

# GREEN'S FUNCTIONS

## A SHORT INTRODUCTION

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## OUTLINE

- 1 Introduction
- 2 Generalized functions
- 3 Direct solution
- 4 Boundary conditions
- 5 Properties of Green's functions
- 6 Spectral methods
- 7 The 3D wave equation
- 8 Conclusion

- This is intended as a small (dense) introduction to Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them.
- Tip: read a lot of different references. Different authors take totally different approaches and it's interesting to see them all. Lots of references are provided at the end.

## INTRODUCTION

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- Fortunately, the basic idea of Green's functions is really simple.
- You've actually used them before!
- What's far more interesting is how to calculate/interpret them.

## WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function  $f(x)$ .
- The Green's function is the solution when the source  $f(x)$  is an impulse located at  $x'$ .
- Can think of it as a generalization of the impulse response from signal processing.

## WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, dx \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, dx$$

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- Once we know the Green's function for a problem, we can find the solution for any source  $f(x)$ .
- Impulses  $\delta(x - x')$  produce a response  $G(x, x')$ .
- We can split the source  $f(x)$  up into a sum (integral) of impulses  $\delta(x - x')$ .
- Then the response to  $f(x)$  is just a weighted sum (integral) of impulse responses.

## WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$u(x) = \int G(x, x') f(x') \, dx$$

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- Once we know the Green's function, we have an explicit formula for the solution  $u(x)$  for any source function  $f(x)$ .

## FAMILIAR GREEN'S FUNCTIONS

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t') dt'$$

E.g., for an RL-circuit:

$$G(t, t') = h(t - t') = u(t - t')e^{-\alpha(t-t')}$$

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- In electrical engineering, we've seen Green's functions before.
- Impulse response  $h(t - t')$  from linear system theory is an example of a Green's function.

$$G(t, t') = h(t - t')$$

- Usually find  $h(t - t')$  using Fourier transform of the transfer function.

## FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \rho(\mathbf{r}') d^3 \mathbf{r}'$$

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- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2}$$

## FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$(\nabla^2 + k^2) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} J_z(\mathbf{r}') d^3\mathbf{r}'$$

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## FAMILIAR GREEN'S FUNCTIONS

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

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- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

## GENERALIZED FUNCTIONS

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### TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of  $\delta(x - x_0)$ :

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

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- Delta functions play a key role in Green’s functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz’s theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

- Often see definitions like this one.
- Often said to imply that  $\delta(x - x_0) = \infty$  at  $x = x_0$ .
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

## GENERALIZED FUNCTIONS

$f(x)$  defines a linear operator  $\phi(x)$  via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

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- Let's see if we can generalize the idea of a "function" so that it includes delta functions.
- Given a function  $f(x)$ , we can use it to define a linear operator (a functional, to be exact) on other functions  $\phi(x)$ .
- $f[\cdot]$  is a linear operator. It takes a function  $\phi(x)$  and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

- If we ensure that  $\phi(x)$  is very well-behaved, then every function  $f(x)$  defines an operator in this way.

## GENERALIZED FUNCTIONS

If we have  $f[\phi]$ , but no  $f(x)$ , then  $f$  is a generalized function.

**Symbolically**, we write

$$f[\phi] \stackrel{s}{=} \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

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- It's possible to have an operator  $f[\phi]$ , but we can't find an  $f(x)$  to implement it via an integral.
- Then  $f(x)$  is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions  $f[\phi]$ .
- We still symbolically write

$$f[\phi] \stackrel{s}{=} \int_{-\infty}^{\infty} f(x)\phi(x) dx$$

but this is just suggestive notation. It is not actually an integral unless  $f(x)$  is a "proper" function!

## DEFINING THE DELTA FUNCTION

$\delta(x - x_0)$  is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) \, dx$$

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## DELTA FUNCTION DERIVATIVES

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{s}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) \phi(x) \, dx$$

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- We can define a simple linear operator via the sifting property  $\delta_{x_0}[\phi] = \phi(x_0)$ .
- There is no actual function  $\delta(x - x_0)$  which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) \, dx = \phi(x_0)$$

so  $\delta(x - x_0)$  is a generalized function and the above integral is purely symbolic.

- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$  is just an operator that picks out the value of the  $n$ th derivative of  $\phi(x)$  at the point  $x_0$ .



## DELTA FUNCTION LIMITS

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \rightarrow 0} f_{\epsilon}[\phi] = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) \phi(x) dx = \phi(0)$$

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- Often useful to show that some set of actual functions  $f_{\epsilon}(x)$  “approach” the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

## DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

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- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

## DELTA FUNCTION LIMITS

A more interesting example:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} dt = \delta(x)$$

because

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{ixt} dt = \delta(x)$$

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- Example of a common, but unintuitive expression for the Delta function.
- Can show that it's true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

## WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)[\phi] = \phi(x') \stackrel{s}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x')) \phi(x) dx$$

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- Technically, the Green's function is a generalized function such that  $\mathcal{L}G$  is the delta function (it has the sifting property).
- In practise, we'll keep using the less-precise way; just remember that there is a more correct way.

## TAKEAWAY

If in doubt, think of  $\delta(x - x_0)$  as an operator, not a function!

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## DIRECT SOLUTION

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- In practise, thinking of  $\delta(x - x_0)$  as a function is usually fine. (We'll even do that for the rest of this presentation.)
- But if anything starts to seem fishy, it's good to remember that  $\delta(x - x_0)$  is actually an operator, and not a function.

- Back to Green's functions!

## A SIMPLE EXAMPLE

Original problem:

$$\frac{d^2 u(x)}{dx^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

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- Let's look at a simple example now.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at  $x = \pm\infty$ .
- If we can find the Green's function, then we can find the solution to the original problem.
- But the Green's function problem looks pretty hard. The point of this example is to demonstrate that we can actually solve it.

