

# GREEN'S FUNCTIONS

## A short introduction

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- 2 Generalized functions
- 3 Direct solution
- 4 Boundary conditions
- 5 Spectral methods
- 6 3D problems
- 7 Properties of the Green's function
- 8 Advanced topics

# INTRODUCTION

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# WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the impulse response:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

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## WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, dx \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, dx$$

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$$\mathcal{L}G(x, x') = \delta(x - x')$$

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Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t - t') \, dt'$$

E.g., for an RL-circuit:

$$G(t, t') = h(t - t') = u(t - t')e^{-\alpha(t-t')}$$

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Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \rho(\mathbf{r}') d^3 \mathbf{r}'$$

Helmholtz equation:

$$(\nabla^2 + k^2) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi |\mathbf{r}-\mathbf{r}'|} J_z(\mathbf{r}') d^3 \mathbf{r}'$$

# FAMILIAR GREEN'S FUNCTIONS

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

# GENERALIZED FUNCTIONS

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## TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of  $\delta(x - x_0)$ :

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

$f(x)$  defines a linear operator  $\phi(x)$  via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) \, dx$$



If we have  $f[\phi]$ , but no  $f(x)$ , then  $f$  is a generalized function.

**Symbolically**, we write

$$f[\phi] \stackrel{s}{=} \int_{-\infty}^{\infty} f(x)\phi(x) \, dx$$

## DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$  is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) \, dx$$

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) \phi(x) \, dx$$

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}[\phi] = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) \phi(x) \, dx = \phi(0)$$

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

A more interesting example:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jxt} dt = \delta(x)$$

because

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt} dt = \delta(x)$$

## WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)[\phi] = \phi(x') \stackrel{s}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x')) \phi(x) \, dx$$

If in doubt, think of  $\delta(x - x_0)$  as an operator, not a function!



## DIRECT SOLUTION

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## A SIMPLE EXAMPLE

Original problem:

$$\frac{d^2 u(x)}{dx^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

## A SIMPLE EXAMPLE

For  $x \neq x'$

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

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$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \rightarrow 0} [G(x' + \epsilon, x') - G(x' - \epsilon, x')] = 0$$

## A SIMPLE EXAMPLE

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

Discontinuity condition:

$$\lim_{\epsilon \rightarrow 0} \left[ \left. \frac{dG}{dx} \right|_{x=x'+\epsilon} - \left. \frac{dG}{dx} \right|_{x=x'-\epsilon} \right] = 1$$

## A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[ \frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') \right] dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x') dx$$

Discontinuity condition:

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## A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of  $G$ :

$$A = B$$

Discontinuity of  $\frac{dG}{dx}$ :

$$kA + kB = 1$$



## A SIMPLE EXAMPLE

At last, our Green's function is

$$G(x, x') = \begin{cases} \frac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \frac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$G(x, x') = \frac{e^{k|x-x'|}}{2k}$$

## A SIMPLE EXAMPLE

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} dx'$$

Properties of  $G(x, x')$ :

- Looks like the sourceless solution except at  $x = x'$ .
- Function is continuous at  $x = x'$ .
- Derivative is discontinuous at  $x = x'$ .

# BOUNDARY CONDITIONS

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# SPECTRAL METHODS

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## 3D PROBLEMS

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# PROPERTIES OF THE GREEN'S FUNCTION

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## ADVANCED TOPICS

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## INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*.  
Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*.  
Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for  
electromagnetic theory*. Great introduction to 1D Green's  
functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and  
quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshbach, *Methods of theoretical physics*. Another big, detailed reference. Emphasis on theory and insights.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.