GREEN'S FUNCTIONS

A short introduction

Chris Deimert

November 20, 2015

Department of Electrical and Computer Engineering, University of Calgary

- This is intended as a small (dense) overview of Green's functions for electrical engineers.
- Basic idea of Green's functions is simple, but there is a huge amount of theory for actually calculating and using them.
- Tip: read a lot of different references. Different authors take totally different approaches and it's interesting to see them all. Lots of references are provided at the end.

OUTLINE

INTRODUCTION

- Fortunately, the basic idea of Green’s functions is really simple.
- You’ve actually used them before!
- What’s far more interesting is how to calculate/interpret them.

Linear equation to solve:

$$\mathcal{L}u(x) = f(x)$$

Green's function is the **impulse response**:

$$\mathcal{L}G(x, x') = \delta(x - x')$$

- Most EM problems are described by linear (differential) equations with some source/driving function $f(x)$.
- The Green's function is the solution when the source $f(x)$ is an impulse located at x'.
- Can think of it as a generalization of the impulse response from signal processing.

WHY IS IT USEFUL?

$$\delta(x - x') \xrightarrow{\mathcal{L}^{-1}} G(x, x')$$

$$f(x) = \int \delta(x - x') f(x') \, \mathrm{d}\, x \xrightarrow{\mathcal{L}^{-1}} \int G(x, x') f(x') \, \mathrm{d}\, x$$

- Once we know the Green's function for a problem, we can find the solution for any source $f(x)$.
- Impulses $\delta(x - x')$ produce a response $G(x, x')$.
- We can split the source $f(x)$ up into a sum (integral) of impulses $\delta(x - x')$.
- Then the response to $f(x)$ is just a weighted sum (integral) of impulse responses.

WHY IS IT USEFUL?

$$\mathcal{L}u(x) = f(x)$$

$$\mathcal{L}G(x, x') = \delta(x - x')$$

$$\boxed{u(x) = \int G(x, x') f(x') \, \mathrm{d}\, x}$$

- Once we know the Green's function, we have an explicit formula for the solution $u(x)$ for any source function $f(x)$.

Impulse response of a LTI system:

$$y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\,\mathrm{d}\,t'$$

E.g., for an RL-circuit:

$$G(t,t') = h(t-t') = u(t-t')e^{-\alpha(t-t')}$$

- In electrical engineering, we've seen Green's functions before.
- Impulse response $h(t-t')$ from linear system theory is an example of a Green's function.

$$G(t,t') = h(t-t')$$

- Usually find $h(t-t')$ using Fourier transform of the transfer function.

Poisson's equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$V(\mathbf{r}) = \iiint \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2} \rho(\mathbf{r}')\, \mathrm{d}^3\, \mathbf{r}'$$

- Green's function for Poisson's equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0 \left|\mathbf{r} - \mathbf{r}'\right|^2}$$

Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A_z(\mathbf{r}) = -J_z(\mathbf{r})$$

$$A_z(\mathbf{r}) = \iiint \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|} J_z\left(\mathbf{r}'\right) \mathrm{d}^3\,\mathbf{r}'$$

- Green's function for the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}$$

Our goal:

- Derive these expressions.
- Generalize to other problems and boundary conditions.

GENERALIZED FUNCTIONS

- Delta functions play a key role in Green’s functions (and electrical engineering in general), but tend to lead to hand-waving.
- Worth seeing how they can be rigorously defined before moving on.
- Machinery for this is Schwartz’s theory of distributions (generalized functions).
- See Folland (1992), *Fourier analysis and its applications*, Chapter 9 for more.

TYPICAL DELTA FUNCTION DEFINITION

Typical “definition” of $\delta(x - x_0)$:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0$$

$$\int\limits_{-\infty}^{\infty} \delta(x - x_0) = 1$$

- Often see definitions like this one.
- Often said to imply that $\delta(x - x_0) = \infty$ at $x = x_0$.
- Might be okay intuitively, but very imprecise mathematically.
- There is no true function which satisfies both of these requirements!

$f(x)$ defines a linear operator $\phi(x)$ via

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- Let’s see if we can generalize the idea of a “function” so that it includes delta functions.
- Given a function $f(x)$, we can use it to define a linear operator (a functional, to be exact) on other functions $\phi(x)$.
- $f[\cdot]$ is a linear operator. It takes a function $\phi(x)$ and returns the number

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- If we ensure that $\phi(x)$ is very well-behaved, then every function $f(x)$ defines an operator in this way.

GENERALIZED FUNCTIONS

If we have $f[\phi]$, but no $f(x)$, then f is a generalized function.

