GREEN'S FUNCTIONS

A short introduction

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OUTLINE

- This is intended as a quick overview of Green's functions for electrical engineers.
- Green's functions are a huge subject: it's easy to get overwhelmed by calculation techniques.
- Focus here will be on intuition/understanding and awareness of some key techniques.
- Lots of further reading provided at the end.
- Tip: read a lot of different references. Different authors take totally different approaches and it's interesting to see them all.

INTRODUCTION

- Fortunately, the basic idea of Green's functions is really simple.
- You've actually used them before!
- What's far more interesting is how to calculate/interpret them.

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.

Familiar Green's functions

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t') h(t-t') \, \mathrm{d}\, t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t') e^{-\alpha(t-t')}$$

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Familiar Green's functions

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r}-\mathbf{r}'\right|^2} \rho(\mathbf{r}') \, \mathrm{d}^3\, \mathbf{r}'$$

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- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t-t')$ from linear system theory is an example of a Green's function.

$$G(t,t') = h(t-t')$$

- Usually find $h(t-t')$ using Fourier transform of the transfer function.

- Green's function for Poisson's equation is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r}-\mathbf{r}'\right|^2}$$

Familiar Green's functions

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

Familiar Green's functions

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}$$

GENERALIZED FUNCTIONS

TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int\limits_{-\infty}^{\infty} \delta(x - x_0) = 1$$

- Delta functions play a key role in Green’s functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz’s theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

- Often see definitions like this one.
- Often said to imply that $\delta(x - x_0) = \infty$ at $x = x_0$.
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- Let's see if we can generalize the idea of a "function" so that it includes delta functions.
- Given a function $f(x)$, we can use it to define a linear operator (a functional, to be exact) on other functions $\phi(x)$.
- $f[\cdot]$ is a linear operator. It takes a function $\phi(x)$ and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- If we ensure that $\phi(x)$ is very well-behaved, then every function $f(x)$ defines an operator in this way.

GENERALIZED FUNCTIONS

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- It's possible to have an operator $f[\phi]$, but we can't find an $f(x)$ to implement it via an integral.
- Then $f(x)$ is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions $f[\phi]$.
- We still symbolically write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

  but this just suggestive notation. It is not actually an integral unless $f(x)$ is a "proper" function!

DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x$$

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- We can define a simple linear operator via the sifting property $\delta_{x_0}[\phi] = \phi(x_0)$.
- There is no actual function $\delta(x - x_0)$ which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x = \phi(x_0)$$

  so $\delta(x - x_0)$ is a generalized function and the above integral is purely symbolic.

DELTA FUNCTION DERIVATIVES

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}\,x$$

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- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$ is just an operator that picks out the value of the nth derivative of $\phi(x)$ at the point x_0.

DELTA FUNCTION LIMITS

$$\lim_{\epsilon\to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon\to 0} f_\epsilon[\phi] = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}\,x = \phi(0)$$

- Often useful to show that some set of actual functions $f_\epsilon(x)$ "approach" the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{t^2+\epsilon^2}$$

- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

DELTA FUNCTION LIMITS

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

- Example of a common, but unintuitive expression for the Delta function.
- Can show that it's true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \overset{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\,\phi(x)\,\mathrm{d}\,x$$

- Technically, the Green's function is a generalized function such that $\mathcal{L}G$ is the delta function (it has the sifting property).
- In practise, we'll keep using the less-precise way; just remember that there is a more correct way.

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

DIRECT SOLUTION

- In practise, thinking of $\delta(x - x_0)$ as a function is usually fine. (We'll even do that for the rest of this presentation.)
- But if anything starts to seem fishy, it's good to remember that $\delta(x - x_0)$ is actually an operator, and not a function.

- Back to Green's functions!

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\,x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') = \delta(x-x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Let's look at a simple example now.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem.
- But the Green's function problem looks pretty hard. The point of this example is to demonstrate that we can actually solve it.

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x,x')$ is discontinuous (like a step function), then $\mathrm{d}\,G/\,\mathrm{d}\,x$ will behave like a delta function and $\mathrm{d}^2\,G/\,\mathrm{d}\,x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x,x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon}\left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x')\right]\mathrm{d}\,x = \int_{x'-\epsilon}^{x'+\epsilon}\delta(x-x')\,\mathrm{d}\,x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'-\epsilon}\right] = 1$$

- But what if the derivative $\mathrm{d}\,G/\,\mathrm{d}\,x$ is discontinuous?
- Then $\mathrm{d}^2\,G/\,\mathrm{d}\,x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x,x')$ is continuous.
- But, we expect $\mathrm{d}\,G/\,\mathrm{d}\,x$ to be discontinous.
- Using fundamental theorem of calculus, we get at a *discontinuity condition for the derivative.* (Key idea for the direct solution method!)

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G}{\mathrm{d}x}$:

$$kA + kB = 1$$

A SIMPLE EXAMPLE

At last, our Green's function is

$$G(x,x') = \begin{cases} \frac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \frac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

- Applying our two conditions, we can solve for A and B.

A SIMPLE EXAMPLE

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

GENERAL APPROACH

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(t)$.

- Listed are the key things to note from that example.
- This approach works quite well for solving 1D Green's function problems.

BOUNDARY CONDITIONS

SIMPLE HARMONIC OSCILLATOR

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d} x^2} + \omega_0^2 u(x) = f(x)$$

$$u(a) = \alpha$$
$$u(b) = \beta$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d} x^2} + \omega_0^2 G(x,x') = \delta(x - x')$$

- We didn't worry about boundary conditions in the last example.
- As it turns out, Green's functions allow us to deal with boundary conditions in an elegant way.

- Now let's look at an example with boundary conditions: a simple harmonic oscillator.
- Question: what should the boundary conditions be for $G(x,x')$?

SIMPLE HARMONIC OSCILLATOR

$$G(x,x')\left[\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\,x^2}+\omega_0^2 u(x)\right]=G(x,x')f(x)$$

$$u(x)\left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2}+\omega_0^2 G(x,x')\right]=u(x)\delta(x-x')$$

- To figure out the best boundary conditions, we use a little bit of trickery.
- First, we multiply the original equation by $G(x,x')$ and the Green's functions equation with by $u(x)$.

SIMPLE HARMONIC OSCILLATOR

$$\int_a^b \left[u(x)\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2}-G(x,x')\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\,x^2}\right]\mathrm{d}\,x$$
$$=\int_a^b \left[u(x)\delta(x-x')-G(x,x')f(x)\right]\mathrm{d}\,x$$

- Next, we subtract and integrate over $[a,b]$.
- The integral over $u(x)\delta(x-x')$ becomes just $u(x')$.
- The left hand side can be dealt with using integration by parts.

SIMPLE HARMONIC OSCILLATOR

$$u(x') = \int_a^b G(x,x')f(x)\,\mathrm{d}\,x +$$
$$+ \left[u(x)\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x} - G(x,x')\frac{\mathrm{d}\,u(x)}{\mathrm{d}\,x}\right]_{x=a}^{b}$$

SPECTRAL METHODS

- We now have a more informative expression for $u(x')$.
- We have the original integral over $G(x,x')f(x)$, but now we also have a boundary term.

3D PROBLEMS

PROPERTIES OF THE GREEN'S FUNCTION

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ADVANCED TOPICS

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

ADVANCED RESOURCES

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.