GREEN'S FUNCTIONS

A SHORT INTRODUCTION

Chris Deimert

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Department of Electrical and Computer Engineering, University of Calgary

OUTLINE

1. Basic idea
2. Finding the Green's function
3. Constructing the solution
4. Special properties
5. Conclusion

BASIC IDEA

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

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Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x')\,\mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x')\,\mathrm{d}x$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}x$$

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}x}$$

(Some conditions apply.)

Impulse response of a linear time-invariant system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive and understand these expressions.
- Generalize to other problems and boundary conditions.

FINDING THE GREEN’S FUNCTION

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

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A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\, \mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x,x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

A SIMPLE EXAMPLE

Solving, our Green's function is

$$G(x, x') = \frac{1}{2k} \begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x, x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}x'$$

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ and its derivatives are continuous or discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

ADJOINT OPERATORS

Inner product:

$$\langle u, v\rangle = \int_a^b u(x)\bar{v}(x)\,\mathrm{d}x$$

Adjoint operator $\mathcal{L}^*$:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \longrightarrow \text{Only for certain } u, v\,!$$

Adjoint boundary conditions:

$$\mathcal{B}_i[u] = 0; \qquad \mathcal{B}_i^*[v] = 0$$

Otherwise

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \underbrace{J(u, \overline{v})}_{\text{Conjunct}}\Big|_a^b$$

$$\mathcal{L}u(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

Want $\mathcal{L}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u'' + k^2 u\right] \overline{v}\, \mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[-u'\overline{v}' + k^2 u\overline{v}\right] \mathrm{d}x + [u'\overline{v}]_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right]\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Looks like

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

with

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \overline{k^2}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

For

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

must have

$$J(u, \overline{v})\Big|_a^b = 0$$
$$\left[u'\overline{v} - u\overline{v}' \right]_a^b = 0$$

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Boundary conditions:

$$\begin{matrix} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u(b) = 0 \end{matrix} \implies \begin{matrix} \mathcal{B}_1^*[v] = v(a) = 0 \\ \mathcal{B}_2^*[v] = v(b) = 0 \end{matrix}$$

$$\mathcal{B}_i = \mathcal{B}_i^*$$

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Initial conditions:

$$\begin{array}{c} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u'(a) = 0 \end{array} \implies \begin{array}{c} \mathcal{B}_1^*[v] = v(b) = 0 \\ \mathcal{B}_2^*[v] = v'(b) = 0 \end{array}$$

$$\mathcal{B}_i \neq \mathcal{B}_i^*$$

ADJOINT OPERATORS: SUMMARY

Adjoint operator:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

Adjoint boundary conditions:

$$\begin{aligned} \mathcal{B}_i[u] &= 0 \\ \mathcal{B}_i^*[v] &= 0 \end{aligned} \implies J(u, \overline{v})\Big|_a^b = 0$$

Otherwise:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

THE ADJOINT GREEN'S FUNCTION

Original problem:

$$\mathcal{L}u(x) = f(x); \qquad \mathcal{B}_i[u(x)] = \alpha_i$$

Green's problem:

$$\mathcal{L}G(x, x') = \delta(x - x'); \qquad \mathcal{B}_i[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^* H(x, x') = \delta(x - x'); \qquad \mathcal{B}_i^*[H(x, x')] = 0$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\, \mathrm{d}x = \int_a^b u(x)\delta(x-x')\, \mathrm{d}x + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)\overline{H}(x,x')\, \mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b}$$

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$\int_a^b \delta(x - x')\overline{H}(x, x'')\, \mathrm{d}x = \int_a^b G(x, x')\delta(x - x'')\, \mathrm{d}x$$

$$\overline{H}(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = \overline{H}(x', x)}$$

$$u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x),\overline{H}(x,x')\big)\Big|_a^b$$

and

$$G(x,x') = \overline{H}(x',x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'),G(x,x')\big)\Big|_{x'=a}^b}$$

Original problem:

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}; \qquad \begin{array}{l} V(a) = V_a \\ V(b) = V_b \end{array}$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{array}{l} G(a, x') = 0 \\ G(b, x') = 0 \end{array}$$

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x,x')\,\mathrm{d}x' - J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

Take $\rho(x) = 0$ for now.

