Notes

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During the past years, there has been a growing interest on the links between several conjectures involving assembly maps. This report will focus on the link between the coarse Baum-Connes conjecture and the Novikov conjecture. If Γ is a finitely generated group, the descent principle assures that if the coarse Baum-Connes assembly map for Γ as a metric space with the word length is an isomorphism, then the Baum-Connes assembly map for Γ is rationnaly injective, thus the Novikov conjecture holds for Γ.

Following ideas of M. Gromov, G. Yu introduced new coarse concepts in the study of these assembly maps. He was able to prove the coarse Baum-Connes conjecture for proper metric spaces with finite asymptotic dimension [11], which is a coarse analogue of the topological covering dimension. Later on, in a paper with Guenter and R. Tessera [1], they defined decomposition complexity for metric spaces, which is a broad generalization of asymptotic dimension. In particular, proper metric spaces with finite asymptotic dimension are of finite decomposition complexity. At the end of [1], as concluding remarks, the authors point out that one should be able to derive a new proof of the coarse Baum-Connes conjecture for spaces with finite decomposition complexity. We should emphasize that this is already known : a space which is finitely decomposable has property (A), hence verifies the coarse Baum-Connes conjecture by the work of G. Yu. [12] But the techniques of this proof is highly analytical, it uses a Dirac-Dual Dirac type construction, which involves infinite dimensional analysis. The suggestion of [1] is to give a geometrical proof, using a coarse Mayer-Vietoris argument in the spirit of the proof of the Baum-Connes conjecture for spaces with finite asymptotic dimension.

Such a proof was given in the setting of algebraic K-theory in a paper of D. A. Ramras, R. Tessera and G. Yu where they established the integral Novikov conjecture for algebraic K-theory of group rings $R[\Gamma]$ when the group Γ has FDC (finite decomposition complexity). Their proof uses the continuously controlled algebraic K-theory groups very intensively : their key lemma is a vanishing theorem of these groups. In a series of papers [3][4], H. Oyono-Oyono and G. Yu developed an analogue of this controlled K-theory for operator algebras, which they named quantitative K-theory. It consists of a family of groups $\hat{K}(A) = (K^{\epsilon,r}(A))$ for $r \geq 0, \epsilon \in (0, \frac{1}{4})$ and A a filtered C^*-algebra, which we shall describe later. They were able to define quantitative assembly maps that factorize the usual ones, and to give equivalence between isomorphisms of the assembly map and quantitative statements.

Following the route of these articles [3][4], we will define quantitative assembly maps for étale groupoids with a proper length. These assembly maps are equivalent to the coarse quantitative assembly maps for proper metric spaces X defined in [4] if one takes $G = G(X)$, the coarse groupoid of X. We give also quantitative statements equivalent to a certain isomorphism. **(rerédiger ce paragraphe de façon plus précise une fois les résultats écrits)**

1 Review of quantitative K-theory

This section presents basic constructions of quantitative K-theory for operator algebras that we shall use. For more details, see the original article of H. Oyono-Oyono and G. Yu.[3] We will refer either to quantitative or controlled K-theory for the same object, namely a family of abelian groups $\hat{K}(A) = (K^{\epsilon,R}(A))$ where $R > 0, 0 < \epsilon < \frac{1}{4}$, defined for a filtered C^*-algebra A. The motivating idea is to keep track of propagation of an operator while taking his (possibly higher) index. The main example is that of Roe algebras.

1.1 Roe algebras and filtration

Let (X, d) be a discrete proper metric space, i.e. its closed ball are compact, that is uniformly bounded, so that for every $R > 0$, there exists an integer $N \geq 0$ such that every ball of radius R contains less than N elements. A X-module is a hilbert space H equiped with a C^*-morphism $\rho : C_0(X) \to \mathcal{L}(H)$. To lighten notations, we write fx instead of $\rho(f)x$ if $f \in C_0(X)$ and $x \in X$. All these definitions can be found in [6]

Définition 1. Let H be a X-module.

- An operator $T \in \mathcal{L}(H)$ is locally compact if for every $f \in C_0(X)$, fT and Tf are compact operators, where f is understood as a multiplication operator.
- An operator $T \in \mathcal{L}(H)$ is of finite propagation bounded by $R > 0$ if for every pair of functions $f, g \in C_0(X)$ such that $d(\text{supp } f, \text{supp } g) > R$, $fTg = 0$.
- We denote by $C_R[X]$ the set of locally compact operators with finite propagation bounded by R. The Roe algebra of X is $C^*(X)$, the closure of $\cup_{R>0} C_R[X]$ in the operator topology of $\mathcal{L}(H)$.

A simple example is given by $l^2(X) \otimes H$ with H a separable Hilbert space, in which case $C_R[X]$ is the algebra of operators $(T_{xy})_{x,y\in X}$ such that $T_{x,y} \in K(H)$ for every $x, y \in X$, and $T_{xy} = 0$ as soon as $d(x, y) > R$.
Remark : one coulde replace Hilbert spaces by Hilbert modules E over a C^*-algebra B in this definition, $\mathcal{L}(H)$ by adjoinable operators $\mathcal{L}_B(E)$ and $K(H)$ by compact operators $K_B(E)$, to obtain $C^*(X, B)$, the Roe algebra with coefficient in B. The Roe algebra $C^*(X, B)$ enjoys functorial properties in B.

This example motivates the following definition.

Définition 2. A C^*-algebra A is said to be filtered if there are closed $*$-stable linear subspaces A_R for every $R > 0$ such that

- $A_s \subset A_r$ when $s \leq r$,
- $\cup_{R>0} A_R$ is dense in A,
- $A_s.A_r \subset A_{s+r}$ for every $r, s \geq 0$,
- $\forall r > 0, 1 \in A_r$ when A is unital.

A C^*-morphism between filtered C^*-algebras $\phi : A \to B$ is filtered if $\phi(A_R) \subset B_R$ for every $R > 0$.

If A is a non-unital C^*-algebra, let A^+ be the unital C^*-algebra containing A as a two-sided ideal, defined as :

$$A^+ = \{(a,\lambda) \in A \times \mathbb{C}\}$$
$$(a,\lambda)(b,\mu) = (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a,\lambda)^* = (a^*, \overline{\lambda})$$

with the norm operator

$$||(a,\lambda)|| = \sup\{||ax + \lambda x|| : x \in A, ||x|| = 1\}.$$

When A is not unital and filtered by $(A_R)_{R>0}$, A^+ is filtered by $A_R^+ = \{(x,\lambda) : x \in A_R, \lambda \in \mathbb{C}\}$.

1.2 Definition of quantitative K-theory

Let A be unital and filtered, and $\epsilon \in (0, \frac{1}{4}), R > 0$.
- p is a ϵ-R-projection if $p \in A_R$ and $||p^2 - p|| < \epsilon$.
- u is a ϵ-R-unitary if $u \in A_R$ and $||u^*u - 1|| < \epsilon$ and $||uu^* - 1|| < \epsilon$.

A ϵ-R-projection has its spectrum localized around 0 and 1, with a spectral gap in between, which allows to define a true projection $\kappa(p)$ by functional calculus.

Let $P_n^{\epsilon,R}(A)$ be the set of ϵ-R-projections and $U_n^{\epsilon,R}(A)$ the set of ϵ-R-unitaries of $M_n(A)$. We can also define the inductive limits $P_\infty^{\epsilon,R}(A) = \varinjlim P_n^{\epsilon,R}(A)$ and $U_\infty^{\epsilon,R}(A) = \varinjlim U_n^{\epsilon,R}(A)$ under the inductive system of morphisms

$$p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \text{ and } u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The following defines equivalence relations on $P_\infty^{\epsilon,R}(A) \times \mathbb{N}$ and $U_\infty^{\epsilon,R}(A)$:
- $(p,m) \sim (q,n)$ if there exists $h \in P_\infty^{\epsilon,R}(A[0,1])$ and $k > 0$ such that $h(0) = \begin{pmatrix} p & 0 \\ 0 & I_{m+k} \end{pmatrix}$ and $h(0) = \begin{pmatrix} q & 0 \\ 0 & I_{n+k} \end{pmatrix}$.
- $u \sim v$ if there exists $h \in U_\infty^{3\epsilon,2R}(A[0,1])$ such that $h(0) = u$ and $h(1) = v$.

Définition 3. Let $\epsilon \in (0, \frac{1}{4}), R > 0$.

- $K_0^{\epsilon,R}(A) = P_\infty^{\epsilon,R}(A) \times \mathbb{N} / \sim$ if A is unital, or $\{[p,n] \in P_\infty^{\epsilon,R}(A^+) \times \mathbb{N}$ such that $\mathrm{tr}\rho_A(\kappa(p)) = n\}$ if A is not unital.
- $K_1^{\epsilon,R}(A) = U_\infty^{\epsilon,R}(A^+) / \sim$ where $A = A^+$ if A is unital.

