MAT 421 Project

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Introduction

In the early 1800's the famous Navier-Stokes Equations were formulated, and almost 300 years later they remain unsolved. This set of equations govern the flow of viscous fluids in any imaginable system. Applications are found within the turbulent flow of aerodynamics, hydrodynamic systems, and even closed fluid flow within a body, e.g., the ocean. Navier-Stokes can be written in many forms and coordinate systems. Here, a specific case will be analyzed and studied. Consider a completely filled and stationary cylinder; where, the bottom face of the cylinder spins and the rest is stagnant. In other words, let the bottom face rotate under the *still* cylinder. Now, let the cylinder have height H, radius R, and the face angular velocity Ω. Suppose the cylinder is filled with an incompressible fluid of kinematic viscosity ν. Then, we know the fluid flow within the cylinder is governed by the Navier-Stokes Equations. Using R as our length scale, and $1/\Omega$ as the time scale, our form of the equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \ \nabla \cdot \mathbf{u} = 0 \tag{1}$$

Where $Re = \Omega R^2/\nu$ is Reynolds number. Also known as the ratio of viscous time scale, R^2/ν to the inertial time scale $1/\Omega$ and aspect ratio: $\Gamma = H/R$. We'll use a cylindrical coordinate system with $\mathbf{u} = (u, v, w)$ as the velocity vector. That is, $u = u(r, \theta, z)$, $v = v(r, \theta, z)$, $w = w(r, \theta, z)$. First, we'll make note of the structure of our Navier-Stokes equations and break them down. This gives us the following clear information:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}(u, v, w)}{\partial t} = \Big(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\Big) \tag{2}$$

$$\mathbf{u} \cdot \nabla \mathbf{u} = \Big(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}, u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{v^2}{r}, u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\Big) \tag{3}$$

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) = \begin{pmatrix} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r}\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \\ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{v}{r^2} \\ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial z} \end{pmatrix}^T \tag{4}$$

$$-\nabla p = -\Big(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\Big) \tag{5}$$

Considering this information, we can analyze and observe (1) in it's full glory.

$$\Big(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}\Big)+\mathbf{u}\cdot\nabla\mathbf{u}=-\Big(\frac{\partial p}{\partial r},\frac{1}{r}\frac{\partial p}{\partial\theta},\frac{\partial p}{\partial z}\Big)+\frac{1}{Re}\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\mathbf{u}}{\partial r}\Big)+\frac{1}{r^2}\frac{\partial^2\mathbf{u}}{\partial\theta^2}+\frac{\partial^2\mathbf{u}}{\partial z^2}\Big]\tag{6}$$

Now, suppose the Reynolds numbers are small ($Re\lesssim 10^3$), then the flow is steady ($\frac{\partial}{\partial t}=0$) and axisymmetric ($\frac{\partial}{\partial\theta}$=0) for aspect ratios $\Gamma\sim O(1)$. If we break this equation down into component form and take note of **only** the axisymmetric property, we have the following:

$$1{:}\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\cancelto{0}{\frac{v}{r}\frac{\partial u}{\partial\theta}}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\frac{1}{Re}\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\Big]\tag{7}$$

Cleaning this up with some simplification and cleaner notation, (7) becomes

$$\partial_t u+u\partial_r u+w\partial_z w-\frac{v^2}{r}=-\partial_r p+\frac{1}{Re}\Big[\partial_r^2 u+\frac{1}{r}\partial_r u+\partial_z^2 u-\frac{u}{r^2}\Big]\tag{8}$$

$$2{:}\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\cancelto{0}{\frac{v}{r}\frac{\partial v}{\partial\theta}}+w\frac{\partial v}{\partial z}+\frac{v^2}{r}=-\cancelto{0}{\frac{1}{r}\frac{\partial p}{\partial\theta}}+\frac{1}{Re}\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\Big]\tag{9}$$

Simplifying, (9) becomes

$$\partial_t v+u\partial_r v+w\partial_r v+\frac{v^2}{r}=\frac{1}{Re}\Big[\partial_r^2 v+\frac{1}{r}\partial_r v+\partial_z^2 v-\frac{v}{r^2}\Big]\tag{10}$$

$$3{:}\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\cancelto{0}{\frac{v}{r}\frac{\partial w}{\partial\theta}}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\Big[\frac{\partial^2 w}{\partial r^2}+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial z}\Big]\tag{11}$$

Leading to

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\Big[\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial z}\Big]\tag{12}$$

Since $\nabla\cdot\mathbf{u}=0$ We end up getting

$$\nabla\cdot\mathbf{u}=\frac{1}{r}\frac{\partial ru}{\partial r}+\frac{\partial w}{\partial z}=0\tag{13}$$

Now, since we have an axisymmetric system, we can define a convenient streamfunction ψ

$$\mathbf{u}=(u,v,w)=\Big(-\frac{1}{r}\frac{\partial\psi}{\partial z},v,\frac{1}{r}\frac{\partial\psi}{\partial r}\Big)$$

Just to make sure we satisfy (13)

$$\begin{aligned}\frac{1}{r}\frac{\partial ru}{\partial r}+\frac{\partial w}{\partial z}&=\frac{1}{r}\frac{\partial}{\partial r}\Big(-r\frac{1}{r}\frac{\partial\psi}{\partial z}\Big)+\frac{1}{r}\frac{\partial}{\partial z}\Big(\frac{\partial\psi}{\partial r}\Big)\\&=-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2\psi}{\partial z\partial r}\\&=0\end{aligned}$$