Symbolically, we write

$$f[\phi] \stackrel{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

- It's possible to have an operator $f[\phi]$, but we can't find an $f(x)$ to implement it via an integral.
- Then $f(x)$ is a generalized function. It is not a function in its own right, but it is defined purely by its action on other functions $f[\phi]$.
- We still symbolically write

$$f[\phi] \overset{S}{=} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}\,x$$

  but this just suggestive notation. It is not actually an integral unless $f(x)$ is a "proper" function!

$\delta(x - x_0)$ is a generalized function defined by the sifting property

$$\delta_{x_0}[\phi] = \phi(x_0) \stackrel{S}{=} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x$$

- We can define a simple linear operator via the sifting property $\delta_{x_0}[\phi] = \phi(x_0)$.
- There is no actual function $\delta(x - x_0)$ which gives

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,\mathrm{d}\,x = \phi(x_0)$$

so $\delta(x - x_0)$ is a generalized function and the above integral is purely symbolic.

We can define derivatives too:

$$\delta_{x_0}^{(n)}[\phi] = (-1)^n \phi^{(n)}(x_0) \stackrel{S}{=} \int\limits_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\phi(x)\, \mathrm{d}\, x$$

- Generalized function theory lets us make sense of the derivatives of the delta function too.
- $\delta_{x_0}^{(n)}$ is just an operator that picks out the value of the nth derivative of $\phi(x)$ at the point x_0.

DELTA FUNCTION LIMITS

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$$

if and only if

$$\lim_{\epsilon \to 0} f_\epsilon[\phi] = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\, \mathrm{d}\, x = \phi(0)$$

- Often useful to show that some set of actual functions $f_\epsilon(x)$ “approach” the delta function in a limit.
- To do this, we need to show that the sifting property is obeyed in the limit.

DELTA FUNCTION LIMITS

Limit of Gaussian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-x^2/2\epsilon^2}$$

Limit of Lorentzian functions:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

- Two examples of delta function limits.
- Confirms our intuition of the delta function as a limit of sharply-peaked functions.
- In fact, basically any limit of sharply-peaked functions of area 1 will work: see Folland (Theorem 9.2).

A more interesting example:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

because

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon^2 t^2} e^{jxt}\,\mathrm{d}\,t = \delta(x)$$

- Example of a common, but unintuitive expression for the Delta function.
- Can show that it’s true by expressing it as a delta function limit. (If you want to go through it, use Theorem 9.2 from Folland.)

WHAT DOES THIS MEAN FOR GREEN'S FUNCTIONS?

$$\mathcal{L}G(x, x') = \delta(x - x')$$

actually means

$$(\mathcal{L}G)\,[\phi] = \phi(x') \stackrel{S}{=} \int_{-\infty}^{\infty} (\mathcal{L}G(x, x'))\,\phi(x)\,\mathrm{d}\,x$$

- Technically, the Green's function is a generalized function such that $\mathcal{L}G$ is the delta function (it has the sifting property).
- In practise, we'll keep using the less-precise way; just remember that there is a more correct way.

TAKEAWAY

If in doubt, think of $\delta(x - x_0)$ as an operator, not a function!

- In practise, thinking of $\delta(x - x_0)$ as a function is usually fine. (We'll even do that for the rest of this presentation.)
- But if anything starts to seem fishy, it's good to remember that $\delta(x - x_0)$ is actually an operator, and not a function.

DIRECT SOLUTION

- Back to Green's functions!

A SIMPLE EXAMPLE

Original problem:

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}\, x^2} - k^2 u(x) = f(x)$$

Green's function problem:

$$\frac{\mathrm{d}^2 G(x, x')}{\mathrm{d}\, x^2} - k^2 G(x, x') = \delta(x - x')$$

- Let's look at a simple example now.
- This problem is similar to a simple harmonic oscillator, but the negative sign means we expect lossy behaviour rather than oscillation.
- We won't worry much about boundary conditions yet, we'll just look for solutions that don't blow up at $x = \pm\infty$.
- If we can find the Green's function, then we can find the solution to the original problem.
- But the Green's function problem looks pretty hard. The point of this example is to demonstrate that we can actually solve it.

A SIMPLE EXAMPLE

For $x \neq x'$

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = 0$$

So we have

$$G(x,x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

- Key thing to notice is that the source is concentrated at $x = x'$.
- So for $x > x'$ and $x < x'$, we expect the solutions to look like those of the source-free equation.
- To keep the solutions finite, we expect exponential growth before $x = x'$ and exponential decay afterward.
- Now, how do we find the constants A and B?

