GREEN'S FUNCTIONS

A SHORT INTRODUCTION

Chris Deimert

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Department of Electrical and Computer Engineering, University of Calgary

OUTLINE

- This is intended as a short introduction to Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them.
- We won't be able to cover much here, but we'll try to focus on building a solid foundation and understanding of Green's functions.
- Suggested further reading is provided at the end.

1. Basic idea of Green's functions.
2. Simplest method for solving the Green's function equation.
3. How to use the Green's function to solve a problem with boundary conditions. (Biggest section!)
4. Useful properties of Green's functions for special types of problems.
5. Summary and suggested further reading.

BASIC IDEA

- The basic idea of Green's functions is really simple. You've actually used them before!

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most electromagnetics problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is set equal to an impulse (delta function) located at x'.
- Can think of it as a generalization of the familiar impulse response from signal processing.

Why is it useful?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}x$$

Why is it useful?

$$\mathcal{L} u(x) = f(x)$$

$$\mathcal{L} G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}x}$$

(Some conditions apply.)

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.
- Beware the fine print! This formula actually only works under certain assumptions about the boundary conditions.
- We'll deal with the more general approach later. For now, we'll use this simple version to get the key idea across.

FAMILIAR GREEN'S FUNCTIONS

Impulse response of a linear time-invariant system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t-t')$ from linear system theory is an example of a Green's function.

$$G(t,t') = h(t-t')$$

- Output $y(t)$ is given by convolution of the impulse $h(t)$ with the input $x(t)$.

- Green's function for Poisson's equation is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|}$$

- The Green's function is the potential created by a point (impulse) charge.

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}$$

- The Green's function is the potential created by a point (impulse) current.

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive and understand these expressions.
- Generalize to other problems and boundary conditions.

- With Green's function theory, we learn how to derive the above expressions and understand them a little more rigorously. (Though we won't have time to derive the 3D ones here.)
- In addition, Green's function theory allows us to deal with different boundary conditions. The solutions to the Poisson and Helmholtz equations above assume free space (boundaries at infinity). Green's functions would allow us to, e.g., find the response to a current source inside a specific waveguide.

FINDING THE GREEN'S FUNCTION

- In this section, we'll look at one of the simplest methods for actually solving the Green's function problem.
- Often called the *direct method*.

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

- Let's start off by looking at a simple example.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem for any $f(x)$.
- But the Green's function problem looks hard! The point of this example is to demonstrate that we can actually solve it.

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x')\right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x,x')$ is discontinuous (like a step function), then $\mathrm{d}G/\,\mathrm{d}x$ will behave like a delta function and $\mathrm{d}^2G/\,\mathrm{d}x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x,x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

- But what if the derivative $\mathrm{d}G/\,\mathrm{d}x$ is discontinuous?
- Then $\mathrm{d}^2G/\,\mathrm{d}x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x,x')$ is continuous.
- The first integral is an integral of a derivative, so we can use the fundamental theorem of calculus. The result is a *discontinuity condition for the derivative.*

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\dfrac{\mathrm{d}G(x,x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

- Applying our two conditions, we can solve for A and B. We find

$$A = B = \frac{1}{2k}$$

A SIMPLE EXAMPLE

Solving, our Green's function is

$$G(x,x') = \frac{1}{2k} \begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(x)$.
- Caution: remember the fine print from before. This solution only works with certain assumptions about boundary conditions. (More on this to come!)

GENERAL APPROACH

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ and its derivatives are continuous or discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

- Write down the source-free solution for $x \neq x'$: usually has a few unknown coefficients.
- Examine the equation to find continuity/discontinuity requirements for $G(x, x')$ and its derivatives. (Most books on Green's functions provide these requirements for general Sturm-Liouville problems.)
- This approach is great if it works. Unfortunately, it doesn't always work (especially in 3D problems).
- We'll briefly look at an alternative solution method later using eigenvalues and eigenfunctions, but this still just scratches the surface. See the references provided at the end.

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

- In the introduction, we showed non-rigorously how to construct a solution from the Green's function. To keep things simpler, we ignored boundary conditions.
- Here, we'll look at how to properly construct a solution from the Green's function when boundary conditions are involved.
- Our approach is quite challenging compared to a lot of books on the subject. The advantage is that we'll deal with some subtleties that can otherwise lead to confusion.
- For approaches similar to the one in this section, see Dudley, Morse and Feshbach, or Gerlach.

ADJOINT OPERATORS

Inner product:

$$\langle u, v\rangle = \int_a^b u(x)\overline{v}(x)\,\mathrm{d}x$$

Adjoint operator $\mathcal{L}^*$:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

- Underpinning our approach is the idea of an adjoint operator.
- Start with an inner product (for 1D problems, usually the one shown). Note that $\overline{v}$ is the complex conjugate of v.
- If $\mathcal{L}$ is a linear operator, then its adjoint $\mathcal{L}^*$ is defined as the operator which satisfies

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$
$$\implies \int_a^b (\mathcal{L}u)v^*\,\mathrm{d}x = \int_a^b u\overline{(\mathcal{L}^* v)}\,\mathrm{d}x$$

- Roughly, $\mathcal{L}^*$ is what appears if we try to move $\mathcal{L}$ into the other slot of the inner product.

ADJOINT BOUNDARY CONDITIONS

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \longrightarrow \text{Only for certain } u, v\,!$$

Adjoint boundary conditions:

$$\mathcal{B}_i[u] = 0; \qquad \mathcal{B}_i^*[v] = 0$$

Otherwise

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \underbrace{J(u, \overline{v})}_{\text{Conjunct}}\Big|_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}u(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

Want $\mathcal{L}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

- The adjoint is only truly the adjoint (i.e., $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$) for certain functions u and v. (Mathematically, when $\mathcal{L}$ and $\mathcal{L}^*$ are unbounded operators, they do not necessarily share the same *domain*, and we need to consider that.)
- This is where boundary conditions come in. Specifically, if u obeys some boundary conditions $\mathcal{B}_i[u] = 0$, then v has to obey some adjoint boundary conditions $\mathcal{B}_i^*[v] = 0$.
- Note on notation: $\mathcal{B}_i[u] = 0$ means that some linear combination of u and its derivatives are set equal to zero at the boundaries.
- If $\mathcal{B}_i[u] = 0$ and $\mathcal{B}_i^*[v] = 0$ are not satisfied, then the adjoint equation almost holds, but we get an extra term $J(u, v^*)$ called the *conjunct*. It's only evaluated at the boundaries.

