GREEN'S FUNCTIONS

A SHORT INTRODUCTION

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OUTLINE

- This is intended as a short introduction to Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them.
- For a short presentation, we can either cover a lot of material incompletely, or a small amount of material thoroughly. The second is chosen here, because I think it leads to a clearer understanding of Green's functions as a whole. Of course, the downside is that we'll miss out on a lot of interesting topics and applications.
- Suggested further reading provided at the end.

1. Basic idea of Green's functions.
2. Simplest method for solving the Green's function equation.
3. How to use the Green's function to solve a problem with boundary conditions. (Biggest section!)
4. Useful properties of Green's functions for special types of problems.
5. Relationship between Green's functions and eigenvalues/eigenfunctions.
6. Summary and suggested further reading.

BASIC IDEA

- The basic idea of Green's functions is really simple. You've actually used them before!

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x')f(x')\,\mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x')f(x')\,\mathrm{d}x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x')f(x')\,\mathrm{d}x}$$

(Some conditions apply.)

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.
- Beware the fine print! This formula actually only works under certain assumptions about the boundary conditions.
- We'll deal with the more general approach later. For now, this gets the key idea across.

FAMILIAR GREEN'S FUNCTIONS

Impulse response of an LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t') h(t - t') \, \mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t - t')$ from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find $h(t - t')$ using Fourier transform of the transfer function.

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|}$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

- With Green's function theory, we learn how to derive the above expressions. (Though we won't have time to do the 3D ones here.)
- More importantly, Green's function theory allows us to deal with different boundary conditions. The solutions to the Poisson and Helmholtz equations above assume free space (boundaries at infinity). Green's functions would allow us to, e.g., find the response to a current source inside a specific waveguide.

FINDING THE GREEN'S FUNCTION

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

- In this section, we'll look at one of the simplest methods for actually solving the Green's function problem.

- Let's start off by looking at a simple example.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem for any $f(x)$.
- But the Green's function problem looks hard! The point of this example is to demonstrate that we can actually solve it.

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x,x')$ is discontinuous (like a step function), then $\mathrm{d}G/\,\mathrm{d}x$ will behave like a delta function and $\mathrm{d}^2G/\,\mathrm{d}x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x,x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x')\right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon}\right] = 1$$

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x,x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

- But what if the derivative $\mathrm{d}G/\mathrm{d}x$ is discontinuous?
- Then $\mathrm{d}^2G/\mathrm{d}x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x,x')$ is continuous.
- The first integral is an integral of a derivative, so we can use the fundamental theorem of calculus. The result is a *discontinuity condition for the derivative.*

- Applying our two conditions, we can solve for A and B. We find

$$A = B = \frac{1}{2k}$$

A SIMPLE EXAMPLE

Solving, our Green's function is

$$G(x,x') = \frac{1}{2k}\begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k}\, \mathrm{d}x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(x)$.
- Caution: remember the fine print from before. This solution only works with certain assumptions about boundary conditions.

GENERAL APPROACH

Second-order problems:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- Derivative of $G(x, x')$ is discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

- This approach works quite well for solving 1D Green's function problems.
- For problems of other orders, will have a different combination of continuity/discontinuity requirements at $x = x'$. E.g., $G(x, x')$ will be discontinuous for a first order problem.

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

ADJOINT OPERATORS

$$\begin{aligned} \text{Original:} \quad & \mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = 0 \\ \text{Adjoint:} \quad & \mathcal{L}^*[v(x)] = f(x); \qquad \mathcal{B}^*[v(x)] = 0 \end{aligned}$$

Defining property:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^*v\rangle \qquad \text{where } \langle u, v\rangle = \int_a^b u(x)v^*(x)\,\mathrm{d}x$$

- In the introduction, we showed non-rigorously how to construct a solution from the Green's function. To keep things simpler, we ignored boundary conditions.
- Here, we'll look at how to properly construct a solution from the Green's function when boundary conditions are involved.
- Our approach is quite challenging compared to a lot of books on the subject. The advantage is that we'll deal with a lot of subtleties that can otherwise lead to confusion.
- For approaches similar to the one in this section, see Dudley, Morse and Feshbach, or Gerlach.