A SIMPLE EXAMPLE

$$\frac{\mathrm{d}^2\, G(x,x')}{\mathrm{d}\, x^2} - k^2 G(x,x') = \delta(x-x')$$

Continuity of the Green's function:

$$\lim_{\epsilon\to 0}\left[G(x'+\epsilon,x') - G(x'-\epsilon,x')\right] = 0$$

- How continuous do we expect our Green's function to be?
- If $G(x, x')$ is discontinuous (like a step function), then $\mathrm{d}\, G / \mathrm{d}\, x$ will behave like a delta function and $\mathrm{d}^2\, G / \mathrm{d}\, x^2$ will behave like a delta function derivative. No good!
- So we expect $G(x, x')$ to be continuous.
- That gives us one condition we can use to find A and B. (In fact, it tells us that $A = B$.)

A SIMPLE EXAMPLE

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}\,x^2} - k^2 G(x,x') \right] \mathrm{d}\,x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}\,x$$

Discontinuity condition:

$$\lim_{\epsilon\to 0} \left[\left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'+\epsilon} - \left.\frac{\mathrm{d}\,G}{\mathrm{d}\,x}\right|_{x=x'-\epsilon} \right] = 1$$

- But what if the derivative $\mathrm{d}\,G/\,\mathrm{d}\,x$ is discontinuous?
- Then $\mathrm{d}^2\,G/\,\mathrm{d}\,x^2$ is like a delta function. But that's fine, because we have a delta function on the right hand side too.
- We can find exactly how discontinuous the derivative is by integrating over a small interval around x'.
- In the limit of $\epsilon \to 0$, the second integral vanishes because $G(x, x')$ is continuous.
- But, we expect $\mathrm{d}\,G/\,\mathrm{d}\,x$ to be discontinous.
- Using fundamental theorem of calculus, we get at a *discontinuity condition for the derivative.* (Key idea for the direct solution method!)

A SIMPLE EXAMPLE

$$G(x, x') = \begin{cases} Ae^{+k(x-x')} & \text{for } x < x' \\ Be^{-k(x-x')} & \text{for } x > x' \end{cases}$$

Continuity of G:

$$A = B$$

Discontinuity of $\frac{\mathrm{d}G}{\mathrm{d}x}$:

$$kA + kB = 1$$

- Applying our two conditions, we can solve for A and B.

At last, our Green's function is

$$G(x,x') = \begin{cases} \dfrac{e^{+k(x-x')}}{2k} & \text{for } x < x' \\ \dfrac{e^{-k(x-x')}}{2k} & \text{for } x > x' \end{cases}$$

or, more compactly

$$\boxed{G(x,x') = \frac{e^{k|x-x'|}}{2k}}$$

A SIMPLE EXAMPLE

Solution:

$$u(x) = \int_{-\infty}^{\infty} f(x') \frac{e^{k|x-x'|}}{2k} \, \mathrm{d}\, x'$$

- Now that we have the Green's function, we can construct the solution to our original problem for any forcing function $f(t)$.

Properties of $G(x, x')$:

- Behaves like source-free solution except at $x = x'$.
- Function is continuous at $x = x'$.
- Derivative is discontinuous at $x = x'$.

- Listed are the key things to note from that example.
- This approach works quite well for solving 1D Green's function problems.

BOUNDARY CONDITIONS

- We didn't worry about boundary conditions in the last example.
- As it turns out, Green's functions allow us to deal with boundary conditions in an elegant way.
- Unfortunately, to derive it, we either have to do a lot of hand-waving or a lot of math. We're going to do a lot of math.
- See Dudley's *Mathematical foundations for electromagnetic theory* for a more thorough discussion.

$$\begin{aligned} &\text{Original:} \quad && \mathcal{L}[u(x)] = f(x); \quad && \mathcal{B}[u(x)] = 0 \\ &\text{Adjoint:} \quad && \mathcal{L}^*[v(x)] = f(x); \quad && \mathcal{B}^*[v(x)] = 0 \end{aligned}$$

Defining property:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle \qquad \text{where } \langle u, v\rangle = \int_a^b u(x) v^*(x)\, \mathrm{d}\, x$$

- To really make sense of boundary conditions, we need the concept of the adjoint problem.
- Suppose we have an original problem defined by operator $\mathcal{L}$ and boundary conditions $\mathcal{B}$.
- Then, $\mathcal{L}^*$ is the adjoint operator and $\mathcal{B}^*$ are the adjoint boundary conditions if $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle$ for all u, v.
- Here $\langle u, v\rangle$ is the inner product as defined on the slide. ($v^*(x)$ is the complex conjugate of $v(x)$.)