- Let's look at an example: the 1D simple harmonic oscillator.
- Let's try to find $\mathcal{L}^*$ without worrying about boundary conditions for now. (So we expect the conjunct to appear.)

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \int_a^b \left[u'' + k^2 u\right] \overline{v} \, \mathrm{d}x$$

$$\langle \mathcal{L}u, v \rangle = \int_a^b \left[-u'\overline{v}' + k^2 u\overline{v}\right] \mathrm{d}x + \left[u'\overline{v}\right]_a^b$$

$$\langle \mathcal{L}u, v \rangle = \int_a^b u \left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

25

- To find the adjoint, let's expand $\langle \mathcal{L}u, v \rangle$.
- Use integration by parts twice.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \int_a^b u \left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Looks like

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + J(u, \overline{v})\Big|_a^b$$

with

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \overline{k^2}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

26

- After integration by parts, we can read off the adjoint operator and the conjunct.
- So in this case, the adjoint operator is the almost the same as the original operator, but there's an extra complex conjugate. If k is real, then $\mathcal{L} = \mathcal{L}^*$.

ADJOINT OPERATORS: EXAMPLE

For

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

must have

$$J(u,\bar{v})\Big|_a^b = 0$$
$$\left[u'\bar{v} - u\bar{v}'\right]_a^b = 0$$

$$u'(b)\bar{v}(b) - u(b)\bar{v}'(b) - u'(a)\bar{v}(a) + u(a)\bar{v}'(a) = 0$$

- Now let's look at adjoint boundary conditions.
- For $\mathcal{L}^*$ to be a true adjoint, we need the conjunct to be zero.
- Let's expand the conjunct for this particular example.

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\bar{v}(b) - u(b)\bar{v}'(b) - u'(a)\bar{v}(a) + u(a)\bar{v}'(a) = 0$$

Boundary conditions:

$$\begin{array}{lcl} \mathcal{B}_1[u] = u(a) = 0 & \Longrightarrow & \mathcal{B}_1^*[v] = v(a) = 0 \\ \mathcal{B}_2[u] = u(b) = 0 & & \mathcal{B}_2^*[v] = v(b) = 0 \end{array}$$

$$\mathcal{B}_i = \mathcal{B}_i^*$$

- Suppose we have the simple boundary conditions $u(a) = u(b) = 0$.
- Then, to make the conjunct zero, we need $\bar{v}(a) = \bar{v}(b) = 0$ or $v(a) = v(b) = 0$.
- So in this case, the adjoint boundary conditions on v are the same as the boundary conditions on u.
- Remember what these boundary conditions mean. $\mathcal{L}^*$ is the true adjoint when $\mathcal{L}$ operates on functions $u(x)$ which are zero at $x = a, b$ and $\mathcal{L}^*$ operates on functions $v(x)$ which are zero at $x = a, b$.

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Initial conditions:

$$\begin{matrix} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u'(a) = 0 \end{matrix} \implies \begin{matrix} \mathcal{B}_1^*[v] = v(b) = 0 \\ \mathcal{B}_2^*[v] = v'(b) = 0 \end{matrix}$$

$$\mathcal{B}_i \neq \mathcal{B}_i^*$$

ADJOINT OPERATORS: SUMMARY

Adjoint operator:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

Adjoint boundary conditions:

$$\begin{matrix} \mathcal{B}_i[u] = 0 \\ \mathcal{B}_i^*[v] = 0 \end{matrix} \implies J(u, \overline{v})\Big|_a^b = 0$$

Otherwise:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

- What if we have initial conditions instead? $u(a) = u'(a) = 0$.
- Then, to make the conjunct zero, we need $v(b) = v'(b) = 0$.
- So, for initial conditions, the adjoint boundary conditions are *final* conditions. $\mathcal{B}_i \neq \mathcal{B}_i^*$.

- The adjoint operator satisfies $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$.
- This only works for certain u, v, though. Specifically, it works when u obeys boundary conditions and v obeys adjoint boundary conditions.
- If u, v do not satisfy these boundary conditions, then $\mathcal{L}^*$ is not truly the adjoint anymore. However, it still nearly obeys the adjoint equation; there's just a leftover conjunct term which depends on the boundary values of u, v and their derivatives.
- For a lot of things (e.g., using eigenfunction bases) we need this conjunct to be zero. But for Green's functions, it will end up being indispensible.

THE ADJOINT GREEN'S FUNCTION

Original problem:

$$\mathcal{L}u(x) = f(x); \qquad \mathcal{B}_i[u(x)] = \alpha_i$$

Green's problem:

$$\mathcal{L}G(x, x') = \delta(x - x'); \qquad \mathcal{B}_i[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^* H(x, x') = \delta(x - x'); \qquad \mathcal{B}_i^*[H(x, x')] = 0$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x, x')\rangle = \langle u(x), \mathcal{L}^* H(x, x')\rangle + J\big(u(x), \overline{H}(x, x')\big)\Big|_a^b$$

$$\langle f(x), H(x, x')\rangle = \langle u(x), \delta(x - x')\rangle + J\big(u(x), \overline{H}(x, x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x, x')\, \mathrm{d}x = \int_a^b u(x)\delta(x - x')\, \mathrm{d}x + J\big(u(x), \overline{H}(x, x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)\overline{H}(x, x')\, \mathrm{d}x - J\big(u(x), \overline{H}(x, x')\big)\Big|_a^b}$$

- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha_i \to 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning! A lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Sometimes the "Green's function" in an expression is really the adjoint Green's function.

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- Then, we use the fact that $\mathcal{L}u(x) = f(x)$ and $\mathcal{L}H(x, x') = \delta(x - x')$.
- After evaluating the inner product terms, we arrive at a fairly general formula which looks somewhat like what we had in the introduction. The difference is that it involves the *adjoint* Green's function $H(x, x')$, and it has an extra conjunct term.
- We'll deal with the conjunct later. For now, let's try to get rid of $H(x, x')$ and express $u(x)$ in terms of $G(x, x')$. To do that, we need a relationship between $H(x, x')$ and $G(x, x')$.

