfer function.

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|}$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

- With Green's function theory, we learn how to derive the above expressions. (Though we won't have time to do the 3D ones here.)
- More importantly, Green's function theory allows us to deal with different boundary conditions. The solutions to the Poisson and Helmholtz equations above assume free space (boundaries at infinity). Green's functions would allow us to, e.g., find the response to a current source inside a specific waveguide.

FINDING THE GREEN'S FUNCTION

- In this section, we'll look at one of the simplest methods for actually solving the Green's function problem.

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = \delta(x - x')$$

- Let's start off by looking at a simple example.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem for any $f(x)$.
- But the Green's function problem looks hard! The point of this example is to demonstrate that we can actually solve it.

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

$$\frac{d^2 G(x,x')}{dx^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x'+\epsilon, x') - G(x'-\epsilon, x') \right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x, x')$ is discontinuous (like a step function), then $\mathrm{d}G/\,\mathrm{d}x$ will behave like a delta function and $\mathrm{d}^2G/\,\mathrm{d}x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x, x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')\, \mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left. \frac{\mathrm{d}G}{\mathrm{d}x} \right|_{x=x'+\epsilon} - \left. \frac{\mathrm{d}G}{\mathrm{d}x} \right|_{x=x'-\epsilon} \right] = 1$$

- But what if the derivative $\mathrm{d}G/\,\mathrm{d}x$ is discontinuous?
- Then $\mathrm{d}^2G/\,\mathrm{d}x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x, x')$ is continuous.
- The first integral is an integral of a derivative, so we can use the fundamental theorem of calculus. The result is a *discontinuity condition for the derivative.*

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x,x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

- Applying our two conditions, we can solve for A and B. We find

$$A = B = \frac{1}{2k}$$

Solving, our Green's function is

$$G(x,x') = \frac{1}{2k}\begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(x)$.
- Caution: remember the fine print from before. This solution only works with certain assumptions about boundary conditions.

GENERAL APPROACH

Second-order problems:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- Derivative of $G(x, x')$ is discontinuous at $x = x'$.

- This approach works quite well for solving 1D Green's function problems.
- For problems of other orders, will have a different combination of continuity/discontinuity requirements at $x = x'$. E.g., $G(x, x')$ will be discontinuous for a first order problem.

CONSTRUCTING THE SOLUTION

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\, \mathrm{d}x'$$

Can we prove/generalize this?

- In the introduction, we showed non-rigorously how to construct a solution from the Green's function. To keep things simpler, we ignored boundary conditions.
- Here, we'll look at how to properly construct a solution from the Green's function when boundary conditions are involved.
- Our approach is quite challenging compared to a lot of books on the subject. The advantage is that we'll deal with a lot of subtleties that can otherwise lead to confusion.
- For approaches similar to the one in this section, see Dudley, Morse and Feshbach, or Gerlach.

Inner product:

$$\langle u, v \rangle = \int_a^b u(x)\bar{v}(x)\,\mathrm{d}x$$

Adjoint operator $\mathcal{L}^*$:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

- Underpinning our approach is the idea of an adjoint operator.
- Start with an inner product (for 1D problems, usually the one shown). Note that $\overline{v}$ is the complex conjugate of v.
- If $\mathcal{L}$ is a linear operator, then its adjoint $\mathcal{L}^*$ is defined as the operator which satisfies

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

$$\implies \quad \int_a^b (\mathcal{L}u) v^* \, \mathrm{d}x = \int_a^b u \overline{(\mathcal{L}^* v)} \, \mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \longrightarrow \text{Only for certain } u, v\,!$$

Adjoint boundary conditions:

$$\mathcal{B}_i[u] = 0; \qquad \mathcal{B}_i^*[v] = 0$$

Otherwise

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \underbrace{J(u, \overline{v})}_{\text{Conjunct}}\Big|_a^b$$

- The adjoint is only truly the adjoint (i.e., $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$) for certain functions u and v. (Mathematically, we need to specify the *domains* of $\mathcal{L}$ and $\mathcal{L}^*$ because these are usually unbounded operators.)