These are abelian groups (see [3],[4]) with laws described by $[p,m]_{\epsilon,R} + [q,n]_{\epsilon,R} = [diag(p,q), m+n]_{\epsilon,R}$ and $[u]_{\epsilon,R} + [v]_{\epsilon,R} = [diag(u,v)]_{\epsilon,R}$.

Définition 4. To be more flexible, we allow our morphisms between quantitative objets to increase propagation, but in a controlled way. That control is implemented by what Oyono-Oyono and Yu call a control pair.
- A control pair (α, h) is a positive number $\alpha > 1$ and a map $h : (0, \frac{1}{4\alpha}) \to \mathbb{R}_*^+$ such that $h \leq F$ for F a non-increasing function. We define a partial order on the control pair by $(\alpha, h) \leq (\beta, k)$ if $\alpha \leq \beta$ and $h \leq k$ on $(0, \frac{1}{4\beta})$.

- A quantitative object is a family of abelian groups $\hat{H} = (H^{\epsilon,R})_{\epsilon\in(0,\frac{1}{4}),R>0}$ and group morphisms $\phi_{\epsilon,R}^{\epsilon',R'} : G^{\epsilon,R} \to G^{\epsilon',R'}$ any time $R \leq R'$ and $\epsilon \leq \epsilon'$, such that $\phi_{\epsilon,R}^{\epsilon,R} = id_{H^{\epsilon,R}}$ and $\phi_{\epsilon_1,R_1}^{\epsilon_2,R_2} \circ \phi_{\epsilon_0,R_0}^{\epsilon_1,R_1} = \phi_{\epsilon,R}^{\epsilon_2,R_2}$.
- For a control pair (α, k) and two quantitative objects H and H', a (α, k)-controlled (quantitative) morphism $F : H \to H'$ is a family of group morphisms
$$F^{\epsilon,R} : H^{\epsilon,R} \to H'^{\alpha\epsilon,k_\epsilon R} \quad \forall\epsilon \in (0, \frac{1}{4\alpha}), R > 0.$$

We have natural morphisms of abelian groups $\iota_{\epsilon,R}^{\epsilon',R'} : K_*^{\epsilon,R}(A) \to K_*^{\epsilon',R'}(A)$ if $\epsilon \leq \epsilon', R \leq R'$ and $\iota_{\epsilon,R} : K_*^{\epsilon,R}(A) \to K_*(A)$, and $\iota_{\epsilon',R'} \circ \iota_{\epsilon,R}^{\epsilon',R'} = \iota_{\epsilon,R}$ holds. This fact gives sense to saying that a controlled morphism induced a morphism in K-theory.

Exemple 1. The basic example of quantitative object is of course quantitative K-theory.
If $\phi : A \to B$ is a filtered C^*-morphism, it naturally induces a $(1, 1)$-controlled morphism $\phi_* : \hat{K}(A) \to \hat{K}(B)$.
Another (important) examples will be that of the controlled Morita equivalence and quantitative boundary maps.

1.3 Morita equivalence

As in classical K-theory, we have an isomorphism which we call the (controlled) Morita equivalence.

Proposition 1. Let A be a filtered C^*-algebra and H a separable Hilbert space. We denote by K_A the C^*-algebra of compact operators of the standard Hilbert module H_A, which is C^*-isomorphic to $A\otimes K(H)$. Let e be any rank-one projection in $K(H)$. Then the C^*-morphism
$$\begin{cases} A & \to & K_A \\ a & \mapsto & a \otimes e \end{cases}$$
induces an $\mathbb{Z}_2$-graded isomorphism
$$M_A^{\epsilon,R} : K^{\epsilon,R}(A) \to K^{\epsilon,R}(K_A)$$
for every $R > 0$ and $0 < \epsilon < \frac{1}{4}$.

1.4 Quantitative boundary maps

Let $0 \to J \to A \to A/J \to 0$ be an extension of C^*-algebras. We denote $\partial_{J,A} : K_*(A/J) \to K_{*+1}(J)$ the associated boundary map.

Proposition 2. There exist a control pair (α_D, k_D) such that for any semi-split extension of filtered C^*-algebras $0 \to J \to A \to A/J \to 0$, there exists a (α_D, k_D)-controlled morphism of odd degree
$$D_{J,A} : \hat{K}(A/J) \to \hat{K}(J) \quad \epsilon \in (1, \frac{1}{4\alpha_D}), R > 0$$
which induces $\partial_{J,A}$ in K-theory.

We have to recall what we will refer as the remark 3.7 of [3], a property of quantitative assembly maps that we shall intensively.
Let $\phi : A \to B$ be a filtered morphism between two filtered C^*-algebras. We suppose we have extensions $0 \to I \to A \to A/I \to 0$ and $0 \to J \to B \to B/J \to 0$. If

- $\phi(I) \subset J$
- ϕ intertwines two completely positive sections of the extensions : $\phi \circ s_A = s_B \circ \phi_I^J$, where ϕ_I^J is the induced map $A/I_rightarrowB/J$.

then $D_{J,B} \circ (\phi_I^J)_* = \phi_* \circ D_{I,A}$.
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2 Quantitative statements

2.1 A reminder on groupoids action

For the rest of the article, $\mathcal{G}$ will denote a topological groupoid, $X = \mathcal{G}^{(0)}$ its space of units, $s, r : \mathcal{G}X$ respectively the source and target maps. The fibers over $x \in X$ are denoted by $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$. In this report, all the groupoids are supposed to be locally compact, σ-compact and étale, that is r is a local homeomorphism, which entails that the fibers are discrete, hence $\mathcal{G}$ is naturally endowed with a Haar system $(\lambda^x)_{x\in X}$.
For any $C(X)$-algebra B, $B_x = B \otimes_{ev_x} \mathbb{C}$ is the fiber over $x \in X$, where ev_x is the evaluation at x. The pull-back r^*B and s^*B are respectively defined to be $B \otimes_r C_0(\mathcal{G})$ and $B \otimes_s C_0(\mathcal{G})$. They are $C(\mathcal{G})$-algebras with fiber over $g \in \mathcal{G}$ equal to $B_{r(g)}$ (resp. $B_{s(g)}$).
An action of $\mathcal{G}$ on B is an isomorphism of $C(\mathcal{G})$-algebras $\alpha : s^*B \to r^*B$ which respects the product : $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$ for all composable pairs (g, g').
For a right B-Hilbert module $\mathcal{E}$, we can also define the pull-backs $r^*\mathcal{E} = \mathcal{E} \otimes_r C_0(\mathcal{G})$ and $s^*\mathcal{E} = \mathcal{E} \otimes_s C_0(\mathcal{G})$: they are respectively right and left s^*B-Hilbert modules (the s^*B-module structure on $r^*\mathcal{E}$ being obtained by the inverse of the action $\alpha^{-1} : r^*B \to s^*B$) with fiber over g given by $\mathcal{E}_{r(g)}$ and $\mathcal{E}_{s(g)}$.
An action of $\mathcal{G}$ on $\mathcal{E}$ is a unitary $U \in \mathcal{L}(s^*\mathcal{E}, r^*\mathcal{E})$ such that $U_g U_{g'} = U_{gg'}$ for all composable pairs (g, g').
We will call any $C(X)$-algebra B endowed with an action of $\mathcal{G}$ a **$\mathcal{G}$-algebra**. If we have at our disposition a family of representations $\mathcal{F}$ of $\mathcal{G}$, we can then form the crossed-product $B \rtimes_{\mathcal{F}} \mathcal{G}$: it is the completion-reduction of $C_c(\mathcal{G}, B)$ for the norm operator $||f||_{\mathcal{F}} = \sup_{\pi\in\mathcal{F}} ||\pi(f)||$. Classical examples are that of the reduced crossed product $B \rtimes_r \mathcal{G}$ for which $\mathcal{F}$ is the left-regular representation, and the maximal crossed-product $B \rtimes_{max} \mathcal{G}$ for which $\mathcal{F}$ is the family of all unitary representations. If $\phi : A \to B$ is a $\mathcal{G}$-morphism, $f \mapsto \phi \circ f$ gives rise to a C^*-morphism $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$. The crossed-product is easily seen to be a functor from $\mathcal{G}$-algebras to C^*-algebras.
The formula
$$\langle \psi, \eta \rangle_x = \int_{\mathcal{G}^x} \langle \psi(g), \eta(g) \rangle d\lambda^x(g)$$
defines a $C(X)$-hermitian product on $C_c(\mathcal{G}, B)$, which we can complete to get the right-$B \rtimes \mathcal{G}$-Hilbert module $\mathcal{E} \rtimes \mathcal{G}$. So the functor $B \mapsto B \rtimes \mathcal{G}$ carries to $\mathcal{E} \mapsto \mathcal{E} \rtimes \mathcal{G}$.

