GREEN'S FUNCTIONS

A short introduction

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- This is intended as a short overview of Green's functions for electrical engineers.
- Green's functions are a huge subject: we will try to demonstrate the most important techniques and ideas.
- Tip: read a lot of different references. Different authors take totally different approaches and it's interesting to see them all. Lots of references are provided at the end.

OUTLINE

INTRODUCTION

- Fortunately, the basic idea of Green’s functions is really simple.
- You’ve actually used them before!
- What’s far more interesting is how to calculate/interpret them.

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x')f(x')\,\mathrm{d}\,x}$$

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.

FAMILIAR GREEN'S FUNCTIONS

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t - t')$ from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find $h(t - t')$ using Fourier transform of the transfer function.

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}') \, \mathrm{d}^3 \, \mathbf{r}'$$

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2}$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi \left|\mathbf{r} - \mathbf{r}'\right|}$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

GENERALIZED FUNCTIONS

- Delta functions play a key role in Green's functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz's theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

- Often see definitions like this one.
- Often said to imply that $\delta(x - x_0) = \infty$ at $x = x_0$.
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- Let's see if we can generalize the idea of a "function" so that it includes delta functions.

- Given a function $f(x)$, we can use it to define a linear operator (a functional, to be exact) on other functions $\phi(x)$.

- $f[\cdot]$ is a linear operator. It takes a function $\phi(x)$ and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- If we ensure that $\phi(x)$ is very well-behaved, then every function $f(x)$ defines an operator in this way.

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- It's possible to have an operator $f[\phi]$, but we can't find an $f(x)$ to implement it via an integral.
- Then $f(x)$ is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions $f[\phi]$.
- We still symbolically write

$$f[\phi] \overset{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

but this just suggestive notation. It is not actually an integral unless $f(x)$ is a "proper" function!

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\, \mathrm{d}\, x$$

- We can define a simple linear operator via the sifting property $\delta_{x_0}[\phi] = \phi(x_0)$.
- There is no actual function $\delta(x - x_0)$ which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x = \phi(x_0)$$

so $\delta(x - x_0)$ is a generalized function and the above integral is purely symbolic.

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \overset{s}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}\,x$$

- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$ is just an operator that picks out the value of the nth derivative of $\phi(x)$ at the point x_0.

DELTA FUNCTION LIMITS

$$\lim_{\epsilon\to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon\to 0} f_\epsilon[\phi] = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}\,x = \phi(0)$$

- Often useful to show that some set of actual functions $f_\epsilon(x)$ "approach" the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

- Example of a common, but unintuitive expression for the Delta function.
- Can show that it’s true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\,\phi(x)\,\mathrm{d}\,x$$

- Technically, the Green's function is a generalized function such that $\mathcal{L}G$ is the delta function (it has the sifting property).
- In practise, we'll keep using the less-precise way; just remember that there is a more correct way.

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

- In practise, thinking of $\delta(x - x_0)$ as a function is usually fine. (We'll even do that for the rest of this presentation.)
- But if anything starts to seem fishy, it's good to remember that $\delta(x - x_0)$ is actually an operator, and not a function.

DIRECT SOLUTION

- Back to Green's functions!

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

- Let's look at a simple example now.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem.
- But the Green's function problem looks pretty hard. The point of this example is to demonstrate that we can actually solve it.

For $x \neq x'$

$$\frac{\mathrm{d}^2\, G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x, x')$ is discontinuous (like a step function), then $\mathrm{d}\, G / \mathrm{d}\, x$ will behave like a delta function and $\mathrm{d}^2\, G / \mathrm{d}\, x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x, x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') \right] \mathrm{d}\, x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\, \mathrm{d}\, x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left. \frac{\mathrm{d}\, G}{\mathrm{d}\, x} \right|_{x=x'+\epsilon} - \left. \frac{\mathrm{d}\, G}{\mathrm{d}\, x} \right|_{x=x'-\epsilon} \right] = 1$$

- But what if the derivative $\mathrm{d}\, G/\, \mathrm{d}\, x$ is discontinuous?
- Then $\mathrm{d}^2\, G/\, \mathrm{d}\, x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x, x')$ is continuous.
- But, we expect $\mathrm{d}\, G/\, \mathrm{d}\, x$ to be discontinous.
- Using fundamental theorem of calculus, we get at a *discontinuity condition for the derivative.* (Key idea for the direct solution method!)

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}\,G}{\mathrm{d}\,x}$:

$$kA + kB = 1$$

- Applying our two conditions, we can solve for A and B.

At last, our Green's function is

$$G(x, x') = \begin{cases} \dfrac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \dfrac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x, x') = \frac{e^{k|x-x'|}}{2k}}$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(t)$.

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

- Listed are the key things to note from that example.
- This approach works quite well for solving 1D Green’s function problems.

BOUNDARY CONDITIONS

- We didn't worry about boundary conditions in the last example.
- As it turns out, Green's functions allow us to deal with boundary conditions in an elegant way.
- Unfortunately, to derive it, we either have to do a lot of hand-waving or a lot of math. We're going to do a lot of math.
- See Dudley's *Mathematical foundations for electromagnetic theory* for a more thorough discussion.