## A SIMPLE EXAMPLE

For  $x \neq x'$

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

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- Key thing to notice is that the source is concentrated at  $x = x'$ .
- So for  $x > x'$  and  $x < x'$ , we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before  $x = x'$  and exponential decay afterward.
- Now, how do we find the constants  $A$  and  $B$ ?

## A SIMPLE EXAMPLE

$$\frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \rightarrow 0} [G(x' + \epsilon, x') - G(x' - \epsilon, x')] = 0$$

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- How continuous do we expect our Green's function to be?
- If  $G(x, x')$  is discontinuous (like a step function), then  $dG/dx$  will behave like a delta function and  $d^2 G/dx^2$  will behave like a delta function derivative. No good!
- So we expect  $G(x, x')$  to be continuous.
- That gives us one condition we can use to find  $A$  and  $B$ . (In fact, it tells us that  $A = B$ .)

## A SIMPLE EXAMPLE

$$\int_{x' - \epsilon}^{x' + \epsilon} \left[ \frac{d^2 G(x, x')}{dx^2} - k^2 G(x, x') \right] dx = \int_{x' - \epsilon}^{x' + \epsilon} \delta(x - x') dx$$

Discontinuity condition:

$$\lim_{\epsilon \rightarrow 0} \left[ \left. \frac{dG}{dx} \right|_{x=x'+\epsilon} - \left. \frac{dG}{dx} \right|_{x=x'-\epsilon} \right] = 1$$

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- But what if the derivative  $dG/dx$  is discontinuous?
- Then  $d^2 G/dx^2$  is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around  $x'$ .
- In the limit of  $\epsilon \rightarrow 0$ , the second integral vanishes because  $G(x, x')$  is continuous.
- But, we expect  $dG/dx$  to be discontinuous.
- Using fundamental theorem of calculus, we get at a *discontinuity condition for the derivative*. (Key idea for the direct solution method!)

## A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of  $G$ :

$$A = B$$

Discontinuity of  $\frac{dG}{dx}$ :

$$kA + kB = 1$$

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## A SIMPLE EXAMPLE

At last, our Green's function is

$$G(x, x') = \begin{cases} \frac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \frac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$G(x, x') = \frac{e^{k|x-x'|}}{2k}$$

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- Applying our two conditions, we can solve for  $A$  and  $B$ .

## A SIMPLE EXAMPLE

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} dx'$$

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- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function  $f(t)$ .

## GENERAL APPROACH

Properties of  $G(x, x')$ :

- Behaves like source-free solution except at  $x = x'$ .
- Function is continuous at  $x = x'$ .
- Derivative is discontinuous at  $x = x'$ .

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- Listed are the key things to note from that example.
- This approach works quite well for solving 1D Green's function problems.

## BOUNDARY CONDITIONS

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## ADJOINT OPERATORS

$$\begin{array}{lll} \text{Original:} & \mathcal{L}[u(x)] = f(x); & \mathcal{B}[u(x)] = 0 \\ \text{Adjoint:} & \mathcal{L}^*[v(x)] = f(x); & \mathcal{B}^*[v(x)] = 0 \end{array}$$

Defining property:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle \quad \text{where } \langle u, v \rangle = \int_a^b u(x)v^*(x) dx$$

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- We didn't worry about boundary conditions in the last example.
- As it turns out, Green's functions allow us to deal with boundary conditions in an elegant way.
- Unfortunately, to derive it, we either have to do a lot of hand-waving or a lot of math. We're going to do a lot of math.
- See Dudley's *Mathematical foundations for electromagnetic theory* for a more thorough discussion.

- To really make sense of boundary conditions, we need the concept of the adjoint problem.
- Suppose we have an original problem defined by operator  $\mathcal{L}$  and boundary conditions  $\mathcal{B}$ .
- Then,  $\mathcal{L}^*$  is the adjoint operator and  $\mathcal{B}^*$  are the adjoint boundary conditions if  $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$  for all  $u, v$ .
- Here  $\langle u, v \rangle$  is the inner product as defined on the slide. ( $v^*(x)$  is the complex conjugate of  $v(x)$ .)



Linear algebra notes:

- The boundary conditions are important because they specify the domains of  $\mathcal{L}$  and  $\mathcal{L}^*$ . (i.e.,  $\mathcal{L}$  operates on the Hilbert space of functions  $u(x)$  which satisfy  $\mathcal{B}[u] = 0$ .)
- So if  $\mathcal{B} \neq \mathcal{B}^*$ , then  $\mathcal{L}$  and  $\mathcal{L}^*$  are operators on different Hilbert spaces.
- If both  $\mathcal{L} = \mathcal{L}^*$  and  $\mathcal{B} = \mathcal{B}^*$ , we say that  $\mathcal{L}$  is self-adjoint.
- If  $\mathcal{L} = \mathcal{L}^*$  but  $\mathcal{B} \neq \mathcal{B}^*$ , we say that  $\mathcal{L}$  is *formally* self-adjoint.

## ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}[u(x)] = \left[ \frac{d^2}{dx^2} + k^2 \right] u(x)$$
$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want  $\mathcal{L}^*$  and  $\mathcal{B}^*$  so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$$

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- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that  $u(a) = u(b) = 0$ .

## ADJOINT OPERATORS: EXAMPLE

$$\begin{aligned}\langle \mathcal{L}u, v \rangle &= \int_a^b [u''(x) + k^2 u(x)] v^*(x) dx \\ \langle \mathcal{L}u, v \rangle &= \int_a^b u(x) [v''(x) + (k^2)^* v(x)]^* dx + \\ &\quad + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b\end{aligned}$$

By inspection:

$$\mathcal{L}^* v(x) = \left[ \frac{d^2}{dx^2} + (k^2)^* \right] v(x)$$

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## ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b$$

Since  $u(a) = u(b) = 0$ ,

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

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- Integrate by parts twice.
- The remaining integral term looks like

$$\int_a^b u(x) [\mathcal{L}^* v(x)]^* dx$$

so let's define  $\mathcal{L}^*$  this way, and hope it will work out (it will).

