Green's functions

A short introduction

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OUTLINE

1. Basic idea
2. Finding the Green's function
3. Constructing the solution
4. Special properties
5. Conclusion
6. Bonus sections!

BASIC IDEA

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x')\, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x')\, \mathrm{d}x$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x')\,\mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x')\,\mathrm{d}x$$

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}x}$$

(Some conditions apply.)

familiar green's functions

Impulse response of a linear time-invariant system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Green's functions let us:

- Derive and understand these expressions.
- Generalize to other problems and boundary conditions.

FINDING THE GREEN'S FUNCTION

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}x^2} - k^2 G(x, x') = \delta(x - x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

A SIMPLE EXAMPLE

$$\frac{d^2G(x,x')}{dx^2} - k^2G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x')\right] = 0$$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left. \frac{\mathrm{d}G}{\mathrm{d}x} \right|_{x=x'+\epsilon} - \left. \frac{\mathrm{d}G}{\mathrm{d}x} \right|_{x=x'-\epsilon} \right] = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')\, \mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x,x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x,x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

Solving, our Green's function is

$$G(x, x') = \frac{1}{2k} \begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x, x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \,\mathrm{d}x'$$

GENERAL APPROACH

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ and its derivatives are continuous or discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

ADJOINT OPERATORS

Inner product:

$$\langle u, v \rangle = \int_a^b u(x)\overline{v}(x)\,\mathrm{d}x$$

Adjoint operator $\mathcal{L}^*$:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

ADJOINT BOUNDARY CONDITIONS

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle \longrightarrow \text{Only for certain } u, v\,!$$

Adjoint boundary conditions:

$$\mathcal{B}_i[u] = 0; \qquad \mathcal{B}_i^*[v] = 0$$

Otherwise

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + \underbrace{J(u, \overline{v})}_{\text{Conjunct}} \Big|_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}u(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

Want $\mathcal{L}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + J(u, \overline{v})\Big|_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[u'' + k^2 u\right] \overline{v}\, \mathrm{d}x$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b \left[-u'\overline{v}' + k^2 u\overline{v}\right] \mathrm{d}x + [u'\overline{v}]_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

$$\langle \mathcal{L}u, v\rangle = \int_a^b u\left[\overline{v}'' + k^2\overline{v}\right]\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Looks like

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

with

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \overline{k^2}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

ADJOINT OPERATORS: EXAMPLE

For

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

must have

$$J(u, \overline{v})\Big|_a^b = 0$$
$$\left[u'\overline{v} - u\overline{v}'\right]_a^b = 0$$

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Boundary conditions:

$$\begin{array}{c} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u(b) = 0 \end{array} \implies \begin{array}{c} \mathcal{B}_1^*[v] = v(a) = 0 \\ \mathcal{B}_2^*[v] = v(b) = 0 \end{array}$$

$$\mathcal{B}_i = \mathcal{B}_i^*$$

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Initial conditions:

$$\begin{array}{c} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u'(a) = 0 \end{array} \implies \begin{array}{c} \mathcal{B}_1^*[v] = v(b) = 0 \\ \mathcal{B}_2^*[v] = v'(b) = 0 \end{array}$$

$$\mathcal{B}_i \neq \mathcal{B}_i^*$$

ADJOINT OPERATORS: SUMMARY

Adjoint operator:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

Adjoint boundary conditions:

$$\begin{aligned} \mathcal{B}_i[u] &= 0 \\ \mathcal{B}_i^*[v] &= 0 \end{aligned} \implies J(u, \overline{v})\Big|_a^b = 0$$

Otherwise:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + J(u, \overline{v})\Big|_a^b$$

THE ADJOINT GREEN'S FUNCTION

Original problem:

$$\mathcal{L}u(x) = f(x); \qquad \mathcal{B}_i[u(x)] = \alpha_i$$

Green's problem:

$$\mathcal{L}G(x, x') = \delta(x - x'); \qquad \mathcal{B}_i[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^* H(x, x') = \delta(x - x'); \qquad \mathcal{B}_i^*[H(x, x')] = 0$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\,\mathrm{d}x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}x + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b}$$