CONSTRUCTING SOLUTIONS: DERIVATION

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$
$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$
$$\int_a^b \delta(x - x')\overline{H}(x, x'')\,\mathrm{d}x = \int_a^b G(x, x')\delta(x - x'')\,\mathrm{d}x$$
$$\overline{H}(x', x'') = G(x'', x')$$
$$\boxed{G(x, x') = \overline{H}(x', x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.
- Note, in the last example, we had to include the conjunct because $\mathcal{B}[u] \neq 0$. The boundary conditions and adjoint boundary conditions have to be set to zero to eliminate the conjunct. In this case, $\mathcal{B}[G] = 0$ and $\mathcal{B}[H] = 0$, so the conjunct is eliminated.
- Also note a surprising result of this: if $G(x, x')$ obeys the boundary conditions with respect to x, then it automatically obeys the *adjoint* boundary conditions with respect to x'. We'll come back to this idea later.

CONSTRUCTING SOLUTIONS: DERIVATION

$$u(x') = \int_a^b f(x)\overline{H}(x, x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x, x')\big)\Big|_a^b$$

and

$$G(x, x') = \overline{H}(x', x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x, x')\,\mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^b}$$

- Let's go back to our expression for $u(x')$ in terms of $H(x, x')$.
- Using our new relationship $G(x, x') = \overline{H}(x, x')$, we can rewrite this as an expression for $u(x)$ in terms of $G(x, x')$. (Note: we switched x and x' to make it look a little nicer.)
- So this is the more correct version of what we saw in the introduction. If the conjunct happens to be zero, then we get what we had before. If not, we have an extra term that depends only on the boundaries.
- In general, the conjunct term deals with the boundary conditions of $u(x)$. It turns out that the boundary conditions act, in some way, like additional sources. We'll look more closely at this now through an example.

EXAMPLE: 1D POISSON EQUATION

Original problem:

$$\frac{\mathrm{d}^2V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}; \qquad \begin{array}{l} V(a) = V_a \\ V(b) = V_b \end{array}$$

Green's problem:

$$\frac{\mathrm{d}^2G(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{array}{l} G(a,x') = 0 \\ G(b,x') = 0 \end{array}$$

EXAMPLE: 1D POISSON EQUATION

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x,x')\,\mathrm{d}x' - J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

Take $\rho(x) = 0$ for now.

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

- Let's look at a simple 1D voltage problem.
- We have both a charge density ρ inside the region $a < x < b$, and we have an applied voltage at the boundaries. Intuitively, both of these will affect the voltage in the region.
- In the Green's function problem, we turn the charge density into an impulse function, and we set the applied voltage to zero.

- From our recent results, we can write down the solution for $V(x)$ in terms of the Green's function.
- Take $\rho = 0$ so that we can focus on the boundary conditions for now.

Example: 1D Poisson equation

Can show

$$\langle \mathcal{L}u, v\rangle = \int_a^b u''\overline{v}\,\mathrm{d}x = \int_a^b u\overline{v}''\,\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Comparing with

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v^*)\big|_a^b$$

we see

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \mathcal{L}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

- To write the solution more explicitly, we need to find the conjunct of the operator $\mathcal{L} = \mathrm{d}^2/\,\mathrm{d}x^2$.
- As before, use integration by parts and compare with the expected formula.

Example: 1D Poisson equation

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) - \\ -V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a)$$

- Now that we know the conjunct, we can write the solution for $V(x)$ more explicitly.
- Problem: we don't know $\mathrm{d}V/\,\mathrm{d}x$ at the boundaries.
- The adjoint problem saves us, because we can show that $G(x,b) = G(x,a) = 0$.

EXAMPLE: 1D POISSON EQUATION

Green's problem:

$$\frac{d^2G(x,x')}{dx^2} = \delta(x-x'); \qquad \begin{array}{l} G(a,x') = 0 \\ G(b,x') = 0 \end{array}$$

Adjoint Green's problem:

$$\frac{d^2H(x,x')}{dx^2} = \delta(x-x'); \qquad \begin{array}{l} H(a,x') = 0 \\ H(b,x') = 0 \end{array}$$

But $G(x,x') = \overline{H}(x',x)$ so

$$\frac{d^2\overline{G}(x,x')}{dx'^2} = \delta(x-x'); \qquad \begin{array}{l} G(x,a) = 0 \\ G(x,b) = 0 \end{array}$$

- Back to the adjoint Green's equation.
- Because of the relationship between $G(x,x')$ and $H(x,x')$ we see that $G^*(x,x')$ obeys the adjoint equation with respect to x'.
- More importantly, we see that $\overline{G}(x,x')$ (and thus $G(x,x')$) obeys the adjoint boundary conditions with respect to x'.
- So not only do we have $G(a,x') = G(b,x') = 0$, we also have $G(x,a) = G(x,b) = 0$. This is not a trivial or obvious result (at least to me).

EXAMPLE: 1D POISSON EQUATION

$$V(x) = V_b\frac{dG(x,b)}{dx'} - \frac{dV(b)}{dx'}G(x,b) -$$
$$-V_a\frac{dG(x,a)}{dx'} + \frac{dV(a)}{dx'}G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b\frac{dG(x,b)}{dx'} - V_a\frac{dG(x,a)}{dx'}}$$

- Using the fact that $G(x,x')$ obeys the adjoint boundary conditions with respect to x', we can eliminate the unknown values and simplify our result.
- Now we have an explicit solution for $V(x)$ given any boundary conditions $V(a) = V_a$ and $V(b) = V_b$.
- Further, we see that the solution only depends on the Green's function. It's as if the non-zero boundary conditions V_a, V_b act like additional sources whose response is given by the Green's function.

EXAMPLE: 1D POISSON EQUATION

Full solution with $\rho(x)$:

$$\boxed{V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x,x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}$$

- Putting back our charge distribution $\rho(x)$, we get a full solution for any charge distribution and boundary conditions.
- The first part gives the voltage produced by the charge distribution $\rho(x)$. The last two parts give the voltage produced by the boundary conditions V_a, V_b.
- The response to *both* of these sources of voltage is given by the Green's function!
- For those familiar with the derivative of the delta function, this can be written in an even more suggestive form:

$$V(x) = \int_a^b \left[-\frac{\rho(x)}{\epsilon_0} - V_b\delta'(x'-b) + V_a\delta'(x'-a)\right] G(x,x')\,\mathrm{d}x'$$

SOLVING PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:}\quad \mathcal{L}u(x) = f(x); \quad \mathcal{B}_i[u(x)] = \alpha_i$$

1. Solve Green's problem
$$\mathcal{L}G(x,x') = \delta(x-x'); \quad \mathcal{B}_i[G(x,x')] = 0$$
2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u,\bar{v})$ from $\langle \mathcal{L}u, v\rangle$.
3. Solution is
$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$
4. Simplify using $\mathcal{B}^*[G(x,x')] = 0$ (with respect to x').