- Underpinning our approach is the concept of an adjoint problem.
- Suppose we have an original problem defined by operator $\mathcal{L}$ and boundary conditions $\mathcal{B}$.
- Then, $\mathcal{L}^*$ is the adjoint operator and $\mathcal{B}^*$ are the adjoint boundary conditions if $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^*v\rangle$ for all u, v.
- Here $\langle u, v\rangle$ is the inner product as defined on the slide. ($v^*(x)$ is the complex conjugate of $v(x)$.)

Linear algebra notes:

- The boundary conditions are important because they specify the domains of $\mathcal{L}$ and $\mathcal{L}^*$. (I.e., $\mathcal{L}$ operates on the Hilbert space of functions $u(x)$ which satisfy $\mathcal{B}[u] = 0$.)
- So if $\mathcal{B} \neq \mathcal{B}^*$, then $\mathcal{L}$ and $\mathcal{L}^*$ are operators on different Hilbert spaces.
- If both $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$, we say that $\mathcal{L}$ is self-adjoint.
- If $\mathcal{L} = \mathcal{L}^*$ but $\mathcal{B} \neq \mathcal{B}^*$, we say that $\mathcal{L}$ is *formally* self-adjoint.

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that $u(a) = u(b) = 0$.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^*\mathrm{d}x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

- To find the adjoint, let's expand $\langle \mathcal{L}u, v\rangle$.
- Use integration by parts twice.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^*\mathrm{d}x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

If we take

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (k^2)^*$$

then we have

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x)\right]_a^b$$

- The remaining integral term looks like

$$\int_a^b u(x)\left[\mathcal{L}^* v(x)\right]^*\mathrm{d}x$$

- If we define $\mathcal{L}^*$ this way, we get closer to our goal. We just need to make the last part zero.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

- We almost have the required $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$, but we need the part on the right to be zero.
- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the last part zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case, $\mathcal{B} = \mathcal{B}^*$.

ADJOINT OPERATORS: EXAMPLE

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$
$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

- What if we use the same operator $\mathcal{L}$, but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the remaining part zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem! $\mathcal{B} \neq \mathcal{B}^*$.

Adjoint operators

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v^*)\Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \Longleftrightarrow J(u, v^*)\Big|_a^b = 0$$

- In the last example, we saw that we had $\langle \mathcal{L}u, v\rangle$ equal to $\langle u, \mathcal{L}^* v\rangle$ plus a leftover term which depended on the boundaries.
- This is true more generally: if we don't specify the boundary conditions of u and v, then we can still *almost* get the adjoint operator equation. We just have a leftover term $J(u, v^*)|_a^b$, which depends only on the boundary conditions. This term is called the conjunct.
- We define adjoint boundary conditions as those boundary conditions which make the conjunct equal to zero.

Adjoint Green's functions

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x, x')] = \delta(x - x'); \qquad \mathcal{B}[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x, x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x, x')] = 0$$

- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha \to 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning! A lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Sometimes the "Green's function" in an expression is really the adjoint Green's function.

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\,\mathrm{d}x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}x + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x) H^*(x,x')\,\mathrm{d}x - J\big(u(x), H^*(x,x')\big)\Big|_a^b}$$

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x,x')$, and apply our knowledge of adjoints and conjuncts.
- Then, we use the fact that $\mathcal{L}u(x) = f(x)$ and $\mathcal{L}H(x,x') = \delta(x-x')$.
- After evaluating the inner product terms, we arrive at a fairly general formula which looks somewhat like what we had in the introduction. The difference is that it involves the *adjoint* Green's function $H(x,x')$, and it has an extra conjunct term.
- We'll deal with the conjunct later. For now, let's try to get rid of $H(x,x')$ and express $u(x)$ in terms of $G(x,x')$. To do that, we need a relationship between $H(x,x')$ and $G(x,x')$.

CONSTRUCTING SOLUTIONS: DERIVATION

How are $G(x,x')$ and $H(x,x')$ related?

$$\langle \mathcal{L}G(x,x'), H(x,x'')\rangle = \langle G(x,x'), \mathcal{L}^* H(x,x'')\rangle$$

$$\langle \delta(x-x'), H(x,x'')\rangle = \langle G(x,x'), \delta(x-x'')\rangle$$

$$\int_a^b \delta(x-x') H^*(x,x'')\,\mathrm{d}x = \int_a^b G(x,x')\delta(x-x'')\,\mathrm{d}x$$

$$H^*(x',x'') = G(x'',x')$$

$$\boxed{G(x,x') = H^*(x',x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x,x')$ and $H(x,x')$.
- Note a surprising result of this: if $G(x,x')$ obeys the boundary conditions with respect to x, then it automatically obeys the *adjoint* boundary conditions with respect to x'. This will be important later.