Linear algebra notes:

- The boundary conditions are important because they specify the domains of $\mathcal{L}$ and $\mathcal{L}^*$. (I.e., $\mathcal{L}$ operates on the Hilbert space of functions $u(x)$ which satisfy $\mathcal{B}[u] = 0$.)
- So if $\mathcal{B} \neq \mathcal{B}^*$, then $\mathcal{L}$ and $\mathcal{L}^*$ are operators on different Hilbert spaces.
- If both $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$, we say that $\mathcal{L}$ is self-adjoint.
- If $\mathcal{L} = \mathcal{L}^*$ but $\mathcal{B} \neq \mathcal{B}^*$, we say that $\mathcal{L}$ is *formally* self-adjoint.

$$\mathcal{L}[u(x)] = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right] u(x)$$

$$\mathcal{B}[u(x)] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Want $\mathcal{L}^*$ and $\mathcal{B}^*$ so that

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$$

- Let's look at an example: the 1D simple harmonic oscillator.
- We'll use boundary conditions so that $u(a) = u(b) = 0$.

$$\langle \mathcal{L}u, v \rangle = \int\limits_a^b \left[u''(x) + k^2 u(x)\right] v^*(x)\,\mathrm{d}\,x$$

$$\langle \mathcal{L}u, v \rangle = \int\limits_a^b u(x)\left[v''(x) + (k^2)^* v(x)\right]^* \mathrm{d}\,x +$$
$$+ \left[u'(x)v^*(x) - u(x)v'^*(x)\right]_a^b$$

By inspection:

$$\mathcal{L}^* v(x) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + (k^2)^*\right] v(x)$$

- Integrate by parts twice.
- The remaining integral term looks like

$$\int_a^b u(x)\,[\mathcal{L}^* v(x)]^*\,\mathrm{d}\,x$$

so let's define $\mathcal{L}^*$ this way, and hope it will work out (it will).

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$

Since $u(a) = u(b) = 0$,

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(a)v^*(a) - u'(b)v^*(b)$$

So, pick

$$v(a) = v(b) = 0$$

- We almost have the required $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle$, but we need the part on the right (called the conjunct) to be zero.
- From the original problem, we have

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- To make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}^*[v] = \begin{bmatrix} v(a) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- So in this case, $\mathcal{B} = \mathcal{B}^*$.

What if $u(a) = u'(a) = 0$ (initial conditions)?

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + [u'(x)v^*(x) - u(x)v^{*\prime}(x)]_a^b$$
$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + u'(b)v^*(b) - u(b)v'^*(b)$$

We must have *final* conditions for v:

$$v(b) = v'(b) = 0$$

- What if we use the same operator $\mathcal{L}$, but we switch from a boundary value problem to an initial condition problem? That is,

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Then, to make the conjunct zero, we need the adjoint boundary conditions to be

$$\mathcal{B}[v] = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- For an initial condition problem, the adjoint problem is a *final* condition problem! $\mathcal{B} \neq \mathcal{B}^*$.

In general:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle + J(u, v)\Big|_a^b$$

Definition of adjoint boundary conditions:

$$\mathcal{B}[u] = \mathcal{B}^*[v] = 0 \Longleftrightarrow J(u, v)\Big|_a^b = 0$$

- In that example, we saw that we always had $\langle \mathcal{L}u, v \rangle$ equal to $\langle u, \mathcal{L}^* v \rangle$ plus a leftover term which depended on the boundaries.
- This is true more generally: if we don't specify the boundary conditions of u and v, then we can still *almost* get the adjoint operator equation. We just have a leftover "conjunct" term $J(u, v)|_a^b$, which depends only on the boundary condtions.
- We define adjoint boundary conditions as those boundary conditions which make the conjunct equal to zero.

Original problem:

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$

Green's problem:

$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$

Adjoint Green's problem:

$$\mathcal{L}^*[H(x,x')] = \delta(x-x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

- Now we'll be able to deal with boundary conditions properly.
- We define $G(x, x')$ to obey the same equation as $u(x)$, but with $f(x) \to \delta(x - x')$ and $\alpha \to 0$. As before, $G(x, x')$ is the impulse response.
- In addition, we define a new function $H(x, x')$ which is called the adjoint Green's function. It obeys the adjoint version of the $G(x, x')$ equation.
- Warning: a lot of textbooks don't distinguish between $H(x, x')$ and $G(x, x')$. Quite often, the "Green's function" is really the adjoint Green's function.

