Questions from Münster

Clément Dell'Aiera

Contents

Chapter 1

Seminar

These are the notes I took co-corganizing with Erik Guentner and Rufus Willett the seminar of Noncommutative Geometry from Fall 2017 up until now (Fall 2018).

1.1 Cartan subalgebras

The goal of this section is... Historical remarks: aVN and Feldman-Moore,...

The first part will detail J. Renault's work [**?**] on Cartan pairs.

Recall that an element $x \in A$ normalizes a self-adjoint subspace B of A if

$$xBx^* \cup x^*Bx \subset B.$$

The normalizer $N_A(B)$ is the set of all the elements of A that normalize B.

Definition 1.1.1. Let A be a C^*-algebra. A sub-C^*-algebra $B \subseteq A$ is called a Cartan subalgebra of A if:

- B is a maximal abelian self-adjoint subalgebra (MASA) of A;
- B contains an approximate unit for A;
- the normaliser of B in A generates A as a C^*-algebra;
- there is a faithful conditional expectation $E : A \to B$.

The pair (A, B) is referred to as a *Cartan pair.*

Examples:

- $D_n \subset M_n(\mathbb{C})$,
- $C(X) \subset C(X) \rtimes \Gamma$,
- $l^\infty(X) \subset C_u^*(X)$,
- $C_0(G^0) \subset C_r^*(G, \Sigma)$.

Renault obtained the following result in [**?**].

Theorem 1.1.2. Any Cartan pair (A, B) is isomorphic to the Cartan pair

$$(C_r^*(G, \Sigma), C_0(G^0)),$$

where G is an étale topologically prinicipal groupoid with base space G^0 and Σ is a twist over G.

This theorem is very uselful. For instance, it implies that a nuclear C^*-algebra with a Cartan subalgebra satisfies the universal coefficient theorem of Rosenberg and Schochet [**?**]. Indeed, the reduced C^*-algebra of an étale groupoid is nuclear iff it is amenable, in which case it belongs to the bootstrap class [**?**].

The first step in the proof of the theorem is to build, for any inclusion of C^*-algebras $A \subseteq B$ with B unital commutative, an action of $N_A(B)$ by partial homeomorphisms on the spectrum of B. A standard construction then give rise to an étale groupoid $\mathcal{G}_B$ (the groupoid of germs of a *pseudogroup*) of this action. The twist is given by the same kind of construction.

For the second step, one defines a generalized Gelfand transform

$$\{$$

1.1.1 Groupoids of germs

Out of any inclusion of C^*-algebras $A \subseteq B$ with A unital commutative, we construct an action of the normalizer of A in B by partial homeomorphism on X the spectrum of A, i.e. a homomorphism of semigroup

$$\alpha : N_B(A) \to SHomeo(X).$$

If $n \in N_B(A)$ and $x \in Spec(A)$, set

$$\langle \alpha_n(x), a \rangle = \langle x, n^*an \rangle.$$

This defines a homeomorphism

$$\alpha_n : U_n \to U_{n^*},$$

where $U_n = \{x \in Spec(A), n^*n(x) > 0\}$ such that $\alpha_{nm} = \alpha_n \circ \alpha_m$.

Lemma 1.1.3. If B is abelian and contains an approximate unit, $\alpha : N_A(B) \to PHomeo(X)$ is a homorphism of inverse-semigroups.

