

GREEN'S FUNCTIONS

A short introduction

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OUTLINE

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- This is intended as a quick overview of Green's functions for electrical engineers.
- Green's functions are a huge subject: it's easy to get overwhelmed by calculation techniques.
- Focus here will be on intuition/understanding and awareness of some key techniques.
- Lots of further reading provided at the end.
- Tip: read a lot of different references. Different authors take totally different approaches and it's interesting to see them all.

INTRODUCTION

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

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- Fortunately, the basic idea of Green's functions is really simple.
- You've actually used them before!
- What's far more interesting is how to calculate/interpret them.

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x' .
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, dx \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, dx$$

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WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$u(x) = \int G(x, x') f(x') \, dx$$

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- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.

FAMILIAR GREEN'S FUNCTIONS

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t') h(t - t') dt'$$

E.g., for an RL-circuit:

$$G(t, t') = h(t - t') = u(t - t') e^{-\alpha(t - t')}$$

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FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \rho(\mathbf{r}') d^3 \mathbf{r}'$$

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- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t - t')$ from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find $h(t - t')$ using Fourier transform of the transfer function.

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2}$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$(\nabla^2 + k^2) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} J_z(\mathbf{r}') d^3\mathbf{r}'$$

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FAMILIAR GREEN'S FUNCTIONS

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

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- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

GENERALIZED FUNCTIONS

TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

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- Delta functions play a key role in Green’s functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz’s theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

- Often see definitions like this one.
- Often said to imply that $\delta(x - x_0) = \infty$ at $x = x_0$.
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

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GENERALIZED FUNCTIONS

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

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- Let's see if we can generalize the idea of a "function" so that it includes delta functions.
- Given a function $f(x)$, we can use it to define a linear operator (a functional, to be exact) on other functions $\phi(x)$.
- $f[\cdot]$ is a linear operator. It takes a function $\phi(x)$ and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

- If we ensure that $\phi(x)$ is very well-behaved, then every function $f(x)$ defines an operator in this way.

- It's possible to have an operator $f[\phi]$, but we can't find an $f(x)$ to implement it via an integral.
- Then $f(x)$ is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions $f[\phi]$.
- We still symbolically write

$$f[\phi] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

but this is just suggestive notation. It is not actually an integral unless $f(x)$ is a "proper" function!

DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) dx$$

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DELTA FUNCTION DERIVATIVES

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) \phi(x) dx$$

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- We can define a simple linear operator via the sifting property $\delta_{x_0}[\phi] = \phi(x_0)$.
- There is no actual function $\delta(x - x_0)$ which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) dx = \phi(x_0)$$

so $\delta(x - x_0)$ is a generalized function and the above integral is purely symbolic.

- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$ is just an operator that picks out the value of the n th derivative of $\phi(x)$ at the point x_0 .

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}[\phi] = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) \phi(x) dx = \phi(0)$$

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- Often useful to show that some set of actual functions $f_{\epsilon}(x)$ “approach” the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

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- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

DELTA FUNCTION LIMITS

A more interesting example:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} dt = \delta(x)$$

because

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{ixt} dt = \delta(x)$$

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WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)[\phi] = \phi(x') \stackrel{s}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x')) \phi(x) dx$$

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- Example of a common, but unintuitive expression for the Delta function.
- Can show that it's true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

- Technically, the Green's function is a generalized function such that $\mathcal{L}G$ is the delta function (it has the sifting property).
- In practise, we'll keep using the less-precise way; just remember that there is a more correct way.

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

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DIRECT SOLUTION

- In practise, thinking of $\delta(x - x_0)$ as a function is usually fine. (We'll even do that for the rest of this presentation.)
- But if anything starts to seem fishy, it's good to remember that $\delta(x - x_0)$ is actually an operator, and not a function.

- Back to Green's functions!

A SIMPLE EXAMPLE

Original problem:

$$\frac{d^2 u(x)}{dx^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

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- Let's look at a simple example now.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem.
- But the Green's function problem looks pretty hard. The point of this example is to demonstrate that we can actually solve it.

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

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- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B ?