EXAMPLE WITH BOUNDARY CONDITIONS

Original problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) u(x) = f(x); \quad \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Green's problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) G(x, x') = \delta(x - x'); \quad \begin{bmatrix} G(a, x') \\ G'(a, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adjoint Green's problem:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + k^2\right) H(x, x') = \delta(x - x'); \quad \begin{bmatrix} H(b, x') \\ H'(b, x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- We'll show how $H(x, x')$ is useful with an example.
- Here we have a driven simple harmonic oscillator with initial conditions. (We'll take k real for simplicity.)
- From before, we know that the adjoint problem will be the same differential equation, but with final conditions instead of initial conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} A\cos(k(x - x')) + B\sin(k(x - x')) & \text{for } x < x' \\ C\cos(k(x - x')) + D\sin(k(x - x')) & \text{for } x > x' \end{cases}$$

$$\begin{aligned} \text{Final conditions } &\implies C = D = 0 \\ \text{Continuity of function } &\implies A = 0 \\ \text{Discontinuity of derivative } &\implies B = \frac{-1}{k} \end{aligned}$$

- We can solve for $H(x, x')$ using a similar approach to before.
- Except at $x = x'$, we write $H(x, x')$ as a solution to the source-free equation.
- Then we find the coefficients using the boundary conditions and continuity/discontinuity requirements.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1}\sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

- So we have our adjoint Green's function.
- Now we just need to figure out how to construct $u(x)$ from it.

EXAMPLE WITH BOUNDARY CONDITIONS

$$\langle \mathcal{L}u(x), H(x,x')\rangle = \langle u(x), \mathcal{L}^* H(x,x')\rangle + J(u,H)\Big|_a^b$$

$$\langle f(x), H(x,x')\rangle = \langle u(x), \delta(x-x')\rangle + J(u,H)\Big|_a^b$$

$$\int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x = \int_a^b u(x)\delta(x-x')\,\mathrm{d}\,x + J(u,H)\Big|_a^b$$

$$\boxed{u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x - J(u,H)\Big|_a^b}$$

- To construct the solution $u(x)$, we take an inner product of $\mathcal{L}u(x)$ with $H(x, x')$, and apply our knowledge of adjoints and conjuncts.
- We arrive at a fairly general formula which looks close to what we expect a Green's function formula to look like, but with an extra conjunct term.
- We'll gain insight into the $J(u, H)$ term by expanding it for this example.

Expand $J(u, H)$:

$$u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x - \left[\frac{\mathrm{d}\,u(x)}{\mathrm{d}\,x}H^*(x,x') - u(x)\frac{\mathrm{d}\,H^*(x,x')}{\mathrm{d}\,x}\right]_a^b$$

Recall:

$$\mathcal{B}[u] = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad \mathcal{B}^*[H] = \begin{bmatrix} H(b,x') \\ H'(b,x') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Now let's expand $J(u, v)$ for our particular example. (We can basically copy it from a previous part of the derivation.)
- We can simplify the conjunct by remembering our boundary conditions.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x + \beta H^*(a, x') - \alpha \frac{\mathrm{d}\, H^*(a, x')}{\mathrm{d}\, x}$$

- We're very close to our final goal.
- The last thing will be to get rid of $H(x, x')$ and replace it with $G(x, x')$.

How are $G(x,x')$ and $H(x,x')$ related?

$$\langle \mathcal{L}G(x,x'), H(x,x'')\rangle = \langle G(x,x'), \mathcal{L}^* H(x,x'')\rangle$$
$$\langle \delta(x-x'), H(x,x'')\rangle = \langle G(x,x'), \delta(x-x'')\rangle$$
$$\int_a^b \delta(x-x')H^*(x,x'')\,\mathrm{d}\,x = \int_a^b G(x,x')\delta(x-x'')\,\mathrm{d}\,x$$
$$H^*(x',x'') = G(x'',x')$$

$$\boxed{G(x,x') = H^*(x',x)}$$

- Using the definition of the adjoint problem, we find that there is a simple relationship between $G(x, x')$ and $H(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$H(x, x') = \begin{cases} -k^{-1} \sin(k(x - x')) & \text{for } x < x' \\ 0 & \text{for } x > x' \end{cases}$$

$$G(x, x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1} \sin(k(x - x')) & \text{for } x > x' \end{cases}$$

- Since we already know the adjoint Green's function $H(x, x')$, we can use find the Green's function via $G(x, x') = H^*(x, x')$.