- This is where boundary conditions come in. Specifically, if u obeys some boundary conditions $\mathcal{B}_i[u] = 0$, then v has to obey some adjoint boundary conditions $\mathcal{B}_i^*[v] = 0$.

- Note on notation: $\mathcal{B}_i[u] = 0$ means that some linear combination of u and its derivatives are set equal to zero at the boundaries.

- If $\mathcal{B}_i[u] = 0$ or $\mathcal{B}_i^*[v] = 0$ are not satisfied, then the adjoint equation almost holds, but we get an extra term $J(u, v^*)$ called the *conjunct*.

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}u(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

Want $\mathcal{L}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

- Let’s look at an example: the 1D simple harmonic oscillator.
- Let’s try to find $\mathcal{L}^*$ without worrying about boundary conditions for now.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u'' + k^2 u\right] \overline{v}\, \mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[-u'\overline{v}' + k^2 u\overline{v}\right] \mathrm{d}x + [u'\overline{v}]_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

- To find the adjoint, let's expand $\langle \mathcal{L}u, v \rangle$.
- Use integration by parts twice.

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Looks like

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

with

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \overline{k^2}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

- After integration by parts, we can read off the adjoint operator and the conjunct.
- So in this case, the adjoint operator is the almost the same as the original operator, but there's an extra complex conjugate. If k is real, then $\mathcal{L} = \mathcal{L}^*$.

ADJOINT OPERATORS: EXAMPLE

For

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

must have

$$J(u, \overline{v})\Big|_a^b = 0$$
$$\left[u'\overline{v} - u\overline{v}'\right]_a^b = 0$$

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

- Now let's look at adjoint boundary conditions.
- For $\mathcal{L}^*$ to be a true adjoint, we need the conjunct to be zero.
- Let's expand the conjunct for this particular example.

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Boundary conditions:

$$\begin{matrix} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u(b) = 0 \end{matrix} \implies \begin{matrix} \mathcal{B}_1^*[v] = v(a) = 0 \\ \mathcal{B}_2^*[v] = v(b) = 0 \end{matrix}$$

$$\mathcal{B}_i = \mathcal{B}_i^*$$

- Suppose we have the simple boundary conditions $u(a) = u(b) = 0$.
- Then, to make the conjunct zero, we need $\overline{v}(a) = \overline{v}(b) = 0$ or $v(a) = v(b) = 0$.
- So in this case, the adjoint boundary conditions on v are the same as the boundary conditions on u.
- Remember what these boundary conditions mean. $\mathcal{L}^*$ is the true adjoint when $\mathcal{L}$ operates on functions $u(x)$ which are zero at $x = a, b$ and $\mathcal{L}^*$ operates on functions $v(x)$ which are zero at $x = a, b$.

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Initial conditions:

$$\begin{array}{c} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u'(a) = 0 \end{array} \implies \begin{array}{c} \mathcal{B}_1^*[v] = v(b) = 0 \\ \mathcal{B}_2^*[v] = v'(b) = 0 \end{array}$$

$$\mathcal{B}_i \neq \mathcal{B}_i^*$$

- What if we have initial conditions instead?
  $u(a) = u'(a) = 0$.
- Then, to make the conjunct zero, we need $v(b) = v'(b) = 0$.
- So, for initial conditions, the adjoint boundary conditions are *final* conditions. $\mathcal{B}_i \neq \mathcal{B}_i^*$.

Adjoint operator:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

Adjoint boundary conditions:

$$\begin{matrix} \mathcal{B}_i[u] = 0 \\ \mathcal{B}_i^*[v] = 0 \end{matrix} \implies J(u, \overline{v})\Big|_a^b = 0$$

Otherwise:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

- The adjoint operator satisfies $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$.
- This only works for certain u, v, though. Specifically, it works when u obeys boundary conditions and v obeys adjoint boundary conditions.
- If u, v do not satisfy these boundary conditions, then $\mathcal{L}^*$ is not truly the adjoint anymore. However, it still nearly obeys the adjoint equation; there's just a leftover conjunct term which depends on the boundary values of u, v and their derivatives.
- This conjunct term messes up a lot of things from linear algebra (e.g. self-adjoint operators having a complete basis of eigenvectors). But for our purposes, it will end up being useful.