- We almost have the required  $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$ , but we need the part on the right (called the conjunct) to be zero.
- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case,  $\mathcal{B} = \mathcal{B}^*$ .

## ADJOINT OPERATORS: EXAMPLE

What if  $u(a) = u'(a) = 0$  (initial conditions)?

$$\begin{aligned}\langle \mathcal{L}u, v \rangle &= \langle u, \mathcal{L}^*v \rangle + [u'(x)v^*(x) - u(x)v'^*(x)]_a^b \\ \langle \mathcal{L}u, v \rangle &= \langle u, \mathcal{L}^*v \rangle + u'(b)v^*(b) - u(b)v'^*(b)\end{aligned}$$

We must have *final* conditions for  $v$ :

$$v(b) = v'(b) = 0$$

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## ADJOINT OPERATORS

In general:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle + J(u, v) \Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \iff J(u, v) \Big|_a^b = 0$$

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- What if we use the same operator  $\mathcal{L}$ , but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem!  $\mathcal{B} \neq \mathcal{B}^*$ .

- In that example, we saw that we always had  $\langle \mathcal{L}u, v \rangle$  equal to  $\langle u, \mathcal{L}^*v \rangle$  plus a leftover term which depended on the boundaries.
- This is true more generally: if we don't specify the boundary conditions of  $u$  and  $v$ , then we can still *almost* get the adjoint operator equation. We just have a leftover "conjunct" term  $J(u, v) \Big|_a^b$ , which depends only on the boundary conditions.
- We define adjoint boundary conditions as those boundary conditions which make the conjunct equal to zero.

## ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \quad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x, x')] = \delta(x - x'); \quad \mathcal{B}[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x, x')] = \delta(x - x'); \quad \mathcal{B}^*[H(x, x')] = 0$$

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- Now we'll be able to deal with boundary conditions properly.
- We define  $G(x, x')$  to obey the same equation as  $u(x)$ , but with  $f(x) \rightarrow \delta(x - x')$  and  $\alpha \rightarrow 0$ . As before,  $G(x, x')$  is the impulse response.
- In addition, we define a new function  $H(x, x')$  which is called the adjoint Green's function. It obeys the adjoint version of the  $G(x, x')$  equation.
- Warning: a lot of textbooks don't distinguish between  $H(x, x')$  and  $G(x, x')$ . Quite often, the "Green's function" is really the adjoint Green's function.

## EXAMPLE WITH BOUNDARY CONDITIONS

Original problem:

$$\left( \frac{d^2}{dx^2} + k^2 \right) u(x) = f(x); \quad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Green's problem:

$$\left( \frac{d^2}{dx^2} + k^2 \right) G(x, x') = \delta(x - x'); \quad \begin{bmatrix} G(a, x') \\ G'(a, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adjoint Green's problem:

$$\left( \frac{d^2}{dx^2} + k^2 \right) H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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- We'll show how  $H(x, x')$  is useful with an example.
- Here we have a driven simple harmonic oscillator with initial conditions. (We'll take  $k$  real for simplicity.)
- From before, we know that the adjoint problem will be the same differential equation, but with final conditions instead of initial conditions.

## EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} A \cos(k(x - x')) + B \sin(k(x - x')) & \text{for } x < x' \\ C \cos(k(x - x')) + D \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

$$\text{Final conditions} \implies C = D = 0$$

$$\text{Continuity of function} \implies A = 0$$

$$\text{Discontinuity of derivative} \implies B = \frac{-1}{k}$$

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- We can solve for  $H(x, x')$  using a similar approach to before.
- Except at  $x = x'$ , we write  $H(x, x')$  as a solution to the source-free equation.
- Then we find the coefficients using the boundary conditions and continuity/discontinuity requirements.

## EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1} \sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

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- So we have our adjoint Green's function.
- Now we just need to figure out how to construct  $u(x)$  from it.

## EXAMPLE WITH BOUNDARY CONDITIONS

$$\begin{aligned}\langle \mathcal{L}u(x), H(x, x') \rangle &= \langle u(x), \mathcal{L}^*H(x, x') \rangle + J(u, H) \Big|_a^b \\ \langle f(x), H(x, x') \rangle &= \langle u(x), \delta(x - x') \rangle + J(u, H) \Big|_a^b \\ \int_a^b f(x)H^*(x, x') \, dx &= \int_a^b u(x)\delta(x - x') \, dx + J(u, H) \Big|_a^b\end{aligned}$$

$$u(x') = \int_a^b f(x)H^*(x, x') \, dx - J(u, H) \Big|_a^b$$

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- To construct the solution  $u(x)$ , we take an inner product of  $\mathcal{L}u(x)$  with  $H(x, x')$ , and apply our knowledge of adjoints and conjuncts.
- We arrive at a fairly general formula which looks close to what we expect a Green's function formula to look like, but with an extra conjunct term.
- We'll gain insight into the  $J(u, H)$  term by expanding it for this example.

## EXAMPLE WITH BOUNDARY CONDITIONS

Expand  $J(u, H)$ :

$$u(x') = \int_a^b f(x)H^*(x, x') \, dx - \left[ \frac{d u(x)}{d x} H^*(x, x') - u(x) \frac{d H^*(x, x')}{d x} \right]_a^b$$

Recall:

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \quad \mathcal{B}^*[H] = \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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- Now let's expand  $J(u, v)$  for our particular example. (We can basically copy it from a previous part of the derivation.)
- We can simplify the conjunct by remembering our boundary conditions.

## EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') dx + \beta H^*(a, x') - \alpha \frac{d H^*(a, x')}{d x}$$

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## ADJOINT GREEN'S FUNCTIONS

How are  $G(x, x')$  and  $H(x, x')$  related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$\int_a^b \delta(x - x') H^*(x, x'') dx = \int_a^b G(x, x') \delta(x - x'') dx$$

$$H^*(x', x'') = G(x'', x')$$

$$G(x, x') = H^*(x', x)$$

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- We're very close to our final goal.
- The last thing will be to get rid of  $H(x, x')$  and replace it with  $G(x, x')$ .