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

EXAMPLE: 1D POISSON EQUATION

Can show

$$\langle \mathcal{L}u, v\rangle = \int_a^b u''\overline{v}\,\mathrm{d}x = \int_a^b u\overline{v}''\,\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Comparing with

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

we see

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \mathcal{L}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) -$$

$$-V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a)$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{aligned} G(a,x') &= 0 \\ G(b,x') &= 0 \end{aligned}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{aligned} H(a,x') &= 0 \\ H(b,x') &= 0 \end{aligned}$$

But $G(x,x') = \overline{H}(x',x)$ so

$$\frac{\mathrm{d}^2 \overline{G}(x,x')}{\mathrm{d}x'^2} = \delta(x-x'); \qquad \begin{aligned} G(x,a) &= 0 \\ G(x,b) &= 0 \end{aligned}$$

EXAMPLE: 1D POISSON EQUATION

$$
\begin{aligned}
V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'} G(x,b) - \\
- V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'} G(x,a)
\end{aligned}
$$

With $G(x,a) = G(x,b) = 0$, we have

$$
\boxed{V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}
$$

Full solution with $\rho(x)$:

$$\boxed{V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x, a)}{\mathrm{d}x'}}$$

$$\text{Given:} \quad \mathcal{L}u(x) = f(x); \quad \mathcal{B}_i[u(x)] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(x, x') = \delta(x - x'); \quad \mathcal{B}_i[G(x, x')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(x) = \int_a^b f(x')G(x, x')\,\mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

4. Simplify using $\mathcal{B}^*[G(x, x')] = 0$ (with respect to x').

SOLVING 3D PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}_i[u(\mathbf{r})] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'); \quad \mathcal{B}_i[G(\mathbf{r}, \mathbf{r}')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} \mathbf{J}\big(u(\mathbf{r}'), G(\mathbf{r}, \mathbf{r}')\big) \cdot \mathrm{d}\mathbf{s}$$

4. Simplify using $\mathcal{B}^*[G(\mathbf{r}, \mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

Comments:

- Calculating Green's functions is not trivial.
- Adjoint approach is difficult, but offers clarity.

SPECIAL PROPERTIES

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B}_i = \mathcal{B}_i^*$, then

$$G(x, x') = \overline{G}(x', x)$$

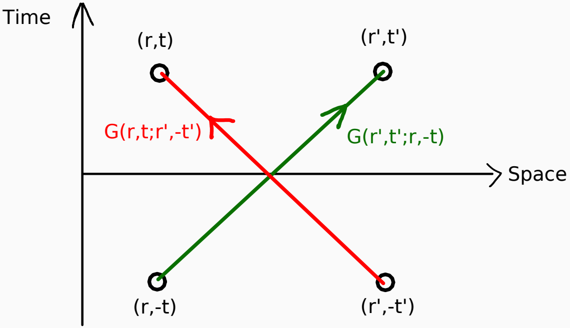
RECIPROCITY

Reciprocity in the frequency-domain:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

Reciprocity in the time-domain:

$$G(\mathbf{r}, t; \mathbf{r}', -t') = G(\mathbf{r}', t'; \mathbf{r}, -t)$$



Causality:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } t < t'$$

Special relativity:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } |\mathbf{r} - \mathbf{r}'| > c(t - t')$$

Time-invariance:

$$G(t, t') = G(t - t')$$

Spatial-invariance:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}')$$

$$(\mathcal{L} - \lambda)u(x) = f(x); \quad \mathcal{B}[u] = 0$$

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$ then

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

SPECTRAL THEORY

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\, \mathrm{d}x'$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x'; \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

$$\begin{aligned} \lambda_n &\longrightarrow \text{ Poles of } G(x, x'; \lambda) \\ \phi_n(x)\phi_n^*(x') &\longrightarrow \text{ Residues of } G(x, x'; \lambda) \end{aligned}$$

CONCLUSION

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Lots of information in the Green's function.
- Green's functions $\iff$ eigenvalues/eigenfunctions.

FURTHER READING

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Gerlach (2010), *Linear mathematics in infinite dimensions*. Nice set of online course notes for quick reference.

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Morse and Feshback, *Methods of theoretical physics*. Big, detailed reference. Great resource for deeper insight and understanding.

FURTHER READING

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Folland (1992), *Fourier analysis and its applications*. Rigorous math book. Chapter on generalized functions is particularly nice.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.