The corresponding vorticity with our axisymmetric condition is

$$
\begin{aligned}
\omega = \nabla \times \mathbf{u} = \frac{1}{r} = \frac{1}{r}\Big[\frac{\partial w}{\partial \theta} - \frac{\partial rv}{\partial z}\Big]\hat{\mathbf{r}} + \Big[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\Big]\hat{\theta} + \frac{1}{r}\Big[\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta}\Big]\hat{\mathbf{z}} \\
= \Big(-\frac{1}{r}\frac{\partial rv}{\partial z}, \eta, \frac{1}{r}\frac{\partial rv}{\partial r}\Big)
\end{aligned}
$$

Where

$$
\begin{aligned}
\eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\
&= -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r}
\end{aligned}
$$

Leading to

$$
r\eta = \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{14}
$$

Now, we want to eliminate the pressure term since it's unclear and presents a challenge, so observing $\partial_z(8)$ and $\partial_r(12)$ we end up with

$$
\partial_z(8) \implies \partial^2_{tz}u + \partial_z(u\partial_r v) + \partial_z(w\partial_z v) + \partial_z\frac{v^2}{r} = -\partial^2_{zr}p + \frac{1}{Re}\Big[\partial^3_z u + \partial_z\partial^2_r u + \frac{1}{r}\partial_z\partial_r u - \partial_z\frac{u}{r^2}\Big] \tag{15}
$$

and

$$
\partial_r(12) \implies \partial^2_{rt}w + \partial_r(u\partial_r w) + \partial_r(w\partial_z w) = -\partial^2_{zr}p + \frac{1}{Re}\Big[\partial^3_r w + \partial_r\partial^2_z w + \partial_r\Big(\frac{1}{r}\partial_z w\Big)\Big] \tag{16}
$$

Now, if we commit to $\partial_z(8)$-$\partial_r(12)$ introducing ψ and η we get

$$
\begin{aligned}
\partial^2_{tz}u + \partial_z(u\partial_r v) + \partial_z(w\partial_z v) + \partial_z\frac{v^2}{r} - \partial^2_{rt}w - \partial_r(u\partial_r w) - \partial_r(w\partial_z w) \\
= \frac{1}{Re}\Big[\partial^3_z u + \partial_z\partial^2_r u + \frac{1}{r}\partial_z\partial_r u - \partial_z\frac{u}{r^2}\Big] - \frac{1}{Re}\Big[\partial^3_r w + \partial_r\partial^2_z w + \partial_r\Big(\frac{1}{r}\partial_z w\Big)\Big]
\end{aligned} \tag{17}
$$

Becomes

$$
\frac{\partial \eta}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \eta}{\partial z} + \frac{\eta}{r^2}\frac{\partial \psi}{\partial z} - \frac{2v}{r}\frac{\partial v}{\partial z} = \frac{1}{Re}\Big[\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2} + \frac{\partial^2 \eta}{\partial z^2}\Big] \tag{18}
$$

With this introduction of ψ and η (10) becomes

$$
\frac{\partial v}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\Big(\frac{\partial v}{\partial r} + \frac{v}{r}\Big) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial z} = \frac{1}{Re}\Big[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\Big] \tag{19}
$$

Moving forward, we look at the intertial-less limit, $Re \to 0$, that is, the flow is now steady, so $\partial/\partial t = 0$. Moving Re to the other side, our modified Navier-Stokes equations simplify once more. Eqs. (18) and (19) respectively become

$$
\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2} + \frac{\partial^2 \eta}{\partial z^2} = 0 \tag{20}
$$

$$
\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{21}
$$

First Application

So, we've laid all the ground work for the system we will be studying. All of this information will be useful and utilized throughout the entirety of this analysis. Now, we will consider a boundary value problem within our swirling flow in a cylinder. A detailed solution will be constructed for (21). Within $r \in [0,1]$ and $z \in [0,\Gamma]$, subject to the boundary conditions $v(0,z) = v(1,z) = v(r,\Gamma) = 0$ and $v(r,0) = r$. Next, we consider additional boundary conditions: "non-slip". In other words, the flow velocity at any solid wall of the cylinder is the velocity of the wall. Well, all of our walls are solid and so they will be $\mathbf{0}$.

$$\text{Stationary sidewall at } r = 1 : (u,v,w) = (0,0,0),\ \psi = 0,\ \eta = -\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}$$

$$\text{Stationary top at } z = \Gamma : (u,v,w) = (0,0,0),\ \psi = 0,\ \eta = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}$$

$$\text{Rotating bottom at } z = 0 : (u,v,w) = (0,0,0), \psi = 0, \eta = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}$$

$$\text{Axis at } r = 0 : (u,v,\frac{\partial w}{\partial r}) = (0,0,0), \psi = 0, \eta = 0$$

Since the flow is axisymmetric, we have a special boundary condition at the axis. Now, consider our streamfunction boundary conditions to be $\psi = C$, with C an arbitrary constant. Due to some magical math stuff, we select, for convenience, $C = 0$. Looking back at Eq. (20) and (21) they appear to have the same format, but subject to different boundary conditions. Recall

$$v(0,z) = 0, v(1,z) = 0, v(r,\Gamma) = 0, v(r,0) = r \tag{22}$$

$$\eta(1,z) = -\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}, \eta(r,\Gamma) = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}, \eta(r,0) = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}, \eta(0,z) = 0 \tag{23}$$