- Using the definition of the adjoint problem, we find that there is a simple relationship between  $G(x, x')$  and  $H(x, x')$ .
- This is somewhat surprising. For one thing, it means that if  $G(x, x')$  obeys the boundary conditions with respect to  $x$ , then it automatically obeys the *adjoint* boundary conditions with respect to  $x'$ . This is a useful property.

## EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1} \sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1} \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

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- Since we already know the adjoint Green's function  $H(x, x')$ , we can use find the Green's function via  $G(x, x') = H^*(x, x')$ .

## EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') dx + \beta H^*(a, x') - \alpha \left. \frac{d H^*(x, x')}{dx} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) dx + \beta G(x', a) - \alpha \left. \frac{d G(x', x)}{dx} \right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{d G(x, x')}{dx'} \right|_{x'=a}$$

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- With our new knowledge that  $G(x, x') = H^*(x', x)$ , we can rewrite the solution  $u(x)$  in terms of  $G(x, x')$ .



## FINAL SOLUTION

$$u(x) = \int_a^b f(x') G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{dG(x, x')}{dx'} \right|_{x'=a}$$

where

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1} \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

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- Finally, we arrive at our solution.
- First, note that the final answer does *not* depend on  $H(x, x')$ . We didn't actually need to ever calculate  $H(x, x')$  from the adjoint Green's equation, we could have just found  $G(x, x')$  from the (non-adjoint) Green's equation.
- However, it would have been very difficult to derive this expression without using  $H(x, x')$  (I couldn't see an easy way). Because of this, a lot of authors stop at the expression for  $u(x')$  in terms of  $H^*(x, x')$ , and they just call  $H^*(x, x')$  the "Green's function."
- By doing the extra work, though, we gain a very nice interpretation for the boundary conditions term.

## INTERPRETATION OF BOUNDARY CONDITIONS

$$u(x) = \int_a^b f(x') G(x, x') dx' + \beta G(x, a) - \alpha \left. \frac{dG(x, x')}{dx'} \right|_{x'=a}$$

$$u(x) = \int_a^b \left[ f(x') + \beta \delta(x' - a) - \alpha \delta'(x' - a) \right] G(x, x') dx'$$

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- We note that our expression for  $u(x)$  looks like it did before (integral over  $f(x')G(x, x')$ ), but now there are extra terms which depend on the boundary conditions.
- We come to a key idea: boundary conditions have the same effect on  $u(x)$  as adding little impulse sources at the boundary. The Green's function can deal with both sources  $f(x)$  and non-zero boundary conditions.
- Be careful, though:  $G(x, x')$  still depends on the *type* of boundary condition. E.g., we use the same  $G(x, x')$  for all initial value problems ( $u(a), u'(a)$  specified), but we'll need a different  $G(x, x')$  for boundary value problems ( $u(a), u(b)$  specified).

## SUMMARY

$$\begin{aligned}\mathcal{L}[u(x)] &= f(x); & \mathcal{B}[u(x)] &= \alpha \\ \mathcal{L}[G(x, x')] &= \delta(x - x'); & \mathcal{B}[G(x, x')] &= 0 \\ \mathcal{L}^*[H(x, x')] &= \delta(x - x'); & \mathcal{B}^*[H(x, x')] &= 0\end{aligned}$$

- ① Find  $G(x, x')$ .
- ② Find  $u(x')$  in terms of  $H(x, x')$ :

$$u(x') = \int_a^b f(x) H^*(x, x') \, dx - J(u(x), H(x, x')) \Big|_a^b$$

- ③ Express  $u(x)$  in terms of  $G(x, x') = H^*(x', x)$ .

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## SUMMARY

$$\begin{aligned}\mathcal{L}[u(x)] &= f(x); & \mathcal{B}[u(x)] &= \alpha \\ \mathcal{L}[G(x, x')] &= \delta(x - x'); & \mathcal{B}[G(x, x')] &= 0\end{aligned}$$

$$u(x) = \int_a^b f(x') G(x, x') \, dx' - J(u(x'), G(x, x')) \Big|_{x'=a}^b$$

Simplify with adjoint boundary conditions!

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- That was a long process, so let's summarize what we did.
- First, we wrote out the original equation, the Green's function equation, and the adjoint Green's function equation. (Take note of the boundary conditions in particular.)
- Next, we solve the Green's function equation for  $G(x, x')$ .
- Next, we found that it's much easier to express  $u(x)$  in terms of  $H(x, x')$ , because we can use inner products. The only tricky part is finding the conjunct. (Usually, just requires integration by parts. See Dudley for a general formula for Sturm-Liouville problems.)
- Finally, we replace  $H(x, x')$  with  $G^*(x', x)$  (which we solved for previously), and we have our final expression for  $u(x)$ .

- We can streamline this process a little bit by removing the intermediate steps involving  $H(x, x')$ .
- Then, we can simply write down our final solution in terms of the Green's function  $G(x, x')$ .
- However, there are still traces of the adjoint problem.
- First, we cannot find the conjunct  $J(u, v)$  without considering the adjoint problem.
- Second, we have to simplify the expression  $J(u, G) \Big|_a^b$  using the adjoint boundary conditions. In particular, use the fact that  $G(x, x')$  obeys the adjoint boundary conditions with respect to  $x'$ .

## PROPERTIES OF GREEN'S FUNCTIONS

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- The Green's function gives us a lot of information about the system we're dealing with.
- Here we'll look at a few of the properties Green's functions can have and what those tell us about our system.

## RECIPROCITY

If  $\mathcal{L}$  is self-adjoint

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

then

$$G(x, x') = G^*(x', x)$$

- For self-adjoint problems,  $G = H$ .
- Using our relationship between  $G$  and  $H$ , we quickly see that the Green's function is complex symmetric (Hermitian) for self-adjoint problems.
- Roughly, putting a source at  $x$  and measuring at  $x'$  is the same as putting a source at  $x'$  and measuring at  $x$ . This is often called reciprocity (though it's not quite the same as reciprocity from time-harmonic E&M, because we're using a true inner product rather than the so-called "reaction inner product").
- Note that we require *both*  $\mathcal{L} = \mathcal{L}^*$  and  $\mathcal{B} = \mathcal{B}^*$ . So this would likely not hold for initial condition problems.

