GREEN'S FUNCTIONS

A short introduction

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OUTLINE

INTRODUCTION

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

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Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

WHY IS IT USEFUL?

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WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

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FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}')\, \mathrm{d}^3\, \mathbf{r}'$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

GENERALIZED FUNCTIONS

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x$$

DELTA FUNCTION DERIVATIVES

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{s}{=} \int\limits_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}\,x$$

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}\,x = \phi(0)$$

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\,\phi(x)\,\mathrm{d}\,x$$

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

DIRECT SOLUTION

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x - x')$$

For $x \neq x'$

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

For $x \neq x'$

$$\frac{\mathrm{d}^2\, G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon}\right] = 1$$

A SIMPLE EXAMPLE

$$\int\limits_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') \right] \mathrm{d}\,x = \int\limits_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}\,x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0} \left[\left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}\,G}{\mathrm{d}\,x}$:

$$kA + kB = 1$$

At last, our Green's function is

$$G(x,x') = \begin{cases} \dfrac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \dfrac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

GENERAL APPROACH

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

BOUNDARY CONDITIONS

$$
\begin{aligned}
\text{Original:} \qquad & \mathcal{L}[u(x)] = f(x); \qquad & \mathcal{B}[u(x)] = 0 \\
\text{Adjoint:} \qquad & \mathcal{L}^*[v(x)] = f(x); \qquad & \mathcal{B}^*[v(x)] = 0
\end{aligned}
$$

Defining property:

$$
\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \qquad \text{where } \langle u, v\rangle = \int_a^b u(x)v^*(x)\,\mathrm{d}\,x
$$

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}\,x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

By inspection:

$$\mathcal{L}^* v(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*\right] v(x)$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}\,x +$$
$$+ \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

By inspection:

$$\mathcal{L}^* v(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*\right] v(x)$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$
$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u,v)\Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \iff J(u,v)\Big|_a^b = 0$$

ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x, x')] = \delta(x - x'); \qquad \mathcal{B}[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x, x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x, x')] = 0$$

EXAMPLE WITH BOUNDARY CONDITIONS

Original problem:

$$\left(\frac{d^2}{d\,x^2} + k^2\right) u(x) = f(x); \quad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Green's problem:

$$\left(\frac{d^2}{d\,x^2} + k^2\right) G(x, x') = \delta(x - x'); \quad \begin{bmatrix} G(a, x') \\ G'(a, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adjoint Green's problem:

$$\left(\frac{d^2}{d\,x^2} + k^2\right) H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x,x') = \begin{cases} A\cos(k(x-x')) + B\sin(k(x-x')) & \text{for } x < x' \\ C\cos(k(x-x')) + D\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

$$\begin{aligned} \text{Final conditions} &\implies C = D = 0 \\ \text{Continuity of function} &\implies A = 0 \\ \text{Discontinuity of derivative} &\implies B = \frac{-1}{k} \end{aligned}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$\langle \mathcal{L}u(x), H(x,x') \rangle = \langle u(x), \mathcal{L}^* H(x,x') \rangle + J(u,H)\Big|_a^b$$

$$\langle f(x), H(x,x') \rangle = \langle u(x), \delta(x-x') \rangle + J(u,H)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x') \,\mathrm{d}\,x = \int_a^b u(x)\delta(x-x') \,\mathrm{d}\,x + J(u,H)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x) H^*(x,x') \,\mathrm{d}\,x - J(u,H)\Big|_a^b}$$

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}x - \left[\frac{\mathrm{d}\, u(x)}{\mathrm{d}\, x} H^*(x, x') - u(x) \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right]_a^b$$

Recall:

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad \mathcal{B}^*[H] = \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}x + \beta H^*(a, x') - \alpha \frac{\mathrm{d}\, H^*(a, x')}{\mathrm{d}x}$$