ADJOINT OPERATORS

$$\begin{aligned} \text{Original:} \qquad & \mathcal{L}[u(x)] = f(x); \qquad & \mathcal{B}[u(x)] = 0 \\ \text{Adjoint:} \qquad & \mathcal{L}^*[v(x)] = f(x); \qquad & \mathcal{B}^*[v(x)] = 0 \end{aligned}$$

Defining property:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \qquad \text{where } \langle u, v\rangle = \int_a^b u(x)v^*(x)\,\mathrm{d}\,x$$

- To really make sense of boundary conditions, we need the concept of the adjoint problem.
- Suppose we have an original problem defined by operator $\mathcal{L}$ and boundary conditions $\mathcal{B}$.
- Then, $\mathcal{L}^*$ is the adjoint operator and $\mathcal{B}^*$ are the adjoint boundary conditions if $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$ for all u, v.
- Here $\langle u, v \rangle$ is the inner product as defined on the slide. ($v^*(x)$ is the complex conjugate of $v(x)$.)
- (Side note: technically, we should call $\mathcal{L}^*$ the *formal* adjoint. A boundary condition like $\mathcal{B}[u] = 0$ restricts u to a subspace of its original Hilbert space. Thus, if $\mathcal{B} \neq \mathcal{B}^*$, the domains of $\mathcal{L}$ and $\mathcal{L}^*$ are not the same, and they are not true adjoints.)

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that $u(a) = u(b) = 0$.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$
$$= \int_a^b \left[v''^*(x) + k^2 v^*(x)\right] u(x)\,\mathrm{d}\,x +$$
$$+ \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

By inspection:

$$\mathcal{L}^* v(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*\right] v(x)$$

- Integrate by parts twice.
- The remaining integral term looks like

$$\int_a^b u(x)\,[\mathcal{L}^* v(x)]^*\,\mathrm{d}\,x$$

  so we can read off $\mathcal{L}^*$.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + \left[u'(x) v^*(x) - u(x) v^{*\prime}(x) \right]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(a) v^*(a) - u'(b) v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

- We almost have the required $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$, but we need the part on the right (called the conjunct) to be zero.

- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case, $\mathcal{B} = \mathcal{B}^*$.

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + \left[u'(x) v^*(x) - u(x) v^{*\prime}(x) \right]_a^b$$
$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(b) v^*(b) - u(b) v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

- What if we use the same operator $\mathcal{L}$, but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem! $\mathcal{B} \neq \mathcal{B}^*$.

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^*v\rangle + J(u, v)$$

where $J(u, v)$ depends on $u, v, u', v', \ldots$ at the boundaries.

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \iff J(u, v) = 0$$

- In that example, we saw that we always had $\langle \mathcal{L}u, v \rangle$ equal to $\langle u, \mathcal{L}^* v \rangle$ plus a “leftover” term called the conjunct.
- This is true in general: we always have a leftover term $J(u, v)$ which depends only on the boundary values of u and v.
- We say $\mathcal{B}[u] = 0$ and $\mathcal{B}^*[v] = 0$ are adjoint boundary conditions if and only if they make $J(u, v)$.

ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x,x')] = \delta(x-x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha \to 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning: a lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Quite often, the "Green's function" is really the adjoint Green's function.

Original problem:

$$\mathcal{L}[u] = \frac{\mathrm{d}^2\, u(x)}{\mathrm{d}\, x^2} + k^2 u(x) = f(x); \qquad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2\, H(x,x')}{\mathrm{d}\, x^2} + k^2 H(x,x') = \delta(x - x'); \qquad \begin{bmatrix} H(b,x') \\ H'(b,x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- We'll show how $H(x, x')$ is useful with an example.
- Here we have a driven simple harmonic oscillator with initial conditions. (We'll take k real for simplicity.)
- From before, we know that the adjoint problem will be the same differential equation, but with final conditions instead of initial conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x,x') = \begin{cases} A\cos(k(x-x')) + B\sin(k(x-x')) & \text{for } x < x' \\ C\cos(k(x-x')) + D\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

$$\begin{aligned} \text{Final conditions} &\implies C = D = 0 \\ \text{Continuity of function} &\implies A = 0 \\ \text{Discontinuity of derivative} &\implies B = \frac{-1}{k} \end{aligned}$$

- We can solve for $H(x, x')$ using a similar approach to before.
- Except at $x = x'$, we write $H(x, x')$ as a solution to the source-free equation.
- Then we find the coefficients using the boundary conditions and continuity/discontinuity requirements.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

- So we have our adjoint Green's function.
- Now we just need to figure out how to construct $u(x)$ from it.