## INVARIANCE

$\mathcal{L}$  is invariant if

$$\mathcal{L}[u(x - \xi)] = \mathcal{L}[u(x)] \Big|_{x=x-\xi}$$

For example,

$$\mathcal{L} = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c$$

is invariant only if  $a$ ,  $b$ , and  $c$  are constants.

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## INVARIANCE

If  $\mathcal{L}$  is invariant in  $x$ , then

$$\begin{aligned}\mathcal{L}[G(x, x')] &= \delta(x - x') \\ \mathcal{L}[G(x - \xi, x' - \xi)] &= \delta(x - x')\end{aligned}$$

$$\implies G(x, x') = G(x - \xi, x' - \xi)$$

$G(x, x') = G(x - x')$

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- An operator is invariant if shifting the input just results in a shifted output.
- Probably familiar from signal processing: linear *time-invariant* systems. Delaying the input signal leads to the same output signal, just delayed by the same amount.
- Invariance is really important in modern physics.
- Maxwell's equations in free space are invariant with respect to  $x, y, z, \phi, \theta$ . That is, at a fundamental level, the laws of electromagnetism do not change if we move to a different location or look in a different direction.

- For invariant problems, we can see that shifting both  $x$  and  $x'$  by the same amount  $\xi$  does not affect the Green's function.
- Taking  $\xi = x'$ , we see that  $G(x, x')$  is actually only a function of the difference  $(x - x')$ .
- That is, the response only depends on the *relative* locations of the source and measurement. This fits nicely with our intuitive understanding of invariance.

## INVARIANCE

Convolution:

$$u(x) = \int_a^b G(x - x')f(x') \, dx' = G(x) * f(x)$$

Frequency domain:

$$\tilde{u}(k) = \tilde{G}(k)\tilde{f}(k)$$

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- For invariant systems (with boundary conditions zero), the solution  $u(x)$  is just given as a convolution of the source function  $f(x)$  with the impulse response  $G(x)$ .
- Taking Fourier transforms, the convolution turns into multiplication.
- Looks familiar from signal processing!

## SPECTRAL METHODS

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- In this section, we'll look at the strong relationship between Green's functions and spectral theory.
- Essentially, eigenfunction expansion allows us to calculate the Green's function when direct methods don't work.
- A basic background in spectral theory can be found in most books covering Green's functions.
- Unfortunately, these are rarely rigorous when dealing with continuous sets of eigenvalues. For fully rigorous spectral theory (not for the faint of heart!), see Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications*.

## EIGENFUNCTION EXPANSION

Problem:

$$(\mathcal{L} - \lambda) u(x) = f(x); \quad \mathcal{B}[u(x)] = 0$$

where  $\mathcal{L}$  is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

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- First we'll review a bit of eigenfunction theory, but we'll quickly see how it relates to Green's functions.
- Set up a similar problem as before, but we've added a complex parameter  $\lambda$  for later convenience.
- Also, for this section we'll insist that  $\mathcal{L}$  be fully self-adjoint so that we can take full advantage of spectral theory.
- A brief discussion of the non-self-adjoint case can be found in Morse and Feshbach under "Non-Hermitian operators: biorthogonal functions".

## EIGENFUNCTION EXPANSION

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n\phi_n(x)$$

With  $\mathcal{L}$  self-adjoint,  $\lambda_n \in \mathbb{R}$  and

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

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- Because  $\mathcal{L}$  is self-adjoint, we know that its eigenvalues  $\lambda_n$  are real.
- We also know that it has a complete orthonormal set of eigenfunctions  $\phi_n$ .
- That is, we can expand any function (in this case  $u(x)$  and  $f(x)$ ) in terms of  $\phi_n(x)$ . (Generalized Fourier series.)

## EIGENFUNCTION EXPANSION

$$\begin{aligned}
 (\mathcal{L} - \lambda)u(x) &= f(x) \\
 (\mathcal{L} - \lambda) \left[ \sum_n \langle u, \phi_n \rangle \phi_n(x) \right] &= \sum_n \langle f, \phi_n \rangle \phi_n(x) \\
 \sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda) \phi_n(x) &= \sum_n \langle f, \phi_n \rangle \phi_n(x) \\
 (\lambda_n - \lambda) \langle u, \phi_n \rangle &= \langle f, \phi_n \rangle
 \end{aligned}$$

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- Going back to our original equation, let's expand  $u(x)$  and  $f(x)$  in terms of eigenfunctions of  $\mathcal{L}$ .
- Using the fact that  $\mathcal{L}$  is linear and  $\mathcal{L}\phi_n = \lambda_n\phi_n$ , we can get rid of  $\mathcal{L}$  (third line).
- Finally, since the  $\phi_n(x)$  are linearly independent, each term in the sums on the RHS and LHS must be equal. So we get an expression for the generalized Fourier coefficients  $\langle u, \phi_n \rangle$ .

## EIGENFUNCTION EXPANSION

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

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- Plugging in our new expression for the Fourier coefficients, we obtain a formula for  $u(x)$  in terms of the eigenfunctions and eigenvalues of  $\mathcal{L}$ .

## EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left( \int_a^b \frac{f(x') \phi_n^*(x')}{\lambda_n - \lambda} dx' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left( \sum_n \frac{f(x) \phi_n^*(x')}{\lambda_n - \lambda} \phi_n(x) \right) dx'$$

$$u(x) = \int_a^b \left( \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') dx'$$

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## SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left( \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') dx'$$

$$G(x, x') = \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

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- Usually, the inner product is defined by an integral.
- If we write this out and do some manipulation, we get something that looks a lot like the Green's function expression.