We will be mainly interested in the properties of the functor $B \mapsto B \rtimes \mathcal{G}$, and the majority of the constructions performed in this section are carriable with "good" crossed product functors. We will discuss the property needed for a crossed-product functor to be "good".
The first property is **semi-split–exactness**. We recall that an extension of $\mathcal{G}$-algebra

$$0 \longrightarrow J \longrightarrow A \xrightarrow{p} A/J \longrightarrow 0$$

is said to be semi-split if there exists a completely positive map $s : A/J \to A$ such that $p \circ s = 1$. The functor $- \rtimes \mathcal{G}$ is said to be semi-split exact if for any such extension,

$$0 \longrightarrow J \rtimes \mathcal{G} \longrightarrow A \rtimes \mathcal{G} \xrightarrow{p_{\mathcal{G}}} A/J \rtimes \mathcal{G} \longrightarrow 0$$

is a semi-split extension of C^*-algebras. The second property implies the first. We say that $- \rtimes \mathcal{G}$ has the CP-property if it preserves completely positive maps, so that from any $\mathcal{G}$-completely positive map $\phi : A \to B$, we can induce a CP map $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$. This assures that the action of $\mathcal{G}$ on any $\mathcal{G}$-module $\mathcal{E}$ descends to an isomorphism $K(\mathcal{E}) \rtimes \mathcal{G} \simeq K(\mathcal{E} \rtimes \mathcal{G})$. **A FINIR, faire une preuve entre autre ...**

2.2 The Baum-Connes conjecture

The more general setting of the Baum-Connes conjecture [8] is that of a locally compact σ-compact Hausdorff groupoid $\mathcal{G}$ endowed with a Haar system, together with a coefficient C^*-algebra B acted upon by $\mathcal{G}$, which give rise to an assembly map

$$\mu_r : K_*^{top}(\mathcal{G}, B) \to K_*(B \rtimes_r \mathcal{G}).$$

The left hand side $K_*^{top}(\mathcal{G}, B)$ is the K-homology of the classifying space $\mathcal{E}\mathcal{G}$ for proper actions of $\mathcal{G}$ with coefficients in B. We give a sketch of the construction when $\mathcal{G}$ is étale. Let $d \geq 0$ and $P_d(\mathcal{G})$ be the Rips complex of $\mathcal{G}$, i.e. the space of probabilities supported on a fiber $\mathcal{G}^x$ for a $x \in \mathcal{G}^{(0)}$

$$P_d(\mathcal{G}) = \{p \in \mathcal{P}(\mathcal{G}) : \exists x \in \mathcal{G}^{(0)}, r^*p = \delta_x, \operatorname{supp} p \subset B(e_x, d)\}.$$

Then $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B)$ is defined to be the inductive limite of $KK^{\mathcal{G}}(C_0(X), B)$ for X $\mathcal{G}$-proper $\mathcal{G}$-spaces (such that X/G is compact). If $d \leq d'$, we have a morphism $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B) \to KK^{\mathcal{G}}(C_0(P_{d'}(\mathcal{G})), B)$ naturally induced by the inclusion $P_d(\mathcal{G}) \subset P_{d'}(\mathcal{G})$, and the K-homology of $\mathcal{G}$ is defined as

$$K_*^{top}(\mathcal{G}, B) = \lim_{d \to \infty} KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B).$$

In his thesis [2], P.-Y. Le Gall constructed the Kasparov transform for the action of a groupoid

$$j_{\mathcal{G}} : KK^{\mathcal{G}}(A, B) \to KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$$

for any $\mathcal{G}$-C^*-algebras A and B. It is also in this paper that equivariant KK-theory for groupoids and the corresponding Kasparov product are defined. One can then give an formula for the assembly map, namely if $z \in KK^G(C_0(X), B)$ for a $\mathcal{G}$-proper $\mathcal{G}$-space X of $P_d(\mathcal{G})$, then

$$\mu_r(z) = [\mathcal{L}_X] \otimes_{C_0(X) \rtimes_r \mathcal{G}} j_{\mathcal{G}}(z) \in K_*(B \rtimes_r \mathcal{G})$$

holds, where $[\mathcal{L}_X]$ is the class of a canonical element associated to X which is to be thought of as a Miscenko bundle over $C_0(X) \rtimes_r \mathcal{G}$.

The remaining of this section will be devoted to the construction of a controlled Kasparov transformation for every $z \in KK^{\mathcal{G}}(A, B)$:

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G})$$

which is of course a controlled morphism which induces right multiplication by $j_{\mathcal{G}}(z)$ in K-theory. This will allow us to define a bunch of quantitative assembly maps

$$\mu_{\mathcal{G}}^{\epsilon,R} : K^{top}(\mathcal{G}, B) \to K^{\epsilon,R}(B \rtimes \mathcal{G})$$

inducing the assembly map in K-theory, and to study the relation between the quantitative Baum-Connes conjecture and the classical one for $\mathcal{G}$.

2.3 Length, propagation and controlled six-terms exact sequence

Let $\mathcal{G}$ be a locally compact groupoid with base $\mathcal{G}^{(0)} = X$, a compact space, endowed with a Haar system $\lambda = (\lambda^x)_{x \in X}$. We suppose that $\mathcal{G}$ comes with a proper length l, that is a family of application $(l^x)_{x \in X}$ defined on the fibers $\mathcal{G}^x$ with values in $\mathbb{R}_+$, such that

$$\begin{array}{l} l^x(e_x) = 0 \\ l^{r(\gamma)}(\gamma) = l^{s(\gamma)}(\gamma^{-1}) \\ l^x(\gamma_1^{-1}\gamma_2) \leq l^x(\gamma_1) + l^x(\gamma_2). \end{array}$$

That length allows us to define a filtration on crossed-product algebras of $\mathcal{G}$ by

$$(A \rtimes \mathcal{G})_r = \{f \in C_c(\mathcal{G}, A) : \text{ supp } f \subset \cup_{x \in X} B_x(r)\}$$

for any $\mathcal{G}$-algebra A. Here, $B_x(r)$ is the ball $\{\gamma \in \mathcal{G}^x : l^x(\gamma) \leq r\}$, and $\rtimes$ can be either the reduced cross-product $\rtimes_r$ or the maximal one $\rtimes_{max}$. Recall that $A \rtimes \mathcal{G}$ is functorial in A, from the category of $\mathcal{G}$-C^*-algebras with $\mathcal{G}$-equivariant C^*-morphisms to the category of C^*-algebras with C^*-morphisms. For $\phi : A \to B$ a $\mathcal{G}$-equivariant C^*-morphism, we denote by $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$ the induced C^*-morphism.

If $\quad 0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad$ is a semi-split exact sequence of $\mathcal{G}$-C^*-algebras,

then $\quad 0 \longrightarrow J \rtimes \mathcal{G} \xrightarrow{\phi_{\mathcal{G}}} A \rtimes \mathcal{G} \xrightarrow{\psi_{\mathcal{G}}} A/J \rtimes \mathcal{G} \longrightarrow 0 \quad$ is a flitered semi-split exact sequence. From this, we can state the following proposition.

Proposition 3. There exists a control pair (λ, h) such that for every semi-split extension of $\mathcal{G}$-C^*-algebras

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad ,$$

the following diagrams (λ, h)-commutes and are (λ, h)-exact

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_r \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_r \mathcal{G}) \end{array} \quad ,$$

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_{max} \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_{max} \mathcal{G}) \end{array} \quad .$$

2.4 The Kasparov transform

Let A and B be two $\mathcal{G}$-C^*-algebras, and H a separable Hilbert space, $l^2(\mathbb{Z})$ for instance, and $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. The standard Hilbert module over B is denoted by $H_B = H_{\mathcal{G}} \otimes B$, and K_B is the algebra of compact operators for H_B, i.e. $K(H) \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Every K-cycle $z \in KK^G(A, B)$ can be represented as a triplet (H_B, π, T) where :

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B.
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator.
- T and π verify the K-cycle condition, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ and $\pi(a)(g.T - T)$ are compact operator over H_B for all $a \in A, g \in \mathcal{G}$.

Set $T_{\mathcal{G}} = T \otimes id_{B \rtimes \mathcal{G}} \in \mathcal{L}_{B \rtimes \mathcal{G}}(H_B \otimes (B \rtimes \mathcal{G})) \simeq \mathcal{L}_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$, and $\pi_G : A \rtimes \mathcal{G}v \to L_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$. Then, according to Le Gall [2], $(H_{B \rtimes \mathcal{G}, \pi_{\mathcal{G}}, T_{\mathcal{G}}})$ represents the K-cycle $j_{\mathcal{G}}(z) \in KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$. Let us construct a controlled morphism associated to z,

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G}),$$

which induces right multiplication by $j_{\mathcal{G}}(z)$ in K-theory.

2.4.1 Odd case

Let us first do the for work for $z \in KK_1^{\mathcal{G}}(A, B)$. Let (H_B, π, T) be a K-cycle representing z. Set $P = \frac{1+T}{2}$ and $P_{\mathcal{G}} = P \otimes id_{B \rtimes \mathcal{G}}$. We define

$$E^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in A \rtimes \mathcal{G}, y \in K_{B \rtimes \mathcal{G}}\}$$

a C^*-algebra which is filtered by

$$E_R^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in (A \rtimes \mathcal{G})_R, y \in K \otimes (B \rtimes \mathcal{G})_R\}$$

which gives us a filtered extension

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow E^{(\pi,T)} \longrightarrow A\rtimes_r\mathcal{G} \longrightarrow 0$$

and semi split by $s : \begin{cases} A\rtimes_r\mathcal{G} & \to & E^{(\pi,T)} \\ x & \mapsto & (x, P_{\mathcal{G}}\pi_{\mathcal{G}}(x)P_{\mathcal{G}}) \end{cases}$.