EXAMPLE WITH BOUNDARY CONDITIONS

$$\langle \mathcal{L}u(x), H(x,x') \rangle = \langle u(x), \mathcal{L}^* H(x,x') \rangle + J(u,H)$$

$$\langle f(x), H(x,x') \rangle = \langle u(x), \delta(x-x') \rangle + J(u,H)$$

$$\int_a^b f(x) H^*(x,x') \, \mathrm{d}\, x = \int_a^b u(x) \delta(x-x') \, \mathrm{d}\, x + J(u,H)$$

$$\boxed{u(x') = \int_a^b f(x) H^*(x,x') \, \mathrm{d}\, x - J(u,H)}$$

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- We arrive at a fairly general formula which looks close to what we expect a Green's function formula to look like, but with an extra conjunct term.
- We'll gain insight into the $J(u, H)$ term by expanding it for this example.

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x)H^*(x, x')\,\mathrm{d}\,x - \left[\frac{\mathrm{d}\,u(x)}{\mathrm{d}\,x}H^*(x, x') - u(x)\frac{\mathrm{d}\,H^*(x, x')}{\mathrm{d}\,x}\right]_a^b$$

Recall:

$$\begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Now let's expand $J(u, v)$ for our particular example. (We can basically copy it from a previous part of the derivation.)
- We can simplify the conjunct by remembering our boundary conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x + \beta H^*(a, x') - \alpha \frac{\mathrm{d}\, H^*(a, x')}{\mathrm{d}\, x}$$

- We're very close to our final goal.
- The last thing will be to get rid of $H(x, x')$ and replace it with $G(x, x')$.

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$
$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$
$$H^*(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = H^*(x', x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x - x')) & \text{for } x > x' \end{cases}$$

- Since we already know the adjoint Green's function $H(x, x')$, we can use find the Green's function via $G(x, x') = H^*(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \,\mathrm{d}x + \beta H^*(a, x') - \alpha \left. \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}x} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) \,\mathrm{d}x + \beta G(x', a) - \alpha \left. \frac{\mathrm{d}\, G(x', x)}{\mathrm{d}x} \right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') \,\mathrm{d}x' + \beta G(x, a) - \alpha \left. \frac{\mathrm{d}\, G(x, x')}{\mathrm{d}x'} \right|_{x'=a}$$

- With our new knowledge that $G(x, x') = H^*(x', x)$, we can rewrite the solution $u(x)$ in terms of $G(x, x')$.

FINAL SOLUTION

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}}$$

where

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

- Finally, we arrive at our solution.
- First, note that the final answer does *not* depend on $H(x, x')$. We didn't actually need to ever calculate $H(x, x')$ from the adjoint Green's equation, we could have just found $G(x, x')$ from the (non-adjoint) Green's equation.
- However, it would have been very difficult to derive this expression without using $H(x, x')$ (I couldn't see an easy way). Because of this, a lot of authors stop at the expression for $u(x')$ in terms of $H^*(x, x')$, and they just call $H^*(x, x')$ the "Green's function."
- By doing the extra work, though, we gain a very nice interpretation for the boundary conditions term.

INTERPRETATION OF BOUNDARY CONDITIONS

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}$$

$$u(x) = \int_a^b \Big[f(x') + \beta\delta(x'-a) - \alpha\delta'(x'-a)\Big]G(x,x')\,\mathrm{d}\,x'$$

- We note that our expression for $u(x)$ looks like it did before (integral over $f(x')G(x,x')$), but now there are extra terms which depend on the boundary conditions.

- We come to a key idea: boundary conditions have the same effect on $u(x)$ as adding little impulse sources at the boundary. The Green's function can deal with both sources $f(x)$ and non-zero boundary conditions.

- Be careful, though: $G(x,x')$ still depends on the *type* of boundary condition. E.g., we use the same $G(x,x')$ for all initial value problems ($u(a), u'(a)$ specified), but we'll need a different $G(x,x')$ for boundary value problems ($u(a), u(b)$ specified).

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$
$$\mathcal{L}^*[H(x,x')] = \delta(x-x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

1. Find $G(x,x')$.
2. Find $u(x')$ in terms of $H(x,x')$:

$$u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x - J(u(x), H(x,x'))$$

3. Express $u(x)$ in terms of $G(x,x') = H^*(x',x)$.

- That was a long process, so let's summarize what we did.
- First, we wrote out the original equation, the Green's function equation, and the adjoint Green's function equation. (Take note of the boundary conditions in particular.)
- Next, we solve the Green's function equation for $G(x, x')$.
- Next, we found that it's much easier to express $u(x)$ in terms of $H(x, x')$, because we can use inner products. The only tricky part is finding the conjunct. (Usually, just requires integration by parts. See Dudley for a general formula for Sturm-Liouville problems.)
- Finally, we replace $H(x, x')$ with $G^*(x', x)$ (which we solved for previously), and we have our final expression for $u(x)$.

SPECTRAL METHODS

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

EIGENFUNCTION EXPANSION

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

Because $\mathcal{L}$ is self-adjoint:

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \, \phi_n(x)$$

3D PROBLEMS

PROPERTIES OF THE GREEN'S FUNCTION

ADVANCED TOPICS

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.