- It turns out that we actually can read off this weird sum as the Green's function.
- So, if we know the eigenvalues and eigenfunctions of  $\mathcal{L}$ , we can immediately construct the Green's function as an infinite series.
- Note also that  $G(x, x') = G^*(x, x')$ , as we expect because this is a self-adjoint problem.



## SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of  $(\mathcal{L} - \lambda)$ :

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

$\lambda_n$  are poles of  $G(x, x', \lambda)$ .

$\phi_n(x)$  can be found by residue integration.

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- It also goes the other way. If we know the Green's function of  $(\mathcal{L} - \lambda)$  for any complex  $\lambda$ , then the eigenvalues of  $\mathcal{L}$  are just the poles of the Green's function with respect to  $\lambda$ .
- Eigenfunctions are a little trickier to read off, but it's possible to find them from the Green's function using residue integration.

## SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda)G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda) \phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x) \phi_n^*(x')}$$

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- Using our Green's function equation, we can also derive an expression for the delta function as a sum of eigenfunctions.
- This expression is useful when solving three-dimensional problems with separation of variables.

## EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left( \frac{d^2}{dx^2} - \lambda \right)}_{\mathcal{L} - \lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of  $\mathcal{L}$ :

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n^2}{a^2}$$

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- Let's do a simple example to illustrate the idea.
- For  $\mathcal{L} = d^2/dx^2$  we know that the eigenfunctions are sines and cosines. The boundary conditions restrict us to just sines with  $\lambda_n = \pi n^2/a^2$ .
- $\sqrt{2/a}$  ensures that the eigenfunctions are normalized.

## EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n^2 - \lambda a^2}$$

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- Using the formula we derived earlier, we can very quickly write out the Green's function as an infinite series.

## EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x, x') = \begin{cases} \frac{\sin(\sqrt{\lambda}(a-x')) \sin(\sqrt{\lambda}x)}{\sqrt{\lambda} \sin(\sqrt{\lambda}a)} & \text{for } x < x' \\ \frac{\sin(\sqrt{\lambda}x') \sin(\sqrt{\lambda}(a-x))}{\sqrt{\lambda} \sin(\sqrt{\lambda}a)} & \text{for } x > x' \end{cases}$$

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## THE 3D WAVE EQUATION

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- We could have also solved this problem directly (Assume  $G(x, x')$  behaves like the source-free solution except at  $x = x'$ . Apply the boundary conditions, continuity and discontinuity requirements to find the coefficient.) The result is shown.
  - The direct solution is a little uglier, but it's much easier to evaluate numerically because it doesn't involve an infinite series. For that reason, direct solution is usually more desirable if it actually works. However, series solutions tend to be needed for solving multi-dimensional problems.
  - Note: there's a trick for evaluating infinite series using residue calculus, and (I think) you could use this to derive the second expression from the first. You can also use residue integration to derive the first from the second.
- Unfortunately we don't have time to really get into 3D problems.
  - However, just to get a taste of it, we'll look at the 3D scalar wave equation in a vacuum.

## THE WAVE EQUATION PROBLEM

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)}_{\mathcal{L}} u(\mathbf{r}, t) = f(\mathbf{r}, t); \quad \mathcal{B}[u] = \alpha$$

In electromagnetism:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

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- The wave equation is a key component of electromagnetism.
- We've spent this entire course learning different ways to solve the Helmholtz equation, which is just a Fourier-transformed version of the wave equation.
- In the time domain (in the Lorentz gauge), each component of the vector potential  $\mathbf{A}$  obeys the scalar wave equation, with a component of  $\mathbf{J}$  as its source. So we would like to find a Green's functions approach to solving the scalar wave equation.

## THE ADJOINT WAVE EQUATION

Find  $\mathcal{L}$ ,  $\mathcal{L}^*$ ,  $\mathcal{B}$  and  $\mathcal{B}^*$  so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^*v \rangle$$

where

$$\langle u, v \rangle = \int_{t_i}^{t_f} \int_V u(\mathbf{r}, t) v^*(\mathbf{r}, t) d^3 \mathbf{r} dt$$

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- To construct  $u(\mathbf{r}, t)$  in terms of the Green's function, we proceed similar to the 1D case.
- The first step is to find the adjoint operators and adjoint boundary conditions.
- Before our inner product was an integral over the single variable  $x$ . Now, it's an integral over all four variables  $x, y, z, t$ .

## THE ADJOINT WAVE EQUATION

$$\langle \mathcal{L}u, v \rangle = \int_{t_i}^{t_f} \int_V \left( \nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \right) v^* d^3 \mathbf{r} dt$$

Use Green's identity

$$\int_V (v^* \nabla^2 u - u \nabla^2 v^*) = \oint_{\partial V} \left( v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) dS$$

and integration by parts

$$\int_{t_i}^{t_f} \left( \frac{\partial^2 u}{\partial t^2} v^* - u \frac{\partial^2 v^*}{\partial t^2} \right) dt = \left[ v^* \frac{\partial u}{\partial t} - u \frac{\partial v^*}{\partial t} \right]_{t_i}^{t_f}$$

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- To find  $\mathcal{L}^*$  and  $\mathcal{B}^*$ , we write out  $\langle \mathcal{L}u, v \rangle$  and then try to rewrite it in the form  $\langle u, \mathcal{L}^*v \rangle$  using theorems from calculus.
- Green's identity is essentially a 3D version of the integration by parts that we used in the 1D case. Note that  $V$  is a volume and  $\partial V$  is the boundary of that volume.
- We use Green's identity to deal with the spatial derivative part  $v^* \nabla^2 u$ .
- We use 1D integration by parts to deal with the time derivative part.