EXAMPLE WITH BOUNDARY CONDITIONS

$$u(x') = \int_a^b f(x) H^*(x, x') \, \mathrm{d}\, x + \beta H^*(a, x') - \alpha \left. \frac{\mathrm{d}\, H^*(x, x')}{\mathrm{d}\, x} \right|_{x=a}$$

$$u(x') = \int_a^b f(x) G(x', x) \, \mathrm{d}\, x + \beta G(x', a) - \alpha \left. \frac{\mathrm{d}\, G(x', x)}{\mathrm{d}\, x} \right|_{x=a}$$

$$u(x) = \int_a^b f(x') G(x, x') \, \mathrm{d}\, x' + \beta G(x, a) - \alpha \left. \frac{\mathrm{d}\, G(x, x')}{\mathrm{d}\, x'} \right|_{x'=a}$$

- With our new knowledge that $G(x, x') = H^*(x', x)$, we can rewrite the solution $u(x)$ in terms of $G(x, x')$.

$$\boxed{u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}}$$

where

$$G(x,x') = \begin{cases} 0 & \text{for } x < x' \\ k^{-1}\sin(k(x-x')) & \text{for } x > x' \end{cases}$$

- Finally, we arrive at our solution.
- First, note that the final answer does *not* depend on $H(x, x')$. We didn't actually need to ever calculate $H(x, x')$ from the adjoint Green's equation, we could have just found $G(x, x')$ from the (non-adjoint) Green's equation.
- However, it would have been very difficult to derive this expression without using $H(x, x')$ (I couldn't see an easy way). Because of this, a lot of authors stop at the expression for $u(x')$ in terms of $H^*(x, x')$, and they just call $H^*(x, x')$ the "Green's function."
- By doing the extra work, though, we gain a very nice interpretation for the boundary conditions term.

$$u(x) = \int_a^b f(x')G(x,x')\,\mathrm{d}\,x' + \beta G(x,a) - \alpha \left.\frac{\mathrm{d}\,G(x,x')}{\mathrm{d}\,x'}\right|_{x'=a}$$

$$u(x) = \int_a^b \left[f(x') + \beta\delta(x'-a) - \alpha\delta'(x'-a)\right] G(x,x')\,\mathrm{d}\,x'$$

- We note that our expression for $u(x)$ looks like it did before (integral over $f(x')G(x,x')$), but now there are extra terms which depend on the boundary conditions.

- We come to a key idea: boundary conditions have the same effect on $u(x)$ as adding little impulse sources at the boundary. The Green's function can deal with both sources $f(x)$ and non-zero boundary conditions.

- Be careful, though: $G(x,x')$ still depends on the *type* of boundary condition. E.g., we use the same $G(x,x')$ for all initial value problems ($u(a), u'(a)$ specified), but we'll need a different $G(x,x')$ for boundary value problems ($u(a), u(b)$ specified).

SUMMARY

$$\mathcal{L}[u(x)] = f(x); \qquad \mathcal{B}[u(x)] = \alpha$$
$$\mathcal{L}[G(x,x')] = \delta(x-x'); \qquad \mathcal{B}[G(x,x')] = 0$$
$$\mathcal{L}^*[H(x,x')] = \delta(x-x'); \qquad \mathcal{B}^*[H(x,x')] = 0$$

1. Find $G(x,x')$.
2. Find $u(x')$ in terms of $H(x,x')$:

$$u(x') = \int_a^b f(x)H^*(x,x')\,\mathrm{d}\,x - J(u(x), H(x,x'))$$

3. Express $u(x)$ in terms of $G(x,x') = H^*(x',x)$.

- That was a long process, so let's summarize what we did.
- First, we wrote out the original equation, the Green's function equation, and the adjoint Green's function equation. (Take note of the boundary conditions in particular.)
- Next, we solve the Green's function equation for $G(x, x')$.
- Next, we found that it's much easier to express $u(x)$ in terms of $H(x, x')$, because we can use inner products. The only tricky part is finding the conjunct. (Usually, just requires integration by parts. See Dudley for a general formula for Sturm-Liouville problems.)
- Finally, we replace $H(x, x')$ with $G^*(x', x)$ (which we solved for previously), and we have our final expression for $u(x)$.

PROPERTIES OF GREEN'S FUNCTIONS

- The Green's function gives us a lot of information about the system we're dealing with.
- Here we'll look at a few of the properties Green's functions can have and what those tell us about our system.

RECIPROCITY

If $\mathcal{L}$ is self-adjoint

$$\mathcal{L} = \mathcal{L}^* \qquad \text{and} \qquad \mathcal{B} = \mathcal{B}^*$$

then

$$G(x, x') = G^*(x', x)$$

- For self-adjoint problems, $G = H$.
- Using our relationship between G and H, we quickly see that the Green's function is complex symmetric (Hermitian) for self-adjoint problems.
- Roughly, putting a source at x and measuring at x' is the same as putting a source at x' and measuring at x. This is often called reciprocity (though it's not quite the same as reciprocity from time-harmonic E&M, because we're using a true inner product rather than the so-called "reaction inner product").
- Note that we require *both* $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{B} = \mathcal{B}^*$. So this would likely not hold for initial condition problems.