Now let's look at the general process for solving a boundary value problem with a source.

1. Set up the Green's function equation by setting the source to $\delta(x-x')$ and the boundary conditions to zero. Solve this to find the Green's function.
2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u,\bar{v})$ by expanding the inner product $\langle \mathcal{L}u, v\rangle$ (usually using integration by parts).
3. Write down the solution.
4. Unknown boundary values of $u(x)$ will appear in the conjunct term. Eliminate them using the fact that $G(x,x')$ obeys the adjoint boundary conditions with respect to x'.

Solving 3D problems with Green's functions

Given: $\mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}_i[u(\mathbf{r})] = \alpha_i$

1. Solve Green's problem

$$\mathcal{L}G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'); \quad \mathcal{B}_i[G(\mathbf{r},\mathbf{r}')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u,\overline{v})$ from $\langle \mathcal{L}u, v\rangle$.
3. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} \mathbf{J}\big(u(\mathbf{r}'), G(\mathbf{r},\mathbf{r}')\big)\cdot \mathrm{d}\mathbf{s}$$

4. Simplify using $\mathcal{B}^*[G(\mathbf{r},\mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

- Great thing about our approach is that it's easily extended to 3D.
- Inner product is now a volume integral over the region of interest (V). Have to use 3D versions of integration by parts (e.g., Green's identities) to find $\mathcal{L}^*$, $J(u,\overline{v})$ and $\mathcal{B}_i^*$.
- The conjunct is now vector-valued, and it must be integrated over the surface of V (denoted ∂V).
- Still have the same interpretation though. The first (volume) integral is the contribution from the source $f(\mathbf{r})$. The second (surface) integral is the contribution from the non-zero boundary conditions.

Constructing the solution

Comments:

- Calculating Green's functions is not trivial.
- Adjoint approach is difficult, but offers clarity.

- Step 1 of our approach (calculate the Green's function) can be difficult: especially in 3D. Learning these techniques is time-consuming, but we now have a solid foundation with which to understand them.
- Our approach was not the easiest, but note the critical role played by the adjoint, both in the derivation and the final solution method.
- Though some authors don't talk about adjoints, they still use these ideas. E.g., often the "Green's function" used in 3D problems is actually the *adjoint* Green's function. (Can be confusing!)

SPECIAL PROPERTIES

SELF-ADJOINTNESS

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B}_i = \mathcal{B}_i^*$, then

$$G(x, x') = \overline{G}(x', x)$$

- The behaviour of the Green's function is shaped by the type of problem we're trying to solve.
- In this section, we'll see how a lot of properties of problems are tied to properties of the Green's function.

- Self-adjointness is equivalent to conjugate symmetry of the Green's function.
- If a problem is self-adjoint, then the adjoint Green's function $H(x, x')$ is the same as the Green's function $G(x, x')$. Then, using a result from before, we find that $G(x, x') = \overline{G}(x, x')$.

RECIPROCITY

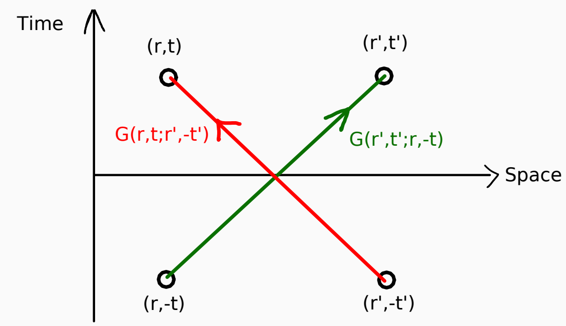
Reciprocity in the frequency-domain:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

RECIPROCITY

Reciprocity in the time-domain:

$$G(\mathbf{r}, t; \mathbf{r}', -t') = G(\mathbf{r}', t'; \mathbf{r}, -t)$$



- The notion of reciprocity in electromagnetism can be tied to this property of the Green's function.
- In the frequency-domain, the Green's function is symmetric under interchange of $\mathbf{r}, \mathbf{r}'$ (no complex conjugate!). It can be shown that this is equivalent to reciprocity. (See Collin for a more thorough discussion using dyadic Green's functions.)
- Mathematically, frequency-domain reciprocity is related to the pseudo-inner product

$$\langle u, v \rangle_p = \iiint u(\mathbf{r}) v(\mathbf{r}) \, \mathrm{d}^3\mathbf{r}$$

and operators which are "self-adjoint" under it:

$$\langle \mathcal{L}u, v \rangle_p = \langle u, \mathcal{L}v \rangle_p$$

- In the time-domain, the Green's function is symmetric, but with an added minus sign on the time variables. This is because of causality (see Morse and Feshbach).
- Think of $G(\mathbf{r}, t; \mathbf{r}', t')$ as having an impulse at $(\mathbf{r}', t')$ and measuring it at $(\mathbf{r}, t)$. Then reciprocity shows that interchanging sources and measurements leads to identical results.
- Mathematically, time-domain reciprocity related to the pseudo-inner product

$$\langle u, v \rangle_p = \int \iiint u(\mathbf{r}, t) v(\mathbf{r}, -t) \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}t$$

and operators which are "self-adjoint" under it:

$$\langle \mathcal{L}u, v \rangle_p = \langle u, \mathcal{L}v \rangle_p$$

CAUSALITY

Causality:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } t < t'$$

Special relativity:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } |\mathbf{r} - \mathbf{r}'| > c(t - t')$$

- The Green's function can tell us if a system is causal or not.
- Causality means that an effect cannot precede a cause, so the impulse response (Green's function) cannot appear before the impulse itself. I.e., the Green's function has to be zero for $t < t'$.
- In special relativity, causality means that information cannot propagate faster than light. There is a corresponding restriction on the Green's function.
- Compare these causal Green's function with the Green's function for a self-adjoint problem: they are incompatible. So we cannot have a causal system which is also self-adjoint in time.