CONSTRUCTING SOLUTIONS: DERIVATION

How are $G(x, x')$ and $H(x, x')$ related?

$$\langle \mathcal{L}G(x, x'), H(x, x'') \rangle = \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle$$

$$\langle \delta(x - x'), H(x, x'') \rangle = \langle G(x, x'), \delta(x - x'') \rangle$$

$$\int_a^b \delta(x - x')\overline{H}(x, x'')\,\mathrm{d}x = \int_a^b G(x, x')\delta(x - x'')\,\mathrm{d}x$$

$$\overline{H}(x', x'') = G(x'', x')$$

$$\boxed{G(x, x') = \overline{H}(x', x)}$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

and

$$G(x,x') = \overline{H}(x',x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'), G(x,x')\big)\Big|_{x'=a}^b}$$

EXAMPLE: 1D POISSON EQUATION

Original problem:

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}; \qquad \begin{aligned} V(a) &= V_a \\ V(b) &= V_b \end{aligned}$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{aligned} G(a,x') &= 0 \\ G(b,x') &= 0 \end{aligned}$$

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x') \, \mathrm{d}x' - J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Take $\rho(x) = 0$ for now.

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

EXAMPLE: 1D POISSON EQUATION

Can show

$$\langle \mathcal{L}u, v\rangle = \int_a^b u''\overline{v}\,\mathrm{d}x = \int_a^b u\overline{v}''\,\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Comparing with

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

we see

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \mathcal{L}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) - $$
$$-V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a)$$

EXAMPLE: 1D POISSON EQUATION

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{aligned} G(a,x') &= 0 \\ G(b,x') &= 0 \end{aligned}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x,x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{aligned} H(a,x') &= 0 \\ H(b,x') &= 0 \end{aligned}$$

But $G(x,x') = \overline{H}(x',x)$ so

$$\frac{\mathrm{d}^2 \overline{G}(x,x')}{\mathrm{d}x'^2} = \delta(x - x'); \qquad \begin{aligned} G(x,a) &= 0 \\ G(x,b) &= 0 \end{aligned}$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'} G(x,b) -$$
$$- V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'} G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b \frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x,a)}{\mathrm{d}x'}}$$

Full solution with $\rho(x)$:

$$\boxed{V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x, a)}{\mathrm{d}x'}}$$

SOLVING PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(x) = f(x); \quad \mathcal{B}_i[u(x)] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(x, x') = \delta(x - x'); \quad \mathcal{B}_i[G(x, x')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(x) = \int_a^b f(x') G(x, x') \, \mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

4. Simplify using $\mathcal{B}^*[G(x, x')] = 0$ (with respect to x').

SOLVING 3D PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}_i[u(\mathbf{r})] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'); \quad \mathcal{B}_i[G(\mathbf{r},\mathbf{r}')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u,\overline{v})$ from $\langle \mathcal{L}u, v\rangle$.
3. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} \mathbf{J}\big(u(\mathbf{r}'), G(\mathbf{r},\mathbf{r}')\big)\cdot \mathrm{d}\mathbf{s}$$

4. Simplify using $\mathcal{B}^*[G(\mathbf{r},\mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

CONSTRUCTING THE SOLUTION

Comments:

- Calculating Green's functions is not trivial.
- Adjoint approach is difficult, but offers clarity.

SPECIAL PROPERTIES

SELF-ADJOINTNESS

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B}_i = \mathcal{B}_i^*$, then

$$G(x, x') = \overline{G}(x', x)$$

RECIPROCITY

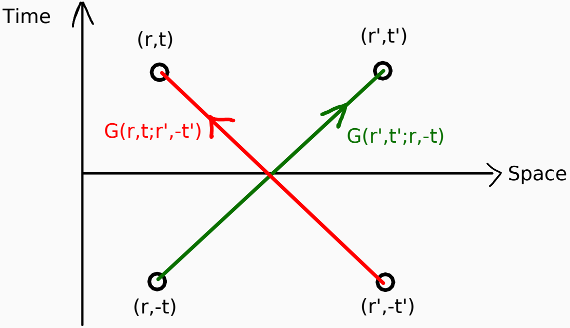
Reciprocity in the frequency-domain:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