ADJOINT GREEN'S FUNCTIONS

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$\int_a^b \delta(x - x') H^*(x, x'') \, \mathrm{d}\, x = \int_a^b G(x, x') \delta(x - x'') \, \mathrm{d}\, x$$

$$H^*(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = H^*(x', x)}$$

$$H(x,x') = \begin{cases} -k^{-1}\sin(k(x-x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \,\mathrm{d}\, x + \beta H^*(a, x') - \alpha \left. \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) \,\mathrm{d}\, x + \beta G(x', a) - \alpha \left. \frac{\mathrm{d}\, G(x', x)}{\mathrm{d}\, x} \right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') \,\mathrm{d}\, x' + \beta G(x, a) - \alpha \left. \frac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x'} \right|_{x'=a}$$

$$u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x + \beta H^*(a,x') - \alpha \left.\frac{\mathrm{d}\,H^*(x,x')}{\mathrm{d}\,x}\right|_{x=a}$$

$$u(x') = \int_a^b f(x)G(x',x)\,\mathrm{d}\,x + \beta G(x',a) - \alpha \left.\frac{\mathrm{d}\,G(x',x)}{\mathrm{d}\,x}\right|_{x=a}$$

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}$$

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \,\mathrm{d}\, x + \beta H^*(a, x') - \alpha \left. \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right|_{x=a}$$

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$$u(x) = \int_a^b f(x') G(x, x') \,\mathrm{d}\, x' + \beta G(x, a) - \alpha \left. \frac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x'} \right|_{x'=a}$$

FINAL SOLUTION

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}}$$

where

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

INTERPRETATION OF BOUNDARY CONDITIONS

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha\,\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\bigg|_{x'=a}$$

$$u(x) = \int_a^b \Big[f(x') + \beta\delta(x'-a) - \alpha\delta'(x'-a)\Big]G(x,x')\,\mathrm{d}\,x'$$

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$
$$\mathcal{L}^*[H(x,x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

1. Find $G(x,x')$.
2. Find $u(x')$ in terms of $H(x,x')$:

$$u(x') = \int_a^b f(x) H^*(x,x') \,\mathrm{d}\, x - J(u(x), H(x,x'))\Big|_a^b$$

3. Express $u(x)$ in terms of $G(x,x') = H^*(x',x)$.

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^{b}}$$

Simplify with adjoint boundary conditions!

PROPERTIES OF GREEN’S FUNCTIONS

If $\mathcal{L}$ is self-adjoint

$$\mathcal{L} = \mathcal{L}^* \qquad \text{and} \qquad \mathcal{B} = \mathcal{B}^*$$

then

$$G(x, x') = G^*(x', x)$$

$\mathcal{L}$ is invariant if

$$\mathcal{L}[u(x-\xi)] = \mathcal{L}[u(x)]\Big|_{x=x-\xi}$$

For example,

$$\mathcal{L} = a\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + b\frac{\mathrm{d}}{\mathrm{d}\,x} + c$$

is invariant only if a, b, and c are constants.

INVARIANCE

If $\mathcal{L}$ is invariant in x, then

$$\mathcal{L}[G(x,x')] = \delta(x-x')$$
$$\mathcal{L}[G(x-\xi,x'-\xi)] = \delta(x-x')$$

$$\implies G(x,x') = G(x-\xi,x'-\xi)$$

$$\boxed{G(x,x') = G(x-x')}$$

INVARIANCE

Convolution:

$$u(x) = \int_a^b G(x - x')f(x')\,\mathrm{d}\,x' = G(x) * f(x)$$

Frequency domain:

$$\tilde{u}(k) = \tilde{G}(k)\tilde{f}(k)$$

SPECTRAL METHODS

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

With $\mathcal{L}$ self-adjoint, $\lambda_n \in \mathbb{R}$ and

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$(\mathcal{L}-\lambda)u(x) = f(x)$$

$$(\mathcal{L}-\lambda)\left[\sum_n \langle u, \phi_n\rangle\, \phi_n(x)\right] = \sum_n \langle f, \phi_n\rangle\, \phi_n(x)$$

$$\sum_n \langle u, \phi_n\rangle\, (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n\rangle\, \phi_n(x)$$

$$(\lambda_n - \lambda)\, \langle u, \phi_n\rangle = \langle f, \phi_n\rangle$$

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}\, x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{f(x)\phi_n^*(x')}{\lambda_n - \lambda} \phi_n(x) \right) \mathrm{d}\, x'$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}\, x'$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}\, x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

