GREEN'S FUNCTIONS

A SHORT INTRODUCTION

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OUTLINE

BASIC IDEA

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHAT IS A GREEN'S FUNCTION?

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x')\,\mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x')\,\mathrm{d}x$$

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}x$$

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x')f(x')\,\mathrm{d}x}$$

(Some conditions apply.)

FAMILIAR GREEN'S FUNCTIONS

Impulse response of a linear time-invariant system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}t'$$

FAMILIAR GREEN'S FUNCTIONS

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|} \rho(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'$$

FAMILIAR GREEN'S FUNCTIONS

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\mathbf{r}'$$

Green's functions let us:

- Derive and understand these expressions.
- Generalize to other problems and boundary conditions.

FINDING THE GREEN'S FUNCTION

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = 0$$

So we have

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A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x - x')$$

Continuity of the Green's function:

$$\lim_{\epsilon \to 0} \left[G(x' + \epsilon, x') - G(x' - \epsilon, x') \right] = 0$$

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') = \delta(x-x')$$

Discontinuity condition:

$$\lim_{\epsilon\to 0}\left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon}\right] = 1$$

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon \to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} - k^2 G(x,x') \right] \mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0} \left[\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=x'-\epsilon} \right] = 1$$

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of $G(x, x')$:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G(x, x')}{\mathrm{d}x}$:

$$kA + kB = 1$$

Solving, our Green's function is

$$G(x, x') = \frac{1}{2k} \begin{cases} e^{+k(x-x')} & \text{for } x < x' \\ e^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Or, more compactly:

$$\boxed{G(x, x') = \frac{e^{k|x-x'|}}{2k}}$$

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - k^2 u(x) = f(x)$$

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}x'$$

Direct solution:

- $G(x, x')$ obeys source-free equation for $x \neq x'$.
- $G(x, x')$ and its derivatives are continuous or discontinuous at $x = x'$.

CONSTRUCTING THE SOLUTION

$$u(x) = \int G(x, x')f(x')\,\mathrm{d}x'$$

Can we prove/generalize this?

ADJOINT OPERATORS

Inner product:

$$\langle u, v \rangle = \int_a^b u(x)\bar{v}(x)\,\mathrm{d}x$$

Adjoint operator $\mathcal{L}^*$:

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

ADJOINT BOUNDARY CONDITIONS

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \longrightarrow \text{Only for certain } u, v\,!$$

Adjoint boundary conditions:

$$\mathcal{B}_i[u] = 0; \qquad \mathcal{B}_i^*[v] = 0$$

Otherwise

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + \underbrace{J(u, \overline{v})}_{\text{Conjunct}}\Big|_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\mathcal{L}u(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right] u(x)$$

Want $\mathcal{L}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + J(u, \overline{v})\Big|_a^b$$

ADJOINT OPERATORS: EXAMPLE

$$\langle \mathcal{L}u, v \rangle = \int_a^b \left[u'' + k^2 u\right] \overline{v} \, \mathrm{d}x$$

$$\langle \mathcal{L}u, v \rangle = \int_a^b \left[-u'\overline{v}' + k^2 u\overline{v}\right] \mathrm{d}x + \left[u'\overline{v}\right]_a^b$$

$$\langle \mathcal{L}u, v \rangle = \int_a^b u \left[\overline{v}'' + k^2\overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

$$\langle \mathcal{L}u, v \rangle = \int_a^b u \left[\overline{v}'' + k^2 \overline{v}\right] \mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Looks like

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle + J(u, \overline{v})\Big|_a^b$$

with

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \overline{k^2}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

For

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

must have

$$J(u, \overline{v})\Big|_a^b = 0$$
$$\left[u'\overline{v} - u\overline{v}'\right]_a^b = 0$$

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Boundary conditions:

$$\begin{array}{c} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u(b) = 0 \end{array} \implies \begin{array}{c} \mathcal{B}_1^*[v] = v(a) = 0 \\ \mathcal{B}_2^*[v] = v(b) = 0 \end{array}$$

$$\mathcal{B}_i = \mathcal{B}_i^*$$

ADJOINT OPERATORS: EXAMPLE

$$u'(b)\overline{v}(b) - u(b)\overline{v}'(b) - u'(a)\overline{v}(a) + u(a)\overline{v}'(a) = 0$$

Initial conditions:

$$\begin{matrix} \mathcal{B}_1[u] = u(a) = 0 \\ \mathcal{B}_2[u] = u'(a) = 0 \end{matrix} \implies \begin{matrix} \mathcal{B}_1^*[v] = v(b) = 0 \\ \mathcal{B}_2^*[v] = v'(b) = 0 \end{matrix}$$

$$\mathcal{B}_i \neq \mathcal{B}_i^*$$

Adjoint operator:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$$

Adjoint boundary conditions:

$$\begin{array}{l} \mathcal{B}_i[u] = 0 \\ \mathcal{B}_i^*[v] = 0 \end{array} \implies J(u, \overline{v})\Big|_a^b = 0$$