RECIPROCITY

Reciprocity in the time-domain:

$$G(\mathbf{r}, t; \mathbf{r}', -t') = G(\mathbf{r}', t'; \mathbf{r}, -t)$$



CAUSALITY

Causality:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } t < t'$$

Special relativity:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } |\mathbf{r} - \mathbf{r}'| > c(t - t')$$

SYMMETRY AND INVARIANCE

Time-invariance:

$$G(t, t') = G(t - t')$$

Spatial-invariance:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}')$$

SPECTRAL THEORY

$$(\mathcal{L} - \lambda)u(x) = f(x); \quad \mathcal{B}[u] = 0$$

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$ then

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

SPECTRAL THEORY

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\, \mathrm{d}x'$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x'; \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

$$\begin{aligned} \lambda_n &\longrightarrow \text{ Poles of } G(x, x'; \lambda) \\ \phi_n(x)\phi_n^*(x') &\longrightarrow \text{ Residues of } G(x, x'; \lambda) \end{aligned}$$

CONCLUSION

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Lots of information in the Green's function.
- Green's functions $\iff$ eigenvalues/eigenfunctions.

FURTHER READING

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Gerlach (2010), *Linear mathematics in infinite dimensions*. Nice set of online course notes for quick reference.

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Morse and Feshback, *Methods of theoretical physics*. Big, detailed reference. Great resource for deeper insight and understanding.

FURTHER READING

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Folland (1992), *Fourier analysis and its applications*. Rigorous math book. Chapter on generalized functions is particularly nice.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.

BONUS SECTIONS!

SPECTRAL METHODS

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

EIGENFUNCTION EXPANSION

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

Since $\mathcal{L}$ is self-adjoint,

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

EIGENFUNCTION EXPANSION

$$(\mathcal{L} - \lambda)u(x) = f(x)$$

$$(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$(\lambda_n - \lambda) \langle u, \phi_n \rangle = \langle f, \phi_n \rangle$$

EIGENFUNCTION EXPANSION

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

EIGENFUNCTION EXPANSION

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}x'$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda)G(x, x')$$

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EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\right)}_{\mathcal{L} - \lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

$$G(x,x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

Compare with direct method:

$$G(x,x') = \begin{cases} \frac{\sin\left(\sqrt{\lambda}(a-x')\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x < x' \\ \frac{\sin\left(\sqrt{\lambda}x'\right)\sin\left(\sqrt{\lambda}(a-x)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}a\right)} & \text{for } x > x' \end{cases}$$

3D wave equation

THE WAVE EQUATION PROBLEM

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\mathcal{L}} u(\mathbf{r}, t) = f(\mathbf{r}, t); \qquad \mathcal{B}[u] = \alpha$$

In electromagnetism:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{J}(\mathbf{r}, t)$$

THE ADJOINT WAVE EQUATION

Find $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{B}$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

where

$$\langle u, v\rangle = \int_{t_i}^{t_f} \int_V u(\mathbf{r}, t)\overline{v}(\mathbf{r}, t)\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}t$$

THE ADJOINT WAVE EQUATION

$$\langle \mathcal{L}u, v\rangle = \int_{t_i}^{t_f} \int_V \left(\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right) \overline{v}\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}t$$

Use Green's identity

$$\int_V \left(\overline{v}\nabla^2 u - u\nabla^2 \overline{v}\right) = \oint_{\partial V} \left(\overline{v}\frac{\partial u}{\partial n} - u\frac{\partial \overline{v}}{\partial n}\right) \mathrm{d}S$$

and integration by parts

$$\int_{t_i}^{t_f} \left(\frac{\partial^2 u}{\partial t^2}\overline{v} - u\frac{\partial^2 \overline{v}}{\partial t^2}\right) \mathrm{d}t = \left[\overline{v}\frac{\partial u}{\partial t} - u\frac{\partial \overline{v}}{\partial t}\right]_{t_i}^{t_f}$$