$\mathcal{L}$ is invariant if

$$\mathcal{L}[u(x-\xi)] = \mathcal{L}[u(x)]\Big|_{x=x-\xi}$$

For example,

$$\mathcal{L} = a\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + b\frac{\mathrm{d}}{\mathrm{d}\,x} + c$$

is invariant only if a, b, and c are constants.

- An operator is invariant if shifting the input just results in a shifted output.
- Probably familiar from signal processing: linear *time-invariant* systems. Delaying the input signal leads to the same output signal, just delayed by the same amount.
- Invariance is really important in modern physics.
- Maxwell's equations in free space are invariant with respect to x, y, z, ϕ, θ. That is, at a fundamental level, the laws of electromagnetism do not change if we move to a different location or look in a different direction.

INVARIANCE

If $\mathcal{L}$ is invariant in x, then

$$\mathcal{L}[G(x,x')] = \delta(x-x')$$
$$\mathcal{L}[G(x-\xi,x'-\xi)] = \delta(x-x')$$

$$\implies G(x,x') = G(x-\xi,x'-\xi)$$

$$\boxed{G(x,x') = G(x-x')}$$

- For invariant problems, we can see that shifting both x and x' by the same amount ξ does not affect the Green's function.
- Taking $\xi = x'$, we see that $G(x, x')$ is actually only a function of the difference $(x - x')$.
- That is, the response only depends on the *relative* locations of the source and measurement. This fits nicely with our intuitive understanding of invariance.

Convolution:

$$u(x) = \int_a^b G(x - x')f(x') \, \mathrm{d}\, x' = G(x) * f(x)$$

Frequency domain:

$$\tilde{u}(k) = \tilde{G}(k)\tilde{f}(k)$$

- For invariant systems (with boundary conditions zero), the solution $u(x)$ is just given as a convolution of the source function $f(x)$ with the impulse response $G(x)$.
- Taking Fourier transforms, the convolution turns into multiplication.
- Looks familiar from signal processing!

SPECTRAL METHODS

- In this section, we'll look at the strong relationship between Green's functions and spectral theory.
- Essentially, eigenfunction expansion allows us to calculate the Green's function when direct methods don't work.
- A basic background in spectral theory can be found in most books covering Green's functions.
- Unfortunately, these are rarely rigorous when dealing with continuous sets of eigenvalues. For fully rigorous spectral theory (not for the faint of heart!), see Naylor and Sell's *Linear operator theory in engineering and science* or Kreyszig's *Introductory functional analysis with applications.*

Problem:

$$(\mathcal{L} - \lambda)\, u(x) = f(x); \qquad \mathcal{B}[u(x)] = 0$$

where $\mathcal{L}$ is self-adjoint:

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad \mathcal{B} = \mathcal{B}^*$$

- First we'll review a bit of eigenfunction theory, but we'll quickly see how it relates to Green's functions.
- Set up a similar problem as before, but we've added a complex parameter λ for later convenience.
- Also, for this section we'll insist that $\mathcal{L}$ be fully self-adjoint so that we can take full advantage of spectral theory.
- A brief discussion of the non-self-adjoint case can be found in Morse and Feshbach under "Non-Hermitian operators: biorthogonal functions".

Eigenfunctions:

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x)$$

With $\mathcal{L}$ self-adjoint, $\lambda_n \in \mathbb{R}$ and

$$u(x) = \sum_n \langle u, \phi_n \rangle \phi_n(x)$$
$$f(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

- Because $\mathcal{L}$ is self-adjoint, we know that its eigenvalues λ_n are real.
- We also know that it has a complete orthonormal set of eigenfunctions ϕ_n.
- That is, we can expand any function (in this case $u(x)$ and $f(x)$) in terms of $\phi_n(x)$. (Generalized Fourier series.)

