# GREEN'S FUNCTIONS

A short introduction

---

Chris Deimert

November 2, 2015

Department of Electrical and Computer Engineering, University of Calgary

- This is intended as a quick overview of Green’s functions for electrical engineers.
- Green’s functions are a huge subject: it’s easy to get overwhelmed by calculation techniques.
- Focus here will be on intuition/understanding and awareness of some key techniques.
- Lots of further reading provided at the end.

# OUTLINE

# INTRODUCTION

- Fortunately, the basic idea of Green’s functions is really simple.
- You’ve actually used them before!

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at $x'$.
- Can think of it as a generalization of the impulse response from signal processing.

# WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.

- Impulses $\delta(x - x')$ produce a response $G(x, x')$.

- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.

- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

# WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t-t')$ from linear system theory is an example of a Green's function.

$$G(t,t') = h(t-t') \tag{1}$$

- Usually find $h(t-t')$ using Fourier transform of the transfer function.

# FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}') \, \mathrm{d}^3\, \mathbf{r}'$$

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2}$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r} - \mathbf{r}'\right|}$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

# GENERALIZED FUNCTIONS

- Delta functions play a key role in Green’s functions (and electrical engineering in general).
- Worth seeing how they are rigorously defined before moving on.
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

Common “definition”:

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

$$\delta(x - x_0) = \begin{cases} 0 & \text{for } x \neq x_0 \\ \infty & \text{for } x = x_0 \end{cases}$$

- Often see definitions like "$\delta(x - x_0)$ is zero for $x \neq x_0$, but the area under it is 1."
- Might be okay intuitively, but very imprecise mathematically.

- $\delta(x - x_0)$ is an operator, **not** a function!
- Define it by the sifting property:

$$\delta_{x_0}[f] = f(x_0)$$

- **Symbolically**, write

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \, \mathrm{d}\, x = f(x_0)$$

- Delta function is rigorously defined using Schwartz distribution theory.
- Basically, the delta function is not a function at all, it is a linear operator which takes a function and returns a number: the value of the function at $x_0$.
- I.e., *the sifting property is the definition of the delta function.*
- Symbolically, we often write it as a function, but it's good to remember the proper definition in case anything fishy shows up.
- Remember, $\delta(x - x')$ has no meaning as a stand-alone function. It only has meaning when it operates on a function.

# DELTA FUNCTION LIMITS

$$\lim_{n\to\infty} \phi_n(x) = \delta(x)$$

if and only if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f(x) \, \mathrm{d}\, x = f(0)$$

- Often, we want to show that regular functions are equivalent to the delta function.
- To do this in a reasonable way, we need to show that the sifting property is obeyed, usually in a limit.

Example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t$$

is a delta-function limit as $\epsilon \to 0$.

- Example of a common, but perplexing expression for the Delta function.
- Can show that it's true by expressing it as a delta function limit. (Multiply by a Gaussian distribution and take the limit as standard deviation goes to $\infty$.)
- Doing this proof is not a bad exercise if you're interested. Theorem 9.2 from Folland will make it manageable.

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.

# SOLUTION METHODS

Boundary condition approaches:

1. Green's function gives particular solution; add homogeneous solution to find boundary conditions. Easier to set up, but requires extra work to deal with BC's.
2. Green's function includes BC's. Harder to set up, but gives full solution including BC's.

Solving Green's function approaches:

1. Direct solution. (Great if it's possible.)
2. Eigenvalue expansion. (Works every time.)

- Time-domain wave equation has a unique solution in the lossless case.
- Frequency-domain wave equation does not.
- Taking infinitesimally small loss is equivalent to assuming $u(x)$ and $u'(x)$ are zero at some initial time.

# APPLICATIONS

# APPLICATIONS

- Born approximation for scattering?
- Perturbation theory?
- Propagator/Huygen’s principle?