Result:

$$\int_{t_i}^{t_f}\int_V (\mathcal{L}u)\,\overline{v}\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t = \int_{t_i}^{t_f}\int_V u\overline{(\mathcal{L}v)}\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}t +$$

$$+\int_{t_i}^{t_f}\oint_{\partial V}\left(\overline{v}\frac{\partial u}{\partial n} - u\frac{\partial \overline{v}}{\partial n}\right)\mathrm{d}S\,\mathrm{d}t -$$

$$-\frac{1}{c^2}\left[\int_V\left(\overline{v}\frac{\partial u}{\partial t} - u\frac{\partial \overline{v}}{\partial t}\right)\mathrm{d}V\right]_{t_i}^{t_f}$$

Original problem:

$$\mathcal{L}u(\mathbf{r},t) = f(\mathbf{r},t); \quad \mathcal{B}[u] = \alpha$$

Green's problem:

$$\mathcal{L}G(\mathbf{r},t;\mathbf{r}',t') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'); \quad \mathcal{B}[G] = 0$$

SOLUTION TO THE WAVE EQUATION

$$
\begin{aligned}
u(\mathbf{r}, t) = & \int_{t_i}^{t_f} \int_{V'} f(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') \, \mathrm{d}^3\mathbf{r}' \, \mathrm{d}t' + \\
& - \int_{t_i}^{t_f} \oint_{\partial V'} \left(G \frac{\partial u}{\partial n'} - u \frac{\partial G}{\partial n'} \right) \mathrm{d}S' \, \mathrm{d}t' - \\
& + \frac{1}{c^2} \left[\int_{V'} \left(G \frac{\partial u}{\partial t'} - u \frac{\partial G}{\partial t'} \right) \mathrm{d}V' \right]_{t_i}^{t_f}
\end{aligned}
$$

FINDING THE GREEN'S FUNCTION

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t; \mathbf{r}', t') = \delta^3(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

Initial conditions:

$$G(\mathbf{r}, 0; \mathbf{r}', t') = \left. \frac{\partial G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t} \right|_{t=0} = 0$$

Boundary conditions:

$$\lim_{|\mathbf{r}| \to \infty} G(\mathbf{r}, t; \mathbf{r}', t') \to 0$$

FINDING THE GREEN'S FUNCTION

Translation invariance:

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r} - \mathbf{r}', t - t')$$

Let $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = c(t - t')$ so that

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

FINDING THE GREEN'S FUNCTION

$$\left(\nabla_R^2 - \frac{\partial^2}{\partial \tau^2}\right) G(\mathbf{R}, \tau) = \delta^3(\mathbf{R})\delta(\tau)$$

Spatial Fourier transform:

$$\left(k^2 - \frac{\partial^2}{\partial \tau^2}\right) \tilde{G}(\mathbf{k}, \tau) = \delta(\tau)$$

so

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

FINDING THE GREEN'S FUNCTION

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} A\cos(k\tau) + B\sin(k\tau) & \text{for } \tau < 0 \\ C\cos(k\tau) + D\sin(k\tau) & \text{for } \tau > 0 \end{cases}$$

- Initial conditions give $A = B = 0$.
- Continuity at $t = t'$ gives $C = 0$.
- Discontinuity at $t = t'$ gives $D = -1/k$.

$$\tilde{G}(\mathbf{k}, \tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ -\dfrac{\sin(k\tau)}{k} & \text{for } \tau > 0 \end{cases}$$

FINDING THE GREEN'S FUNCTION

Inverse Fourier transform for $\tau > 0$:

$$G(\mathbf{R},\tau) = \frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3} \tilde{G}(\mathbf{k},\tau)e^{j\mathbf{k}\cdot\mathbf{R}}\,\mathrm{d}^3\mathbf{k}$$

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int\limits_{\mathbb{R}^3} \frac{\sin(k\tau)}{k}e^{j\mathbf{k}\cdot\mathbf{R}}\,\mathrm{d}^3\mathbf{k}$$

Use spherical coordinates in $\mathbf{k}$: (k,θ,ϕ).