CONSTRUCTING SOLUTIONS: DERIVATION

$$u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}x - J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

and

$$G(x,x') = H^*(x',x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^b}$$

- Let's go back to our expression for $u(x')$ in terms of $H(x,x')$.
- Using our new relationship $G(x,x') = H^*(x,x')$, we can rewrite this as an expression for $u(x)$ in terms of $G(x,x')$. (Note: we switched x and x' to make it look a little nicer.)
- So this is the more correct version of what we saw in the introduction. If the conjunct happens to be zero, then we get what we had before. If not, we have an extra term that depends only on the boundaries.
- In general, the conjunct term deals with the boundary conditions of $u(x)$. It turns out that the boundary conditions act, in some way, like additional sources. We'll look more closely at this now through an example.

EXAMPLE: 1D POISSON EQUATION

Original problem:

$$\frac{\mathrm{d}^2V(x)}{\mathrm{d}x^2} = -\frac{\rho}{\epsilon_0}; \qquad \begin{bmatrix} V(a) \\ V(b) \end{bmatrix} = \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

Green's problem:

$$\frac{\mathrm{d}^2G(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{bmatrix} G(a,x') \\ G(b,x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Let's look at a simple 1D voltage problem.
- We have both a charge density ρ inside the region $a < x < b$, and we have an applied voltage at the boundaries. Intuitively, both of these will affect the voltage in the region.
- In the Green's function problem, we turn the charge density into an impulse function, and we set the applied voltage to zero.

EXAMPLE: 1D POISSON EQUATION

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' - J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Take $\rho = 0$ for now.

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

EXAMPLE: 1D POISSON EQUATION

Can show

$$\int_a^b \left(\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\right) v^*(x)\,\mathrm{d}x = \int_a^b u(x) \left(\frac{\mathrm{d}^2 v^*(x)}{\mathrm{d}x^2}\right) \mathrm{d}x + \\ + \left[\frac{\mathrm{d}u}{\mathrm{d}x} v^*(x) - u(x)\frac{\mathrm{d}v^*(x)}{\mathrm{d}x}\right]_a^b$$

So $\mathcal{L} = \mathcal{L}^*$, $\mathcal{B} = \mathcal{B}^*$, and

$$J(u, v^*) = \frac{\mathrm{d}u(x)}{\mathrm{d}x} v^*(x) - u(x)\frac{\mathrm{d}v^*(x)}{\mathrm{d}x}$$

Example: 1D Poisson equation

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) - \\ -V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a)$$

- Problem: we don't know $\mathrm{d}V/\,\mathrm{d}x$ at the boundaries. Our solution for the unknown $V(x)$ includes the unknown $V(x)$.
- But we could solve this if we could show that $G(x,a) = G(x,b) = 0$.

Example: 1D Poisson equation

Adjoint Green's problem:

$$\frac{\mathrm{d}^2H(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{bmatrix} H(a,x') \\ H(b,x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $G(x,x') = H^*(x',x)$ so

$$\frac{\mathrm{d}^2G^*(x,x')}{\mathrm{d}x'^2} = \delta(x-x'); \qquad \begin{bmatrix} G(x,a) \\ G(x,b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Back to the adjoint Green's equation.
- Because of the relationship between $G(x,x')$ and $H(x,x')$ we see that $G^*(x,x')$ obeys the adjoint equation with respect to x'.
- More importantly, we see that $G^*(x,x')$ (and thus $G(x,x')$) obeys the adjoint boundary conditions with respect to x'.
- So not only do we have $G(a,x') = G(b,x') = 0$, we also have $G(x,a) = G(x,b) = 0$. This is not a trivial or obvious result (at least to me).