In our case, given a Cartan pair (A, B), and $X = Spec(B)$, one defines:

- Σ_B as the quotient of

$$\{(x, n) \in X \times N_A(B) \text{ s.t. } n^*n(x) > 0\}$$

  by the equivalence relation $(x, n) \sim (x, n')$ when there exist $b, b' \in B$ such that $nb = n'b'$;

- $\mathcal{G}_B$ as the groupoid of germs of the pseudogroup $\alpha(N_A(B))$;

1.1.2 Generalized Gelfand transform

If $(x,n) \in X \times N_A(B)$ such that $n^*n(x) > 0$, and $a \in A$ then

$$\frac{E(n^*a)(x)}{\sqrt{n^*n(x)}}$$

only depends on the class of (x,n) in Σ_B, hence defines a continuous section $\hat{a}$ of the twist Σ_B. The map extends to a $*$-homomorphism

$$\Psi : A \to C_r^*(G_B, \Sigma_B)$$

which is always linear injective and respects the Cartan algebras. Moreover, restricted to B, Ψ coincides with the Gelfand transform $B \to C_0(X)$.

When (A,B) is a Cartan pair, Ψ is an $*$-isomorphism.

1.1.3 Roe algebras

In the case of uniform Roe algebras, White and Willett have obtained in [**?**] rigidity results. The questions are:

- What form can a Cartan subalgebra of $C_u^*(X)$ take?
- Can we describe when it is unique up to unitary equivalence?

The answers they have are the following. If X is an infinite countable metric space with bounded geometry, any Cartan subalgebra of $C_u^*(X)$ is non separable and contains a complete family of orthogonal projections. Interesting examples show that Cartan subalgebras of uniform Roe algebras need not be isomorphic to l^∞. Let us recall:

Definition 1.1.4. A sub-C^*-algebra B of A is a Roe Cartan pair if:

- A is unital;
- A contains the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space as an essential ideal;
- B is a co-separable Cartan subalgebra of A abstractly isomorphic to $l^\infty(\mathbb{N})$. (co-separable means that there is a countable subset of A which generates A together with B)

Theorem 1.1.5. Let (A,B) a Roe Cartan pair. Then there exists a metric space with bounded geometry X such that for any irreducible faithful representation of A on a Hilbert space H, there exists a unitary $u : l^2(X) \to H$ that conjugates A with $C_u^*(X)$, and B with $l^\infty(X)$.
Moreover, if $A = C_u^*(Y)$ for some bounded geometry metric spaxce Y with property A, then X and Y are coarsely isomorphic.

1.2 Dynamical Property (T)

The first thing I will try to do is to justify the use of groupoids. My opinion is that these objects are not loved as much as they deserve. People who very much like short and concise definitions enjoy to say that *groupoids are small categories in which all morphisms are invertible.* This is true, but maybe does not shed light on the reasons people look at such objects.

Groupoids can be thought as a generalisation of both groups and spaces. In that effect, a groupoid G is made of two parts, in our case, two spaces, the *group-like* part G and the *space-like* part G^0. Usually G is called the space of arrows, and G^0 the base space, seen as a subset of G. Any arrow $g \in G$ has a starting point $x \in G^0$ and an ending point $y \in G^0$. This is encoded by two maps $s, r : G \rightrightarrows G^0$ called source and range. Two arrows can be composed as long as the ending point of the first coincides with the starting point of the second. The points of the base space act as units, and every arrow as an inverse with respect to this partial multiplication.

$$\begin{tikzcd} s(g) \bullet \arrow[r, bend left, "g"] & \bullet r(g) \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

In our setting, all the spaces will be topological spaces and the maps will be continuous. We will even simplify greatly our life by only looking at second countable, locally compact, étale groupoids with compact base space. From now on, we will only say *étale*, forgetting about all other technical assumptions to gain in clarity.

Being étale means that the range map $r : G \to G^0$ is a local homeomorphism, i.e. for every $g \in G$, there exists a neighborhood U of g such that $r_{|U}$ is a homeomorphism. This implies in particular that every fiber $G^x = r^{-1}(x)$ and $G_x = s^{-1}(x)$ are discrete. When the base space G^0 has the additional property of being totally disconnected, we will say that G is *ample*. Here is a list of examples of étale groupoids.

- A (nice) compact space X defines a trivial groupoid $G = G^0 = X$ and source and target are the identity; in the opposite direction if the base space is a point, the groupoid is a group. One can already see how the notion of groupoid generalises both spaces and groups as promised.