Otherwise:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, \overline{v})\Big|_a^b$$

THE ADJOINT GREEN'S FUNCTION

Original problem:

$$\mathcal{L}u(x) = f(x); \qquad \mathcal{B}_i[u(x)] = \alpha_i$$

Green's problem:

$$\mathcal{L}G(x, x') = \delta(x - x'); \qquad \mathcal{B}_i[G(x, x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^* H(x, x') = \delta(x - x'); \qquad \mathcal{B}_i^*[H(x, x')] = 0$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\int_a^b f(x) H^*(x,x')\,\mathrm{d}x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}x + J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x), \overline{H}(x,x')\big)\Big|_a^b}$$

How are $G(x, x')$ and $H(x, x')$ related?

$$\begin{aligned}
\langle \mathcal{L}G(x, x'), H(x, x'') \rangle &= \langle G(x, x'), \mathcal{L}^* H(x, x'') \rangle \\
\langle \delta(x - x'), H(x, x'') \rangle &= \langle G(x, x'), \delta(x - x'') \rangle \\
\int_a^b \delta(x - x')\overline{H}(x, x'')\,\mathrm{d}x &= \int_a^b G(x, x')\delta(x - x'')\,\mathrm{d}x \\
\overline{H}(x', x'') &= G(x'', x')
\end{aligned}$$

$$\boxed{G(x, x') = \overline{H}(x', x)}$$

CONSTRUCTING SOLUTIONS: DERIVATION

$$u(x') = \int_a^b f(x)\overline{H}(x,x')\,\mathrm{d}x - J\big(u(x),\overline{H}(x,x')\big)\Big|_a^b$$

and

$$G(x,x') = \overline{H}(x',x)$$

so

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}x' - J\big(u(x'),G(x,x')\big)\Big|_{x'=a}^b}$$

Original problem:

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}; \qquad \begin{aligned} V(a) &= V_a \\ V(b) &= V_b \end{aligned}$$

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x - x'); \qquad \begin{aligned} G(a,x') &= 0 \\ G(b,x') &= 0 \end{aligned}$$

Solution:

$$V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' - J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Take $\rho(x) = 0$ for now.

$$V(x) = -J\big(V(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

Can show

$$\langle \mathcal{L}u, v\rangle = \int_a^b u''\overline{v}\,\mathrm{d}x = \int_a^b u\overline{v}''\,\mathrm{d}x + \left[u'\overline{v} - u\overline{v}'\right]_a^b$$

Comparing with

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v^*)\Big|_a^b$$

we see

$$\mathcal{L}^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \mathcal{L}$$

$$J(u, \overline{v}) = u'\overline{v} - u\overline{v}'$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = -J\big(V(x'), G(x,x')\big)\Big|_{x'=a}^{b}$$

$$V(x) = \left[V(x')\frac{\mathrm{d}G(x,x')}{\mathrm{d}x'} - \frac{\mathrm{d}V(x')}{\mathrm{d}x'}G(x,x')\right]_{x'=a}^{b}$$

$$V(x) = V_b\frac{\mathrm{d}G(x,b)}{\mathrm{d}x'} - \frac{\mathrm{d}V(b)}{\mathrm{d}x'}G(x,b) -$$

$$-V_a\frac{\mathrm{d}G(x,a)}{\mathrm{d}x'} + \frac{\mathrm{d}V(a)}{\mathrm{d}x'}G(x,a)$$

Example: 1D Poisson equation

Green's problem:

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{aligned} G(a,x') &= 0 \\ G(b,x') &= 0 \end{aligned}$$

Adjoint Green's problem:

$$\frac{\mathrm{d}^2 H(x,x')}{\mathrm{d}x^2} = \delta(x-x'); \qquad \begin{aligned} H(a,x') &= 0 \\ H(b,x') &= 0 \end{aligned}$$

But $G(x,x') = \overline{H}(x',x)$ so

$$\frac{\mathrm{d}^2 \overline{G}(x,x')}{\mathrm{d}x'^2} = \delta(x-x'); \qquad \begin{aligned} G(x,a) &= 0 \\ G(x,b) &= 0 \end{aligned}$$

EXAMPLE: 1D POISSON EQUATION

$$V(x) = V_b \frac{dG(x,b)}{dx'} - \frac{dV(b)}{dx'} G(x,b) - \\ -V_a \frac{dG(x,a)}{dx'} + \frac{dV(a)}{dx'} G(x,a)$$