A SIMPLE EXAMPLE

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \rightarrow 0} [G(x' + \epsilon, x') - G(x' - \epsilon, x')] = 0$$

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- How continuous do we expect our Green's function to be?
- If $G(x, x')$ is discontinuous (like a step function), then dG/dx will behave like a delta function and d^2G/dx^2 will behave like a delta function derivative. No good!
- So we expect $G(x, x')$ to be continuous.
- That gives us one condition we can use to find A and B . (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x' - \epsilon}^{x' + \epsilon} \left[\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') \right] dx = \int_{x' - \epsilon}^{x' + \epsilon} \delta(x - x') dx$$

Discontinuity condition:

$$\lim_{\epsilon \rightarrow 0} \left[\left. \frac{dG}{dx} \right|_{x=x'+\epsilon} - \left. \frac{dG}{dx} \right|_{x=x'-\epsilon} \right] = 1$$

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- But what if the derivative dG/dx is discontinuous?
- Then d^2G/dx^2 is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x' .
- In the limit of $\epsilon \rightarrow 0$, the second integral vanishes because $G(x, x')$ is continuous.
- But, we expect dG/dx to be discontinuous.
- Using fundamental theorem of calculus, we get at a *discontinuity condition for the derivative*. (Key idea for the direct solution method!)

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G :

$$A = B$$

Discontinuity of $\frac{dG}{dx}$:

$$kA + kB = 1$$

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A SIMPLE EXAMPLE

At last, our Green's function is

$$G(x, x') = \begin{cases} \frac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \frac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$G(x, x') = \frac{e^{k|x-x'|}}{2k}$$

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- Applying our two conditions, we can solve for A and B .

A SIMPLE EXAMPLE

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} dx'$$

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GENERAL APPROACH

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

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- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(t)$.

- Listed are the key things to note from that example.
- This approach works quite well for solving 1D Green's function problems.

BOUNDARY CONDITIONS

ADJOINT OPERATORS

$$\text{Original: } \mathcal{L}[u(x)] = f(x); \quad \mathcal{B}[u(x)] = 0$$

$$\text{Adjoint: } \mathcal{L}^*[v(x)] = f(x); \quad \mathcal{B}^*[v(x)] = 0$$

Defining property:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle \quad \text{where } \langle u, v \rangle = \int_a^b u(x)v^*(x) dx$$

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- We didn't worry about boundary conditions in the last example.
- As it turns out, Green's functions allow us to deal with boundary conditions in an elegant way.
- Unfortunately, to derive it, we either have to do a lot of hand-waving or a lot of math. We're going to do a lot of math.
- See Dudley's *Mathematical foundations for electromagnetic theory* for a more thorough discussion.

- To really make sense of boundary conditions, we need the concept of the adjoint problem.
- Suppose we have an original problem defined by operator \mathcal{L} and boundary conditions \mathcal{B} .
- Then, \mathcal{L}^* is the adjoint operator and \mathcal{B}^* are the adjoint boundary conditions if $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$ for all u, v .
- Here $\langle u, v \rangle$ is the inner product as defined on the slide. ($v^*(x)$ is the complex conjugate of $v(x)$.)
- (Side note: technically, we should call \mathcal{L}^* the *formal* adjoint. A boundary condition like $\mathcal{B}[u] = 0$ restricts u to a subspace of its original Hilbert space. Thus, if $\mathcal{B} \neq \mathcal{B}^*$, the domains of \mathcal{L} and \mathcal{L}^* are not the same, and they are not true adjoints.)

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}[u(x)] = \left[\frac{d^2}{dx^2} + k^2 \right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want \mathcal{L}^* and \mathcal{B}^* so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$$

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ADJOINT OPERATORS: EXAMPLE

$$\begin{aligned} \langle \mathcal{L}u, v \rangle &= \int_a^b [u''(x) + k^2 u(x)] v^*(x) dx \\ &= \int_a^b [v''^*(x) + k^2 v^*(x)] u(x) dx + \\ &\quad + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b \end{aligned}$$

By inspection:

$$\mathcal{L}^*v(x) = \left[\frac{d^2}{dx^2} + (k^2)^* \right] v(x)$$

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- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that $u(a) = u(b) = 0$.