- As an intermediate situation between these two cases, consider a discrete group Γ acting by homeomorphisms on a compact space X. Define the *action groupoid* as follow. Topologically, it is the space $G = X \times \Gamma \rightrightarrows G^0 = X$. The multiplication encodes the action

$$\begin{tikzcd} x \bullet \arrow[r, bend left, "{(x,g)}"] & \bullet\, g.x \arrow[l, bend left, "{(g.x,g^{-1})}"] \arrow[r, bend left, "{(g.x,g')}"] & \bullet g'g.x \end{tikzcd}$$

and this picture gives every element to reconstruct the groupoid.

- If $R \subseteq X \times X$ is an equivalence relation, then R as a canonical structure of groupoid with the base space being the diagonal $R^0 = \{(x,x) \mid x \in X\}$ and the multiplication being the only one possible

$$(x,y)(y,z) = (x,z).$$

- More interesting is the *coarse groupoid* $G(X)$ associated to a discrete countable metric space (X,d) with bounded geometry, that is

$$\sup_{x \in X} |B(x,R)| < \infty \quad \forall R > 0.$$

A nice way of thinking about this condition is to imagine yourself looking at the space with a magnifying glass of prescribed radius, but as great as you wish. Then you should not observe more and more points in your sight as you move around. In other words, the points fitting in the radius of your glass is uniformly bounded.

Now consider the R-diagonals:

$$\Delta_R = \{(x,y) \mid d(x,y) < \infty\} \subseteq X \times X$$

and take their closure $\overline{\Delta_R}$ in $\beta(X \times X)$ (βY being the Stone-Čech compactification of Y). The coarse groupoid is defined topologically as

$$G(X) = \cup_{R>0} \overline{\Delta_R} \rightrightarrows \beta X,$$

and is endowed with the structure of an *ample* groupoid which extend the groupoid $X \times X \rightrightarrows X$ associated with the coarsest equivalence relation on X. The topological property of this groupoid encodes the metric or *coarse* property of the space. For instance, X has property A iff $G(X)$ is amenable, X is coarsely embeddable into a Hilbert space iff $G(X)$ has Haagerup's property, etc.

- The last construction is associated to what is often referred as an *approximated group*, which is the data of $\mathcal{N} = \{\Gamma, \{N_k\}\}$ where Γ is a discrete group, and the N_k's are a tower of finite index normal subgroups with trivial intersection, i.e.

$$N_1 \lhd N_2 \lhd \ldots \quad \text{s.t.} \quad \cap_k N_k = \{e_\Gamma\} \text{ and } [\Gamma : N_k] < \infty.$$

Then the Γ_k's are finite groups. Set $\Gamma_\infty = \Gamma$ for convenience (which is not usually finite!). For any discrete group Λ, there exists a left-invariant proper metric, which is unique up to coarse equivalence (take any word metric if the group is finitely generated). Let us denote by $|\Lambda|$ the coarse class thus obtained. Then the first object of interest in that case is the coarse space $X_{\mathcal{N}}$ defined as the *coarse disjoint union*

$$X_{\mathcal{N}} = \coprod_k |\Gamma_k|.$$

Here the metric is such that $d(|\Gamma_i|, |\Gamma_j|) \to \infty$ as $i+j$ goes to ∞, $i \neq j$.