SYMMETRY AND INVARIANCE

Time-invariance:

$$G(t, t') = G(t - t')$$

Spatial-invariance:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}')$$

- A time-invariant system is one whose behaviour doesn't change over time. That is, if we delay our input, we'll get the exact same output, just delayed by an equal amount.
- In that case, can show that the Green's function only depends on the difference between t and t'. (See Gerlach.)
- Recall: in linear time-invariant (LTI) systems, the impulse response is written as $h(t - t')$: this is why!
- A similar thing applies to systems whose behaviour doesn't change from place to place.
- These properties can make it easier to find the Green's function (e.g., the free-space wave equation). Also, solution is just given by convolution $u(t) = G(t) * f(t)$.

SPECTRAL THEORY

$$(\mathcal{L} - \lambda)u(x) = f(x); \quad \mathcal{B}[u] = 0$$

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$ then

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

SPECTRAL THEORY

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}x'$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

- For a self-adjoint problem, we can write out the solution as a sum of eigenfunctions of $\mathcal{L}$, where $\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$.
- With some work (not shown), we can directly write out the solution in terms of $f(x)$ as a generalized Fourier series. $\langle f, \phi_n \rangle$ are the projections of f onto the normalized eigenfunction basis.
- (Technically, we're also assuming here that $\mathcal{L}$ is a *bounded* linear operator. Unbounded operators have continuous sets of eigenvalues, and the theory behind them is more delicate. See, e.g., Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications.*)

- We can rewrite the last solution, and read off the Green's function.
- So we can write down the Green's function directly if we know the eigenfunctions/eigenvalues!
- It's not the nicest form because we have to sum an infinite series. Finding a closed-form version like we did before would be preferable, but in 3D separation of variable problems, we won't usually have a choice.

SPECTRAL THEORY

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x'; \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

$$\begin{aligned} \lambda_n &\longrightarrow \text{ Poles of } G(x, x'; \lambda) \\ \phi_n(x)\phi_n^*(x') &\longrightarrow \text{ Residues of } G(x, x'; \lambda) \end{aligned}$$

CONCLUSION

- We can also get the eigenvalues/eigenfunctions from the Green's function!
- Specifically, we need to know the Green's function of $\mathcal{L} - \lambda$ for $\lambda \in \mathbb{C}$.
- The eigenvalues are simply the poles of $G(x, x'; \lambda)$ with respect to λ.
- The eigenfunctions $\phi_n(x)$ are more difficult, but if there are no repeated eigenvalues, then they can be found from the residues of $G(x, x'; \lambda)$ at $\lambda = \lambda_n$.
- Important point is that the Green's function can tell us a lot about spectral quantities.

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Lots of information in the Green's function.
- Green's functions $\Longleftrightarrow$ eigenvalues/eigenfunctions.

FURTHER READING

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Gerlach (2010), *Linear mathematics in infinite dimensions*. Nice set of online course notes for quick reference.

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Morse and Feshback, *Methods of theoretical physics*. Big, detailed reference. Great resource for deeper insight and understanding.

FURTHER READING

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Folland (1992), *Fourier analysis and its applications*. Rigorous math book. Chapter on generalized functions is particularly nice.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.

BONUS SECTIONS!

- These bonus slides were removed from the original presentation, which was far too long.
- The wave-equation derivation is somewhat unique, piecing together different ideas from different places. One weakness with many derivations is that they discard the "anti-causal" Green's function in a seemingly arbitrary way. In this derivation, causality automatically follows because initial conditions are used.
- The generalized function section is interesting because it shows how to rigorously deal with the delta function. We use the delta function so frequently, yet we rarely see a proper definition.

SPECTRAL METHODS

- In this section, we'll look at the strong relationship between Green's functions and spectral theory.
- Essentially, eigenfunction expansion allows us to calculate the Green's function when direct methods don't work.
- A basic background in spectral theory can be found in most books covering Green's functions.

EIGENFUNCTION EXPANSION

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

- First we'll review a bit of eigenfunction theory, but we'll quickly see how it relates to Green's functions.
- Set up a problem similar to before, but we've added a complex parameter λ for later convenience.
- For this section we'll insist that $\mathcal{L}$ be fully self-adjoint so that we can take full advantage of spectral theory. (A brief discussion of the non-self-adjoint case can be found in Morse and Feshbach.)
- Technically, we're also assuming here that $\mathcal{L}$ is a *bounded* linear operator. Unbounded operators have continuous sets of eigenvalues, and the theory behind them is much more delicate. See, e.g., Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications*.

EIGENFUNCTION EXPANSION

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

Since $\mathcal{L}$ is self-adjoint,

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

- Since $\mathcal{L}$ is self-adjoint, we know that it has a complete orthonormal set of eigenfunctions ϕ_n.
- That is, we can expand any function (in this case $u(x)$ and $f(x)$) in terms of $\phi_n(x)$. (Generalized Fourier series.)

EIGENFUNCTION EXPANSION

$$(\mathcal{L} - \lambda)u(x) = f(x)$$
$$(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$
$$\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$
$$(\lambda_n - \lambda) \langle u, \phi_n \rangle = \langle f, \phi_n \rangle$$

- Going back to our original equation, let's expand $u(x)$ and $f(x)$ in terms of eigenfunctions of $\mathcal{L}$.
- Using the fact that $\mathcal{L}$ is linear and $\mathcal{L}\phi_n = \lambda_n \phi_n$, we can get rid of $\mathcal{L}$ (third line).
- Finally, since the $\phi_n(x)$ are linearly independent, each term in the sums on the RHS and LHS must be equal. So we get an expression for the generalized Fourier coefficients $\langle u, \phi_n \rangle$.

EIGENFUNCTION EXPANSION

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}x'$$

- Plugging in our new expression for the Fourier coefficients, we obtain a formula for $u(x)$ in terms of the eigenfunctions and eigenvalues of $\mathcal{L}$.

- Usually, the inner product is defined by an integral.
- If we write this out and do some manipulation, we get something that looks a lot like the Green's function expression.

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\, \mathrm{d}x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

- This sum really is the Green's function.
- So, if we know the eigenvalues and eigenfunctions of $\mathcal{L}$, we can immediately construct the Green's function as an infinite series.
- Note also that $G(x, x') = G^*(x, x')$, as we expect because this is a self-adjoint problem.