So, the velocity component v of $\mathbf{u}$ is considered the 'primary flow'. Implying its strictly driven by the rotating bottom. The other velocity components, related to η and ψ, are the secondary meridional flow. Meaning they're driven indirectly from the axial gradients within v (i.e. the $-\frac{2v}{r}\frac{\partial v}{\partial z}$ term in (18).) Following, the primary flow in (21) with (22) can be solved by separation of variables (Note: the boundary conditions are not homogeneous). To begin, first, let

$$v(r,z) = f(r,z) + g(r,z)$$

where $f(r,z)$ is a bi-linear function (i.e. $\partial^2 f/\partial r^2 = \partial^2 f/\partial z^2 = 0$) that satisfies the boundary conditions $f(r,\Gamma) = 0$ and $f(r,0) = r$. From this, we then get that $g(r,\Gamma) = g(r,0) = 0$. We cannot yet make a conclusion about the other boundary conditions given. So, our bi-linear function has the form

$$f(r,z) = a + br + cz + drz$$

with arbitrary constants $a, b, c, d \in \mathbb{R}$. With this, we begin looking at our boundary conditions. Apply the boundary condition at the rotating bottom, $z = 0$, gives us

$$f(r,0) = a + br + c(0) + dr(0) = a + br = r \implies a = 0, b = 1$$

Applying the boundary condition at the stationary top, $z = \Gamma$ and our new found constants

$$f(r,\Gamma) = r + c\Gamma + dr\Gamma = c\Gamma + r(1 + d\Gamma) = 0 \implies c = 0, d = -1/\Gamma$$

Hence,

$$f(r,z) = r(1 - z/\Gamma) \tag{24}$$

So $f(r,z)$ solves PDE (21), but in order for $v = f + g$ to solve (21) with boundary conditions (22), g must also satisfy the following boundary conditions:

$$\begin{aligned}
g(r,0) &= v(r,0) - f(r,0) = r - r = 0 \\
g(r,\Gamma) &= v(r,\Gamma) - f(r,\Gamma) = 0 - 0 = 0 \\
g(1,z) &= v(1,z) - f(1,z) = 0 - (1 - z/\Gamma) = z/\Gamma - 1 \\
g(0,z) &= v(0,z) - f(0,z) = 0 - 0 = 0
\end{aligned}$$

Let's assume it's separable, $g(r,z) = R(r)Z(z)$. Then (21) with $g(r,z)$ becomes

$$\begin{aligned}
0 &= \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{g}{r^2} + \frac{\partial^2 g}{\partial z^2} \\
&= \frac{\partial^2}{\partial r^2}RZ + \frac{1}{r}\frac{\partial}{\partial r}RZ - \frac{RZ}{r^2} + \frac{\partial^2}{\partial z^2}RZ \\
&= ZR'' + Z\frac{R'}{r} - \frac{RZ}{r^2} + RZ'' \\
&= \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} + \frac{Z''}{Z} \implies \\
&\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = -\frac{Z''}{Z} = \alpha
\end{aligned}$$

We'll consider the $Z(z)$ ODE BVP first:

$$Z'' = -Z\alpha \implies Z'' + Z\alpha = 0,\ Z(0) = Z(\Gamma) = 0$$

We have three cases to follow now, as usual. So, let's look at $\alpha = 0, \alpha = -\lambda^2 < 0, \alpha = \lambda^2 > 0$

Case 1: $\alpha = 0$

If $\alpha = 0$, then $Z'' = 0 \implies Z = az + b$. Given our boundary conditions, then $Z(0) = b = 0 \implies b = 0$. Additionally, $Z(\Gamma) = a\Gamma = 0 \implies a = 0$. This then means $Z = 0$ which is the trivial solution.

Case 2: $\alpha = -\lambda^2 < 0$

If $\alpha = -\lambda^2 < 0$, we have $Z'' - \lambda^2 Z = 0$. Luckily, we've seen this many times and know the solution to be $Z = ae^{\lambda z} + be^{-\lambda z}$. Utilizing our boundary conditions, $Z(0) = a + b = 0 \implies a = -b$ and then $Z(\Gamma) = b(-e^{\lambda\Gamma} + e^{-\lambda\Gamma}) = 0 \implies b = 0 \implies a = 0$. Once again, we get that $Z = 0$, the trivial solution.

Case 3: $\alpha = \lambda^2 > 0$

If $\alpha = \lambda^2 > 0$, we get $Z'' + \lambda^2 Z = 0$. We've also seen this before, and the solution is known to be of the form, $Z = a\sin(\lambda z) + b\cos(\lambda z)$. Applying boundary conditions gives us $Z(0) = b = 0$ and then $Z(\Gamma) = a\sin(\lambda\Gamma) = 0$. It's possible that $a = 0$; however, we would then get the trivial solution once more. Naturally, it then turns out $\sin(\lambda\Gamma) = 0$ must be the case. So, the eigenvalues that make this true are

$$\lambda_n = \frac{n\pi}{\Gamma}, n \in \mathbb{N}$$

No conclusions can be made about a other than $a = a_n$ and so we can only say for sure that

$$Z_n(z) \propto \sin(\frac{n\pi z}{\Gamma})$$

We'll come back to $Z(z)$ in a moment. Now, we'll analyze the $R(r)$ ODE with found $\alpha = \lambda_n^2 = \dfrac{n^2\pi^2}{\Gamma^2}$. Our modified (21) of g will become

$$\begin{aligned} 0 &= r^2R'' + rR' - R - \alpha R r^2 \\ &= r^2R'' + rR' - [1 + \lambda_n^2 r^2]R \\ &= r^2R'' + rR' - \left[1 + \left(\frac{n^2\pi^2}{\Gamma^2}\right)r^2\right]R \end{aligned}$$