Let us show that the controlled boundary map of this extension does not depend on the representant chosen, but only on the class z.
Let $(H_B,\pi_j,T_j), j=0,1$ two K-cycles which are homotopic via $(H_{B[0,1]},\pi,T)$. We denote e_t the evaluation at $t\in[0,1]$ for an element of $B[0,1]$, and set $y_t = e_t(y)$ for such a y. The $*$-morphism

$$\phi : \begin{cases} E^{(\pi,T)} & \to & E^{(\pi_t,T_t)} \\ (x,y) & \mapsto & (x,y_t) \end{cases}$$

satisfies $\phi(K_{B[0,1]\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$ and makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & K_{B[0,1]\rtimes_r\mathcal{G}} \arrow[r] \arrow[d, "\phi_{|K_{B[0,1]\rtimes_r\mathcal{G}}}"] & E^{(\pi,T)} \arrow[r] \arrow[d, "\phi"] & A\rtimes_r\mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & K_{B\rtimes_r\mathcal{G}} \arrow[r] & E^{(\pi_t,T_t)} \arrow[r] & A\rtimes_r\mathcal{G} \arrow[r] & 0 \end{tikzcd}.$$

According to [3], remark 3.7., the following holds

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_t,T_t)}} = \phi_* \circ D_{K_{B[0,1]\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As $id\otimes e_t$ gives a homotopy between $id\otimes e_0$ and $id\otimes e_1$, and as if two $*$-morphisms are homotopic, then they are equal in controlled K-theory,

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_0,T_0)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_1,T_1)}}$$

holds, and the boundary of the extension $E^{(\pi,T)}$ depends only on z.

Définition 5. The controlled Kasparov transform of an element $z\in KK_1^{\mathcal{G}}(A,B)$ is defined as the compostion

$$J_{red,\mathcal{G}}(z) = \mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}} \circ D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As the boundary map is a (α_D,k_D)-controlled morphism and the Morita equivalence preserves the filtration, $J_{red,\mathcal{G}(z)}$ is (α_D,k_D)-controlled.

Proposition 4. Let A and B two $\mathcal{G}$-C^*-algebras. There exists a control pair (α_J,k_J) such that for every $z\in KK_1^{\mathcal{G}}(A,B)$, there exists a (α_J,k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}_*(A\rtimes_r\mathcal{G}) \to \hat{K}_{*+1}(B\rtimes_r\mathcal{G})$$

such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z+z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_1^G(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_1^G(A, B_1)$.

(v) Let $0 \to J \to A \to A/J \to 0$ be a semi-split equivariant extension of $\mathcal{G}$-algebras and $[\partial_J] \in KK_1^{\mathcal{G}}(A/J, J)$ be its boundary element. Then

$$J_{\mathcal{G}}([\partial_J]) = D_{J \rtimes G, A \rtimes \mathcal{G}}.$$

Preuve 1. (i) The K-cycle $[\partial_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}] \in KK_1(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ implementing the boundary of the extension $E^{(\pi,T)}$ induces the map $j_{red,\mathcal{G}}$ by definition, and modulo Morita equivalence, which immediately gives the first point.

(ii) If z, z' are elements of $KK_1^G(A, B)$, represented by two K-cycles (H_B, π_j, T_j), and if (H_B, π, T) is a K-cycle representing the sum $z + z'$, then $E^{(\pi,T)}$ is naturally isomorphic to the extension sum of the $E_j := E^{(\pi_j,T_j)}$, namely

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow D \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j, E_i) \right\}.$$

Naturality of the controlled boundary maps [3] ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result.

(iii) Let $z \in KK_1^{\mathcal{G}}(A_2, B)$, represented by a cycle (H_B, π, T). Representing $f^*(z)$ is $(H_B, f^*\pi, T)$ with off course $f^*\pi = \pi \circ f$. The map

$$\phi : \begin{cases} E^{f^*(\pi,T)} & \to \quad E^{(\pi,T)} \\ (x, P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) & \to \quad (f_{\mathcal{G}}(x), P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) \end{cases}$$

satisfies

- $\phi(K_{B\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$, and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{f^*(\pi,T)} & \longrightarrow & A_1 \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\phi} & & \downarrow{f_{\mathcal{G}}} & & \\ 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A_2 \rtimes_r \mathcal{G} & \longrightarrow & 0 \end{array}.$$

- It intertwines the sections of the two extensions.

Remark 3.7 of [3] assures that

$$D_{K_{B\rtimes_r\mathcal{G}},E^{f^*(\pi,T)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}} \circ f_{\mathcal{G},*}$$

, and the claim is clear from composition by $\mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}}$.

(iv) Let $\mathcal{E} = H_{B_1} \otimes_g B_2$, which is a countably generated Hilbert B_2-module. The homomorphism $g : B_1 \to B_2$ gives rise to $g_* : \mathcal{L}_{B_1}(H_{B_1}) \to \mathcal{L}_{B_2}(\mathcal{E})$, which preserves compact operators : $g_*(K_{B_1}) \subset K(\mathcal{E})$. We have a similar statement for $g_G : B_1 \rtimes \mathcal{G} \to B_2 \rtimes \mathcal{G}$. We denote $\mathcal{E}_G$ the Hilbert $B_2 \rtimes \mathcal{G}$-module $\mathcal{E} \rtimes \mathcal{G} \simeq H_{B_1\rtimes\mathcal{G}} \otimes_g (B_2 \rtimes \mathcal{G})$.

Let $z \in KK^{\mathcal{G}}(A, B_1)$ be represented by the K-cycle (H_{B_1}, π, T). Then $(H_{B_1} \otimes_g B_2, g_* \circ \pi, g_*(T)) = (\mathcal{E}, \tilde{\pi}, \tilde{T})$ represents $g_*(z)$.

The map $(x, y) \mapsto (x, (g_G)_*(y))$ induces $\Psi : E^{(\pi,T)} \to E^{g_*(\pi,T)}$ such that

$$\Psi(x, P_G\pi_G(x)P_G + y) \mapsto (x, \tilde{P}_G\tilde{\pi}_G(x)\tilde{P}_G + (g_G)_*(y)).$$

Indeed, the crossed-product functor commutes with pull-back by $\mathcal{G}$-morphisms, and $(g_G) * \circ\pi_G = (g_* \circ \pi)_G = \tilde{\pi}_G$ and $(g_G)_*(P_G) = g_*(P)_G = \tilde{P}_G$ so that

$$(g_G)_*(P_G\pi_G(x)P_G) = \tilde{P}_G\tilde{\pi}_G(x)\tilde{P}_G.$$

Now, by the equivariant stabilisation lemma of Le Gall [2], we know that the countably generated Hilbert module $\mathcal{E}_G$ sits as a complemented module of $H_{B_2\rtimes\mathcal{G}}$, and there exists a projection $p \in L(H_{B_2\rtimes\mathcal{G}})$ such that $pH_{B_2\rtimes\mathcal{G}} \simeq \mathcal{E}_{\mathcal{G}}$ and $pK_{B_2\rtimes\mathcal{G}}p \simeq K(\mathcal{E}_G)$. Let ψ be the composition $K_{B_1\rtimes\mathcal{G}} \to_{(g_G)_*} K(\mathcal{E}_{\mathcal{G}}) \to K_{B_2\rtimes\mathcal{G}}$. In this particular case, we can give an explicit description of ψ. The map defined on basic tensor products $(x_j)_j \otimes b \mapsto (g(x_j)b)_j$ extends to an isometric embedding $\mathcal{E}_{\mathcal{G}} \to H_{B_2\rtimes\mathcal{G}}$, under which $b\theta_{e_i,e_j}$ is mapped to $g(b)\theta_{u_i,u_j}$, where $\{e_j\}$ and $\{u_j\}$ are respectively the canonical orthogonal basis of $H_{B_1\rtimes\mathcal{G}}$ and $H_{B_2\rtimes\mathcal{G}}$. This gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{B_1\rtimes\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A \rtimes_r \mathcal{G} & \longrightarrow & 0 \\
 & & \Big\downarrow \psi & & \Big\downarrow \Psi & & \Big\downarrow = & & \\
0 & \longrightarrow & K_{B_2\rtimes\mathcal{G}} & \longrightarrow & E^{g_*(\pi,T)} & \longrightarrow & A \rtimes \mathcal{G} & \longrightarrow & 0
\end{array}.$$

and Ψ intertwines the two filtered sections by the previous relation. Moreover, $\Psi_{|K_{B_1\rtimes\mathcal{G}}} \subset K_{B_2\rtimes\mathcal{G}}$, so that we can again apply the remark 3.7 of [3] to state

$$D_{K_{B_2\rtimes\mathcal{G}},E^{g_*(\pi,T)}} = \psi_* \circ D_{K_{B_1\rtimes\mathcal{G}},E^{(\pi,T)}},$$

which we compose by the Morita equivalence on the left $M^{-1}_{B_2\rtimes\mathcal{G}}$

$$J_{\mathcal{G}}(g_*(z)) = M^{-1}_{B_2\rtimes\mathcal{G}} \circ g_{G,*} \circ D_{K_{B_1\rtimes\mathcal{G}},E^{(\pi,T)}}.$$

The homomorphisms inducing the Morita equivalence make the following diagram commutes,

$$\begin{tikzcd}
B_1 \rtimes \mathcal{G} \arrow[r, "g_{\mathcal{G}}"] \arrow[d] & B_2 \rtimes \mathcal{G} \arrow[d] \\
K_{B_1 \rtimes \mathcal{G}} \arrow[r, "\psi"] & K_{B_2 \rtimes \mathcal{G}}
\end{tikzcd},$$

and $J_{\mathcal{G}}(g_*(z)) = g_{G,*} \circ M^{-1}_{B_1 \rtimes \mathcal{G}} \circ D_{K_{B_1 \rtimes \mathcal{G}}, E^{(\pi,T)}} = g_{G,*} \circ J_{\mathcal{G}}(z)$.