- It also goes the other way. If we know the Green's function of $(\mathcal{L} - \lambda)$ for any complex λ, then the eigenvalues of $\mathcal{L}$ are just the poles of the Green's function with respect to lambda.
- Eigenfunctions are a little trickier to read off, but it's possible to find them from the Green's function using residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x-x') = (\mathcal{L}-\lambda)G(x,x')$$
$$\delta(x-x') = (\mathcal{L}-\lambda)\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n-\lambda}$$
$$\delta(x-x') = \sum_n \frac{(\lambda_n-\lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n-\lambda}$$
$$\boxed{\delta(x-x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

- Using our Green's function equation, we can also derive an expression for the delta function as a sum of eigenfunctions.
- This expression is useful when solving three-dimensional problems with separation of variables.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\lambda\right)}_{\mathcal{L}-\lambda} u(x) = f(x); \quad u(0)=u(a)=0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

- Let's do a simple example to illustrate the idea.
- For $\mathcal{L} = \mathrm{d}^2/\,\mathrm{d}x^2$ we know that the eigenfunctions are sines and cosines. The boundary conditions restrict us to just sines with $\lambda_n = \pi n/a$.
- $\sqrt{2/a}$ ensures that the eigenfunctions are normalized.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x,x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x,x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

- Using the formula we derived earlier, we can very quickly write out the Green's function as an infinite series.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x,x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x,x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda}x'\right)\sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x > x' \end{cases}$$

- We could have also solved this problem directly (Assume $G(x,x')$ behaves like the source-free solution except at $x = x'$. Apply the boundary conditions, continuity and discontinuity requirements to find the coefficient.) The result is shown.
- The direct solution is a little uglier, but it's much easier to evaluate numerically because it doesn't involve an infinite series. For that reason, direct solution is usually more desireable if it actually works. However, series solutions tend to be needed for solving multi-dimensional problems.
- Note: there's a trick for evaluating infinite series using residue calculus, and (I think) you could use this to derive the second expression from the first. You can also use residue integration to derive the first from the second.

3D wave equation

The wave equation problem

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\mathcal{L}} u(\mathbf{r}, t) = f(\mathbf{r}, t); \qquad \mathcal{B}[u] = \alpha$$

In electromagnetism:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

- To get a taste of 3D problems, let's look at the 3D scalar wave equation in a vacuum.
- We'll use the time-domain because it's something that's less-frequently covered in electrical engineering books.
- The time domain also gives insight to some delicate issues of causality which are less clear in the frequency domain.

- The wave equation is a key component of electromagnetism. Most of this course was spent learning different ways to solve the Helmholtz equation, which is just a Fourier transformed version of the wave equation.
- In the time domain (in the Lorentz gauge), each component of the vector potential $\mathbf{A}$ obeys the scalar wave equation, with a component of $\mathbf{J}$ as its source.
- We can solve a huge number of electromagnetics problems if we can solve the scalar wave equation (or the scalar Helmholtz equation).

THE ADJOINT WAVE EQUATION

Find $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{B}$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

where

$$\langle u, v\rangle = \int_{t_i}^{t_f} \int_V u(\mathbf{r}, t)\overline{v}(\mathbf{r}, t)\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t$$

- To construct $u(\mathbf{r}, t)$ in terms of the Green's function, we proceed similar to the 1D case.
- The first step is to find the adjoint operators and adjoint boundary conditions for the wave equation.
- Before our inner product was an integral over the single variable x. Now, it's an integral over all four variables x, y, z, t.

THE ADJOINT WAVE EQUATION

$$\langle \mathcal{L}u, v\rangle = \int_{t_i}^{t_f} \int_V \left(\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right)\overline{v}\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t$$

Use Green's identity

$$\int_V \left(\overline{v}\nabla^2 u - u\nabla^2\overline{v}\right) = \oint_{\partial V} \left(\overline{v}\frac{\partial u}{\partial n} - u\frac{\partial \overline{v}}{\partial n}\right)\mathrm{d}S$$

and integration by parts

$$\int_{t_i}^{t_f} \left(\frac{\partial^2 u}{\partial t^2}\overline{v} - u\frac{\partial^2 \overline{v}}{\partial t^2}\right)\mathrm{d}t = \left[\overline{v}\frac{\partial u}{\partial t} - u\frac{\partial \overline{v}}{\partial t}\right]_{t_i}^{t_f}$$

- To find $\mathcal{L}^*$ and $\mathcal{B}^*$, we write out $\langle \mathcal{L}u, v\rangle$ and then try to rewrite it in the form $\langle u, \mathcal{L}^* v\rangle$ using theorems from calculus.
- Green's identity is essentially a 3D version of the integration by parts that we used in the 1D case. Note that V is a volume and ∂V is the boundary of that volume.
- We use Green's identity to deal with the spatial derivative part $\overline{v}\nabla^2 u$.
- We use 1D integration by parts to deal with the time derivative part.

THE ADJOINT WAVE EQUATION

Result:

$$\int_{t_i}^{t_f}\int_V (\mathcal{L}u)\,\overline{v}\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t = \int_{t_i}^{t_f}\int_V u\,(\mathcal{L}v)^*\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t +$$

$$+\int_{t_i}^{t_f}\oint_{\partial V}\left(\overline{v}\frac{\partial u}{\partial n} - u\frac{\partial \overline{v}}{\partial n}\right)\mathrm{d}S\,\mathrm{d}t -$$

$$-\frac{1}{c^2}\left[\int_V\left(\overline{v}\frac{\partial u}{\partial t} - u\frac{\partial \overline{v}}{\partial t}\right)\mathrm{d}V\right]_{t_i}^{t_f}$$

- Not pretty! Important thing is that it's similar to what we had in the 1D case.
- The first term is $\langle \mathcal{L}u, v\rangle$ and the second term is $\langle u, \mathcal{L}v\rangle$. So, for the wave equation $\mathcal{L} = \mathcal{L}^*$.
- The last two terms are just a big ugly conjunct. The first one depends only on the spatial boundary conditions (∂V is the boundary of the volume V), while the second one depends only on the temporal boundary conditions.
- Given boundary conditions on u, the adjoint boundary conditions are the ones that make these last two terms go to zero. E.g., if $u(\mathbf{r}, t) = 0$ over the whole boundary, then we would need $v(\mathbf{r}, t) = 0$ over the whole boundary as well to cancel out the second-last integral.