Example: 1D Poisson equation

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'} G(x,b) - \\ - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'} G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}$$

Example: 1D Poisson equation

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}$$

Can be written as

$$V(x) = \int_a^b \left[-V_a \delta'(x'-a) + V_b \delta'(x'-b)\right] G(x,x')\,\mathrm{d}x'$$

- Using delta function theory, we see that the non-zero boundary conditions can really be treated like sources.
- E.g., the boundary condition at $x = a$ is like a source $f(x) = V_a \delta'(x-a)$.
- $\delta'(x-a)$ denotes the delta function derivative defined by the derivative sifting property

$$\int \delta'(x-a) f(x)\,\mathrm{d}x = -f'(a)$$

EXAMPLE: 1D POISSON EQUATION

Full solution with $\rho(x)$:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x,x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} + V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}$$

CONSTRUCTING SOLUTIONS

1. Given

$$\mathcal{L}u(x) = f(x); \quad \mathcal{B}[u(x)] = \alpha$$

2. Solve Green's problem

$$\mathcal{L}G(x,x') = \delta(x-x'); \quad \mathcal{B}[G(x,x')] = 0$$

3. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u,v)$ from $\langle \mathcal{L}u, v\rangle$.
4. Solution is

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

5. Simplify using $\mathcal{B}^*[G(x,x')] = 0$ (with respect to x').

- Putting back our charge distribution $\rho(x)$, we get a full solution for any charge distribution and boundary conditions.

- Now let's look at the general process for solving a boundary value problem with a source.
- The first step is to set up the Green's function equation by setting the source to $\delta(x-x')$ and the boundary conditions to zero. Solve this to find the Green's function.
- Next, we find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u,v)$ by expanding the inner product $\langle \mathcal{L}u, v\rangle$ (usually using integration by parts).
- Then, we can write down the solution.
- Unfortunately, this solution will still include unknown boundary conditions of $u(x)$. Eliminate these using the fact that $G(x,x')$ obeys the adjoint boundary conditions with respect to x'.

Comments:

- Type of boundary condition is important. In the last example, if we used initial conditions instead of boundary conditions, then the final solution would have been in terms of the known initial conditions $V(a)$ and $V'(a)$.
- *We can't ignore the adjoint problem.* Though we've eliminated the adjoint Green's function $H(x, x')$, the adjoint Green's function is still a key part of the solution method. In particular, we need to find $\mathcal{B}^*$ and $J(u, v)$.

CONSTRUCTING SOLUTIONS IN 3D

1. Given
$$\mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}[u(\mathbf{r})] = \alpha$$
2. Solve Green's problem
$$\mathcal{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'); \quad \mathcal{B}[G(\mathbf{r}, \mathbf{r}')] = 0$$
3. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, v)$ from $\langle \mathcal{L}u, v \rangle$.
4. Solution is
$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} J\big(u(\mathbf{r}'), G(\mathbf{r}, \mathbf{r}')\big)\,\mathrm{d}s$$
5. Simplify using $\mathcal{B}^*[G(\mathbf{r}, \mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

- Great thing about our approach is that it's easily extended to 3D.
- Mostly, we just replace x and x' with $\mathbf{r}$ and $\mathbf{r}'$.
- One difference is the conjunct term. If V is the region of interest, then we have to integrate the conjunct over the surface of V (denoted ∂V).
- We still have the same interpretation, though: the non-zero boundary conditions on the surface act like additional sources.

CONSTRUCTING SOLUTIONS IN 3D

Finding $\mathcal{L}^*$, $\mathcal{B}^*$ and $J(u,v)$ in 3D:

$$\langle \mathcal{L}u, v\rangle = \int_V [\mathcal{L}u(\mathbf{r})]\, v^*(\mathbf{r})\, \mathrm{d}^3\mathbf{r}$$

Green's second identity:

$$\int_V \left(u\nabla^2 v - v\nabla^2 u\right) \mathrm{d}^3\mathbf{r} = \oint_{\partial V} \left(u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u\right) \cdot \mathrm{d}\mathbf{s}$$

SPECTRAL METHODS

- To find $\mathcal{L}^*$, $\mathcal{B}^*$ and $J(u,v)$ in 3D, we follow a similar process of expanding the inner product. (This time the inner product is a volume integral.)
- Then we do a 3D version of integration by parts to expand the integral.
- For the Laplacian operator ∇^2 (often what we deal with), Green's second identity gives the integration by parts rule.

- We spent the last section learning how to construct a final solution from the Green's function.
- Before that, we learned a simple way to actually calculating the Green's functions.
- Unfortunately, that simple method doesn't always work, especially when dealing with separation of variables in 3D.
- In this section, we'll look at another method for constructing Green's functions. In the process, we'll see the close tie between Green's functions and eigenvalues/eigenfunctions.