Recall that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (n^2 - x^2)y = 0$$

is known as Bessel's equation of order n. Here, we have a modified form of Bessel's equation, and so the solution takes the form

$$R_n(r) = AI_1\left(\frac{n\pi r}{\Gamma}\right) + BK_1\left(\frac{n\pi r}{\Gamma}\right)$$

where $I_1\left(\frac{n\pi r}{\Gamma}\right)$ and $K_1\left(\frac{n\pi r}{\Gamma}\right)$ are also modified Bessel's functions of the first and second kind respectively. Note, our equation and functions are of order one. Now, let's apply the following boundary condition of $r = 0$:

$$R_n(0) = 0 = AI_1(0) + BK_1(0)$$

Taking for granted, $K_1(0)$ is unbounded, that is $K_1(r) \to \infty$, it must be that $B = 0$ for a non-singular solution. From this, we can say for certain, our eigenfunctions are

$$I_1\left(\frac{n\pi r}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right),$$

Thus, our general solution that satisfies boundary conditions at $r = 0, z = 0$, and $z = \Gamma$ with $A_n = A \cdot a_n$ turns out to be

$$g(r,z) = \sum_{n=1}^{\infty} A_n I_1\left(\frac{n\pi r}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right) \tag{25}$$

Our next step is to determine A_n. We proceed by imposing the boundary condition at $r = 1$:

$$g(1,z) = \frac{z}{\Gamma} - 1 = \sum_{n=1}^{\infty} A_n I_1\left(\frac{n\pi}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right)$$

Some Fourier analysis is required to determine our coefficients and so

$$\int_0^\Gamma \left(\frac{z}{\Gamma} - 1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz = \sum_{n=1}^{\infty} A_n I_1\left(\frac{n\pi}{\Gamma}\right)\int_0^\Gamma \sin\left(\frac{n\pi z}{\Gamma}\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz \tag{26}$$

Recall orthogonality of our sine functions

$$\int_{-a}^{a} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ a & \text{if } m = n \end{cases}$$

In our case, Γ is fixed in the denominator and so we shall integrate from $0 \to 2\Gamma$ to compensate for this, while the value for $m = n$ is then scaled down by the same factor of 2

For clarity,

$$\int_0^{\Gamma} \sin\left(\frac{n\pi z}{\Gamma}\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz = \frac{1}{2}\int_0^{2\Gamma} \sin\left(\frac{n\pi z}{\Gamma}\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz$$

Continuing, we then have

$$\begin{aligned}
\int_0^{\Gamma}\left(\frac{z}{\Gamma}-1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz &= \sum_{n=1}^{\infty} A_n I_1\left(\frac{n\pi}{\Gamma}\right)\frac{\Gamma}{2}\delta_{mn} \\
&= A_m I_1\left(\frac{m\pi}{\Gamma}\right)\frac{\Gamma}{2}
\end{aligned}$$

The right hand side of (26) can be calculated using integration by parts on the first term. Fortunately, it's well known, so

$$\begin{aligned}
\int_0^{\Gamma}\left(\frac{z}{\Gamma}-1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz &= \int_0^{\Gamma}\frac{z}{\Gamma}\sin\left(\frac{m\pi z}{\Gamma}\right) - \int_0^{\Gamma}\sin\left(\frac{m\pi z}{\Gamma}\right)dz \\
&= \frac{1}{\Gamma}\Big[-\frac{\Gamma z}{m\pi}\cos\left(\frac{m\pi z}{\Gamma}\right) + \frac{\Gamma^2}{m^2\pi^2}\sin\left(\frac{m\pi z}{\Gamma}\right)\Big]_0^{\Gamma} - \Big[-\frac{\Gamma}{m\pi}\cos\left(\frac{m\pi z}{\Gamma}\right)\Big]_0^{\Gamma} \\
&= -\frac{\Gamma}{m\pi}\cos(m\pi) + \frac{\Gamma}{m^2\pi^2}\cancelto{0}{\sin(m\pi)} + 0 - \cancelto{0}{\sin(0)} + \frac{\Gamma}{m\pi}\cos(m\pi) - \frac{\Gamma}{m\pi} \\
&= -\frac{\Gamma}{m\pi}
\end{aligned}$$

and so (26) can finally be simplified to

$$-\frac{\Gamma}{m\pi} = A_m I_1\left(\frac{m\pi}{\Gamma}\right)\frac{\Gamma}{2} \implies$$

$$A_n = \frac{-2}{n\pi I_1(\frac{n\pi}{\Gamma})}$$

Finally, the solution $g(r,z)$ can be written

$$g(r,z) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{I_1(n\pi r/\Gamma)}{nI_1(n\pi/\Gamma)}\sin\left(\frac{n\pi z}{\Gamma}\right) \tag{27}$$

We're still not complete though. Recall $v(r,z) = f(r,z) + g(r,z)$ and so the true solution to our system of (21) under the given conditions (22) is

$$\boxed{v(r,z) = r\left(1-\frac{z}{\Gamma}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{I_1(n\pi r/\Gamma)}{nI_1(n\pi/\Gamma)}\sin\left(\frac{n\pi z}{\Gamma}\right)} \tag{28}$$

Now, we'll examine some iso-contour plots with different aspect ratios Γ and see how v changes in different geometries of our cylinder.

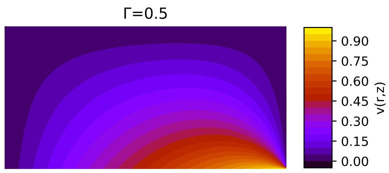


Figure 1: A perfectly clean image of a cylinder with a 0.5 aspect ratio showing distinct regions of the various velocities.

