GREEN'S FUNCTIONS

A short introduction

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OUTLINE

1. Introduction
2. Generalized functions
3. Direct solution
4. Boundary conditions
5. Spectral methods
6. 3D problems
7. Properties of the Green's function
8. Advanced topics

INTRODUCTION

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

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Green's function is the **impulse response**:

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WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

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WHY IS IT USEFUL?

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$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

FAMILIAR GREEN'S FUNCTIONS

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

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FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}') \, \mathrm{d}^3 \, \mathbf{r}'$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

GENERALIZED FUNCTIONS

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x) \, \mathrm{d}\, x$$

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\, \mathrm{d}\, x$$

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}\,x = \phi(0)$$

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} \left(\mathcal{L}G(x, x')\right) \phi(x) \, \mathrm{d}\, x$$

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

DIRECT SOLUTION

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

For $x \neq x'$

$$\frac{d^2\, G(x, x')}{d\, x^2} - k^2 G(x, x') = 0$$

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$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x')\right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x - x')$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'-\epsilon}\right] = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') \right] \mathrm{d}\, x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\, \mathrm{d}\, x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left. \frac{\mathrm{d}\, G}{\mathrm{d}\, x} \right|_{x=x'+\epsilon} - \left. \frac{\mathrm{d}\, G}{\mathrm{d}\, x} \right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}\,G}{\mathrm{d}\,x}$:

$$kA + kB = 1$$

At last, our Green's function is

$$G(x,x') = \begin{cases} \dfrac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \dfrac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

BOUNDARY CONDITIONS

ADJOINT OPERATORS

$$\begin{aligned}
\text{Original:} \qquad & \mathcal{L}[u(x)] = f(x); \qquad & \mathcal{B}[u(x)] = 0 \\
\text{Adjoint:} \qquad & \mathcal{L}^*[v(x)] = f(x); \qquad & \mathcal{B}^*[v(x)] = 0
\end{aligned}$$

Defining property:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle \qquad \text{where } \langle u, v \rangle = \int_a^b u(x) v^*(x) \, \mathrm{d}\, x$$

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$

$$= \int_a^b \left[v''^*(x) + k^2 v^*(x)\right] u(x)\,\mathrm{d}\,x +$$

$$+ \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

By inspection:

$$\mathcal{L}^* v(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*\right] v(x)$$

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ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x) \right]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$
$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v)$$

where $J(u, v)$ depends on $u, v, u', v', \ldots$ at the boundaries.

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \Longleftrightarrow J(u, v) = 0$$

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x,x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

Original problem:

$$\mathcal{L}[u] = \frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} + k^2 u(x) = f(x); \qquad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x, x')}{\mathrm{d}\, x^2} + k^2 H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x,x') = \begin{cases} A\cos(k(x-x')) + B\sin(k(x-x')) & \text{for } x < x' \\ C\cos(k(x-x')) + D\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

$$\begin{aligned} \text{Final conditions} &\implies C = D = 0 \\ \text{Continuity of function} &\implies A = 0 \\ \text{Discontinuity of derivative} &\implies B = \frac{-1}{k} \end{aligned}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$\langle \mathcal{L}u(x), H(x,x') \rangle = \langle u(x), \mathcal{L}^* H(x,x') \rangle + J(u,H)$$

$$\langle f(x), H(x,x') \rangle = \langle u(x), \delta(x-x') \rangle + J(u,H)$$

$$\int_a^b f(x) H^*(x,x') \, \mathrm{d}\, x = \int_a^b u(x)\delta(x-x') \, \mathrm{d}\, x + J(u,H)$$

$$\boxed{u(x') = \int_a^b f(x) H^*(x,x') \, \mathrm{d}\, x - J(u,H)}$$

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x)H^*(x, x')\,\mathrm{d}\,x - \left[\frac{\mathrm{d}\,u(x)}{\mathrm{d}\,x}H^*(x, x') - u(x)\frac{\mathrm{d}\,H^*(x, x')}{\mathrm{d}\,x}\right]_a^b$$

Recall:

$$\begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x + \beta H^*(a, x') - \alpha \frac{\mathrm{d}\, H^*(a, x')}{\mathrm{d}\, x}$$

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$
$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$
$$H^*(x', x'') = G(x'', x')$$

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$$\boxed{G(x, x') = H^*(x', x)}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x,x') = \begin{cases} -k^{-1}\sin(k(x-x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

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EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x')\, \mathrm{d}\, x + \beta H^*(a, x') - \alpha \left. \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x)\, \mathrm{d}\, x + \beta G(x', a) - \alpha \left. \frac{\mathrm{d}\, G(x', x)}{\mathrm{d}\, x} \right|_{x=a}$$

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FINAL SOLUTION

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}}$$

where

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}$$

$$u(x) = \int_a^b \Big[f(x') + \beta\delta(x'-a) - \alpha\delta'(x'-a)\Big] G(x,x')\,\mathrm{d}\,x'$$

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$
$$\mathcal{L}^*[H(x,x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

1. Find $G(x, x')$.
2. Find $u(x')$ in terms of $H(x, x')$:

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x - J(u(x), H(x, x'))$$

3. Express $u(x)$ in terms of $G(x, x') = H^*(x', x)$.

SPECTRAL METHODS

3D problems

PROPERTIES OF THE GREEN'S FUNCTION

ADVANCED TOPICS

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.