green's functions

A short introduction

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OUTLINE

BASIC IDEA

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

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Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x')\, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x')\, \mathrm{d}x$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}x$$

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}x}$$

(Some conditions apply.)

FAMILIAR GREEN'S FUNCTIONS

Impulse response of an LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t') h(t - t') \, \mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

FINDING THE GREEN'S FUNCTION

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = 0$$

So we have

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

$$\frac{d^2G(x,x')}{dx^2} - k^2G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x'+\epsilon, x') - G(x'-\epsilon, x') \right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x')\right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\, \mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$\int\limits_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x')\right] \mathrm{d}x = \int\limits_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon}\right] = 1$$

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x, x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x, x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

Solving, our Green's function is

$$G(x,x') = \frac{1}{2k}\begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k}\, \mathrm{d}x'$$

GENERAL APPROACH

Second-order problems:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ is continuous at $x = x'$.
- Derivative of $G(x, x')$ is discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

$$
\begin{aligned}
\text{Original:} \quad & \mathcal{L}[u(x)] = f(x); \quad & \mathcal{B}[u(x)] = 0 \\
\text{Adjoint:} \quad & \mathcal{L}^*[v(x)] = f(x); \quad & \mathcal{B}^*[v(x)] = 0
\end{aligned}
$$

Defining property:

$$
\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \qquad \text{where } \langle u, v\rangle = \int_a^b u(x) v^*(x)\, \mathrm{d}x
$$

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}x + \\ + \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

If we take

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (k^2)^*$$

then we have

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \left[u'(x)v^*(x) - u(x)v^{*\prime}(x)\right]_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$
$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v^*)\Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \Longleftrightarrow J(u, v^*)\Big|_a^b = 0$$

ADJOINT GREEN'S FUNCTIONS

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x,x')] = \delta(x - x'); \qquad \mathcal{B}[G(x,x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x,x')] = \delta(x - x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\,\mathrm{d}x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}x + J\big(u(x), H^*(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x) H^*(x,x')\,\mathrm{d}x - J\big(u(x), H^*(x,x')\big)\Big|_a^b}$$

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'')\rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'')\rangle$$
$$\langle \delta(x - x'), H(x, x'')\rangle = \langle G(x, x'), \delta(x - x'')\rangle$$
$$\int_a^b \delta(x - x')H^*(x, x'')\,\mathrm{d}x = \int_a^b G(x, x')\delta(x - x'')\,\mathrm{d}x$$
$$H^*(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = H^*(x', x)}$$

$$u(x') = \int_a^b f(x) H^*(x, x')\, \mathrm{d}x - J\big(u(x), H^*(x, x')\big)\Big|_a^b$$

and

$$G(x, x') = H^*(x', x)$$

so

$$\boxed{u(x) = \int_a^b f(x') G(x, x')\, \mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^b}$$

EXAMPLE: 1D POISSON EQUATION

Original problem:

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho}{\epsilon_0}; \quad \begin{bmatrix} V(a) \\ V(b) \end{bmatrix} = \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} = \delta(x - x'); \quad \begin{bmatrix} G(a, x') \\ G(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\, \mathrm{d}x' - J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Take $\rho = 0$ for now.

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

EXAMPLE: 1D POISSON EQUATION

Can show

$$\int_a^b \left(\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\right) v^*(x)\,\mathrm{d}x = \int_a^b u(x) \left(\frac{\mathrm{d}^2 v^*(x)}{\mathrm{d}x^2}\right) \mathrm{d}x + \\ + \left[\frac{\mathrm{d}u}{\mathrm{d}x} v^*(x) - u(x)\frac{\mathrm{d}v^*(x)}{\mathrm{d}x}\right]_a^b$$

So $\mathcal{L} = \mathcal{L}^*$, $\mathcal{B} = \mathcal{B}^*$, and

$$J(u, v^*) = \frac{\mathrm{d}u(x)}{\mathrm{d}x} v^*(x) - u(x)\frac{\mathrm{d}v^*(x)}{\mathrm{d}x}$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x, x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x, x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x, b) -$$
$$-V_a\frac{\mathrm{d}G(x, a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x, a)$$