## THE ADJOINT WAVE EQUATION

Result:

$$\begin{aligned} \int_{t_i}^{t_f} \int_V (\mathcal{L}u) v^* d^3 \mathbf{r} dt &= \int_{t_i}^{t_f} \int_V u (\mathcal{L}v)^* d^3 \mathbf{r} dt + \\ &+ \int_{t_i}^{t_f} \oint_{\partial V} \left( v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) dS dt - \\ &- \frac{1}{c^2} \left[ \int_V \left( v^* \frac{\partial u}{\partial t} - u \frac{\partial v^*}{\partial t} \right) dV \right]_{t_i}^{t_f} \end{aligned}$$

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- Not pretty! Important thing is that it's similar to what we had in the 1D case.
- The first term is  $\langle \mathcal{L}u, v \rangle$  and the second term is  $\langle u, \mathcal{L}v \rangle$ . So, for the wave equation  $\mathcal{L} = \mathcal{L}^*$ .
- The last two terms are just a big ugly conjunct. The first one depends only on the spatial boundary conditions ( $\partial V$  is the boundary of the volume  $V$ ), while the second one depends only on the temporal boundary conditions.
- Given boundary conditions on  $u$ , the adjoint boundary conditions are the ones that make these last two terms go to zero. E.g., if  $u(\mathbf{r}, t) = 0$  over the whole boundary, then we would need  $v(\mathbf{r}, t) = 0$  over the whole boundary as well to cancel out the second-last integral.

## GREEN'S PROBLEMS

Original problem:

$$\mathcal{L}u(\mathbf{r}, t) = f(\mathbf{r}, t); \quad \mathcal{B}[u] = \alpha$$

Green's problem:

$$\mathcal{L}G(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}')\delta(t - t'); \quad \mathcal{B}[G] = 0$$

Adjoint Green's problem:

$$\mathcal{L}H(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}')\delta(t - t'); \quad \mathcal{B}^*[H] = 0$$

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## SOLUTION TO THE WAVE EQUATION

$$\begin{aligned} u(\mathbf{r}', t') = & \int_{t_i}^{t_f} \int_V f(\mathbf{r}, t) H^*(\mathbf{r}, t; \mathbf{r}', t') d^3 \mathbf{r} dt + \\ & - \int_{t_i}^{t_f} \oint_{\partial V} \left( H^* \frac{\partial u}{\partial n} - u \frac{\partial H^*}{\partial n} \right) dS dt - \\ & + \frac{1}{c^2} \left[ \int_V \left( H^* \frac{\partial u}{\partial t} - u \frac{\partial H^*}{\partial t} \right) dV \right]_{t_i}^{t_f} \end{aligned}$$

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- As in the 1D case, we set up three problems.
- For the lossless wave equation  $\mathcal{L} = \mathcal{L}^*$ , so the only difference between the  $G$  and  $H$  equations is the boundary conditions.

## SOLUTION TO THE WAVE EQUATION

This time,  $G(\mathbf{r}, t; \mathbf{r}', t') = H^*(\mathbf{r}', t'; \mathbf{r}, t)$  so

$$u(\mathbf{r}, t) = \int_{t_i}^{t_f} \int_{V'} f(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') d^3 \mathbf{r}' dt' + \\ - \int_{t_i}^{t_f} \oint_{\partial V'} \left( G \frac{\partial u}{\partial n'} - u \frac{\partial G}{\partial n'} \right) dS' dt' - \\ + \frac{1}{c^2} \left[ \int_{V'} \left( G \frac{\partial u}{\partial t'} - u \frac{\partial G}{\partial t'} \right) dV' \right]_{t_i}^{t_f}$$

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- Not very pretty, but we have a full solution for  $u(\mathbf{r}, t)$  in terms of the Green's function.
- The first integral gives the effect of the source term  $f(\mathbf{r}, t)$ .
- The second integral gives the effect of the boundary conditions on  $u$ .
- The third integral gives the effect of the initial conditions on  $u$ .
- Note that  $G(\mathbf{r}, t; \mathbf{r}', t')$  obeys the boundary conditions with respect to  $\mathbf{r}, t$  and the *adjoint* boundary conditions with respect to  $\mathbf{r}', t'$ . Use this for a given problem to simplify the last two terms.

## FINDING THE GREEN'S FUNCTION

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Initial conditions:

$$G(\mathbf{r}, 0; \mathbf{r}', t') = \left. \frac{\partial G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t} \right|_{t=0} = 0$$

Boundary conditions:

$$\lim_{|\mathbf{r}| \rightarrow \infty} G(\mathbf{r}, t; \mathbf{r}', t') \rightarrow 0$$

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- Now let's look at how we can actually solve the Green's function problem.
- We'll assume that we have an initial condition problem, and that we are interested in all of space (boundaries are at infinity).

## FINDING THE GREEN'S FUNCTION

Translation invariance:

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r} - \mathbf{r}', t - t')$$

Let  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  and  $\tau = c(t - t')$  so that

$$\left( \nabla_{\mathbf{R}}^2 - \frac{\partial^2}{\partial \tau^2} \right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R}) \delta(\tau)$$

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## FINDING THE GREEN'S FUNCTION

$$\left( \nabla_{\mathbf{R}}^2 - \frac{\partial^2}{\partial \tau^2} \right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R}) \delta(\tau)$$

Spatial Fourier transform:

$$\left( k^2 - \frac{\partial^2}{\partial \tau^2} \right) \tilde{G}(\mathbf{k}, \tau) = \delta(\tau)$$

so

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A \cos(k\tau) + B \sin(k\tau) & \text{for } \tau < 0 \\ C \cos(k\tau) + D \sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

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- Because the wave equation is translation invariant, we can simplify the Green's function to make the problem easier to solve.
- We define new variables  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  and  $\tau = t - t'$ .
- Note that  $\nabla^2$  is the laplacian with respect to  $\mathbf{R}$ .



## FINDING THE GREEN'S FUNCTION

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A \cos(k\tau) + B \sin(k\tau) & \text{for } \tau < 0 \\ C \cos(k\tau) + D \sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

- Initial conditions give  $A = B = 0$ .
- Continuity at  $t = t'$  gives  $C = 0$ .
- Discontinuity at  $t = t'$  gives  $D = -1/k$ .