EIGENFUNCTION EXPANSION

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

EIGENFUNCTION EXPANSION

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

Since $\mathcal{L}$ is self-adjoint,

$$u(x) = \sum_n \langle u, \phi_n \rangle\, \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle\, \phi_n(x)$$

- First we'll review a bit of eigenfunction theory, but we'll quickly see how it relates to Green's functions.
- Set up a problem similar to before, but we've added a complex parameter λ for later convenience.
- For this section we'll insist that $\mathcal{L}$ be fully self-adjoint so that we can take full advantage of spectral theory. (A brief discussion of the non-self-adjoint case can be found in Morse and Feshbach.)
- Technically, we're also assuming here that $\mathcal{L}$ is a *bounded* linear operator. Unbounded operators have continuous sets of eigenvalues, and the theory behind them is much more delicate. See, e.g., Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications*.

- Since $\mathcal{L}$ is self-adjoint, we know that it has a complete orthonormal set of eigenfunctions ϕ_n.
- That is, we can expand any function (in this case $u(x)$ and $f(x)$) in terms of $\phi_n(x)$. (Generalized Fourier series.)

EIGENFUNCTION EXPANSION

$$(\mathcal{L} - \lambda)u(x) = f(x)$$

$$(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$(\lambda_n - \lambda) \langle u, \phi_n \rangle = \langle f, \phi_n \rangle$$

- Going back to our original equation, let's expand $u(x)$ and $f(x)$ in terms of eigenfunctions of $\mathcal{L}$.
- Using the fact that $\mathcal{L}$ is linear and $\mathcal{L}\phi_n = \lambda_n\phi_n$, we can get rid of $\mathcal{L}$ (third line).
- Finally, since the $\phi_n(x)$ are linearly independent, each term in the sums on the RHS and LHS must be equal. So we get an expression for the generalized Fourier coefficients $\langle u, \phi_n \rangle$.

EIGENFUNCTION EXPANSION

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}\phi_n(x)}$$

- Plugging in our new expression for the Fourier coefficients, we obtain a formula for $u(x)$ in terms of the eigenfunctions and eigenvalues of $\mathcal{L}$.

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}x'$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

- Usually, the inner product is defined by an integral.
- If we write this out and do some manipulation, we get something that looks a lot like the Green's function expression.

- This sum really is the Green's function.
- So, if we know the eigenvalues and eigenfunctions of $\mathcal{L}$, we can immediately construct the Green's function as an infinite series.
- Note also that $G(x, x') = G^*(x, x')$, as we expect because this is a self-adjoint problem.

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

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- It also goes the other way. If we know the Green's function of $(\mathcal{L} - \lambda)$ for any complex λ, then the eigenvalues of $\mathcal{L}$ are just the poles of the Green's function with respect to lambda.
- Eigenfunctions are a little trickier to read off, but it's possible to find them from the Green's function using residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda)G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda)\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

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- Using our Green's function equation, we can also derive an expression for the delta function as a sum of eigenfunctions.
- This expression is useful when solving three-dimensional problems with separation of variables.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\right)}_{\mathcal{L}-\lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

- Let's do a simple example to illustrate the idea.
- For $\mathcal{L} = \mathrm{d}^2/\,\mathrm{d}x^2$ we know that the eigenfunctions are sines and cosines. The boundary conditions restrict us to just sines with $\lambda_n = \pi n/a$.
- $\sqrt{2/a}$ ensures that the eigenfunctions are normalized.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

- Using the formula we derived earlier, we can very quickly write out the Green's function as an infinite series.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x,x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x,x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda}x'\right)\sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x > x' \end{cases}$$

CONCLUSION

- We could have also solved this problem directly (Assume $G(x,x')$ behaves like the source-free solution except at $x = x'$. Apply the boundary conditions, continuity and discontinuity requirements to find the coefficient.) The result is shown.
- The direct solution is a little uglier, but it’s much easier to evaluate numerically because it doesn’t involve an infinite series. For that reason, direct solution is usually more desireable if it actually works. However, series solutions tend to be needed for solving multi-dimensional problems.
- Note: there’s a trick for evaluating infinite series using residue calculus, and (I think) you could use this to derive the second expression from the first. You can also use residue integration to derive the first from the second.

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
  - Source-free behaviour for $x \neq x'$.
  - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
  - Systematic method using adjoint equation.
  - Non-zero boundary conditions behave like sources.
- Green's functions $\Longleftrightarrow$ eigenvalues/eigenfunctions.

FURTHER READING

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Fully rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

FURTHER READING

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.