$$(\mathcal{L} - \lambda)u(x) = f(x)$$

$$(\mathcal{L} - \lambda)\left[\sum_n \langle u, \phi_n \rangle \phi_n(x)\right] = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$\sum_n \langle u, \phi_n \rangle (\lambda_n - \lambda)\phi_n(x) = \sum_n \langle f, \phi_n \rangle \phi_n(x)$$

$$(\lambda_n - \lambda) \langle u, \phi_n \rangle = \langle f, \phi_n \rangle$$

- Going back to our original equation, let's expand $u(x)$ and $f(x)$ in terms of eigenfunctions of $\mathcal{L}$.
- Using the fact that $\mathcal{L}$ is linear and $\mathcal{L}\phi_n = \lambda_n \phi_n$, we can get rid of $\mathcal{L}$ (third line).
- Finally, since the $\phi_n(x)$ are linearly independent, each term in the sums on the RHS and LHS must be equal. So we get an expression for the generalized Fourier coefficients $\langle u, \phi_n \rangle$.

$$\langle u, \phi_n \rangle = \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

So

$$u(x) = \sum_n \langle u, \phi_n \rangle \, \phi_n(x)$$

$$\boxed{u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)}$$

- Plugging in our new expression for the Fourier coefficients, we obtain a formula for $u(x)$ in terms of the eigenfunctions and eigenvalues of $\mathcal{L}$.

$$u(x) = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

$$u(x) = \sum_n \left(\int_a^b \frac{f(x')\phi_n^*(x')}{\lambda_n - \lambda} \, \mathrm{d}\, x' \right) \phi_n(x)$$

$$u(x) = \int_a^b \left(\sum_n \frac{f(x)\phi_n^*(x')}{\lambda_n - \lambda} \phi_n(x) \right) \mathrm{d}\, x'$$

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x') \, \mathrm{d}\, x'$$

- Usually, the inner product is defined by an integral.
- If we write this out and do some manipulation, we get something that looks a lot like the Green's function expression.

SPECTRAL FORM OF THE GREEN'S FUNCTION

$$u(x) = \int_a^b \left(\sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda} \right) f(x')\,\mathrm{d}\,x'$$

$$\boxed{G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}}$$

- It turns out that we actually can read off this weird sum as the Green's function.
- So, if we know the eigenvalues and eigenfunctions of $\mathcal{L}$, we can immediately construct the Green's function as an infinite series.
- Note also that $G(x, x') = G^*(x, x')$, as we expect because this is a self-adjoint problem.

SPECTRAL FORM OF THE GREEN'S FUNCTION

Green's function of $(\mathcal{L} - \lambda)$:

$$G(x, x', \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

λ_n are poles of $G(x, x', \lambda)$.

$\phi_n(x)$ can be found by residue integration.

- It also goes the other way. If we know the Green's function of $(\mathcal{L} - \lambda)$ for any complex λ, then the eigenvalues of $\mathcal{L}$ are just the poles of the Green's function with respect to lambda.
- Eigenfunctions are a little trickier to read off, but it's possible to find them from the Green's function using residue integration.

SPECTRAL FORM OF THE DELTA FUNCTION

$$\delta(x - x') = (\mathcal{L} - \lambda) G(x, x')$$

$$\delta(x - x') = (\mathcal{L} - \lambda) \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\delta(x - x') = \sum_n \frac{(\lambda_n - \lambda)\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$\boxed{\delta(x - x') = \sum_n \phi_n(x)\phi_n^*(x')}$$

- Using our Green's function equation, we can also derive an expression for the delta function as a sum of eigenfunctions.
- This expression is useful when solving three-dimensional problems with separation of variables.

$$\underbrace{\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} - \lambda\right)}_{\mathcal{L}-\lambda} u(x) = f(x); \quad u(0) = u(a) = 0$$

Eigenfunctions of $\mathcal{L}$:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right); \quad \lambda_n = \frac{\pi n}{a}$$

EXAMPLE

$$G(x, x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n - \lambda}$$

$$G(x, x') = \sum_{n=0}^{\infty} \frac{2\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right)}{\pi n - \lambda a}$$

- Not as good as the direct method (have to add up an infinite sum to calculate numerical values), but this may be necessary for 3D problems.

3D PROBLEMS

ADVANCED TOPICS

Balanis (2012), *Advanced engineering electromagnetics*. Less rigorous, but good for getting the key ideas.

Folland (1992), *Fourier analysis and its applications*. Rigorous. Chapter on generalized functions is particularly nice.

Dudley (1994), *Mathematical foundations for electromagnetic theory*. Great introduction to 1D Green's functions: deals with subtleties that others ignore.

Byron and Fuller (1992), *Mathematics of classical and quantum physics*. Interesting alternative approach.

Collin (1990), *Field theory of guided waves.* Huge chapter on Green's functions. Emphasis on dyadics.

Morse and Feshback, *Methods of theoretical physics.* Another big, detailed reference. Great resource for deeper insight and understanding.

Warnick (1996), "Electromagnetic Green functions using differential forms." For the differential forms inclined.