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\infty} \frac{\sin(k\tau)}{k}e^{jkR\cos(\theta)}k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$G(\mathbf{R},\tau) = \frac{-1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\sin(k\tau)}{k}e^{jkR\cos(\theta)}k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\int_0^{\pi}\sin(\theta)e^{jkR\cos(\theta)}\,\mathrm{d}\theta\right]\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{4\pi^2}\int_0^{\infty}k\sin(k\tau)\left[\frac{2\sin(kR)}{kR}\right]\mathrm{d}k$$

$$G(\mathbf{R},\tau) = \frac{-1}{2\pi^2 R}\int_0^{\infty}\sin(k\tau)\sin(kR)\,\mathrm{d}k$$

FINDING THE GREEN'S FUNCTION

$$G(\mathrm{R},\tau) = \frac{-1}{2\pi^2 R} \int_0^\infty \sin(k\tau)\sin(kR)\,\mathrm{d}k$$

$$G(\mathrm{R},\tau) = \frac{-1}{4\pi^2 R} \int_{-\infty}^\infty \sin(k\tau)\sin(kR)\,\mathrm{d}k$$

$$G(\mathrm{R},\tau) = \frac{-1}{4\pi^2 R} \int_{-\infty}^\infty \left(\frac{e^{jk\tau} - e^{-jk\tau}}{2j}\right)\left(\frac{e^{jkR} - e^{-jkR}}{2j}\right)\mathrm{d}k$$

$$G(\mathrm{R},\tau) = \frac{1}{16\pi^2 R} \int_{-\infty}^\infty \left(e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}\right)\mathrm{d}k$$

FINDING THE GREEN'S FUNCTION

$$G(\mathrm{R}, \tau) = \frac{1}{16\pi^2 R} \int_{-\infty}^{\infty} \left(e^{jk(\tau+R)} - e^{-jk(\tau-R)} - e^{jk(\tau-R)} + e^{-jk(\tau+R)}\right) \mathrm{d}k$$

$$G(\mathrm{R}, \tau) = \frac{1}{16\pi^2 R} 2\pi \left[\delta(\tau + R) - \delta(\tau - R) - \delta(\tau - R) + \delta(\tau + R)\right]$$

But $\tau > 0$ and $R > 0$ so $\delta(\tau + R) = 0$ and we have

$$G(\mathrm{R}, \tau) = \frac{-\delta(\tau - R)}{4\pi R}$$

GREEN'S FUNCTION FOR THE WAVE EQUATION

$$G(\mathbf{r},t;\mathbf{r}',t') = \begin{cases} -\frac{\delta(c(t-t')-|\mathbf{r}-\mathbf{r'}|)}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

With zero boundary/initial conditions:

$$u(\mathbf{r},t) = \int_{t_i}^{t_f} \int_{\mathbb{R}^3} G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')\, \mathrm{d}^3\mathbf{r}'\, \mathrm{d}t'$$

$$u(\mathbf{r},t) = \int_{\mathbb{R}^3} \frac{-f\left(\mathbf{r}, t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}^3\mathbf{r'}$$

VECTOR POTENTIAL: INITIAL CONDITION PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathsf{A}(\mathbf{r},t) = -\mathsf{J}(\mathbf{r},t)$$

For initial conditions, we have the "retarded potential":

$$\mathsf{A}(\mathbf{r},t) = \int\limits_{\mathbb{R}^3} \frac{\mathsf{J}\left(\mathbf{r}, t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

$$\tilde{\mathsf{A}}(\mathbf{r},\omega) = \int\limits_{\mathbb{R}^3} \tilde{\mathsf{J}}\,(\mathbf{r},\omega)\,\frac{e^{-j\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

VECTOR POTENTIAL: FINAL VALUE PROBLEM

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\mathbf{J}(\mathbf{r},t)$$

For final conditions, we have the “advanced potential”:

$$\mathbf{A}(\mathbf{r},t) = \int\limits_{\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}, t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \int\limits_{\mathbb{R}^3} \tilde{\mathbf{J}}\left(\mathbf{r},\omega\right)\frac{e^{+j\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

GENERALIZED FUNCTIONS

TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

GENERALIZED FUNCTIONS

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$$

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}x$$

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\,\mathrm{d}x$$

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,\mathrm{d}x = \phi(0)$$

DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}t = \delta(x)$$

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\, \phi(x)\, \mathrm{d}x$$

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator!