GREEN'S FUNCTIONS

A short introduction

Chris Deimert

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Department of Electrical and Computer Engineering, University of Calgary

OUTLINE

INTRODUCTION

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

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Green's function is the **impulse response**:

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WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

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$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \,\mathrm{d}\, x}$$

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

FAMILIAR GREEN'S FUNCTIONS

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Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}')\, \mathrm{d}^3\, \mathbf{r}'$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

GENERALIZED FUNCTIONS

TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

GENERALIZED FUNCTIONS

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int\limits_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x$$

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \overset{S}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}\,x$$

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}\,x = \phi(0)$$

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x,x') = \delta(x-x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x,x'))\,\phi(x)\,\mathrm{d}\,x$$

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

DIRECT SOLUTION

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

A SIMPLE EXAMPLE

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A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2\, G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x-x')$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'-\epsilon}\right] = 1$$

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') \right] \mathrm{d}\, x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\, \mathrm{d}\, x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\, G}{\mathrm{d}\, x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G}{\mathrm{d}x}$:

$$kA + kB = 1$$

At last, our Green's function is

$$G(x,x') = \begin{cases} \dfrac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \dfrac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

BOUNDARY CONDITIONS

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} + \omega_0^2 u(x) = f(x)$$

$$u(a) = \alpha$$
$$u(b) = \beta$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} + \omega_0^2 G(x, x') = \delta(x - x')$$

SIMPLE HARMONIC OSCILLATOR

$$G(x,x')\left[\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\,x^2}+\omega_0^2 u(x)\right]=G(x,x')f(x)$$

$$u(x)\left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2}+\omega_0^2 G(x,x')\right]=u(x)\delta(x-x')$$

$$\int_a^b \left[u(x)\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - G(x,x')\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\,x^2}\right]\mathrm{d}\,x$$

$$= \int_a^b \left[u(x)\delta(x-x') - G(x,x')f(x)\right]\mathrm{d}\,x$$

SIMPLE HARMONIC OSCILLATOR

$$u(x') = \int_a^b G(x,x')f(x)\,\mathrm{d}\,x + \\ + \left[u(x)\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x} - G(x,x')\frac{\mathrm{d}\,u(x)}{\mathrm{d}\,x}\right]_{x=a}^{b}$$

SPECTRAL METHODS

3D problems

PROPERTIES OF THE GREEN'S FUNCTION

ADVANCED TOPICS

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves.* Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics.* Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.