(v) Let $q : A \to A/J$ be the quotient map and (H_J, π, T) be a cycle representing $[\partial_J]$. Then we apply remark 3.7 of [3] to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \rtimes \mathcal{G} \arrow[r] \arrow[d] & A \rtimes \mathcal{G} \arrow[r] \arrow[d, "s \circ q_{\mathcal{G}}"] & A/J \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & K_{J \rtimes \mathcal{G}} \arrow[r] & E^{(\pi,T)} \arrow[r] & A/J \rtimes \mathcal{G} \arrow[r] & 0
\end{tikzcd},$$

where the first vertical arrow is the canonical mapping that induces the Morita equivalence.

□

2.4.2 Even case

We can now define $J_{\mathcal{G}}$ for even K-cycles. Let A and B be two $\mathcal{G}$-algebras. Let $[\partial_{SB}] \in KK_1(B, SB)$ be the K-cycle implementing the boundary of the extension $0 \to SB \to CB \to B \to 0$, and $[\partial] \in KK_1(\mathbb{C}, S)$ be the Bott generator. As $z \otimes_B [\partial_{SB}]$ is an odd K-cycle, we can define

$$J_{\mathcal{G}}(z) := \tau_{B \rtimes \mathcal{G}}([\partial]^{-1}) \circ J_{\mathcal{G}}(z \otimes [\partial_{SB}]).$$

Here τ_D refers to the (α_τ, k_τ)-controlled map $\hat{K}(A_1 \otimes D) \to \hat{K}(A_2 \otimes D)$, that H. Oyono-Oyono and G. Yu constructed in [3] for any C^*-algebras D, A_1, A_2 and $z \in KK_*(A_1, A_2)$. It enjoys many natural properties, and induces right multiplication by $\tau_D(z) \in KK(A_1 \otimes D, A_2 \otimes D)$ in K-theory. We can see that, if we set $\alpha_J = \alpha_\tau \alpha_D$ and $k_J = k_\tau * k_D$, $J_{\mathcal{G}}(z)$ is (α_J, k_J)-controlled.

Proposition 5. Let A and B two $\mathcal{G}$-C^*-algebras. For every $z \in KK_*^{\mathcal{G}}(A, B)$, there exists a control pair (α_J, k_J) and a (α_J, k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}(A \rtimes_r \mathcal{G}) \to \hat{K}(B \rtimes_r \mathcal{G})$$

of the same degree as z, such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z + z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_*^G(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_*^G(A, B_1)$.

(v) $J_G([id_A]) \sim_{(\alpha_J,k_J)} id_{\hat{K}(A\rtimes\mathcal{G})}$

Preuve 2. The point (iii) is a consequence of the previous proposition 4, and of the equality $f^*(x) \otimes y = f^*(x \otimes y)$.

□

We now show that the controlled Kasparov transform respects in a quantitative way the Kasparov product.

Proposition 6. There exists a control pair (α_K, k_K) such that for every $\mathcal{G}$-C^*-algebra A, B and C, and every $z \in KK^{\mathcal{G}}(A, B)$, $z' \in KK^{\mathcal{G}}(B, C)$, the controlled equality

$$J_{\mathcal{G}}(z \otimes_B z') \sim_{\alpha_K,k_K} J_{\mathcal{G}}(z') \circ J_{\mathcal{G}}(z)$$

holds.

Preuve 3. We will use the following fact : there exists a positive integer d such that every cycle $z \in KK^{\mathcal{G}}(A, B)$ has decomposition property (d). For more details, we send to the appendice of the article of V. Lafforgue [5] where H. Oyono-Oyono shows that claim. We just need to know that z satisfies the decomposition property (d) if there exist $d + 1$ $\mathcal{G}$-C^*-algebras A_j and d cycles $\alpha_j \in KK^{\mathcal{G}}(A_{j-1}, A_j), j = 1, d$ such that $A_0 = A$, $A_d = B$ and each α_j is either coming from a $*$-morphism $A_{j-1} \to A_j$, or there is a $*$-morphism $\theta_j : A_j \to A_{j-1}$ such that $\alpha_j \otimes_{A_j} [\theta_j] = 1$ in $KK^G(A_{j-1}, A_{j-1})$.

This property reduces the proof to the special case of α being the inverse of a morphism in $KK^{\mathcal{G}}$-theory : $\alpha \otimes [\theta] = 1$, then :

$$\begin{array}{rcl}
J_{\mathcal{G}}(\alpha \otimes z) & \sim_{\alpha_J^2,k_J*k_J} & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\alpha \otimes [\theta]) \\
& \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\theta_*(\alpha)) \\
& \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ \theta_{\mathcal{G},*} \circ J_{\mathcal{G}}(\alpha) \\
& \sim & J_{\mathcal{G}}(\theta^*(\alpha \otimes z)) \circ J_{\mathcal{G}}(\alpha) \\
& \sim & J_{\mathcal{G}}(z) \circ J_{\mathcal{G}}(\alpha)
\end{array}$$

because $\theta^*(\alpha \otimes z) = \theta^*(\alpha) \otimes z = 1 \otimes z = z$. The control on the propagation of the first line follows from remark 2.5 of [3] and point (v), the other lines are equal by points (iii) and (iv). As d is uniform for all locally compact groupoids with Haar systems, a simple induction concludes, and (α_K, k_K) can be taken to be $(d\alpha_J^{2d}, (k_J * k_J)^{*d})$. □

2.5 Quantitative assembly maps

Following the article of J.-L. Tu [8], we recall that a locally compact, σ-compact and Hausdorff groupoid G, endowed with a Haar system λ, is said to be proper if there exists a cut-off function $c : G^{(0)} \to \mathbb{R}_+$ continuous such that

- for all compact subset K of $G^{(0)}$, supp $c \cap s(G^K)$ is compact,
- $\int_{G^x} c(s(g))d\lambda^x(g) = 1, \forall x \in G^{(0)}$.

If moreover $G^{(0)}/G$ is also compact, reducing the first condition to "supp c compact", then $g \mapsto \sqrt{c(r(g))c(s(g))}$ defines a projection in $C_c(G)$ for convolution, which gives an element $[\mathcal{L}_G] \in K_0(C^*G)$.

Now when X is a locally compact space which is $\mathcal{G}$-proper and $\mathcal{G}$-compact, the groupoid $X \rtimes \mathcal{G}$ is proper with a compact orbit base space ($X/\mathcal{G}$ is compact). We can then define $[\mathcal{L}_X] \in K_0(C_0(X) \rtimes_r \mathcal{G})$ as the class of the projector for $G = X \rtimes \mathcal{G}$.

2.5.1 Classifying space for proper actions

We remind the construction of a classifying space for proper actions for a σ-compact étale groupoid, which can be found in [9] and [4].
If $d \geq 0$, we set

$$P_d(\mathcal{G}) = \{p \in Prob(\mathcal{G}) : \exists x \in \mathcal{G}^{(0)}, r^*p = \delta_x \text{ and } l^x(g) \leq d, \ \forall g \in \text{supp } p\}$$

endowed with the $*$-weak topology, and with the natural action of $\mathcal{G}$ by translation.
If $p \in P_d(\mathcal{G})$ such that $r^*p = \delta_x$ for a certain $x \in \mathcal{G}^{(0)}$, we can write

$$p = \sum_{g \in \mathcal{G}^x} \lambda_g(p)\delta_g.$$

If we set $\phi^2(p) = \lambda_{e_x}(p) \geq 0$, we have $\phi \in C_0(P_d(\mathcal{G}))$ and $(g.\phi^2)(p) = \lambda_g(p)$. Now define :

$$\mathcal{L}_d = \sum_{g \in \mathcal{G}^x} \phi.(g.\phi) \in C(X, C_0(P_d(\mathcal{G}))) \subset C(\mathcal{G}, C_0(P_d(\mathcal{G})))$$

because X is compact. **? ?** $\mathcal{L}_d$ is a projection of $C_0(P_d(\mathcal{G})) \rtimes_r \mathcal{G}$ without propagation, and defines a class $[\mathcal{L}_d]_{\epsilon,R} \in K_0^{\epsilon,R}(C_0(P_d(\mathcal{G})) \rtimes_r \mathcal{G}))$ for all $R > 0, 0 < \epsilon < 1/4$.