Original problem:

$$\mathcal{L}u(x) = f(x); \qquad \mathcal{B}_i[u(x)] = \alpha_i$$

Green's problem:

$$\mathcal{L}G(x, x') = \delta(x - x'); \qquad \mathcal{B}_i[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^* H(x, x') = \delta(x - x'); \qquad \mathcal{B}_i^*[H(x, x')] = 0$$

- Now we'll be able to deal with boundary conditions properly.

- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha_i \to 0$. As before, $G(x, x')$ is the impulse response.

- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.

- Warning! A lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Sometimes the "Green's function" in an expression is really the adjoint Green's function.

$$\langle \mathcal{L}u(x), H(x,x') \rangle = \langle u(x), \mathcal{L}^* H(x,x') \rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x') \rangle = \langle u(x), \delta(x-x') \rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\,\mathrm{d}x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}x + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b}$$

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- Then, we use the fact that $\mathcal{L}u(x) = f(x)$ and $\mathcal{L}H(x, x') = \delta(x - x')$.
- After evaluating the inner product terms, we arrive at a fairly general formula which looks somewhat like what we had in the introduction. The difference is that it involves the *adjoint* Green's function $H(x, x')$, and it has an extra conjunct term.
- We'll deal with the conjunct later. For now, let's try to get rid of $H(x, x')$ and express $u(x)$ in terms of $G(x, x')$. To do that, we need a relationship between $H(x, x')$ and $G(x, x')$.

How are $G(x, x')$ and $H(x, x')$ related?

$$\begin{aligned}
\langle \mathcal{L}G(x, x'), H(x, x'') \rangle &= \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle \\
\langle \delta(x - x'), H(x, x'') \rangle &= \langle G(x, x'), \delta(x - x'') \rangle \\
\int_a^b \delta(x - x') \overline{H}(x, x'') \, \mathrm{d}x &= \int_a^b G(x, x') \delta(x - x'') \, \mathrm{d}x \\
\overline{H}(x', x'') &= G(x'', x')
\end{aligned}$$

$$\boxed{G(x, x') = \overline{H}(x', x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.
- Note a surprising result of this: if $G(x, x')$ obeys the boundary conditions with respect to x, then it automatically obeys the *adjoint* boundary conditions with respect to x'. This will be important later.

CONSTRUCTING SOLUTIONS: DERIVATION

$$u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

and

$$G(x,x') = \overline{H}(x',x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^b}$$

- Let's go back to our expression for $u(x')$ in terms of $H(x, x')$.
- Using our new relationship $G(x, x') = \overline{H}(x, x')$, we can rewrite this as an expression for $u(x)$ in terms of $G(x, x')$. (Note: we switched x and x' to make it look a little nicer.)
- So this is the more correct version of what we saw in the introduction. If the conjunct happens to be zero, then we get what we had before. If not, we have an extra term that depends only on the boundaries.
- In general, the conjunct term deals with the boundary conditions of $u(x)$. It turns out that the boundary conditions act, in some way, like additional sources. We'll look more closely at this now through an example.

EXAMPLE: 1D POISSON EQUATION

Original problem:

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}; \qquad \begin{aligned} V(a) &= V_a \\ V(b) &= V_b \end{aligned}$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{aligned} G(a, x') &= 0 \\ G(b, x') &= 0 \end{aligned}$$

- Let’s look at a simple 1D voltage problem.
- We have both a charge density ρ inside the region $a < x < b$, and we have an applied voltage at the boundaries. Intuitively, both of these will affect the voltage in the region.
- In the Green’s function problem, we turn the charge density into an impulse function, and we set the applied voltage to zero.

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' - J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Take $\rho(x) = 0$ for now.

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

- From our recent results, we can write down the solution for $V(x)$ in terms of the Green's function.
- Take $\rho = 0$ so that we can focus on the boundary conditions for now.