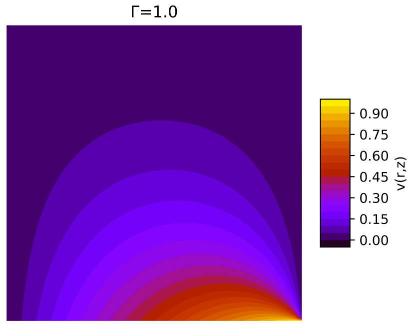


Figure 2: A well defined image of a cylinder of 1.0 aspect ratio.

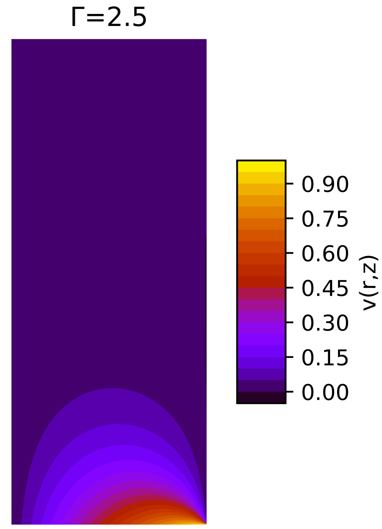


Figure 3: Notice how such a large aspect ratio gives us minimal motion anywhere away from the rotating bottom.

The L2 norm, which will be used later, decays rapidly exponentially and reaches a horizontal asymptote. Letting us know, as we approach the steady state solution, we have a very stable solution to the PDE. This corresponds to the number of steps we used in the solution, as the higher the amount of sums, the more accurate of an answer we get as the L2 norm stabalizes. For this problem, it took about 30 steps to reach a stable point, but closer to 50 for a perfect asmyptote

Second Application

Now, given all the information we unveiled in Part 1 of the project, we can move on to consider the same equation as before (21) under the same boundary conditions. However we'll be using a different method of attack in solving this equation. Now, we will attempt to find a solution numerically using second-order centered finite differences for $\Gamma = 0.5$, $\Gamma = 1.0$, $\Gamma = 2.5$. First, recall

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{29}$$

with boundary conditions

$$v(0,z) = 0,\ v(1,z) = 0,\ v(r,\Gamma) = 0,\ v(r,0) = r$$

Now, let $f(x)$ be a smooth and continuous function. Then, it follows that its Taylor series approximation is

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2} f''(x) + \cdots + \frac{\delta_x^n}{n!} f^{(n)}(x) + \ldots$$

Now, we'll consider this equation up to $O(\delta_x^3)$. To isolate the derivative of second order of the function we'll have to look at the sum and difference of our argument and see what happens.

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2} f''(x) + O(\delta_x^3)$$

$$f(x-\delta_x) = f(x) - \delta_x f'(x) + \frac{\delta_x^2}{2} f''(x) + O(\delta_x^3)$$

Take the difference of the two equations as we get,

$$f(x+\delta_x) - f(x-\delta_x) = 2\delta_x f'(x) + O(\delta_x^3) \implies f'(x) = \frac{f(x+\delta_x) - f(x-\delta_x)}{2\delta_x} + O(\delta_x^2)$$

Now, to get the second derivative, centered and second order, we expand just a bit further and repeat the process

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2} f''(x) + \frac{\delta_x^3}{6} f'''(x) + O(\delta_x^4)$$

$$f(x-\delta_x) = f(x) - \delta_x f'(x) + \frac{\delta_x^2}{2} f''(x) - \frac{\delta_x^3}{6} f'''(x) + O(\delta_x^4)$$

Instead of taking the difference, we'll add these two functions

$$f(x+\delta_x) + f(x-\delta_x) = 2f(x) + \delta_x^2 f''(x) + O(\delta_x^4) \implies f''(x) = \frac{f(x+\delta_x) - 2f(x) + f(x-\delta_x)}{\delta_x^2} + O(\delta_x^2)$$

With this information we will now analyze our system with a different configuration, called spatial discetizating:
The meridional plane $(r,z) = [0,1] \times [0,\Gamma]$ is discretized into a uniform grid $(r_i, z_j) = (i\delta_r, j\delta_z)$ for $i = 0, 1, 2, \dots, n_r$ and $j = 0, 1, 2, \dots, n_z$ with $\delta_r = 1/n_r$ and $\delta_z = \Gamma/n_z$.
Let's use the notation $v_{i,j} = v(r_i, z_j)$.
The grid points (r_i, z_j) for $i = 1 \to (n_r - 1)$ and $j = 1 \to (n_z - 1)$ are the interior grid points;
$i = 0$ is the axis at $r = 0$;
$i = nr$ is the stationary cylinder wall at $r = 1$;
$j = 0$ is the rotating bottom at $z = 0$; and
$j = n_z$ is the stationary top at $z = \Gamma$
Now, we can observe that (21) can be written as

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Big)v + \Big(\frac{\partial^2}{\partial z^2}\Big)v = 0$$

and descritized on each interior grid point using second order centered differences, to get

$$\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\delta_r^2} + \frac{v_{i+1,j} - v_{i-1,j}}{2i\delta_r^2} - \frac{v_{i,j}}{i^2\delta_r^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\delta_z^2} = 0$$

and then this can be regrouped into

$$\frac{1}{\delta_r^2}\Big[\Big(1 - \frac{1}{2i}\Big)v_{i-1,j} - \Big(2 + \frac{1}{i^2}\Big)v_{i,j} + \Big(1 + \frac{1}{2i}\Big)v_{i+1,j}\Big] + \frac{1}{\delta_z^2}[v_{i,j-1} - 2v_{i,j} + v_{i,j+1}] = 0, \qquad (30)$$

for $i = 1 \to (n_r - 1)$ and $j = 1 \to (n_z - 1)$.
Now, make note that for $i = 1$, $i = nr - 1$, $j = 1$ and $j = n_z - 1$, the equations involve the boundary

values of $v_{0,j} = 0$, $v_{n_r,j} = 0$, $v_{i,0} = i\delta_r$, and $v_{i,nz} = 0$
Also, note, that the set of difference equations at each interior grid point (29) can be written using matrix notation as

$$A_{nn}V_{nm} + V_{nm}B_{mm} = F_{nm} \tag{31}$$

where V_{nm} is an $n \times m = (n_r - 1) \times (n_z - 1)$ matrix with entries $v_{i,j}$;