SPECTRAL FORM OF THE DELTA FUNCTION

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SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} - \lambda\right)}_{\mathcal{L}-\lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

$$G(x,x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x,x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda}x'\right)\sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x > x' \end{cases}$$

THE 3D WAVE EQUATION

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\mathcal{L}} u(\mathbf{r}, t) = f(\mathbf{r}, t); \qquad \mathcal{B}[u] = \alpha$$

In electromagnetism:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

Find $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{B}$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

where

$$\langle u, v\rangle = \int_{t_i}^{t_f} \int_V u(\mathbf{r}, t) v^*(\mathbf{r}, t)\, \mathrm{d}^3\,\mathbf{r}\, \mathrm{d}\,t$$

THE ADJOINT WAVE EQUATION

$$\langle \mathcal{L}u, v\rangle = \int_{t_i}^{t_f} \int_V \left(\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \right) v^* \, \mathrm{d}^3\, \mathbf{r} \, \mathrm{d}\, t$$

Use Green's identity

$$\int_V \left(v^* \nabla^2 u - u \nabla^2 v^* \right) = \oint_{\partial V} \left(v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) \mathrm{d}\, S$$

and integration by parts

$$\int_{t_i}^{t_f} \left(\frac{\partial^2 u}{\partial t^2} v^* - u \frac{\partial^2 v^*}{\partial t^2} \right) \mathrm{d}\, t = \left[v^* \frac{\partial u}{\partial t} - u \frac{\partial v^*}{\partial t} \right]_{t_i}^{t_f}$$

Result:

$$\int_{t_i}^{t_f} \int_V (\mathcal{L}u)\, v^* \,\mathrm{d}^3\, \mathbf{r} \,\mathrm{d}\, t = \int_{t_i}^{t_f} \int_V u\, (\mathcal{L}v)^* \,\mathrm{d}^3\, \mathbf{r} \,\mathrm{d}\, t +$$

$$+ \int_{t_i}^{t_f} \oint_{\partial V} \left(v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) \mathrm{d}\, S \,\mathrm{d}\, t -$$

$$- \frac{1}{c^2} \left[\int_V \left(v^* \frac{\partial u}{\partial t} - u \frac{\partial v^*}{\partial t} \right) \mathrm{d}\, V \right]_{t_i}^{t_f}$$

Original problem:

$$\mathcal{L}u(\mathbf{r},t) = f(\mathbf{r},t); \quad \mathcal{B}[u] = \alpha$$

Green's problem:

$$\mathcal{L}G(\mathbf{r},t;\mathbf{r}',t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'); \quad \mathcal{B}[G] = 0$$

Adjoint Green's problem:

$$\mathcal{L}H(\mathbf{r},t;\mathbf{r}',t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'); \quad \mathcal{B}^*[H] = 0$$

SOLUTION TO THE WAVE EQUATION

$$
\begin{aligned}
u(\mathbf{r}', t') = & \int_{t_i}^{t_f} \int_V f(\mathbf{r}, t) H^*(\mathbf{r}, t; \mathbf{r}', t') \, \mathrm{d}^3 \, \mathbf{r} \, \mathrm{d}\, t + \\
& - \int_{t_i}^{t_f} \oint_{\partial V} \left(H^* \frac{\partial u}{\partial n} - u \frac{\partial H^*}{\partial n} \right) \mathrm{d}\, S \, \mathrm{d}\, t - \\
& + \frac{1}{c^2} \left[\int_V \left(H^* \frac{\partial u}{\partial t} - u \frac{\partial H^*}{\partial t} \right) \mathrm{d}\, V \right]_{t_i}^{t_f}
\end{aligned}
$$

This time, $G(\mathbf{r}, t; \mathbf{r}', t') = H^*(\mathbf{r}', t'; \mathbf{r}, t)$ so

$$u(\mathbf{r}, t) = \int_{t_i}^{t_f} \int_{V'} f(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') \, \mathrm{d}^3 \, \mathbf{r}' \, \mathrm{d}\, t' + \\ - \int_{t_i}^{t_f} \oint_{\partial V'} \left(G \frac{\partial u}{\partial n'} - u \frac{\partial G}{\partial n'} \right) \mathrm{d}\, S' \, \mathrm{d}\, t' - \\ + \frac{1}{c^2} \left[\int_{V'} \left(G \frac{\partial u}{\partial t'} - u \frac{\partial G}{\partial t'} \right) \mathrm{d}\, V' \right]_{t_i}^{t_f}$$