- Integrate by parts twice.
- The remaining integral term looks like

$$\int_a^b u(x) [\mathcal{L}^*v(x)]^* dx$$

so we can read off \mathcal{L}^* .

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

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ADJOINT OPERATORS: EXAMPLE

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b$$

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v :

$$v(b) = v'(b) = 0$$

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- We almost have the required $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$, but we need the part on the right (called the conjunct) to be zero.
- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case, $\mathcal{B} = \mathcal{B}^*$.

- What if we use the same operator \mathcal{L} , but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem! $\mathcal{B} \neq \mathcal{B}^*$.

ADJOINT OPERATORS

In general:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + J(u, v)$$

where $J(u, v)$ depends on u, v, u', v', \dots at the boundaries.

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \iff J(u, v) = 0$$

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- In that example, we saw that we always had $\langle \mathcal{L}u, v \rangle$ equal to $\langle u, \mathcal{L}^*v \rangle$ plus a “leftover” term called the conjunct.
- This is true in general: we always have a leftover term $J(u, v)$ which depends only on the boundary values of u and v .
- We say $\mathcal{B}[u] = 0$ and $\mathcal{B}^*[v] = 0$ are adjoint boundary conditions if and only if they make $J(u, v)$.

ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \quad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x, x')] = \delta(x - x'); \quad \mathcal{B}[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x, x')] = \delta(x - x'); \quad \mathcal{B}^*[H(x, x')] = 0$$

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- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \rightarrow \delta(x - x')$ and $\alpha \rightarrow 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning: a lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Quite often, the “Green's function” is really the adjoint Green's function.

EXAMPLE WITH BOUNDARY CONDITIONS

Original problem:

$$\mathcal{L}[u] = \frac{d^2 u(x)}{dx^2} + k^2 u(x) = f(x); \quad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Adjoint Green's problem:

$$\frac{d^2 H(x, x')}{dx^2} + k^2 H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} A \cos(k(x - x')) + B \sin(k(x - x')) & \text{for } x < x' \\ C \cos(k(x - x')) + D \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

$$\text{Final conditions} \implies C = D = 0$$

$$\text{Continuity of function} \implies A = 0$$

$$\text{Discontinuity of derivative} \implies B = \frac{-1}{k}$$

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- We'll show how $H(x, x')$ is useful with an example.
- Here we have a driven simple harmonic oscillator with initial conditions. (We'll take k real for simplicity.)
- From before, we know that the adjoint problem will be the same differential equation, but with final conditions instead of initial conditions.

- We can solve for $H(x, x')$ using a similar approach to before.
- Except at $x = x'$, we write $H(x, x')$ as a solution to the source-free equation.
- Then we find the coefficients using the boundary conditions and continuity/discontinuity requirements.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1} \sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

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EXAMPLE WITH BOUNDARY CONDITIONS

$$\begin{aligned} \langle \mathcal{L}u(x), H(x, x') \rangle &= \langle u(x), \mathcal{L}^*H(x, x') \rangle + J(u, H) \\ \langle f(x), H(x, x') \rangle &= \langle u(x), \delta(x - x') \rangle + J(u, H) \\ \int_a^b f(x)H^*(x, x') \, dx &= \int_a^b u(x)\delta(x - x') \, dx + J(u, H) \end{aligned}$$

$$u(x') = \int_a^b f(x)H^*(x, x') \, dx - J(u, H)$$

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- So we have our adjoint Green's function.
- Now we just need to figure out how to construct $u(x)$ from it.

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- We arrive at a fairly general formula which looks close to what we expect a Green's function formula to look like, but with an extra conjunct term.
- We'll gain insight into the $J(u, H)$ term by expanding it for this example.

EXAMPLE WITH BOUNDARY CONDITIONS

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x) H^*(x, x') dx - \left[\frac{d u(x)}{d x} H^*(x, x') - u(x) \frac{d H^*(x, x')}{d x} \right]_a^b$$

Recall:

$$\begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') dx + \beta H^*(a, x') - \alpha \frac{d H^*(a, x')}{d x}$$

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- Now let's expand $J(u, v)$ for our particular example. (We can basically copy it from a previous part of the derivation.)
- We can simplify the conjunct by remembering our boundary conditions.

- We're very close to our final goal.
- The last thing will be to get rid of $H(x, x')$ and replace it with $G(x, x')$.

ADJOINT GREEN'S FUNCTIONS

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^*H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$H^*(x', x'') = G(x'', x')$$

$$G(x, x') = H^*(x', x)$$

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EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1} \sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1} \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

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- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.

- Since we already know the adjoint Green's function $H(x, x')$, we can use find the Green's function via $G(x, x') = H^*(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') dx + \beta H^*(a, x') - \alpha \left. \frac{d H^*(x, x')}{dx} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) dx + \beta G(x', a) - \alpha \left. \frac{d G(x', x)}{dx} \right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{d G(x, x')}{dx'} \right|_{x'=a}$$

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FINAL SOLUTION

$$u(x) = \int_a^b f(x') G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{d G(x, x')}{dx'} \right|_{x'=a}$$

where

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1} \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

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- With our new knowledge that $G(x, x') = H^*(x', x)$, we can rewrite the solution $u(x)$ in terms of $G(x, x')$.

- Finally, we arrive at our solution.
- First, note that the final answer does *not* depend on $H(x, x')$. We didn't actually need to ever calculate $H(x, x')$ from the adjoint Green's equation, we could have just found $G(x, x')$ from the (non-adjoint) Green's equation.
- However, it would have been very difficult to derive this expression without using $H(x, x')$ (I couldn't see an easy way). Because of this, a lot of authors stop at the expression for $u(x')$ in terms of $H^*(x, x')$, and they just call $H^*(x, x')$ the "Green's function."
- By doing the extra work, though, we gain a very nice interpretation for the boundary conditions term.

INTERPRETATION OF BOUNDARY CONDITIONS

$$u(x) = \int_a^b f(x')G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{dG(x, x')}{dx'} \right|_{x'=a}$$

$$u(x) = \int_a^b \left[f(x') + \beta \delta(x' - a) - \alpha \delta'(x' - a) \right] G(x, x') dx'$$

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- We note that our expression for $u(x)$ looks like it did before (integral over $f(x')G(x, x')$), but now there are extra terms which depend on the boundary conditions.
- We come to a key idea: boundary conditions have the same effect on $u(x)$ as adding little impulse sources at the boundary. The Green's function can deal with both sources $f(x)$ and non-zero boundary conditions.
- Be careful, though: $G(x, x')$ still depends on the *type* of boundary condition. E.g., we use the same $G(x, x')$ for all initial value problems ($u(a), u'(a)$ specified), but we'll need a different $G(x, x')$ for boundary value problems ($u(a), u(b)$ specified).

SUMMARY

$$\begin{aligned} \mathcal{L}[u(x)] &= f(x); & \mathcal{B}[u(x)] &= \alpha \\ \mathcal{L}[G(x, x')] &= \delta(x - x'); & \mathcal{B}[G(x, x')] &= 0 \\ \mathcal{L}^*[H(x, x')] &= \delta(x - x'); & \mathcal{B}^*[H(x, x')] &= 0 \end{aligned}$$

- 1 Find $G(x, x')$.
- 2 Find $u(x')$ in terms of $H(x, x')$:

$$u(x') = \int_a^b f(x)H^*(x, x') dx - J(u(x), H(x, x'))$$

- 3 Express $u(x)$ in terms of $G(x, x') = H^*(x', x)$.

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- That was a long process, so let's summarize what we did.
- First, we wrote out the original equation, the Green's function equation, and the adjoint Green's function equation. (Take note of the boundary conditions in particular.)
- Next, we solve the Green's function equation for $G(x, x')$.
- Next, we found that it's much easier to express $u(x)$ in terms of $H(x, x')$, because we can use inner products. The only tricky part is finding the adjoint (usually, just use integration by parts). See Dudley for a general formula for Sturm-Liouville problems.
- Finally, we replace $H(x, x')$ with $G^*(x', x)$ (which we solved for previously), and we have our final expression for $u(x)$.

SPECTRAL METHODS

3D PROBLEMS

PROPERTIES OF THE GREEN'S FUNCTION

ADVANCED TOPICS

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*.

Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*.

Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

ADVANCED RESOURCES

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshbach, *Methods of theoretical physics*.

Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.