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ -\frac{\sin(k\tau)}{k} & \text{for } \tau > 0 \end{cases}$$

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## FINDING THE GREEN'S FUNCTION

Inverse Fourier transform for  $\tau > 0$ :

$$G(\mathbf{R}, \tau) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \tilde{G}(\mathbf{k}, \tau) e^{i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{k}$$

$$G(\mathbf{R}, \tau) = \frac{-1}{8\pi^3} \int_{\mathbb{R}^3} \frac{\sin(k\tau)}{k} e^{i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{k}$$

Use spherical coordinates in  $\mathbf{k}$ :  $(k, \theta, \phi)$ .

$$G(\mathbf{R}, \tau) = \frac{-1}{8\pi^3} \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{\sin(k\tau)}{k} e^{ikR \cos(\theta)} k^2 \sin(\theta) dk d\theta d\phi$$

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## FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R}, \tau) = \frac{-1}{8\pi^3} \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{\sin(k\tau)}{k} e^{jkR \cos(\theta)} k^2 \sin(\theta) \, dk \, d\theta \, d\phi$$

$$G(\mathbf{R}, \tau) = \frac{-1}{4\pi^2} \int_0^\infty k \sin(k\tau) \left[ \int_0^\pi \sin(\theta) e^{jkR \cos(\theta)} \, d\theta \right] \, dk$$

$$G(\mathbf{R}, \tau) = \frac{-1}{4\pi^2} \int_0^\infty k \sin(k\tau) \left[ \frac{2 \sin(kR)}{kR} \right] \, dk$$

$$G(\mathbf{R}, \tau) = \frac{-1}{2\pi^2 R} \int_0^\infty \sin(k\tau) \sin(kR) \, dk$$

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## FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R}, \tau) = \frac{-1}{2\pi^2 R} \int_0^\infty \sin(k\tau) \sin(kR) \, dk$$

$$G(\mathbf{R}, \tau) = \frac{-1}{4\pi^2 R} \int_{-\infty}^\infty \sin(k\tau) \sin(kR) \, dk$$

$$G(\mathbf{R}, \tau) = \frac{-1}{4\pi^2 R} \int_{-\infty}^\infty \left( \frac{e^{jk\tau} - e^{-jk\tau}}{2j} \right) \left( \frac{e^{jkR} - e^{-jkR}}{2j} \right) \, dk$$

$$G(\mathbf{R}, \tau) = \frac{1}{16\pi^2 R} \int_{-\infty}^\infty (e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}) \, dk$$

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## FINDING THE GREEN'S FUNCTION

$$G(\mathbf{R}, \tau) = \frac{1}{16\pi^2 R} \int_{-\infty}^{\infty} (e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}) dk$$

$$G(\mathbf{R}, \tau) = \frac{1}{16\pi^2 R} 2\pi [\delta(\tau + R) - \delta(\tau - R) - \delta(\tau - R) + \delta(\tau + R)]$$

But  $\tau > 0$  and  $R > 0$  so  $\delta(\tau + R) = 0$  and we have

$$G(\mathbf{R}, \tau) = \frac{-\delta(\tau - R)}{4\pi R}$$

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## GREEN'S FUNCTION FOR THE WAVE EQUATION

$$G(\mathbf{r}, t; \mathbf{r}', t') = \begin{cases} -\frac{\delta(c(t-t') - |\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

With zero boundary/initial conditions:

$$u(\mathbf{r}, t) = \int_{t_i}^{t_f} \int_{\mathbb{R}^3} G(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t') d^3 \mathbf{r}' dt'$$

$$u(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{-f\left(\mathbf{r}, t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

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## VECTOR POTENTIAL: INITIAL CONDITION PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

For initial conditions, we have the “retarded potential”:

$$\mathbf{A}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \int_{\mathbb{R}^3} \tilde{\mathbf{J}}(\mathbf{r}, \omega) \frac{e^{-j\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

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## VECTOR POTENTIAL: FINAL VALUE PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

For final conditions, we have the “advanced potential”:

$$\mathbf{A}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}', t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

Violation of causality?

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## CONCLUSION

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## TAKEAWAYS

Introduction:

- The Green's function is the impulse response
- Use it to construct solutions. Ignoring boundary conditions:

$$u(x) = \int f(x') G(x, x') dx'$$

## TAKEAWAYS

Generalized functions:

- The delta function is an operator, not an actual function.

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## TAKEAWAYS

Direct solution:

- $G(x, x')$  obeys source-free equation for  $x \neq x'$ .
- $G(x, x')$  is continuous at  $x = x'$ .
- $\frac{dG(x, x')}{dx}$  is discontinuous at  $x = x'$ .

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## TAKEAWAYS

Boundary conditions:

- Non-zero boundary conditions act like impulse sources.
- Best way to construct  $u(x)$  from  $G(x, x')$  is to use the adjoint problem.

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## TAKEAWAYS

Properties of Green's functions:

- Self-adjoint problems:  $G(x, x') = G^*(x', x)$  (reciprocity).
- Invariant problems:  $G(x, x') = G(x - x')$ .

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## TAKEAWAYS

Spectral methods:

- Green's function can be found from eigenfunctions/eigenvalues:

$$G(x, x') = \sum_n \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n - \lambda}$$

- Eigenfunctions/eigenvalues can be found from Green's function.
- Delta function can be written in terms of eigenfunctions:

$$\delta(x - x') = \sum_n \phi_n(x) \phi_n^*(x')$$

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## TAKEAWAYS

The 3D wave equation:

- In general, use adjoint problem to write  $u(\mathbf{r}, t)$  in terms of  $G(\mathbf{r}, t; \mathbf{r}', t')$ .
- For free space initial condition problem, solution is the so-called “retarded potential.”

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## FURTHER TOPICS

- Sturm-Liouville problems
- Complex contour integration
- Separation of variables (3D problems)
- Dyadic Green's functions

## INTRODUCTORY RESOURCES

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Fully rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

## ADVANCED RESOURCES

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshbach, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.