The second interesting object attached to $\mathcal{N}$ is the HLS (after Higson-Lafforgue-Skandalis [?], where it was first defined to build counter-examples to the Baum-Connes conjecture) groupoid. The base space is the Alexandrov compactification of the integers

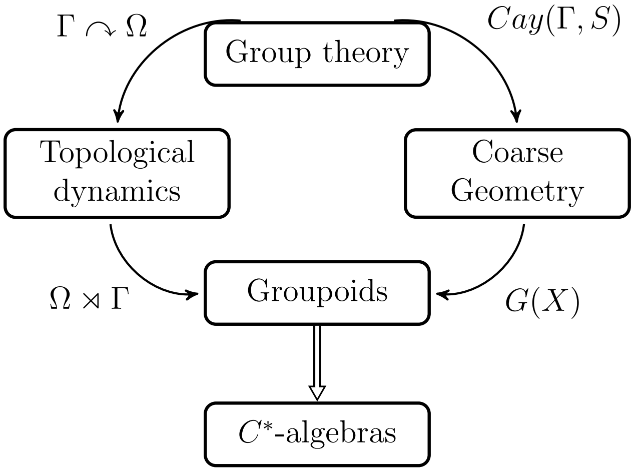
$$G^0_{\mathcal{N}} = \overline{\mathbb{N}},$$

and $G_{\mathcal{N}}$ is a bundle of groups with the fiber of k being Γ_k. The topology is taken to be discrete over the finite base points, and a basis of neighborhood of (∞, γ) is given by

$$\mathcal{V}_{\gamma,N} = \{(k, q_k(\gamma)) \mid k \geq N\} \quad N \in \mathbb{N},$$

where $q_k : \Gamma \to \Gamma_k$ is the quotient map.

One of the reasons we use groupoids is that they are convenient to build interesting C^*-algebras. To see their relevance, one may start with the question *What are operator algebraists doing?* A possible answer is that part of Noncommutative Geometry and Operator Algebras are devoted to the construction of interesting classes of C^*-algebras. For instance, *nuclearity* was naturally introduced after Grothendieck's work, followed by a C^*-algebraic formulation. Arises then the question *does there exist nonnuclear C^*-algebras?* A now classical result states that, when Γ is a discrete group, the reduced $C^*_r(\Gamma)$ is nuclear iff Γ is amenable. Calling out a nonamenable group, like any nonabelian free group, produces then a nonnuclear C^*-algebra. This game revealed itself to be very fruitful: study a property in some field and try to apply it to C^*-algebras to see what exotic being can be built out of it. The most common fields that have natural C^*-algbras associated to them are traditionally group theory, coarse geometry and dynamical systems (there are others like foliations etc, but let me just limit myself to these ones). This can be summarized in the following diagram.



Another interesting strategy is to try and translate a property in one of those upper boxes directly in terms of groupoids. Then the property can either be used to build C^*-algebras, either give a new definition in the case of other upper boxes. For instance, that is what we tried to do with Rufus Willett in our work on property T. Property T is originally a group property defined in terms of its unitary representations. In [?], Willett and Yu defined a geometric property T for monogenic discrete metric spaces with bounded geometry. Following their work, our first goal was to try and define a property T for (nice enough) topological groupoids so that in the case of groups and coarse groupoids, it reduces to these notions of property T. It gives then a notion of property T for dynamical systems, by considering property T for the action groupoid $X \rtimes \Gamma$. The second part of the work is dedicated to go down the last arrow, that is studying implications of property T for G to its reduced and maximal C^*-algebras, and even more general completions of $C_c(G)$.

Let us first recall what is property T for discrete groups.

If $\pi : \Gamma \to B(H)$ is a unitary representation of Γ on a separable Hilbert space, say that π almost has invariant vectors if for every pair (F, ε) where F is a finite subset of the group and ε a positive number, there exists a unit vector $\xi \in H$ such that

$$\|s.\xi - \xi\| < \varepsilon \quad \forall s \in F.$$

Definition 1.2.1. A group Γ has property T if every representation that almost has invariant vectors admits a nonzero invariant vector.

This definition is not the original one. Indeed property T was defined by Kazhdan in order to prove that *some* lattices in *some* Lie groups were finitely generated. It seemed a very specific property and application, but it turned out that property T gave very nice applications. Here are some of the most spectacular the author is aware of.