EXAMPLE: 1D POISSON EQUATION

Can show

$$\langle \mathcal{L}u, v\rangle = \int_a^b u''\overline{v}\,\mathrm{d}x = \int_a^b u\overline{v}''\,\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Comparing with

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v^*)\Big|_a^b$$

we see

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \mathcal{L}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

- To write the solution more explicitly, we need to find the conjunct of the operator $\mathcal{L} = \mathrm{d}^2/\,\mathrm{d}x^2$.
- As before, use integration by parts and compare with the expected formula.

EXAMPLE: 1D POISSON EQUATION

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$\begin{aligned} V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) - \\ -V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a) \end{aligned}$$

- Now that we know the conjunct, we can write the solution for $V(x)$ more explicitly.
- Problem: we don't know $\mathrm{d}V/\,\mathrm{d}x$ at the boundaries.
- The adjoint problem saves us, because we can show that $G(x,b) = G(x,a) = 0$.

EXAMPLE: 1D POISSON EQUATION

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{array}{l} G(a,x') = 0 \\ G(b,x') = 0 \end{array}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{array}{l} H(a,x') = 0 \\ H(b,x') = 0 \end{array}$$

But $G(x,x') = \overline{H}(x',x)$ so

$$\frac{\mathrm{d}^2 \overline{G}(x,x')}{\mathrm{d}x'^2} = \delta(x-x'); \qquad \begin{array}{l} G(x,a) = 0 \\ G(x,b) = 0 \end{array}$$

- Back to the adjoint Green's equation.
- Because of the relationship between $G(x, x')$ and $H(x, x')$ we see that $G^*(x, x')$ obeys the adjoint equation with respect to x'.
- More importantly, we see that $\overline{G}(x, x')$ (and thus $G(x, x')$) obeys the adjoint boundary conditions with respect to x'.
- So not only do we have $G(a, x') = G(b, x') = 0$, we also have $G(x, a) = G(x, b) = 0$. This is not a trivial or obvious result (at least to me).

EXAMPLE: 1D POISSON EQUATION

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'} G(x,b) -$$
$$- V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'} G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}$$

- Using the fact that $G(x, x')$ obeys the adjoint boundary conditions with respect to x', we can eliminate the unknown values and simplify our result.
- Now we have an explicit solution for $V(x)$ given any boundary conditions $V(a) = V_a$ and $V(b) = V_b$.
- Further, we see that the solution only depends on the Green's function. It's as if the non-zero boundary conditions V_a, V_b act like additional sources whose response is given by the Green's function.

EXAMPLE: 1D POISSON EQUATION

Full solution with $\rho(x)$:

$$\boxed{V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x, a)}{\mathrm{d}x'}}$$

- Putting back our charge distribution $\rho(x)$, we get a full solution for any charge distribution and boundary conditions.
- The first part gives the voltage produced by the charge distribution $\rho(x)$. The last two parts give the voltage produced by the boundary conditions V_a, V_b.
- The response to *both* of these sources of voltage is given by the Green's function!

SOLVING PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(x) = f(x); \quad \mathcal{B}_i[u(x)] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(x, x') = \delta(x - x'); \quad \mathcal{B}_i[G(x, x')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u, \bar{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(x) = \int_a^b f(x')G(x, x')\,\mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

4. Simplify using $\mathcal{B}^*[G(x, x')] = 0$ (with respect to x').

Now let's look at the general process for solving a boundary value problem with a source.

1. Set up the Green's function equation by setting the source to $\delta(x - x')$ and the boundary conditions to zero. Solve this to find the Green's function.

2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u, \bar{v})$ by expanding the inner product $\langle \mathcal{L}u, v \rangle$ (usually using integration by parts).

3. Write down the solution.

4. Unknown boundary values of $u(x)$ may appear in the conjunct term. Eliminate these using the fact that $G(x, x')$ obeys the adjoint boundary conditions with respect to x'.

SOLVING 3D PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}_i[u(\mathbf{r})] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'); \quad \mathcal{B}_i[G(\mathbf{r}, \mathbf{r}')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} \mathbf{J}\big(u(\mathbf{r}'), G(\mathbf{r}, \mathbf{r}')\big) \cdot \mathrm{d}\mathbf{s}$$

4. Simplify using $\mathcal{B}^*[G(\mathbf{r}, \mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

- Great thing about our approach is that it's easily extended to 3D.