Définition 6. Let B be a $\mathcal{G}$-algebra, and $R > 0, 0 < \epsilon < 1/4, d > 0$. The local quantitative assembly map for $\mathcal{G}$ is defined as the composition of $J_{\mathcal{G}}$ with the evaluation at $[\mathcal{L}_d]$:

$$\mu_B^{\epsilon,R,d} \begin{cases} KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B) & \to & K_*^{\epsilon,R}(B \rtimes \mathcal{G}) \\ z & \mapsto & J_{\mathcal{G}}^{\epsilon,R}(z)([\mathcal{L}_d]_{\epsilon,R}) \end{cases}$$

Remarks

(1) The assembly map is defined for all reasonnable crossed-products by $\mathcal{G}$. In particular for the reduced one and the maximal one, so that we have two different assembly, which we would distinguish writing $J_{\mathcal{G},r}$ and $J_{\mathcal{G},max}$ if necessary.

(2) The bunch of assembly maps $\mu_B^{\epsilon,R,d}$ induces the Baum-Connes assembly map for $\mathcal{G}$ in K-theory : the following diagram commutes

$$\begin{tikzcd} KK^G(C_0(P_d(\mathcal{G})),B) \arrow[r, "\mu_B^{\epsilon,R,d}"] \arrow[dr, "\mu_{\mathcal{G}}^d"] & K_*^{\epsilon,R}(B \rtimes \mathcal{G}) \arrow[d, "\iota_{\epsilon,R}"] \\ & K_*(B \rtimes \mathcal{G}) \end{tikzcd}$$

because $J_{\mathcal{G}}(z)$ induces the right multiplication by $j_{\mathcal{G}}(z)$ and also $\mu_{\mathcal{G}}^d(z) = [\mathcal{L}_d] \otimes j_{\mathcal{G}}(z)$. But, as $\mathcal{L}_{d'|P_d(\mathcal{G})} = \mathcal{L}_d$ as soon as $d \leq d'$, this diagram commutes with inductive limit over d.

(3) In [4], H. Oyono-Oyono and G. Yu defined a bunch of local quantitative coarse assembly maps for a metric space X. For the sake of simplicity, we take X to be discrete and uniformly bounded. Let $\mathcal{C}$ be its coarse structure, that is the set of all its controlled subsets. Then, for any C^*-algebras A and B and a K-cycle $z \in KK(A,B)$, they construct a controlled morphism

$$\sigma_X(z) : \hat{K}(C^*(X,A)) \to \hat{K}(C^*(X,B)).$$

There exists a projection P_X without propagation, and the local quantitative assembly map is defined as

$$A_{X,B}^{\epsilon,r,d}(z) = \sigma_X^{\epsilon,r}(z)([P_X]_{\epsilon,r})$$

for $z \in KK(C_0(P_d(X)),B)$, where $P_d(X)$ is the classical Rips complex of X. This bunch of assembly maps induce the usual coarse assembly map of X

$$A_{X,B} : KX_*(X,B) \to K_*(C^*(X,B)$$

in K-theory. Now let $\mathcal{G}$ be the coarse groupoid of X. It is an étale groupoid with compact base space $\mathcal{G}^{(0)} = \beta X$, the Stone-Cech compactification of X defined as

$$\mathcal{G} := \cup_{E \in \mathcal{C}} \overline{E},$$

where $\overline{E}$ is the closure of E in $\beta(X \times X)$. A classical result of G. Skandalis, J.-L. Tu and G. Yu [7] claims that the coarse Baum-Connes conjecture for X with coefficients in B is equivalent to the Baum-Connes conjecture for the groupoid G with coefficient in $l^\infty(X,K_B)$. More precisely, there is an isomorphism of C^*-algebras $\Psi_B : l^\infty(X,K_B) \rtimes_r \mathcal{G} \simeq C^*(X,B)$ and the following diagram commutes :

$$\begin{tikzcd} KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G}),l^\infty(X,K_B))) \arrow[r, "\mu^d_{\mathcal{G},l^\infty(X,K_B)}"] \arrow[d, "\iota^*"] & K_*(l^\infty(X,K_B) \rtimes_r G) \arrow[d, "(\Psi_B)_*"] \\ KK_*(C_0(P_d(X),B) \arrow[r, "A^d_{X,B}"] & K_*(C^*(X,B)) \end{tikzcd}$$

where the left vertical arrow comes from the inclusion of groupoid $\iota : \{x\} \to \mathcal{G}$ for any $x \in X$. We claim that we can prove a controlled analogue of this result which induces it in K-theory.

To prove this, we shall describe Ψ more precisely. For any C^*-algebra B, let $\tilde{B} = l^\infty(X,K_B)$. It is naturaly a $\mathcal{G}$-algebra, and the fiber over any $x \in \beta X$ is

easily seen to be $\tilde{B}_x = B$. Now, if $f \in C_c(\mathcal{G}, \tilde{B})$, as $\overline{E}$ are the compact-open of $\mathcal{G}$, f is continuous over a $\overline{E}$, so it is just a bounded function over E.
Define for $g = (x, y) \in X \times X \subset \mathcal{G}$, $\Psi_B(f)_{xy} = f(g)(x) \in \tilde{B}$, so that $\Psi_B(f) = (\Psi_B(f)_{xy})_{x,y\in X}$ is a locally compact operator of finite propagation (its support is in E). This is a $*$-morphism which extends to the annouced isomorphism. Moreover, $\tilde{B}$ is naturally a C^*-subalgebra of both $\tilde{B} \rtimes_r \mathcal{G}$ and $C^*(X, B)$, and the two inclusion commute modulo Ψ_B. We have a commutative diagram :

$$\begin{tikzcd}
 & B \arrow[ddr, bend left, "\iota_3^B"] & \\
 & \tilde{B} \arrow[u, "ev_x"] \arrow[dl, hook, "\iota_1^B"'] \arrow[dr, hook, "\iota_2^B"] & \\
\tilde{B} \rtimes_r \mathcal{G} \arrow[rr, "\Psi_B"] & & C^*(X, B)
\end{tikzcd}$$

Off course, Ψ_B induces $\Psi_{B*} : \mathcal{L}_{\tilde{B}\rtimes_r\mathcal{G}}(H_{B\rtimes_r\mathcal{G}}) \to \mathcal{L}_{C^*(X,B)}(\mathcal{E})$ where $\mathcal{E} = H_{B\rtimes_r\mathcal{G}} \otimes_{\Psi_B} C^*(X, B)$.
Let A and B be two C^*-algebra and $z \in KK_1^{\mathcal{G}}(\tilde{A}, \tilde{B})$, represented by $(H_{\tilde{B}}, \psi, T)$. As $T_{\mathcal{G}} = (\iota_1)_*(T)$, we have $(\Psi_B)_*(T_{\mathcal{G}}) = (\iota_2)_*(T) = (T_x)_X$. Also, the relations $(\iota_1^A)_* \circ \psi = \psi_{\mathcal{G}} \circ \iota_1^A$ and $(\iota_2^A)_* \circ \psi_x = (\psi_x)_X \circ \iota_2^A$ are easy to derive, which lead to $(\Psi_B)_* \circ \psi_G \circ \iota_1^A = (\iota_2^B)_* \circ \psi_x = (\psi_x)_X \circ \Psi_A \circ \iota_1^A$. By extending $\mathcal{G}$-equivariantly to $\tilde{A} \rtimes_r G$, we have $(\Psi_B)_*(\psi_{\mathcal{G}}(a)) = (\psi_x)_X(\Psi_A(a))$. The map $(x, y) \mapsto (\Psi_A(x), (\Psi_B)_*(y))$ induces a morphism $\Psi_E : E^{(\psi,T)} \to E^{((\psi_x,T_x))}$ which sends $(x, P_{\mathcal{G}}\psi_{\mathcal{G}}(x)P_{\mathcal{G}} + y)$ to $(\Psi_A(x), (P_x)_X(\psi_x)_X(\Psi_A(x))(P_x)_X + (\Psi_B)_*(y))$ by the previous computations. This map makes the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & K_{\tilde{B}\rtimes\mathcal{G}} \arrow[r] \arrow[d, "(\Psi_B)_*"] & E^{(\psi,T)} \arrow[r] \arrow[d, "\Psi_E"] & \tilde{A} \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "\Psi_A"] & 0 \\
0 \arrow[r] & K_{C^*(X,B)} \arrow[r] & E^{(\psi_x,T_x)} \arrow[r] & C^*(X, A) \arrow[r] & 0
\end{tikzcd}.$$

Now the remark 3.7 of [3] gives $((\Psi_B)_*)_* \circ D_{\tilde{A}\rtimes_r\mathcal{G}}^{K_{\tilde{B}\rtimes\mathcal{G}}} = D_{C^*(X,A)}^{K_{C^*(X,B)}} \circ (\Psi_A)_*$, and if we compose by the Morita equivalence, we get

$$\sigma(\iota^*(z)) \circ (\Psi_A)_* = (\Psi_B)_* \circ J_{\mathcal{G}}(z),$$

where $\iota^*(z)$ is indeed the class of (H_B, ψ_x, T_x).