- Margulis supperrigidity theorem (about this, see Monod's [**?**] beautiful generalization, which Erik called the most beautiful paper he ever read);

- existence of expander: for any infinite approximated group (in the sense of the examples above) Γ, the space $X_{\mathcal{N}}$ is an expander;

- existence of Kazdhan projections which are very wild objects one should only approach with care;

- more generally, property T was for a long time an obstruction to the Baum-Connes conjecture, up until the work of Lafforgue ([**?**], [**?**]). It still gives interesting properties for diverse crossed-product constructions as we will see.

One can prove easily that finite groups have T. Indeed, in that case, take the finite subset to be the whole group and look intensely at the identity

$$\|s.\xi - \xi\|^2 = 2(1 - Re\langle s.\xi, \xi\rangle).$$

If ξ is (Γ, ε)-invariant for ε sufficiently small, then the above identity implies that $\frac{1}{|\Gamma|}\sum_{s\in\Gamma} s.\xi$ is nonzero because its inner-product with ξ will have real part close to 1. But ξ is invariant.

Now take $\Gamma = \mathbb{Z}$ and look at the left-regular representation, i.e. $H = l^2\Gamma$ and

$$(s.\xi)(x) = \xi(s^{-1}x).$$

Then if $\xi_n = \frac{1}{|F_n|}\chi_{F_n} \in H$ is the characteristic function of F_n normalized to be a unit vector, one can check that

$$\sup_{s \in F} \|s.\xi_n - \xi\| \to 0 \text{ as } n \to \infty$$

so that the regular representation always almost has invariant vectors. But it never has nonzero invariant ones, so that $\mathbb{Z}$ does not have T. This proof actually works for every infinite amenable group.

The moral of this story is that if one wants to find infinite groups with property T, one has to look at nonamenable groups. Maybe $\mathbb{F}_2$ or $SL(2,\mathbb{Z})$? Actually not: they both surject to $\mathbb{Z}$ which does not have T, and this is an obstruction to having T as is obvious from the definition.

Finding infinite groups with property T is actually a hard problem. Here are some examples, without any proofs since these would go out of scope for these notes.

- $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$ if $n \geq 3$;

- $Sp(n,1)$ and its lattices, which gives examples of infinite hyperbolic (in the sense of Gromov) groups having property T;

- $Aut(\mathbb{F}_5)$ and $Out(\mathbb{F}_5)$ by a recent result of Nowak and Ozawa [**?**]. Their proof is interesting in that they use numerical computations to reach their result using a previous result of Ozawa [**?**];

- $SO(p,q)$ with $p > q \geq 2$ and $SO(p,p)$ with $p \geq 3$. More generally, any real Lie group with real rank at least two, and all their lattices. Also, any simple algebraic group over a local field of rank at least two have T.

To define property T for groupoids, we need to choose what kind of representations we are looking at, and to decide what are the invariant vectors.

A representation will be a $*$-homomorphism $\pi : C_c(G) \to B(H)$. A vector $\xi \in H$ is called invariant if

$$f.\xi = \Psi(f).\xi \quad \forall f \in C_c(G).$$

The subspace of invariant vectors is denoted by H^π and its orthogonal complement, the space of coinvariants, is denoted by H_π.

Here Psi... Groups

Let $\mathcal{F}$ be a family of representations.

Definition 1.2.2. *G has property T if there exists a pair (K,ε) where $K \subseteq G$ is compact and $\varepsilon > 0$ such that, for every $\pi \in \mathcal{F}$, there exists $f \in C_K(G)$ such that $\|f\|_I \leq 1$ and*

$$\|f.\xi - \Psi(f).\xi\| < \varepsilon\|\xi\| \quad \forall \xi \in H_\pi.$$

The first thing we did was to study what were the relationships between groupoid property T and other property T.