FINDING THE GREEN'S FUNCTION

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

Initial conditions:

$$G(\mathbf{r}, 0; \mathbf{r}', t') = \left.\frac{\partial G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t}\right|_{t=0} = 0$$

Boundary conditions:

$$\lim_{|\mathbf{r}|\to\infty} G(\mathbf{r}, t; \mathbf{r}', t') \to 0$$

FINDING THE GREEN'S FUNCTION

Translation invariance:

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r} - \mathbf{r}', t - t')$$

Let $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = c(t - t')$ so that

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

FINDING THE GREEN'S FUNCTION

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

Spatial Fourier transform:

$$\left(k^2 - \frac{\partial^2}{\partial \tau^2}\right) \tilde{G}(\mathbf{k}, \tau) = \delta(\tau)$$

so

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

FINDING THE GREEN'S FUNCTION

$$\tilde{G}(\mathbf{k},\tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

- Initial conditions give $A = B = 0$.
- Continuity at $t = t'$ gives $C = 0$.
- Discontinuity at $t = t'$ gives $D = -1/k$.

$$\tilde{G}(\mathbf{k},\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ -\dfrac{\sin(k\tau)}{k} & \text{for } \tau > 0 \end{cases}$$

FINDING THE GREEN'S FUNCTION

Inverse Fourier transform for $\tau > 0$:

$$G(\mathbf{R}, \tau) = \frac{1}{8\pi^3} \int\limits_{\mathbb{R}^3} \tilde{G}(\mathbf{k}, \tau) e^{j\mathbf{k}\cdot\mathbf{R}} \, \mathrm{d}^3 \, \mathbf{k}$$

$$G(\mathbf{R}, \tau) = \frac{-1}{8\pi^3} \int\limits_{\mathbb{R}^3} \frac{\sin(k\tau)}{k} e^{j\mathbf{k}\cdot\mathbf{R}} \, \mathrm{d}^3 \, \mathbf{k}$$

Use spherical coordinates in $\mathbf{k}$: (k, θ, ϕ).

$$G(\mathbf{R}, \tau) = \frac{-1}{8\pi^3} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{\infty} \frac{\sin(k\tau)}{k} e^{jkR\cos(\theta)} k^2 \sin(\theta) \, \mathrm{d}\, k \, \mathrm{d}\, \theta \, \mathrm{d}\, \phi$$

FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\sin(k\tau)}{k}e^{jkR\cos(\theta)}k^2\sin(\theta)\,\mathrm{d}\,k\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\int_0^{\pi}\sin(\theta)e^{jkR\cos(\theta)}\,\mathrm{d}\,\theta\right]\mathrm{d}\,k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\frac{2\sin(kR)}{kR}\right]\mathrm{d}\,k$$

$$G(\mathbf{R},\tau) = \frac{-1}{2\pi^2 R}\int_0^{\infty}\sin(k\tau)\sin(kR)\,\mathrm{d}\,k$$

FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R},\tau) = \frac{-1}{2\pi^2 R}\int_0^\infty \sin(k\tau)\sin(kR)\,\mathrm{d}\,k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2 R}\int_{-\infty}^\infty \sin(k\tau)\sin(kR)\,\mathrm{d}\,k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2 R}\int_{-\infty}^\infty \left(\frac{e^{jk\tau}-e^{-jk\tau}}{2j}\right)\left(\frac{e^{jkR}-e^{-jkR}}{2j}\right)\mathrm{d}\,k$$

$$G(\mathbf{R},\tau) = \frac{1}{16\pi^2 R}\int_{-\infty}^\infty \left(e^{jk(\tau+R)}-e^{-jk(\tau-R)}-e^{jk(\tau-R)}+e^{-jk(\tau+R)}\right)\mathrm{d}\,k$$

$$G(\mathsf{R}, \tau) = \frac{1}{16\pi^2 R} \int_{-\infty}^{\infty} \left(e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}\right) \mathrm{d}\, k$$

$$G(\mathsf{R}, \tau) = \frac{1}{16\pi^2 R} 2\pi \left[\delta(\tau + R) - \delta(\tau - R) - \delta(\tau - R) + \delta(\tau + R)\right]$$