EXAMPLE: 1D POISSON EQUATION

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x,x')}{\mathrm{d}x^2} = \delta(x - x'); \quad \begin{bmatrix} H(a,x') \\ H(b,x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $G(x,x') = H^*(x',x)$ so

$$\frac{\mathrm{d}^2 G^*(x,x')}{\mathrm{d}x'^2} = \delta(x - x'); \quad \begin{bmatrix} G(x,a) \\ G(x,b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'} G(x,b) -$$
$$-V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'} G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}$$

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}$$

Can be written as

$$V(x) = \int_a^b \left[-V_a \delta'(x'-a) + V_b \delta'(x'-b)\right] G(x,x')\,\mathrm{d}x'$$

Full solution with $\rho(x)$:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} + V_a \frac{\mathrm{d}G(x, a)}{\mathrm{d}x'}$$

CONSTRUCTING SOLUTIONS

1. Given

$$\mathcal{L}u(x) = f(x); \quad \mathcal{B}[u(x)] = \alpha$$

2. Solve Green's problem

$$\mathcal{L}G(x, x') = \delta(x - x'); \quad \mathcal{B}[G(x, x')] = 0$$

3. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, v)$ from $\langle \mathcal{L}u, v \rangle$.
4. Solution is

$$u(x) = \int_a^b f(x')G(x, x')\,\mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

5. Simplify using $\mathcal{B}^*[G(x, x')] = 0$ (with respect to x').

CONSTRUCTING SOLUTIONS IN 3D

1. Given

$$\mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}[u(\mathbf{r})] = \alpha$$

2. Solve Green's problem

$$\mathcal{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'); \quad \mathcal{B}[G(\mathbf{r}, \mathbf{r}')] = 0$$

3. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, v)$ from $\langle \mathcal{L}u, v \rangle$.
4. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} J\big(u(\mathbf{r}'), G(\mathbf{r}, \mathbf{r}')\big)\,\mathrm{d}s$$

5. Simplify using $\mathcal{B}^*[G(\mathbf{r}, \mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

constructing solutions in 3d

Finding $\mathcal{L}^*$, $\mathcal{B}^*$ and $J(u,v)$ in 3D:

$$\langle \mathcal{L}u, v\rangle = \int_V \left[\mathcal{L}u(\mathbf{r})\right] v^*(\mathbf{r})\,\mathrm{d}^3\mathbf{r}$$

Green's second identity:

$$\int_V \left(u\nabla^2 v - v\nabla^2 u\right) \mathrm{d}^3\mathbf{r} = \oint_{\partial V} \left(u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u\right)\cdot \mathrm{d}\mathbf{s}$$

SPECTRAL METHODS

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

Since $\mathcal{L}$ is self-adjoint,

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \, \phi_n(x)$$

EIGENFUNCTION EXPANSION

$$
\begin{aligned}
(\mathcal{L} - \lambda)u(x) &= f(x) \\
(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] &= \sum_n \langle f, \phi_n \rangle \phi_n(x) \\
\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) &= \sum_n \langle f, \phi_n \rangle \phi_n(x) \\
(\lambda_n - \lambda) \langle u, \phi_n \rangle &= \langle f, \phi_n \rangle
\end{aligned}
$$

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}x'$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda)G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda)\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda)G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda)\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

SPECTRAL FORM OF THE DELTA FUNCTION

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$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\right)}_{\mathcal{L} - \lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2 \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x, x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right) \sin\left(\sqrt{\lambda} x\right)}{\sqrt{\lambda} \sin\left(\sqrt{\lambda} a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda} x'\right) \sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda} \sin\left(\sqrt{\lambda} a\right)} & \text{for } x > x' \end{cases}$$

CONCLUSION

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Green's functions $\Longleftrightarrow$ eigenvalues/eigenfunctions.

FURTHER READING

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Fully rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves.* Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics*. Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.