- if $G = \Gamma$ is a discrete group, Γ has property T iff G has property T (in the groupoid sense);

- if X is a coarsely geodesic metric space, then X has geometric property T iff $G(X)$ has property T;

- in the case of a topological action, $X \rtimes \Gamma$ has property T iff Γ has T w.r.t. the family $\mathcal{F}_X$ of representations $\pi : \mathbb{C}[\Gamma] \to B(H)$ s.t. there exists a representation $\rho : C(X) \to B(H)$ such that (ρ, π) is covariant. This hypothesis simplifies in the case where there exists a invariant ergodic probability measure on X; in that case property T for $X \rtimes \Gamma$ and for Γ are equivalent;

- in the case of an approximated group Γ, then $G_{\mathcal{N}}$ has property T iff Γ has T. This may sound disappointing, but if one refines the result, one gets the nice following property: Γ has property τ w.r.t. $\mathcal{N}$ iff $G_{\mathcal{N}}$ has T w.r.t. the family of representations that extend to the reduced C^*-algebra of G.

The last part of the work is devoted to the existence of Kazdhan projections. Recall, if $\mathcal{F}$ is a family of representations, $C^*_{\mathcal{F}}(G)$ is the C^*-algebra obtained as the completion of $C_c(G)$ w.r.t. the norm

$$\|a\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \{\|\pi(a)\|\}.$$

A Kazdhan projection $p \in C^*_{\mathcal{F}}(G)$ is a projection such that its image in any of the representations in $\mathcal{F}$ is the orthogonal projection on the invariant vectors.

Theorem 1.2.3. Let G be compactly generated. Then if G has property T w.r.t. $\mathcal{F}$, there exists a Kazdhan projection $p \in C^*_{\mathcal{F}}(G)$.

This gives an obstruction to inner-exactness. Denote by F the closed G-invariant subset

$$\{x \in G^0 \mid G^x \text{ is infinite }\}$$

and U its complement.

Theorem 1.2.4. Let G be compactly generated and with property T. If one can find a sequence of points $(x_i)_i \subset U$ such that, for every compact subset $K \subset G$, K only intersects a finite number of orbits $G.x_i = r(s^{-1}(x_i))$, then G is not inner-exact. In fact it is not K-inner-exact. in particular, at least one of the groupoids G, $G_{|U}$ or $G_{|U}$ does not satisfy the Baum-Connes conjecture.

1.2.1 Kazdhan projections and failure of K-exactness

For $K \subset G$, $C_K(G)$ denotes the continuous functions supported in K.

Theorem 1.2.5. Let G be an étale groupoid whose reduced C^*-algebra contains a non trivial Kazdhan projection p. Suppose there exists an invariant probability measure on G^0 and that there exists an open subset $U \subset G^0$ not equal to G^0 containing a sequence of points (x_i) such that:

- x_i has finite orbit ($x_i \in G^0_{fin}$);
- for every compact $K \subset U$, the orbits $Gx_i = r(G_{x_i})$ ultimately don't intersect K;

then $C^*_r(G)$ is not K-exact.

Proof. Denote by M_i the finite dimensional C^*-algebra $B(l^2 G_{x_i})$ and $\lambda_i : C^*_r(G) \to M_i$ the corresponding left regular representation. We will show that the sequence

$$0 \longrightarrow C^*_r(G) \otimes \oplus M_i \longrightarrow C^*_r(G) \otimes \prod M_i \xrightarrow{q} C^*_r(G) \otimes \prod M_i / \oplus M_i \longrightarrow 0$$

is not exact in K-theory. We shall call q the last map in this diagram.

Define the following $*$-morphism

$$\phi \left\{ \begin{array}{rcl} C^*_r(G) & \to & C^*_r(G) \otimes (\prod M_i) \\ x & \mapsto & x \otimes (\lambda_i(x))_i \end{array} \right.$$

Claim: the image of ϕ is contained in the kernel of q.