A_{nn} is the $r-$difference matrix, of size $n \times n$; it is tridiagonal with main diagonal entries $-(2 + \frac{1}{i^2})/\delta_r^2, i \in [1, n]$, subdiagonal entries $(1 - \frac{1}{2i})/\delta_r^2, i \in [2, n]$ and super diagonal entries of $(1 + \frac{1}{2i})/\delta_r^2, i \in [1, n-1]$;
B_{mm} is the $z-$difference matrix, of size $m \times m$; it is tridiagonal with constant main diagonal entries $-2/\delta_z^2$, and constant sub/super-diagonal entries $1/\delta_z^2$; and
F_{nm} is the right-hand matrix of size $n \times m$; all its entries are 0 except for the first column which has entries $f_{i,1} = -i\delta_r/\delta_z^2$ for $i \in [1, n]$, which is to account for the boundary values of v on the rotating bottom: $v_{i,0} = i\delta_r$.
To solve our matrix equation (31), we should diagonalize A_{nn} using a similiarity transformation, that is, $Z_{nn}^{-1}A_{nn}Z_{nn} = E_{nn}$ with $E_{nn} = \mathbf{diag}\{e_i\}$ is a diagonal matrix with entries e_i that are the eigenvalues of A_{nn}. Then, Z_{nn} has the corresponding eigenvectors ξ_i as its columns and Z_{nn}^{-1} is its inverse.
Moving forward, we then plug $V_{nm} = Z_{nn}U_{nm}$ into (31), where U_{nm} is to be determined, and we get

$$A_{nn}Z_{nn}U_{nm} + Z_{nn}U_{nm}B_{mm} = F_{nm}$$

left-multiplying through by Z_{nn}^{-1} we get

$$\begin{aligned} Z_{nn}^{-1}A_{nn}Z_{nn}U_{nm} + Z_{nn}^{-1}Z_{nn}U_{nm}B_{mm} &= Z_{nn}^{-1}F_{nm} \\ \implies E_{nn}U_{nm} + U_{nm}B_{mm} &= Z_{nn}^{-1}F_{nm} \end{aligned}$$

Now, taking the transpose, and recalling that $E_{nn}^T = E_{nn}$ because it's diagonal, and $B_{mm}^T = B_{mm}$ because its symmetric, we then have

$$B_{mm}U_{nm}^T + U_{nm}^T E_{nn} = F_{nm}^T (Z_{nn}^{-1})^T = H_{mn} \tag{32}$$

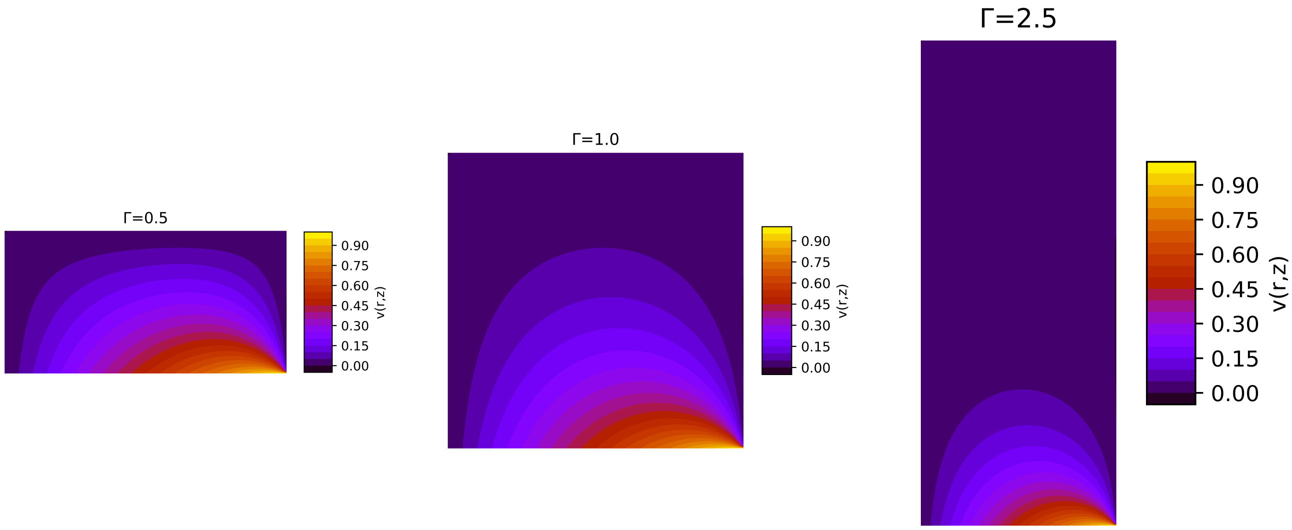
Now, let $\mathbf{u}_i$ be the rows of U_{nm} (n vectors with m components), $\mathbf{h}_i$ be the columns of H_{mn} (n vectors with m components), and remembering that e_i are the n eigenvalues of A_{nn}, that is, the large $[(m(n \times m) \times m(n \times m)]$ matrix in (32) can be written as a system of n decoupled $(m \times m)$ matrix equations:

