GREEN'S FUNCTIONS

A SHORT INTRODUCTION

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- This is intended as a short introduction to Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them. Unfortunately means that this presentation will be missing a lot.
- We'll only cover a few topics, but we'll try to cover them fairly rigorously. Should give a solid foundation for reading books on the topic.
- Suggested further reading provided at the end.

OUTLINE

1. Basic idea of Green's functions.
2. Simplest method for solving the Green's function equation.
3. How to use the Green's function to solve a problem with boundary conditions. (Biggest section!)
4. Useful properties of Green's functions for special types of problems.
5. Relationship between Green's functions and eigenvalues/eigenfunctions.
6. Summary and suggested further reading.

INTRODUCTION

- The basic idea of Green's functions is really simple. You've actually used them before!

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

(Some conditions apply.)

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.
- Beware the fine print! This formula actually only works under certain assumptions about the boundary conditions.
- We'll deal with the more general approach later. For now, this gets the key idea across.

Impulse response of an LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t') h(t - t') \, \mathrm{d}\, t'$$

- In electrical engineering, we've seen Green's functions before.

- Impulse response $h(t - t')$ from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find $h(t - t')$ using Fourier transform of the transfer function.

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3 \, \mathbf{r}'$$

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|}$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

- With Green's function theory, we learn how to derive the above expressions. (Though we won't have time to do the 3D ones here.)
- More importantly, Green's function theory allows us to deal with different boundary conditions. The solutions to the Poisson and Helmholtz equations above assume free space (boundaries at infinity). Green's functions would allow us to, e.g., find the response to a current source inside a specific waveguide.

FINDING THE GREEN’S FUNCTION

- In this section, we'll look at one of the simplest methods for actually solving the Green's function problem.

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

- Let's start off by looking at a simple example.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem for any $f(x)$.
- But the Green's function problem looks hard! The point of this example is to demonstrate that we can actually solve it.

For $x \neq x'$

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x, x')$ is discontinuous (like a step function), then $\mathrm{d}\,G/\,\mathrm{d}\,x$ will behave like a delta function and $\mathrm{d}^2\,G/\,\mathrm{d}\,x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x, x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x') \, \mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

- But what if the derivative $\mathrm{d}\,G/\,\mathrm{d}\,x$ is discontinuous?
- Then $\mathrm{d}^2\,G/\,\mathrm{d}\,x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x, x')$ is continuous.
- The first integral is an integral of a derivative, so we can use the fundamental theorem of calculus. The result is a *discontinuity condition for the derivative.*

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x}$:

$$kA + kB = 1$$

- Applying our two conditions, we can solve for A and B. We find

$$A = B = \frac{1}{2k}$$

Solving, our Green's function is

$$G(x,x') = \frac{1}{2k}\begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \,\mathrm{d}\, x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(x)$.
- Caution: remember the fine print from before. This solution only works with certain assumptions about boundary conditions.

Second-order problems:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- Derivative of $G(x, x')$ is discontinuous at $x = x'$.

- This approach works quite well for solving 1D Green's function problems.
- For problems of other orders, will have a different combination of continuity/discontinuity requirements at $x = x'$. E.g., $G(x, x')$ will be discontinuous for a first order problem.

CONSTRUCTING SOLUTIONS

- In the introduction, we showed non-rigorously how to construct a solution from the Green's function. To keep things simpler, we ignored boundary conditions.
- Here, we'll look at how to properly construct a solution from the Green's function when boundary conditions are involved.
- Our approach is quite challenging compared to a lot of books on the subject. The advantage is that we'll deal with a lot of subtleties that can otherwise lead to confusion.
- For approaches similar to the one in this section, see Dudley, Morse and Feshbach, or Gerlach.

ADJOINT OPERATORS

$$\begin{aligned} \text{Original:} \qquad & \mathcal{L}[u(x)] = f(x); \qquad & \mathcal{B}[u(x)] = 0 \\ \text{Adjoint:} \qquad & \mathcal{L}^*[v(x)] = f(x); \qquad & \mathcal{B}^*[v(x)] = 0 \end{aligned}$$

Defining property:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle \qquad \text{where } \langle u, v \rangle = \int_a^b u(x) v^*(x) \, \mathrm{d}\, x$$

- Underpinning our approach is the concept of an adjoint problem.
- Suppose we have an original problem defined by operator $\mathcal{L}$ and boundary conditions $\mathcal{B}$.
- Then, $\mathcal{L}^*$ is the adjoint operator and $\mathcal{B}^*$ are the adjoint boundary conditions if $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$ for all u, v.
- Here $\langle u, v \rangle$ is the inner product as defined on the slide. ($v^*(x)$ is the complex conjugate of $v(x)$.)