Let $x \in C^*_r(G)$ and $\epsilon > 0$. Let $K \subset G$ be a compact subset and $a \in C_K(G)$ such that $\|x - a\|_r < \varepsilon$. Let ϕ_i be the $*$-homomorphism defined in the same fashion as ϕ only with the first i components of $\phi(x)$ equated to zero. Denote by $\overline{x}$ the class of x in $C^*_r(G) \otimes \prod M_i / \oplus M_i$. Then $\overline{\phi(x)} = \overline{\phi_i(x)}$. Also, as the orbits G_{x_i} are ultimately disjoint, there is a i_0 such that $\lambda_i(a) = 0$ and thus $\phi_i(a) = 0$ for all $i > i_0$. This ensures

$$\|\overline{\phi(x)}\| = \|\overline{\phi_i(x)}\| = \|\overline{\phi_i(x)} - \overline{\phi_i(a)}\| < \varepsilon$$

hence $\overline{\phi(x)} = 0$.

Let $p \in C^*_r(G)$ the Kazdhan projection. Then $P = \phi(p)$ goes to zero in the right side of the sequence above. Let us show that its class in K-theory does not come form an element in $K_0(C^*_r(G) \otimes \oplus M_i)$.

The invariant probability measure on G^0 induces a trace τ on $C^*_r(G)$. Define τ_i to be the trace $\tau \otimes tr$ on $C^*(G) \otimes M_i$, where tr is the normalized trace on M_i. It is easy to see that $\tau_n(P) = \tau(p) > 0$. But if $z \in K_0(C^*_r(G) \otimes \oplus M_i)$, $\tau_n(z)$ is ultimatley zero. This implies that the non triviality of P ensures the non K-exactness of the sequence above in K-theory.

□

This result gives interesting examples of non K-exact C^*-algebras:

- if X is an expander, the coarse groupoid of X satisfies the hypothesis above, so that the uniform Roe algebra $C_u^*(X) \cong C_r^*(G)$ is not K-exact; in particular, if Γ contains an expander almost isometrically, its reduced errrrr no?
- if Γ is a residually finite group with property (τ), then any HLS groupoid associated to an approximating sequence of Γ satisfies the hypothesis above so that $C_r^*(G)$ is not K-exact.

1.3 Classification and the UCT

For A a simple unital C^*-algebra, the Elliot invariant is:

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \to S(K_0(A)))\,,$$

here $T(A)$ is the trace space and r_A the paring $r_A(\tau)([p]) = [\tau(p)]$.

Elliot's conjecture: Separable, simple, nuclear are classifiable by Elliot's invariants.

Theorem 1.3.1. Separable, simple, unital, nuclear, $\mathcal{Z}$-stable, UCT algebras are classifiable by Elliot's invariants.

An example of a classification theorem: Elliot's theorem,

Theorem 1.3.2. Let A and B unital AF-algebras and

$$\alpha : K_0(A) \to K_1(A)$$

a unital order isomorphism, i.e.

$$\alpha(K_0(A)_+) \subseteq K_0(B)_+ \quad \text{and} \quad \alpha([1_A]) = [1_B].$$

Then there exists a unital $*$-isomorphism $\phi; A \to B$ such that $\phi_* = \alpha$.

1.4 C^*-simplicity

1.4.1 General introduction

Let Γ be a discrete group. We will recall two equivalence relations on the set(?) of unitary representations of Γ, which are group homomorphisms

$$\pi : \Gamma \to U(H_\pi)$$

where $U(H_\pi)$ stands for the unitary group of a complex Hilbert space H_π. We will refer to such a representation as (π, H_π) or even just π or H_π if no confusion is possible.

Let π and σ be two representations of Γ.