As $C_0(P_d(\mathcal{G}))$ is a $\mathcal{G}$-algebra whose fiber over any $w \in \beta X$ is isomorphic to $C_0(P_d(X))$, if $A = C_0(P_d(\mathcal{G}))$, then $(\Psi_A)_*[\mathcal{L}_d] \in K_0^{\epsilon,R}(C^*(X, C_0(P_d(X)))$ which is equal to $[P_X]$, and gives the result.

$$(\Psi_B)_* \circ \mu_{\mathcal{G}}^{\epsilon,R}(z) = A^{\epsilon,R}(\iota^*(z)).$$

This, passing to K-theory, implies the result of [7].

2.6 Quantitative statements

Proposition 7. Let A be a $\mathcal{G}$-algebra.
If the following statement is true :

•(Quantitative Injectivity) $\forall d \geq 0$, there exists $\epsilon \in (0, \frac{1}{4})$ such that, for all $r \geq r_{d,\epsilon}$, there exists $d' \geq d$ such that if $x \in KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G})), A)$ satisfies $\mu_{\mathcal{G}}^{\epsilon,R,d}(x) = 0 \in K^{\epsilon,R}$, then $x = 0$ in $KK^{\mathcal{G}}(C_0(P_{d'}(\mathcal{G})), A)$;

then $\mu_{\mathcal{G},A}$ is injective.

On the other hand, if this statement is true :

•(Quantitative Surjectivity) there exists $\epsilon' \in (0, \frac{1}{4})$ such that $\forall r' \geq r_{d,\epsilon}, \exists \epsilon, r$ such that $\epsilon' \leq \epsilon < \frac{1}{4}$ and $r_{d,\epsilon} \leq r \leq r'$, such that for all $y \in K_*^{\epsilon',r'}(A \rtimes \mathcal{G}), \exists x \in KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G})), A)$ such that $\mu_{\mathcal{G},A}^{\epsilon',r',d} = \iota_{\epsilon,r}^{\epsilon',r'}(y)$;

then $\mu_{\mathcal{G},A}$ is surjective.

Preuve 4. Let $x \in KK(C_0(P_d(\mathcal{G})), A)$ which satisfies $\mu_{\mathcal{G},A}(x) = 0$, then $\iota_{\epsilon,r} \circ \mu_{\mathcal{G},A}^{\epsilon,r,d}(x) = 0$. By remark 1.18 of [3], there exists a universal $\lambda > 0$ and a certain $r' > 0$ such that

$$\begin{aligned} 0 &= \iota_{\epsilon,r}^{\lambda\epsilon,r'} \circ \mu_{\mathcal{G},A}^{\epsilon,r,d}(x) \\ &= \iota_{\epsilon,r}^{\lambda\epsilon,r'}(J_{\mathcal{G}}^{\epsilon,r}(x)([\mathcal{L}_d]_{\epsilon,r})) \\ &= J_{\mathcal{G}}^{\lambda\epsilon,r'}(x)([\mathcal{L}_d]_{\lambda\epsilon,r'}) \\ &= \mu_{\mathcal{G},A}^{\lambda\epsilon,r',d}(x). \end{aligned}$$

But then the quantitative injectivity condition assures that $x = 0$ in $KK^{\mathcal{G}}(C_0(P_{d'}), A)$ and $x = 0$ in the inductive limite over d $K^{top}(G, A)$.
The second point is immediate.

□

This kind of statement leads us to define the following proprieties, following [4].

- $QI_{\mathcal{G},B}(d, d', R, \epsilon)$: for any $x \in KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B)$, $\mu_{\mathcal{G}}^{\epsilon,R}(x) = 0$ implies $\iota_d^{d'}(x) = 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), B)$.
- $QS_{\mathcal{G},B}(d, R, R', \epsilon, \epsilon')$: for any $y \in K^{\epsilon,R}(B \rtimes \mathcal{G})$, there exists $x \in KK^{\mathcal{G}}(P_d(\mathcal{G}), B)$ such that $\mu_{\mathcal{G}}^{\epsilon',R'}(x) = \iota_{\epsilon,R}^{\epsilon',R'}(y)$.

Théorème 1. Let B a $\mathcal{G}$-algebra, and $\tilde{B} = l^\infty(X, K_B)$. Then $\mu_{\mathcal{G},\tilde{B}}$ is injective if and only if for all $d, \epsilon, r \geq r_{\mathcal{G},d,\epsilon}$, there exists $d' \geq d$ such that $QI_{\mathcal{G},B}(d, d', \epsilon, R)$.

To prove the theorem, we will need a serie of lemmas.

Lemme 1. Let Z be a $\mathcal{G}$-compact proper $\mathcal{G}$-space such that the anchor map $p : Z \to \mathcal{G}^{(0)}$ is locally injective, and let (B_j) be a countable family of $\mathcal{G}$-algebras. Then the projection $\prod_j B_j \otimes K \to B_j \otimes K$ induces an isomorphism

$$KK^{\mathcal{G}}(C_0(Z), \prod_j B_j \otimes K) \to \prod_j KK^{\mathcal{G}}(C_0(Z), B_j \otimes K).$$

Preuve 5. Let $B_\infty = \prod B_j \otimes K$ and $p_k : B\infty \to B_k$ the projection. Let $(\mathcal{E}, \varphi, F) \in E^{\mathcal{G}}(C_0(Z), B_\infty)$ be a cycle such that every

$$(\mathcal{E}_k, \varphi_k, F_k) = (p_k)_*(\mathcal{E}, \varphi, F)$$

is homotopic to 0. According to [4], we can choose a homotopy which is C-Lipschitz on the Calkin algebra for a universal constant $C > 0$, hence ([10], Lemma 17.3.3) we can find a family of compact operators $T_{s,t} \in K(\mathcal{E})$ such that $||F_s - F_t + T_{s,t}|| \leq C|s-t|$. But $t \mapsto F'_t = F_t + T_{0,t}$ is a compact perturbation of $s \mapsto F_s$ in $\mathcal{L}(\mathcal{E})$ which is C-Lipschitzian. Up to replace $(F_s)_s$ with (F'_t), we can suppose the homotopies are uniformly Lipschitzian, and $\tilde{F} = \prod F_j$ defines a bounded operator.
We now use an idea of [9], lemma 3.6. Namely, using the local injectivity of p, we show that F can be supposed to commute with φ and $\mathcal{G}$. For the sake of completness, we recall the proof. First, choose a finite open cover $(U_j)_j$ of a compact fundamental domain K for the action of $\mathcal{G}$ such that $p_{|U_j}$ is injective, and take compactly supported continuous functions $\phi_j : Z \to \mathbb{R}_+$ such that supp $\phi_j \subset U_j$ and $K \subset \cup\phi_j^{-1}(0,+\infty)$. We can suppose $\sum_{j,g\in\mathcal{G}^{p(z)}} \phi_j(zg) = 1, \forall z \in Z$. Now define $F'_x = \sum_{j,g\in\mathcal{G}^x} \alpha_g(\phi^{\frac{1}{2}} F_{s(g)} \phi^{\frac{1}{2}})$. It is an $\mathcal{G}$-invariant operator which commutes with the action of $C_0(Z)$.
Now we can see that $(\prod_j \mathcal{E}, \prod \varphi_j, \prod_j F_j)$ defines a cycle as $[\varphi(a), \tilde{F}] = 0$ and $\varphi(a)(\alpha_g(F_{s(g)} - \tilde{F}_{r(g)}) = 0$. Moreover it is unitarly equivalent to $(\mathcal{E}, \varphi, \tilde{F})$, and homotopic to 0.
For the surjectivity, just take $[(\prod \mathcal{E}_j, \prod \varphi_j, \prod F_j)]$ as a preimage of $\prod_j[(\mathcal{E}_j, \varphi_j, F_j)]$, using the previous construction.