GREEN'S PROBLEMS

Original problem:

$$\mathcal{L}u(\mathbf{r}, t) = f(\mathbf{r}, t); \quad \mathcal{B}[u] = \alpha$$

Green's problem:

$$\mathcal{L}G(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}')\delta(t - t'); \quad \mathcal{B}[G] = 0$$

- Set up the Green's function problem.

SOLUTION TO THE WAVE EQUATION

$$u(\mathbf{r},t) = \int_{t_i}^{t_f}\int_{V'} f(\mathbf{r}',t')G(\mathbf{r},t;\mathbf{r}',t')\,\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' +$$

$$-\int_{t_i}^{t_f}\oint_{\partial V'}\left(G\frac{\partial u}{\partial n'} - u\frac{\partial G}{\partial n'}\right)\mathrm{d}S'\,\mathrm{d}t' -$$

$$+\frac{1}{c^2}\left[\int_{V'}\left(G\frac{\partial u}{\partial t'} - u\frac{\partial G}{\partial t'}\right)\mathrm{d}V'\right]_{t_i}^{t_f}$$

- Write down the solution using the conjunct. (Can derive this using inner products, similar to what we did in 1D).
- The first integral gives the effect of the source term $f(\mathbf{r},t)$.
- The second integral gives the effect of the boundary conditions on u.
- The third integral gives the effect of the initial conditions on u.
- Note that $G(\mathbf{r},t;\mathbf{r}',t')$ obeys the boundary conditions with respect to $\mathbf{r},t$ and the *adjoint* boundary conditions with respect to $\mathbf{r}',t'$. Use this for a given problem to simplify the last two terms.

FINDING THE GREEN'S FUNCTION

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Initial conditions:

$$G(\mathbf{r},0;\mathbf{r}',t') = \left.\frac{\partial G(\mathbf{r},t;\mathbf{r}',t')}{\partial t}\right|_{t=0} = 0$$

Boundary conditions:

$$\lim_{|\mathbf{r}|\to\infty} G(\mathbf{r},t;\mathbf{r}',t') \to 0$$

- Now let's look at how we can actually solve the Green's function problem.
- We'll assume that we have an initial condition problem, and that we are interested in all of space (boundaries are at infinity).

FINDING THE GREEN'S FUNCTION

Translation invariance:

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r} - \mathbf{r}', t - t')$$

Let $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = c(t - t')$ so that

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

- Because the wave equation is translation invariant, we can simplify the Green's function to make the problem easier to solve.
- We define new variables $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = t - t'$.
- Note that ∇^2 is the laplacian with respect to $\mathbf{R}$.

FINDING THE GREEN'S FUNCTION

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

Spatial Fourier transform:

$$\left(k^2 - \frac{\partial^2}{\partial \tau^2}\right) \tilde{G}(\mathbf{k}, \tau) = \delta(\tau)$$

so

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

- Apply the spatial Fourier transform:

$$\tilde{G}(\mathbf{k}, \tau) = \iiint_{\mathbb{R}^3} G(\mathbf{R}, \tau) e^{-j\mathbf{k}\cdot\mathbf{R}} \, \mathrm{d}^3\mathbf{R}$$

- This reduces it to a 1D initial condition problem in time.
- Solve this using the fact that $\tilde{G}(\mathbf{k}, \tau)$ obeys the source-free equation except at $\tau = 0$.

FINDING THE GREEN'S FUNCTION

$$\tilde{G}(\mathbf{k},\tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

- Initial conditions give $A = B = 0$.
- Continuity at $t = t'$ gives $C = 0$.
- Discontinuity at $t = t'$ gives $D = -1/k$.

$$\tilde{G}(\mathbf{k},\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ -\dfrac{\sin(k\tau)}{k} & \text{for } \tau > 0 \end{cases}$$

- Apply initial conditions and continuity requirements to find the coefficients.
- To apply initial conditions: $\tilde{G}(\mathbf{k},\tau)$ and its derivative must be zero when $t = 0$ or $\tau = -ct'$. Remember that that we are only seeking a solution for $t > 0$, and so $t' > 0$.

FINDING THE GREEN'S FUNCTION

Inverse Fourier transform for $\tau > 0$:

$$G(\mathbf{R},\tau) = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \tilde{G}(\mathbf{k},\tau)e^{j\mathbf{k}\cdot\mathbf{R}}\,\mathrm{d}^3\mathbf{k}$$

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int_{\mathbb{R}^3} \frac{\sin(k\tau)}{k}e^{j\mathbf{k}\cdot\mathbf{R}}\,\mathrm{d}^3\mathbf{k}$$

Use spherical coordinates in $\mathbf{k}$: (k,θ,ϕ).

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\sin(k\tau)}{k}e^{jkR\cos(\theta)}k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$$

- To recover the Green's function from $\tilde{G}(\mathbf{k},\tau)$, apply an inverse Fourier transform. (Only really need to do it for $\tau > 0$: just remember that $G = 0$ for $\tau < 0$.)
- Use spherical coordinates in $\mathbf{k}$-space to evaluate the integral. Define $k = |\mathbf{k}|$. Define θ as the angle between $\mathbf{k}$ and $\mathbf{R}$ so that $\mathbf{k}\cdot\mathbf{R} = kR\cos(\theta)$. Define ϕ as the remaining angle required to complete the coordinate system.

Finding the Green's function

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\sin(k\tau)}{k}e^{jkR\cos(\theta)}k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\int_0^{\pi}\sin(\theta)e^{jkR\cos(\theta)}\,\mathrm{d}\theta\right]\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\frac{2\sin(kR)}{kR}\right]\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{2\pi^2 R}\int_0^{\infty}\sin(k\tau)\sin(kR)\,\mathrm{d}k$$

Finding the Green's function

$$G(\mathbf{R},\tau) = \frac{-1}{2\pi^2 R}\int_0^{\infty}\sin(k\tau)\sin(kR)\,\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2 R}\int_{-\infty}^{\infty}\sin(k\tau)\sin(kR)\,\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2 R}\int_{-\infty}^{\infty}\left(\frac{e^{jk\tau}-e^{-jk\tau}}{2j}\right)\left(\frac{e^{jkR}-e^{-jkR}}{2j}\right)\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{1}{16\pi^2 R}\int_{-\infty}^{\infty}\left(e^{jk(\tau+R)}-e^{-jk(\tau-R)}-e^{jk(\tau-R)}+e^{-jk(\tau+R)}\right)\mathrm{d}k$$

- Do the θ and ϕ integrals.