With $G(x,a) = G(x,b) = 0$, we have

$$\boxed{V(x) = V_b \frac{dG(x,b)}{dx'} - V_a \frac{dG(x,a)}{dx'}}$$

Full solution with $\rho(x)$:

$$\boxed{V(x) = \int_a^b -\frac{\rho(x')}{\epsilon_0} G(x, x')\,\mathrm{d}x' + V_b \frac{\mathrm{d}G(x, b)}{\mathrm{d}x'} - V_a \frac{\mathrm{d}G(x, a)}{\mathrm{d}x'}}$$

SOLVING PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(x) = f(x); \quad \mathcal{B}_i[u(x)] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(x, x') = \delta(x - x'); \quad \mathcal{B}_i[G(x, x')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}_i^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(x) = \int_a^b f(x')G(x, x')\,\mathrm{d}x' - J\big(u(x'), G(x, x')\big)\Big|_{x'=a}^{b}$$

4. Simplify using $\mathcal{B}^*[G(x, x')] = 0$ (with respect to x').

SOLVING 3D PROBLEMS WITH GREEN'S FUNCTIONS

$$\text{Given:} \quad \mathcal{L}u(\mathbf{r}) = f(\mathbf{r}); \quad \mathcal{B}_i[u(\mathbf{r})] = \alpha_i$$

1. Solve Green's problem

$$\mathcal{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'); \quad \mathcal{B}_i[G(\mathbf{r}, \mathbf{r}')] = 0$$

2. Find $\mathcal{L}^*$, $\mathcal{B}^*$, and $J(u, \overline{v})$ from $\langle \mathcal{L}u, v \rangle$.
3. Solution is

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}' - \oint_{\partial V} \mathbf{J}\big(u(\mathbf{r}'), G(\mathbf{r}, \mathbf{r}')\big) \cdot \mathrm{d}\mathbf{s}$$

4. Simplify using $\mathcal{B}^*[G(\mathbf{r}, \mathbf{r}')] = 0$ (with respect to $\mathbf{r}'$).

Comments:

- Calculating Green's functions is not trivial.
- Adjoint approach is difficult, but offers clarity.

SPECIAL PROPERTIES

SELF-ADJOINTNESS

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B}_i = \mathcal{B}_i^*$, then

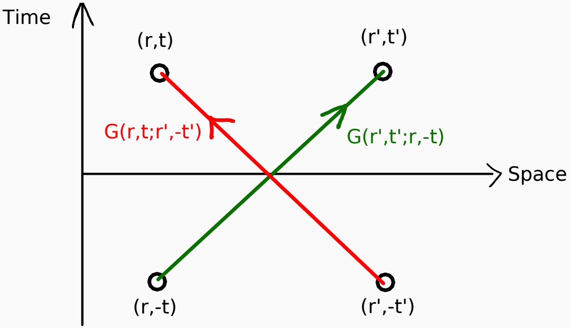
$$G(x, x') = \overline{G}(x', x)$$

Reciprocity in the frequency-domain:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

Reciprocity in the time-domain:

$$G(\mathbf{r}, t; \mathbf{r}', -t') = G(\mathbf{r}', t'; \mathbf{r}, -t)$$



CAUSALITY

Causality:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } t < t'$$

Special relativity:

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \text{for } |\mathbf{r} - \mathbf{r}'| > c(t - t')$$

Time-invariance:

$$G(t, t') = G(t - t')$$

Spatial-invariance:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}')$$

$$(\mathcal{L} - \lambda)u(x) = f(x); \quad \mathcal{B}[u] = 0$$

If $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$ then

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

SPECTRAL THEORY

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}x'$$

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x'; \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x)}{\lambda_n - \lambda}$$

$$\begin{aligned} \lambda_n &\longrightarrow \text{ Poles of } G(x, x'; \lambda) \\ \phi_n(x)\phi_n^*(x') &\longrightarrow \text{ Residues of } G(x, x'; \lambda) \end{aligned}$$

CONCLUSION

TAKEAWAYS

- Green's function is the impulse response.
- Finding Green's function:
    - Source-free behaviour for $x \neq x'$.
    - Continuity/discontinuity requirements at $x = x'$.
- Constructing solutions:
    - Systematic method using adjoint equation.
    - Non-zero boundary conditions behave like sources.
- Lots of information in the Green's function.
- Green's functions $\Longleftrightarrow$ eigenvalues/eigenfunctions.

FURTHER READING

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Gerlach (2010), *Linear mathematics in infinite dimensions*. Nice set of online course notes for quick reference.

Balanis (2012), *Advanced engineering electromagnetics*. Not very rigorous, but decent for getting the key ideas.

Morse and Feshback, *Methods of theoretical physics*. Big, detailed reference. Great resource for deeper insight and understanding.

FURTHER READING

Collin (1990), *Field theory of guided waves*. Huge chapter on Green's functions. Emphasis on dyadics.

Folland (1992), *Fourier analysis and its applications*. Rigorous math book. Chapter on generalized functions is particularly nice.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.