Linear algebra notes:

- The boundary conditions are important because they specify the domains of $\mathcal{L}$ and $\mathcal{L}^*$. (I.e., $\mathcal{L}$ operates on the Hilbert space of functions $u(x)$ which satisfy $\mathcal{B}[u] = 0$.)
- So if $\mathcal{B} \neq \mathcal{B}^*$, then $\mathcal{L}$ and $\mathcal{L}^*$ are operators on different Hilbert spaces.
- If both $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$, we say that $\mathcal{L}$ is self-adjoint.
- If $\mathcal{L} = \mathcal{L}^*$ but $\mathcal{B} \neq \mathcal{B}^*$, we say that $\mathcal{L}$ is *formally* self-adjoint.

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that $u(a) = u(b) = 0$.

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}\,x +$$
$$+ \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

- To find the adjoint, let's expand $\langle \mathcal{L}u, v \rangle$.
- Use integration by parts twice.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}\,x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

If we take

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*$$

then we have

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x)\right]_a^b$$

- The remaining integral term looks like

$$\int_a^b u(x)\,[\mathcal{L}^* v(x)]^*\,\mathrm{d}\,x$$

- If we define $\mathcal{L}^*$ this way, we get closer to our goal. We just need to make the last part zero.

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x)\right]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

- We almost have the required $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$, but we need the part on the right to be zero.

- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the last part zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case, $\mathcal{B} = \mathcal{B}^*$.

ADJOINT OPERATORS: EXAMPLE

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x)\right]_a^b$$
$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

- What if we use the same operator $\mathcal{L}$, but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem! $\mathcal{B} \neq \mathcal{B}^*$.

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v)\Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \iff J(u, v)\Big|_a^b = 0$$

- In the last example, we saw that we had $\langle \mathcal{L}u, v\rangle$ equal to $\langle u, \mathcal{L}^* v\rangle$ plus a leftover term which depended on the boundaries.

- This is true more generally: if we don't specify the boundary conditions of u and v, then we can still *almost* get the adjoint operator equation. We just have a leftover term $J(u, v)|_a^b$, which depends only on the boundary conditions. This term is called the conjunct.

- We define adjoint boundary conditions as those boundary conditions which make the conjunct equal to zero.

ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x,x')] = \delta(x-x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha \to 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning: a lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Quite often, the "Green's function" is really the adjoint Green's function.

EXAMPLE WITH BOUNDARY CONDITIONS

Original problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) u(x) = f(x); \quad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Green's problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) G(x, x') = \delta(x - x'); \quad \begin{bmatrix} G(a, x') \\ G'(a, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adjoint Green's problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- We'll show how $H(x, x')$ is useful with an example.
- Here we have a driven simple harmonic oscillator with initial conditions. (We'll take k real for simplicity.)
- From before, we know that the adjoint problem will be the same differential equation, but with final conditions instead of initial conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x,x') = \begin{cases} A\cos(k(x-x')) + B\sin(k(x-x')) & \text{for } x < x' \\ C\cos(k(x-x')) + D\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

$$\begin{aligned} \text{Final conditions} &\implies C = D = 0 \\ \text{Continuity of function} &\implies A = 0 \\ \text{Discontinuity of derivative} &\implies B = \frac{-1}{k} \end{aligned}$$

- We can solve for $H(x, x')$ using a similar approach to before.
- Except at $x = x'$, we write $H(x, x')$ as a solution to the source-free equation.
- Then we find the coefficients using the boundary conditions and continuity/discontinuity requirements.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

- So we have our adjoint Green's function.
- Now we just need to figure out how to construct $u(x)$ from it.

EXAMPLE WITH BOUNDARY CONDITIONS

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J(u,H)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J(u,H)\Big|_a^b$$

$$\int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}\,x + J(u,H)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x - J(u,H)\Big|_a^b}$$

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- We arrive at a fairly general formula which looks close to what we expect a Green's function formula to look like, but with an extra conjunct term.
- We'll gain insight into the $J(u, H)$ term by expanding it for this example.

EXAMPLE WITH BOUNDARY CONDITIONS

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x - \left[\frac{\mathrm{d}\, u(x)}{\mathrm{d}\, x} H^*(x, x') - u(x) \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right]_a^b$$

Recall:

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad \mathcal{B}^*[H] = \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Now let's expand $J(u, v)$ for our particular example. (We can basically copy it from a previous part of the derivation.)
- We can simplify the conjunct by remembering our boundary conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x)H^*(x, x')\,\mathrm{d}\,x + \beta H^*(a, x') - \alpha \frac{\mathrm{d}\,H^*(a, x')}{\mathrm{d}\,x}$$

- We're very close to our final goal.
- The last thing will be to get rid of $H(x, x')$ and replace it with $G(x, x')$.

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$\int_a^b \delta(x - x') H^*(x, x'') \, \mathrm{d}\, x = \int_a^b G(x, x') \delta(x - x'') \, \mathrm{d}\, x$$

$$H^*(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = H^*(x', x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.
- This is somewhat surprising. For one thing, it means that if $G(x, x')$ obeys the boundary conditions with respect to x, then it automatically obeys the *adjoint* boundary conditions with respect to x'. This is a useful property.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x - x')) & \text{for } x > x' \end{cases}$$

- Since we already know the adjoint Green's function $H(x, x')$, we can use find the Green's function via $G(x, x') = H^*(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \,\mathrm{d}\, x + \beta H^*(a, x') - \alpha \left.\frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x}\right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) \,\mathrm{d}\, x + \beta G(x', a) - \alpha \left.\frac{\mathrm{d}\, G(x', x)}{\mathrm{d}\, x}\right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') \,\mathrm{d}\, x' + \beta G(x, a) - \alpha \left.\frac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x'}\right|_{x'=a}$$

- With our new knowledge that $G(x, x') = H^*(x', x)$, we can rewrite the solution $u(x)$ in terms of $G(x, x')$.

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}}$$

where

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

- Finally, we arrive at our solution.
- First, note that the final answer does *not* depend on $H(x, x')$. We didn't actually need to ever calculate $H(x, x')$ from the adjoint Green's equation, we could have just found $G(x, x')$ from the (non-adjoint) Green's equation.
- However, it would have been very difficult to derive this expression without using $H(x, x')$ (I couldn't see an easy way). Because of this, a lot of authors stop at the expression for $u(x')$ in terms of $H^*(x, x')$, and they just call $H^*(x, x')$ the "Green's function."
- By doing the extra work, though, we gain a very nice interpretation for the boundary conditions term.

INTERPRETATION OF BOUNDARY CONDITIONS

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha\,\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\bigg|_{x'=a}$$

$$u(x) = \int_a^b \Big[f(x') + \beta\delta(x'-a) - \alpha\delta'(x'-a)\Big]G(x,x')\,\mathrm{d}\,x'$$

- We note that our expression for $u(x)$ looks like it did before (integral over $f(x')G(x,x')$), but now there are extra terms which depend on the boundary conditions.
- We come to a key idea: boundary conditions have the same effect on $u(x)$ as adding little impulse sources at the boundary. The Green's function can deal with both sources $f(x)$ and non-zero boundary conditions.
- Be careful, though: $G(x,x')$ still depends on the *type* of boundary condition. E.g., we use the same $G(x,x')$ for all initial value problems ($u(a), u'(a)$ specified), but we'll need a different $G(x,x')$ for boundary value problems ($u(a), u(b)$ specified).

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$
$$\mathcal{L}^*[H(x,x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

1. Find $G(x,x')$.
2. Find $u(x')$ in terms of $H(x,x')$:

$$u(x') = \int_a^b f(x) H^*(x,x') \, \mathrm{d}\, x - J(u(x), H(x,x'))\Big|_a^b$$

3. Express $u(x)$ in terms of $G(x,x') = H^*(x',x)$.

- That was a long process, so let's summarize what we did.
- First, we wrote out the original equation, the Green's function equation, and the adjoint Green's function equation. (Take note of the boundary conditions in particular.)
- Next, we solve the Green's function equation for $G(x, x')$.
- Next, we found that it's much easier to express $u(x)$ in terms of $H(x, x')$, because we can use inner products. The only tricky part is finding the conjunct. (Usually, just requires integration by parts. See Dudley for a general formula for Sturm-Liouville problems.)
- Finally, we replace $H(x, x')$ with $G^*(x', x)$ (which we solved for previously), and we have our final expression for $u(x)$.

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^{b}}$$

Simplify with adjoint boundary conditions!

- We can streamline this process a little bit by removing the intermediate steps involving $H(x, x')$.
- Then, we can simply write down our final solution in terms of the Green's function $G(x, x')$.
- However, there are still traces of the adjoint problem.
- First, we cannot find the conjunct $J(u, v)$ without considering the adjoint problem.
- Second, we have to simplify the expression $J(u, G)|_a^b$ using the adjoint boundary conditions. In particular, use the fact that $G(x, x')$ obeys the adjoint boundary conditions with respect to x'.

PROPERTIES OF GREEN'S FUNCTIONS

- The Green’s function gives us a lot of information about the system we’re dealing with.
- Here we’ll look at a few of the properties Green’s functions can have and what those tell us about our system.

RECIPROCITY

If $\mathcal{L}$ is self-adjoint

$$\mathcal{L} = \mathcal{L}^* \qquad \text{and} \qquad \mathcal{B} = \mathcal{B}^*$$

then

$$G(x, x') = G^*(x', x)$$

- For self-adjoint problems, $G = H$.
- Using our relationship between G and H, we quickly see that the Green's function is complex symmetric (Hermitian) for self-adjoint problems.
- Roughly, putting a source at x and measuring at x' is the same as putting a source at x' and measuring at x. This is often called reciprocity (though it's not quite the same as reciprocity from time-harmonic E&M, because we're using a true inner product rather than the so-called "reaction inner product").
- Note that we require *both* $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$. So this would likely not hold for initial condition problems.

$\mathcal{L}$ is invariant if

$$\mathcal{L}[u(x-\xi)] = \mathcal{L}[u(x)]\Big|_{x=x-\xi}$$

For example,

$$\mathcal{L} = a\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + b\frac{\mathrm{d}}{\mathrm{d}\,x} + c$$

is invariant only if a, b, and c are constants.

- An operator is invariant if shifting the input just results in a shifted output.
- Probably familiar from signal processing: linear *time-invariant* systems. Delaying the input signal leads to the same output signal, just delayed by the same amount.
- Invariance is really important in modern physics.
- Maxwell's equations in free space are invariant with respect to x, y, z, ϕ, θ. That is, at a fundamental level, the laws of electromagnetism do not change if we move to a different location or look in a different direction.

If $\mathcal{L}$ is invariant in x, then

$$\mathcal{L}[G(x, x')] = \delta(x - x')$$
$$\mathcal{L}[G(x - \xi, x' - \xi)] = \delta(x - x')$$

$$\implies G(x, x') = G(x - \xi, x' - \xi)$$

$$\boxed{G(x, x') = G(x - x')}$$

- For invariant problems, we can see that shifting both x and x' by the same amount ξ does not affect the Green's function.
- Taking $\xi = x'$, we see that $G(x, x')$ is actually only a function of the difference $(x - x')$.
- That is, the response only depends on the *relative* locations of the source and measurement. This fits nicely with our intuitive understanding of invariance.

Convolution:

$$u(x) = \int_a^b G(x - x')f(x')\,\mathrm{d}\,x' = G(x) * f(x)$$

Frequency domain:

$$\tilde{u}(k) = \tilde{G}(k)\tilde{f}(k)$$

- For invariant systems (with boundary conditions zero), the solution $u(x)$ is just given as a convolution of the source function $f(x)$ with the impulse response $G(x)$.
- Taking Fourier transforms, the convolution turns into multiplication.
- Looks familiar from signal processing!

SPECTRAL METHODS

- In this section, we'll look at the strong relationship between Green's functions and spectral theory.
- Essentially, eigenfunction expansion allows us to calculate the Green's function when direct methods don't work.
- A basic background in spectral theory can be found in most books covering Green's functions.
- Unfortunately, these are rarely rigorous when dealing with continuous sets of eigenvalues. For fully rigorous spectral theory (not for the faint of heart!), see Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications*.

EIGENFUNCTION EXPANSION

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

- First we'll review a bit of eigenfunction theory, but we'll quickly see how it relates to Green's functions.
- Set up a similar problem as before, but we've added a complex parameter λ for later convenience.
- Also, for this section we'll insist that $\mathcal{L}$ be fully self-adjoint so that we can take full advantage of spectral theory.
- A brief discussion of the non-self-adjoint case can be found in Morse and Feshbach under "Non-Hermitian operators: biorthogonal functions".

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

With $\mathcal{L}$ self-adjoint, $\lambda_n \in \mathbb{R}$ and

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

- Because $\mathcal{L}$ is self-adjoint, we know that its eigenvalues λ_n are real.
- We also know that it has a complete orthonormal set of eigenfunctions ϕ_n.
- That is, we can expand any function (in this case $u(x)$ and $f(x)$) in terms of $\phi_n(x)$. (Generalized Fourier series.)

EIGENFUNCTION EXPANSION

$$(\mathcal{L} - \lambda)u(x) = f(x)$$

$$(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$(\lambda_n - \lambda) \langle u, \phi_n \rangle = \langle f, \phi_n \rangle$$

- Going back to our original equation, let's expand $u(x)$ and $f(x)$ in terms of eigenfunctions of $\mathcal{L}$.
- Using the fact that $\mathcal{L}$ is linear and $\mathcal{L}\phi_n = \lambda_n \phi_n$, we can get rid of $\mathcal{L}$ (third line).
- Finally, since the $\phi_n(x)$ are linearly independent, each term in the sums on the RHS and LHS must be equal. So we get an expression for the generalized Fourier coefficients $\langle u, \phi_n \rangle$.

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

- Plugging in our new expression for the Fourier coefficients, we obtain a formula for $u(x)$ in terms of the eigenfunctions and eigenvalues of $\mathcal{L}$.

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}\, x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{f(x)\phi_n^*(x')}{\lambda_n - \lambda} \phi_n(x) \right) \mathrm{d}\, x'$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}\, x'$$

- Usually, the inner product is defined by an integral.
- If we write this out and do some manipulation, we get something that looks a lot like the Green's function expression.

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}\,x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

- It turns out that we actually can read off this weird sum as the Green's function.
- So, if we know the eigenvalues and eigenfunctions of $\mathcal{L}$, we can immediately construct the Green's function as an infinite series.
- Note also that $G(x, x') = G^*(x, x')$, as we expect because this is a self-adjoint problem.

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

- It also goes the other way. If we know the Green's function of $(\mathcal{L} - \lambda)$ for any complex λ, then the eigenvalues of $\mathcal{L}$ are just the poles of the Green's function with respect to lambda.
- Eigenfunctions are a little trickier to read off, but it's possible to find them from the Green's function using residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

- Using our Green's function equation, we can also derive an expression for the delta function as a sum of eigenfunctions.
- This expression is useful when solving three-dimensional problems with separation of variables.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} - \lambda\right)}_{\mathcal{L} - \lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

- Let's do a simple example to illustrate the idea.
- For $\mathcal{L} = \mathrm{d}^2 / \mathrm{d}\, x^2$ we know that the eigenfunctions are sines and cosines. The boundary conditions restrict us to just sines with $\lambda_n = \pi n/a$.
- $\sqrt{2/a}$ ensures that the eigenfunctions are normalized.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

- Using the formula we derived earlier, we can very quickly write out the Green's function as an infinite series.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x, x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda}x'\right)\sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x > x' \end{cases}$$

- We could have also solved this problem directly (Assume $G(x, x')$ behaves like the source-free solution except at $x = x'$. Apply the boundary conditions, continuity and discontinuity requirements to find the coefficient.) The result is shown.

- The direct solution is a little uglier, but it's much easier to evaluate numerically because it doesn't involve an infinite series. For that reason, direct solution is usually more desireable if it actually works. However, series solutions tend to be needed for solving multi-dimensional problems.

- Note: there's a trick for evaluating infinite series using residue calculus, and (I think) you could use this to derive the second expression from the first. You can also use residue integration to derive the first from the second.

CONCLUSION

TAKEAWAYS

Introduction:

- The Green's function is the impulse response
- Use it to construct solutions. Ignoring boundary conditions:

$$u(x) = \int f(x')G(x, x')\,\mathrm{d}\,x'$$

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- $\dfrac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x}$ is discontinuous at $x = x'$.

Boundary conditions:

- Non-zero boundary conditions act like impulse sources.
- Best way to construct $u(x)$ from $G(x, x')$ is to use the adjoint problem.

TAKEAWAYS

Properties of Green's functions:

- Self-adjoint problems: $G(x, x') = G^*(x', x)$ (reciprocity).
- Invariant problems: $G(x, x') = G(x - x')$.

TAKEAWAYS

Spectral methods:

- Green's function can be found from eigenfunctions/eigenvalues:

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

- Eigenfunctions/eigenvalues can be found from Green's function.
- Delta function can be written in terms of eigenfunctions:

$$\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')$$

FURTHER TOPICS

- Sturm-Liouville problems
- Complex contour integration
- 3D problems: separation of variables
- Dyadic Green's functions
- Generalized functions (delta function)

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Fully rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

ADVANCED RESOURCES

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.