- Note that $\sin(k\tau)\sin(kR)$ is even with respect to k. So we can divide by 2 and take the integral over $-\infty < k < \infty$.
- Then, write out the complex exponentials.

FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R},\tau) = \frac{1}{16\pi^2 R}\int_{-\infty}^{\infty} \left(e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}\right) \mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{1}{16\pi^2 R} 2\pi\left[\delta(\tau+R) - \delta(\tau-R) - \delta(\tau-R) + \delta(\tau+R)\right]$$

But $\tau > 0$ and $R > 0$ so $\delta(\tau+R) = 0$ and we have

$$G(\mathbf{R},\tau) = \frac{-\delta(\tau-R)}{4\pi R}$$

- Use the identity

$$\int_{-\infty}^{\infty} e^{jk\xi}\,\mathrm{d}k = 2\pi\delta(\xi)$$

- Note that $\tau > 0$ (otherwise $G = 0$, from before) and $R = |\mathbf{r} - \mathbf{r}'| > 0$. So $\delta(\tau+R) = 0$.
- Reducing, we obtain the Green's function for $\tau > 0$.

GREEN'S FUNCTION FOR THE WAVE EQUATION

$$G(\mathbf{r},t;\mathbf{r}',t') = \begin{cases} -\frac{\delta(c(t-t')-|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

With zero boundary/initial conditions:

$$u(\mathbf{r},t) = \int_{t_i}^{t_f}\int_{\mathbb{R}^3} G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')\,\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t'$$

$$u(\mathbf{r},t) = \int_{\mathbb{R}^3} \frac{-f\left(\mathbf{r}, t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

- Plug back in expressions for τ and R: we obtain the full Green's function for the wave equation.
- Assuming the boundary/initial conditions are zero, the conjunct terms cancel.
- We are left with an integral formula for $u(\mathbf{r},t)$.
- Note that, because of the delta function, the integral over time disappears. We are left with the so-called "retarded" expression for $u(\mathbf{r},t)$.
- Essentially, when measuring a response at $(\mathbf{r},t)$, we only "see" the source as it appeared at $t - |\mathbf{r}-\mathbf{r}'|/c$. I.e., there is a speed of light delay in the propagation of information!

VECTOR POTENTIAL: INITIAL CONDITION PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\mathbf{J}(\mathbf{r},t)$$

For initial conditions, we have the "retarded potential":

$$\mathbf{A}(\mathbf{r},t) = \int_{\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}, t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \int_{\mathbb{R}^3} \tilde{\mathbf{J}}(\mathbf{r},\omega)\,\frac{e^{-j\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

VECTOR POTENTIAL: FINAL VALUE PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\mathbf{J}(\mathbf{r},t)$$

For final conditions, we have the "advanced potential":

$$\mathbf{A}(\mathbf{r},t) = \int_{\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}, t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \int_{\mathbb{R}^3} \tilde{\mathbf{J}}(\mathbf{r},\omega)\,\frac{e^{+j\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

- Concrete example is the vector potential, which obeys the wave equation with the current density $\mathbf{J}$ as its source. (In the Lorentz gauge.)
- Note, the time-dependent vector potential is the same as the static vector potential, but the source is just evaluated with a speed-of-light delay!
- Taking a Fourier transform, we get the expression from Harrington.

- For final conditions problems, we get something weird: we get an "advanced potential" where the field $\mathbf{A}$ depends on the source $\mathbf{J}$ at some time in the future.
- So it seems that causality is tied to our use of initial conditions (rather than final conditions) in time.
- This is also related to uniqueness breaking down for lossless time-harmonic fields. Harrington fixes this by defining "lossless" as the limit of low loss (not very satisfactory, since the lossless case is more fundamental). Alternatively, we can fix uniqueness by insisting that we have *initial* conditions in the time domain.
- For more, see the book chapter "Causation in classical mechanics" by Sheldon Smith.

GENERALIZED FUNCTIONS

TYPICAL DELTA FUNCTION DEFINITION

Typical "definition" of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

- Delta functions play a key role in Green's functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz's theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

- Often see definitions like this one.
- Often said to imply that $\delta(x - x_0) = \infty$ at $x = x_0$.
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

- Let's see if we can generalize the idea of a "function" so that it includes delta functions.
- Given a function $f(x)$, we can use it to define a linear operator (a functional, to be exact) on other functions $\phi(x)$.
- $f[\cdot]$ is a linear operator. It takes a function $\phi(x)$ and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

- If we ensure that $\phi(x)$ is very well-behaved, then every function $f(x)$ defines an operator in this way.

GENERALIZED FUNCTIONS

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

- It's possible to have an operator $f[\phi]$, but we can't find an $f(x)$ to implement it via an integral.
- Then $f(x)$ is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions $f[\phi]$.
- We still symbolically write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

  but this just suggestive notation. It is not actually an integral unless $f(x)$ is a "proper" function!

DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}x$$

- We can define a simple linear operator via the sifting property $\delta_{x_0}[\phi] = \phi(x_0)$.
- There is no actual function $\delta(x - x_0)$ which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}x = \phi(x_0)$$

so $\delta(x - x_0)$ is a generalized function and the above integral is purely symbolic.

DELTA FUNCTION DERIVATIVES

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}x$$

- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$ is just an operator that picks out the value of the nth derivative of $\phi(x)$ at the point x_0.

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}x = \phi(0)$$

- Often useful to show that some set of actual functions $f_\epsilon(x)$ “approach” the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

Delta function limits

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}t = \delta(x)$$

What does this mean for Green's functions?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\,\phi(x)\,\mathrm{d}x$$

- Example of a common, but unintuitive expression for the Delta function.
- Can show that it's true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

- Technically, the Green's function is a generalized function such that $\mathcal{L}G$ is the delta function (it has the sifting property).

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator!

- In practise, thinking of $\delta(x - x_0)$ as a function is usually fine.
- But if anything starts to seem fishy, it's good to remember that $\delta(x - x_0)$ is actually an operator $\delta_{x_0}[\phi]$, and not a function.