$$(B_{mm} + e_i I_{mm})\mathbf{u}_i = \mathbf{h}_i, \text{ with } i \in [1, n]$$

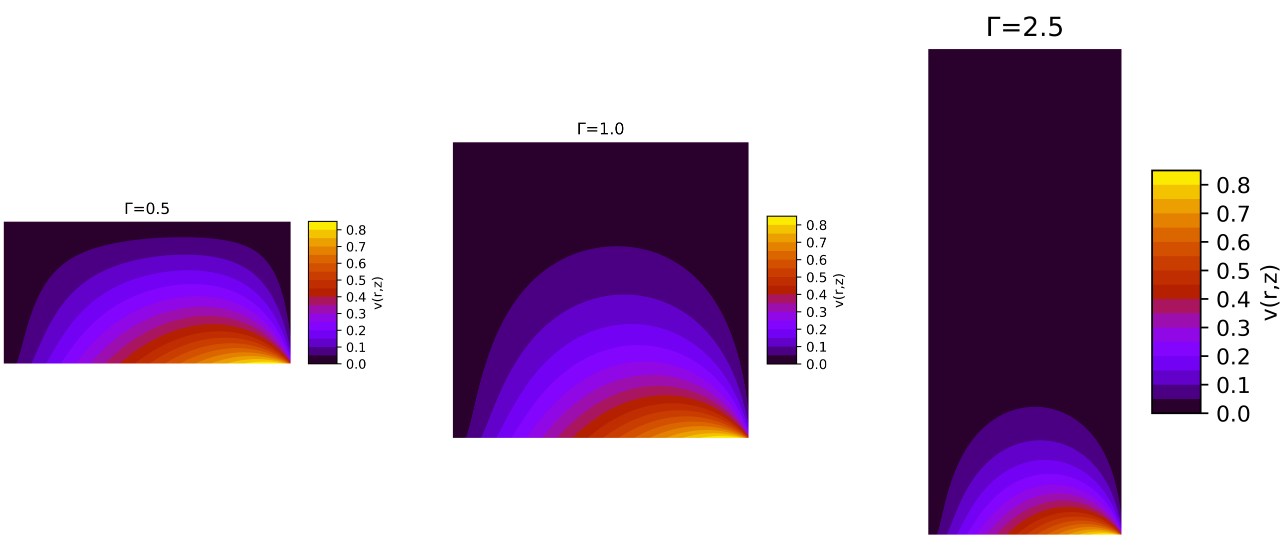
This system of n equations can be solved independently, and they are tridiagonal systems. Upon solving this system, the solution vectors $\mathbf{u}_i$ form the rows of U_{nm}, and thus, $V_{nm} = Z_{nn}U_{nm}$ contains the solution to (29) with our given boundary conditions.

To solve this numerically, I constructed a program in Python and then generated arrays, which acted as matrices. Then, I performed the operations described in the text above on the respective matrices. Fortunately, Python has some built in functions and mathematical applications/packages for us to use. Some of the packages we took advantage of were the *linear algebra* class built into the NumPy package, and also the SciPy package which contains the different types of *Bessel's Functions*. We have the following figures, which give us a flawless visual description of what's happening in our system, even theoretically. Let's take a look.

Exact/Analytic Solutions: These following three figures are the same from Part 1 of the project.



Numerical Solutions: These following three figures are a result of our matrix operations. Notice how they are identical to the analytic solution plots above.



Here, the grid size of $n \times m$ determines the accuracy of our numerical solutions and much like the L2 of part one, we reach a nice stable L2 at $n = 50$. So our number of iterations, or dimensions of matrices are the reason we have a more accurate answer. If we lowered the number, there would be variations and instability; of course, if we increased our matrix size, we would have a more accurate picture.

Numerical Transient Solution

Up to this point, we have found the steady state distribution of the azimuthal velocity v under the slow rotation of the bottom face of our cylinder. Now, we wish to describe how the flow arrives at the steady state starting from some initial conditions, say, $v = 0$ at $t = 0$.

Let's ignore nonlinear processes and retain the time-derivative of v, by taking the small inertia limit, i.e, Re is sufficiently small. In the fluid, the diffusive processes dominate the nonlinear advection processes for $Re \leq O(1)$. We will solve

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\Big[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\Big] \tag{33}$$

With boundary conditions on v still being

$$v(1, z, t) = 0,\ \ v(r, \Gamma, t) = 0,\ \ v(r, 0, t) = r,\ \ v(0, z, t) = 0, \forall t > 0, \tag{34}$$

and the initial condition

$$v(r, z, 0) = 0 \text{ for } r \in [0, 1],\ \ z \in [0, \Gamma]. \tag{35}$$

Notice how (33) resembles a type of heat equation. Thus, we can solve this transient problem analytically. As before, we can split our v into a steady-state and transient solution:

$$v(r, z, t) = f(r, z) + g(r, z, t),$$

with $f(r, z)$ satisfying the non-homogeneous boundary conditions, and $g(r, z, t)$ satisfying the homogeneous boundary conditions and the initial condition.
Recall from earlier, that our $f(r, z)$ is the same solution to the original steady state problem we already solved:

$$f(r, z) = r\Big(1 - \frac{z}{\Gamma}\Big) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{I_1(n\pi r/\Gamma)}{nI_1(n\pi/\Gamma)}\sin\Big(\frac{n\pi z}{\Gamma}\Big).$$