□

Lemme 2. Let $\mathcal{G}$ be a locally compact, σ-compact étale groupoid, $\{B_j\}_{j\geq 0}$ a family of $\mathcal{G}$-algebras and K the algebra of compact operators over a separable Hilbert space. Set $\Delta = P_d(\mathcal{G})$, then we have an $\mathbb{Z}_2$-graded isomorphism

$$KK^{\mathcal{G}}(C_0(\Delta), \prod_j B_j \otimes K) \simeq \prod_j KK^{\mathcal{G}}(C_0(\Delta), B_j)$$

Preuve 6. For all j and any locally compact $\mathcal{G}$-space X, the projection $\prod_j B_j \otimes K \to B_j \otimes K$ induces a morphism

$$\Theta^X : KK^G(C_0(X), \prod_j B_j \otimes K) \to \prod_j KK^G(C_0(X), B_j \otimes K).$$

Let $X_0 \subset X_1 \subset ... \subset X_n$ be the n-skeleton decomposition associated to the simplicial structure of the Rips complex Δ and let $Z_j = C_0(X_j)$, $Z_{j-1}^j = C_0(X_j - X_{j-1})$ and $\Theta_j = \Theta^{X_j}$. We will show the claim by induction on the dimension of Δ.

The extension of $\mathcal{G}$-algebras $0 \to Z_{j-1}^j \to Z_j \to Z_{j-1} \to 0$ gives a commutative diagram with exact lines :

$$\begin{tikzcd}
KK_*(Z^j_{j-1}, \prod_j B_j \otimes K) \arrow[r, "\delta"] \arrow[d, "\Theta^j_{j-1}"] & KK_*(Z_{j-1}, \prod_j B_j \otimes K) \arrow[r] \arrow[d, "\Theta_{j-1}"] & KK_*(Z_j, \prod_j B_j \otimes K) \arrow[r] \arrow[d, "\Theta_j"] & KK_*(Z^j_{j-1}, \prod_j \arrow[d, "\Theta^j_{j}"] \\
\prod_j KK_*(\tilde{Z}^j_{j-1}, B_j \otimes K) \arrow[r, "\delta"] & \prod_j KK_*(\tilde{Z}_{j-1}, B_j \otimes K) \arrow[r] & \prod_j KK_*(\tilde{Z}_j, B_j \otimes K) \arrow[r] & \prod_j KK_*(\tilde{Z}^j_{j-1},
\end{tikzcd}$$

The five lemma assures that if Θ_{j-1} and Θ^j_{j-1} are isomorphisms, then so is Θ_j. Moreover, because Δ is a typed simplicial simplex (see [9]), $X_j - X_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where $\mathring{\sigma}_j$ denotes the interior of the standard simplex, and Σ_j is the set of centers of j-simplices of X_j. Bott periodicty assures then that, if Θ_{j-1} is an isomorphism, then so is Θ^j_{j-1}. By induction, proving that Θ_0 is an isomorphism concludes the proof, which is essentially the content of lemma 1 : X_0 is a $\mathcal{G}$-compact proper $\mathcal{G}$-space, and its anchor map is just the target map $r : \mathcal{G} \to \mathcal{G}^{(0)}$, which is supposed to be étale, so locally injective.

$\square$

We can now prove the theorem 1.

Preuve 7. Let $x \in KK^{\mathcal{G}}(P_d(\mathcal{G}), \tilde{B})$ such that $\mu_{G,\tilde{B}}(x) = 0$. Then, as the quantitative assembly maps factorize $\mu_{G,\tilde{B}}$, there exist $\epsilon > 0$ and $R \geq r_{\mathcal{G},\tilde{B},d}$, such that $\mu^{\epsilon,R}_{\mathcal{G},\tilde{B}}(x) = 0$. Using the isomorphism of lemma 2 and the Morita equivalence, we can identify x with $(x_j)_j$ under $KK^{\mathcal{G}}(P_d(\mathcal{G}), \tilde{A}) \simeq \prod_j KK^{\mathcal{G}}(P_d(\mathcal{G}), A)$. Now let $d' \geq d$ such that $QI_A(d, d', \epsilon, R)$ holds. That assures that $x_j = 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), B)$, and $x = 0$.

For the converse, suppose one can find d, ϵ, R such that $QI_{\mathcal{G},A}(d, d', \epsilon, R)$ is NOT true for all $d' \geq d$. Then one can extract a increasing sequence d_j diverging to $+\infty$ and $x_j \in KK^{\mathcal{G}}(P_d(\mathcal{G}), A)$ such that $\mu^{\epsilon,R}_{\mathcal{G},\tilde{B}}(x_j) = 0$ and $x_j \neq 0$ in $KK^{\mathcal{G}}(P_{d_j}, A)$. Let $x \in KK^{\mathcal{G}}(P_d, \tilde{A})$ be the image of $(x_j) \in \prod KK^{\mathcal{G}}(P_d, A)$. We have $\mu_{\mathcal{G},\tilde{A}}(x) = 0$, and $x \neq 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), \tilde{A})$ for all $d' \geq d$, so $\mu_{\mathcal{G},\tilde{A}}$ is not injective.

$\square$

We also have a theorem relating quantitative surjectivity for $\mu_{\mathcal{G},B}$ and surjectivity of $\mu_{\mathcal{G},\tilde{B}}$.

Théorème 2. Let B a $\mathcal{G}$-algebra, and $\tilde{B} = l^\infty(X, K_B)$. Then there exists $\lambda > 1$ such that $\mu_{\mathcal{G},\tilde{B}}$ is onto if and only if for any $0 < \epsilon < \frac{1}{4\lambda}$ and $R > 0$, there exist $R' \geq \max(R, r_{\mathcal{G},d,\epsilon})$ and $d > 0$ such that $QS_{B,\mathcal{G}}(d, R, R', \epsilon, \lambda\epsilon)$ holds.

Preuve 8. Let $\lambda > 0$ the universal constant of remark 1.18 of [3] : for any C^*-algebra and $x, y \in K^{\epsilon,R}(A)$ such that $\iota_{\epsilon,R} x = \iota_{\epsilon,R} y$, there exists $R' \geq R$ such that $\iota^{\lambda\epsilon,R'}_{\epsilon,R} x = \iota^{\lambda\epsilon,R'}_{\epsilon,R} y$.

Let $y \in K_*(\tilde{B} \rtimes \mathcal{G})$, and take $z \in K^{\epsilon,R}$, where $R > 0, \epsilon < \frac{1}{4\lambda}$, such that $\iota_{\epsilon,R} z = y$. The projection on the j^{th} component $\tilde{B} \to K_B$ used in K-theory then composed with Morita equivalence gives a map $K^{\epsilon,R}(\tilde{B} \rtimes \mathcal{G}) \to K^{\epsilon,R}(B \rtimes \mathcal{G})$, and z_j denotes the image of z under this map. We can pick d and $R' \geq \max(r_{\mathcal{G},d,\epsilon})$ such that $QS(d, R, R', \epsilon, \lambda\epsilon)$: for every j, there exists $x_j \in KK^{\mathcal{G}}(P_d, B)$ such that $\mu^{d,\lambda\epsilon,R'}_{\mathcal{G},B}(x_j) = \iota^{\lambda\epsilon,R'}_{\epsilon,R} z_j$. As $KK^{\mathcal{G}}(P_d, \tilde{B}) \simeq \prod_j KK^{\mathcal{G}}(P_d, B)$, (x_j) can be taken as an element x of $KK^{\mathcal{G}}(P_d, \tilde{B})$. Naturality of the assembly maps, and compatibility of quantitative assembly maps with the usual one assures that $\mu_{\mathcal{G},\tilde{B}}(x) = z$, whereby $\mu_{\mathcal{G},\tilde{B}}$ is onto.

Suppose that there exist $0 < \epsilon < \frac{1}{4\lambda}$ and $R > 0$ such that for every positive numbers $d > 0$ and $R' \geq \max(r_{\mathcal{G},B}, R)$, $QS(d, R, R', \epsilon, \lambda\epsilon)$ does not hold. Let (d_j) and (R_j) be unbounded increasing sequences of positive numbers and $y_j \in K^{\epsilon,R}(B \rtimes \mathcal{G})$ such that $\iota^{\lambda\epsilon,R_j}_{\epsilon,R}(y_j)$ is not in the range of $\mu^{d_j,\lambda\epsilon,R_j}_{\mathcal{G},B}$. Let $y \in K^{\epsilon,R}(\tilde{B} \rtimes \mathcal{G})$ be an element such that its image with the previous map coincides with y_j. If there exists $x \in KK^{\mathcal{G}}(P_s, \tilde{B})$ for a $s \geq d$ such that $\iota_{\epsilon,R}(y) = \mu^s_{\mathcal{G},\tilde{B}}(x)$ then there would exists a $R' \geq R$ such that

$$\iota^{\lambda\epsilon,R'}_{\epsilon,R}(y) = \mu^{s,\lambda\epsilon,R'}_{\mathcal{G},\tilde{B}}(x) = \iota^{\lambda\epsilon,R'}_{\epsilon,R} \circ \mu^{s,\epsilon,R}_{\mathcal{G},\tilde{B}}(x).$$

Now choose j such that $d_j \geq s$ and $R_j > R'$, and compose the previous equality with $\iota^{\lambda\epsilon,R_j}_{\lambda\epsilon,R'}$ and $q^{d_j}_s$ to obtain $\iota^{\lambda\epsilon,R_j}_{\epsilon,R}(y_j) = \mu^{d_j,\lambda\epsilon,R_j}_{\mathcal{G},B}$ which contradicts our assumption.

$\square$

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