- Inner product is now a volume integral over the region of interest (V). Have to use 3D versions of integration by parts (e.g., Green's identities) to find $\mathcal{L}^*$, $J(u, \overline{v})$ and $\mathcal{B}_i^*$.

- The conjunct is now vector-valued, and it must be integrated over the surface of V (denoted ∂V).

- Still have the same interpretation though. The first (volume) integral is the contribution from the source $f(\mathbf{r})$. The second (surface) integral is the contribution from the non-zero boundary conditions.

CONSTRUCTING THE SOLUTION

Comments:

- Calculating Green's functions is not trivial.
- Adjoint approach is difficult, but offers clarity.

- Step 1 of our approach (calculate the Green's function) can be difficult: especially in 3D. Learning these techniques is time-consuming, but we now have a solid foundation with which to understand them.

- Our approach was not the easiest, but note the critical role played by the adjoint, both in the derivation and the final solution method.

- Though some authors don't talk about adjoints, they still use these ideas. E.g., often the "Green's function" used in 3D problems is actually the *adjoint* Green's function. (Can be confusing!)

SPECIAL PROPERTIES

- The behaviour of the Green's function is shaped by the type of problem we're trying to solve.
- In this section, we'll see how a lot of properties of problems are tied to properties of the Green's function.

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B}_i = \mathcal{B}_i^*$, then

$$G(x, x') = \overline{G}(x', x)$$

- Self-adjointness is equivalent to conjugate symmetry of the Green's function.
- If a problem is self-adjoint, then the adjoint Green's function $H(x, x')$ is the same as the Green's function $G(x, x')$. Then, using a result from before, we find that $G(x, x') = \overline{G}(x, x')$.

RECIPROCITY

Reciprocity in the frequency-domain:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

- The notion of reciprocity in electromagnetism can be tied to this property of the Green's function.
- In the frequency-domain, the Green's function is symmetric under interchange of $\mathbf{r}, \mathbf{r}'$ (no complex conjugate!). It can be shown that this is equivalent to reciprocity. (See Collin for a more thorough discussion using dyadic Green's functions.)
- Mathematically, frequency-domain reciprocity is related to the pseudo-inner product

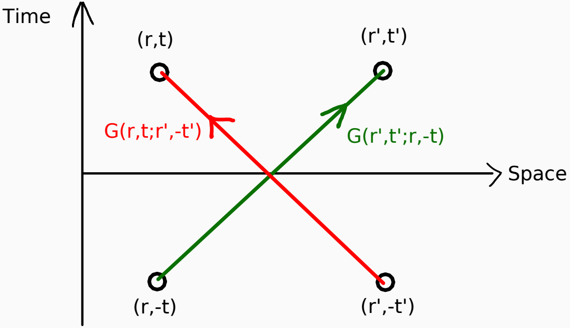
  $$\langle u, v \rangle_p = \iiint u(\mathbf{r}) v(\mathbf{r}) \, \mathrm{d}^3\mathbf{r}$$

  and operators which are "self-adjoint" under it:

  $$\langle \mathcal{L} u, v \rangle_p = \langle u, \mathcal{L} v \rangle_p$$

Reciprocity in the time-domain:

$$G(\mathbf{r}, t; \mathbf{r}', -t') = G(\mathbf{r}', t'; \mathbf{r}, -t)$$



- In the time-domain, the Green's function is symmetric, but with an added minus sign on the time variables. This is because of causality (see Morse and Feshbach).
- Think of $G(\mathbf{r}, t; \mathbf{r}', t')$ as having an impulse at $(\mathbf{r}', t')$ and measuring it at $(\mathbf{r}, t)$. Then reciprocity shows that interchanging sources and measurements leads to identical results.
- Mathematically, time-domain reciprocity related to the pseudo-inner product

$$\langle u, v \rangle_p = \int \iiint u(\mathbf{r}, t) v(\mathbf{r}, -t) \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}t$$

  and operators which are "self-adjoint" under it:

$$\langle \mathcal{L} u, v \rangle_p = \langle u, \mathcal{L} v \rangle_p$$

CAUSALITY

Causality:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } t < t'$$

Special relativity:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } |\mathbf{r} - \mathbf{r}'| > c(t - t')$$