But $\tau > 0$ and $R > 0$ so $\delta(\tau + R) = 0$ and we have

$$G(\mathsf{R}, \tau) = \frac{-\delta(\tau - R)}{4\pi R}$$

GREEN'S FUNCTION FOR THE WAVE EQUATION

$$G(\mathbf{r}, t; \mathbf{r}', t') = \begin{cases} -\frac{\delta(c(t-t')-|\mathbf{r}-\mathbf{r'}|)}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

With zero boundary/initial conditions:

$$u(\mathbf{r}, t) = \int_{t_i}^{t_f} \int_{\mathbb{R}^3} G(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t') \, \mathrm{d}^3 \mathbf{r}' \, \mathrm{d}\, t'$$

$$u(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{-f\left(\mathbf{r}, t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3 \mathbf{r'}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathsf{A}(\mathsf{r}, t) = -\mathsf{J}(\mathsf{r}, t)$$

For initial conditions, we have the “retarded potential”:

$$\mathsf{A}(\mathsf{r}, t) = \int\limits_{\mathbb{R}^3} \frac{\mathsf{J}\left(\mathsf{r}, t - \frac{|\mathsf{r}-\mathsf{r}'|}{c}\right)}{4\pi|\mathsf{r}-\mathsf{r}'|}\,\mathrm{d}^3\,\mathsf{r}'$$

$$\tilde{\mathsf{A}}(\mathsf{r}, \omega) = \int\limits_{\mathbb{R}^3} \tilde{\mathsf{J}}\,(\mathsf{r}, \omega)\,\frac{e^{-j\frac{\omega}{c}|\mathsf{r}-\mathsf{r}'|}}{4\pi|\mathsf{r}-\mathsf{r}'|}\,\mathrm{d}^3\,\mathsf{r}'$$

VECTOR POTENTIAL: FINAL VALUE PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathsf{A}(\mathbf{r}, t) = -\mathsf{J}(\mathbf{r}, t)$$

For final conditions, we have the “advanced potential”:

$$\mathsf{A}(\mathbf{r}, t) = \int\limits_{\mathbb{R}^3} \frac{\mathsf{J}\left(\mathbf{r}, t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\,\mathbf{r}'$$

Violation of causality?

CONCLUSION

TAKEAWAYS

Introduction:

- The Green's function is the impulse response
- Use it to construct solutions. Ignoring boundary conditions:

$$u(x) = \int f(x')G(x, x')\,\mathrm{d}\,x'$$

TAKEAWAYS

Generalized functions:

- The delta function is an operator, not an actual function.

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- $\dfrac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x}$ is discontinuous at $x = x'$.

TAKEAWAYS

Boundary conditions:

- Non-zero boundary conditions act like impulse sources.
- Best way to construct $u(x)$ from $G(x, x')$ is to use the adjoint problem.

TAKEAWAYS

Properties of Green's functions:

- Self-adjoint problems: $G(x, x') = G^*(x', x)$ (reciprocity).
- Invariant problems: $G(x, x') = G(x - x')$.

TAKEAWAYS

Spectral methods:

- Green's function can be found from eigenfunctions/eigenvalues:

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

- Eigenfunctions/eigenvalues can be found from Green's function.
- Delta function can be written in terms of eigenfunctions:

$$\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')$$

The 3D wave equation:

- In general, use adjoint problem to write $u(\mathbf{r}, t)$ in terms of $G(\mathbf{r}, t; \mathbf{r}', t')$.
- For free space initial condition problem, solution is the so-called "retarded potential."

FURTHER TOPICS

- Sturm-Liouville problems
- Complex contour integration
- Separation of variables (3D problems)
- Dyadic Green's functions

INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Fully rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

ADVANCED RESOURCES

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.