Then transient solution is then the solution of

$$\frac{\partial v}{\partial t} - \frac{1}{Re}\Big[\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{g}{r^2} + \frac{\partial^2 g}{\partial z^2}\Big] = \frac{1}{Re}\Big[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2} + \frac{\partial^2 f}{\partial z^2}\Big] = 0 \tag{36}$$

and the boundary conditions on g are

$$g(1, r, t) = 0,\ \ g(r, \Gamma, t) = 0,\ \ g(r, 0, t) = 0,\ \ g(0, z, t) = 0, \forall t > 0 \tag{37}$$

BUT our initial condition has changed to

$$g(r, z, 0) = v(r, z, 0) - f(r, z). \tag{38}$$

Earlier we determined the evolution of v started with an initial condition and evolved into a steady-state f. However, this time, our evolution problem for g starts with the initial condition of $-f$ and evolves to zero as $t \to \infty$.
As usual, let's attempt to solve our transient problem using separation of variables, that is,
Let $g(r, z, t) = R(r)Z(z)T(t)$ so (33) becomes

$$\frac{T'}{T} = \frac{1}{Re}\Big[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} + \frac{Z''}{Z}\Big] = -\omega^2$$

We want our separation constant to be negative in order for it to decay to zero as $t \to \infty$; so we choose $-\omega^2$

So, we shall solve our transient equation with given conditions numerically, using the *method of lines*. We take the same approach as we did in our last section. We descritize our space (r, z), using 2^{nd} order centered finite differences.

At each interior grid point, $(r_i, z_j) = (i\delta_r, j\delta_z)$ for $i \in [1, n_r - 1]$ and $j \in [1, n_j - 1]$, we have the discrete version of (33):

$$\frac{dv_{i.j}}{dt} = \frac{1}{Re}(\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\delta_r^2} + \frac{v_{i+1,j} - v_{i-1,j}}{2i\delta_r^2} - \frac{v_{i,j}}{i^2\delta_r^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\delta_z^2}) = \mathbf{RHS}(v_{i,j}) \quad (39)$$

This is a system of 1^{st} order ODE's, and each ODE determines who v evolves on each interior grid point. Thus, these equations are subject to boundary and initial conditions (34) and (35). Starting with our discrete initial conditions on $v^0_{i,j}$, evaluate $\mathbf{RHS}^0_{i,j}$. Note that when evaluating $\mathbf{RHS}_{i,j}$ at any time, the boundary conditions are used. In time, they are constant and since $v_{i,0} = i\delta_r \neq 0$, and $\mathbf{RHS}^0_{i,j}$ will not be zero everywhere - due to the impulsive start of the bottom wall rotation.

So, numerically, we will integrate (39) using Heun's method - a 2 stage predictor-corrector method:

$$\begin{aligned} \text{Predictor stage: } v^p_{i,j} &= v^k_{i,j} + \delta_t\mathbf{RHS}(v^k_{i,j}) \\ \text{Corrector stage: } v^{k+1}_{i,j} &= v^k_{i,j} + 0.5\delta_t[\mathbf{RHS}(v^k_{i,j}) + \mathbf{RHS}(v^p_{i,j})] \end{aligned} \quad (40)$$

The function **RHS** takes as input an entire array $\mathbf{v}$ (interior and boundary values of $v_{i,j}$) at the stage/time indicated by the superscript on v, and the subscripts (i, j) indicating the grid point at which the right-hand side of (39) is evaluated and returned as the value of **RHS**.

The Heun loop (40) is initiated at $k = 0$ (corresponding to $t = 0; t = k\delta_t$) with $v^0_{i,j} = 0$ for $i \in [0, n_r] j \in [1, n_z]$ and $v^0_{i,0} = i\delta_r$ for $i \in [0, n_r]$; and is evaluated until steady state is asymptotically reached.

Then, to determine when the steady state is asymptotically reached, we have to compute the discrete L_2-norm of $\mathbf{v}$ every nk time steps. At time $t = l\delta t$, the norm is

$$v_{\text{norm}}(l) = \sqrt{\frac{1}{(n_r + 1)(n_z + 1)} \sum_{i=0}^{n_r} \sum_{j=0}^{n_z} (v^l_{i,j})^2}$$

We will plot this versus $t = k\delta_t$, to see when it approaches a constant. Otherwise, we can automate the process and stop the loop (39) when

$$\frac{v_{\text{norm}}(l) - v_{\text{norm}}(l-1)}{v_{\text{norm}}(l)} < \sigma$$

for sufficiently small σ (10^{-5})

Now, we'll use Python as before utilizing the same special packages they've made available. We'll be solving this iterative process and giving results as animations. For analysis, we can notice that if we increase our *Re* from 1 to 10, and then again to 100. The time it takes for the program to process, increases greatly, but also the iso-contour plots and animations grow much slower at higher Reynolds numbers. Of course, this makes sense, as Reynolds number is the ratio of viscous to inertial time scales. Thus, it would make sense that the velocity takes much more time to grow, and the fluid take longer to move. Additionally, the further away from the rotating base, the lower the velocity of the fluid. However, the rotating base has less impact, even closer than it would for a lower Reynold's number. From what it looks like, it seems to be a very viscous or thick fluid, as the number is increased. Naturally, the amount of iterations it takes to reach a steady state solution within tolerance, is greatly increased. Please look at the labeled animation for this part. For $Re = 1$, and $\Gamma = 1.5$ it takes a little over $30,000$ steps to reach within tolerance of the L2 norm. The code file is included in the folder, as well as the animation.

$\square$