- The Green's function can tell us if a system is causal or not.
- Causality means that an effect cannot precede a cause, so the impulse response (Green's function) cannot appear before the impulse itself. I.e., the Green's function has to be zero for $t < t'$.
- In special relativity, causality means that information cannot propagate faster than light. There is a corresponding restriction on the Green's function.
- Compare these causal Green's function with the Green's function for a self-adjoint problem: they are incompatible. So we cannot have a causal system which is also self-adjoint in time.

Time-invariance:

$$G(t, t') = G(t - t')$$

Spatial-invariance:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}')$$

- A time-invariant system is one whose behaviour doesn't change over time. That is, if we delay our input, we'll get the exact same output, just delayed by an equal amount.
- In that case, can show that the Green's function only depends on the difference between t and t'.
- Shouldn't be a surprise from linear time-invariant (LTI) system theory, where the impulse response is usually written as $h(t - t')$.
- A similar thing applies to systems whose behaviour doesn't change from place to place.
- These properties can make it easier to find the Green's function (e.g., the free-space wave equation).

$$(\mathcal{L} - \lambda)u(x) = f(x); \quad \mathcal{B}[u] = 0$$

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$ then

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

- For a self-adjoint problem, we can write out the solution as a sum of eigenfunctions of $\mathcal{L}$, where $\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$.
- With some work (not shown), we can directly write out the solution in terms of $f(x)$ as a generalized Fourier series. $\langle f, \phi_n \rangle$ are the projections of f onto the normalized eigenfunction basis.
- (Technically, we're also assuming here that $\mathcal{L}$ is a *bounded* linear operator. Unbounded operators have continuous sets of eigenvalues, and the theory behind them is more delicate. See, e.g., Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications*.)

SPECTRAL THEORY

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\, \mathrm{d}x'$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

- We can rewrite the last solution, and read off the Green's function.
- So we can write down the Green's function directly if we know the eigenfunctions/eigenvalues!
- It's not the nicest form because we have to sum an infinite series. Finding a closed-form version like we did before would be preferable, but in 3D separation of variable problems, we won't usually have a choice.

Green's function of $(\mathcal{L}-\lambda)$:

$$G(x,x';\lambda)=\sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n-\lambda}$$

$$\begin{aligned}\lambda_n &\longrightarrow \text{ Poles of } G(x,x';\lambda)\\ \phi_n(x)\phi_n^*(x') &\longrightarrow \text{ Residues of } G(x,x';\lambda)\end{aligned}$$

- We can also get the eigenvalues/eigenfunctions from the Green's function!
- Specifically, we need to know the Green's function of $\mathcal{L} - \lambda$ for $\lambda \in \mathbb{C}$.
- The eigenvalues are simply the poles of $G(x, x'; \lambda)$ with respect to λ.
- The eigenfunctions $\phi_n(x)$ are more difficult, but if there are no repeated eigenvalues, then they can be found from the residues of $G(x, x'; \lambda)$ at $\lambda = \lambda_n$.
- Important point is that the Green's function can tell us a lot about spectral quantities.

CONCLUSION

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Lots of information in the Green's function.
- Green's functions $\iff$ eigenvalues/eigenfunctions.

FURTHER READING

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Gerlach (2010), *Linear mathematics in infinite dimensions*. Nice set of online course notes for quick reference.

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Morse and Feshback, *Methods of theoretical physics*. Big, detailed reference. Great resource for deeper insight and understanding.

FURTHER READING

Collin (1990), *Field theory of guided waves.* Huge chapter on Green's functions. Emphasis on dyadics.

Folland (1992), *Fourier analysis and its applications.* Rigorous math book. Chapter on generalized functions is particularly nice.

Byron and Fuller (1992), *Mathematics of classical and quantum physics.* Interesting alternative approach.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.