- $\pi \simeq \sigma$ iff there exists a unitary $u : H_\pi \to H_\sigma$ such that
$$u\pi_\gamma u^* = \sigma_\gamma \quad \forall \gamma \in \Gamma.$$
- $\pi \approx \sigma$ iff there exists a sequence of unitaries $u_n : H_\pi \to H_\sigma$ such that
$$\|u_n \pi_\gamma u_n^* - \sigma_\gamma\| \to 0 \quad \forall \gamma \in \Gamma.$$

Fact: It turns out that for a lot of groups (e.g. finite, abelian, compact, simple Lie groups,...), these two notions coincide

$$\pi \approx \sigma \quad \text{iff} \quad \pi \simeq \sigma \quad \text{for } \pi, \sigma \text{ irreducible.}$$

Let $\hat{\Gamma}$ be the collection of all representations of Γ. A very hard problem is to describe

$$\hat{\Gamma}/\approx .$$

It can be done sometimes, e.g. for $\mathbb{Z}$ the irreducible representations are given by the circle, and any representation decomposes more or less uniquely into these.

Let us recall that the (left) regular representation

$$\lambda : \Gamma \to U(l^2\Gamma)$$

is defined by $\lambda_g(\delta_h) = \delta_{gh}$. The reduced C^*-algebra $C_r^*(\Gamma)$ is the closure under the operator norm of the image of the regular representation, i.e.

$$C_r^*(\Gamma) = \overline{span\{\lambda_\gamma\}_{\gamma\in\Gamma}}.$$

A representation π is tempered if it extends to a $*$-representation of $C_r^*(\Gamma)$. This happens iff the linear extension

$$\pi : \mathbb{C}[\Gamma] \to B(H_\pi)$$

satisfies $\|\pi(a)\| \leq \|\lambda(a)\|, \forall a \in \mathbb{C}[\Gamma]$.

Fact: All representations are tempered iff the group is amenable.

Another (very hard) problem is to describe

$$\hat{\Gamma}_r/\approx .$$

Definition 1.4.1. Γ is C^*-simple if $C^*_r(\Gamma)$ is simple, i.e. admits no proper two sided closed ideal.

Theorem 1.4.2 (Voiculescu)**.** Γ is C^*-simple iff $\hat{\Gamma}_r/\approx$ is a point.

Examples of C^*-simple groups:

- Non abelian free groups;
- Torsion free hyperbolic groups;
- $PSL(n, \mathbb{Z})$;
- Thompson's group V.

Non C^*-simple examples

Recall that a group Γ if the trivial representation

$$1_\Gamma : \Gamma \to U(\mathbb{C}) = \mathbb{S}^1; \gamma \mapsto id = 1;$$

is tempered. As a consequence, non trivial amenable groups are not C^*-simple as $1 \approx \lambda$ $(dim(l^2\Gamma) \neq 1)$.

More generally if there exists an amenable normal subgroup $K \lhd \Gamma$, then the quasi regular representation

$$\lambda_{\Gamma/K} : \Gamma \to U(l^2(\Gamma/K)); \ \lambda_{\Gamma/K}(\gamma)(\delta_{xK}) = \delta_{\gamma xK};$$

is tempered, hence if K is not trivial, Γ is not C^*-simple. In particular any semi-direct product $K \rtimes H$ with K amenable and non trivial is not C^*-simple.

Amenability being stable by extensions and increasing unions, any group has a largest normal amenable subgroup $R \lhd \Gamma$ called the amenable radical. The previous discussion shows that if Γ is C^*-simple, then $R = \{e\}$. The converse does not hold and was completely answered by Kennedy et al.

How to prove C^*-simplicity?

á la Powers [**?**].

Definition 1.4.3. A group Γ is a *Powers group* if for every finite subset $F \subset \Gamma$ there exists a partition

$$\Gamma = C \coprod D$$

and a finite number of elements $\gamma_1, ..., \gamma_n \in \Gamma$ with

- $\gamma C \cap C = \emptyset$ for every $\gamma in F$:
- $\gamma_i D \cap \gamma_j D = \emptyset$ for every $i \neq j$.

Examples:

- The free group on two generators $\mathbb{F}_2$ (Powers [**?**]);

- Many other examples using "North-South" type dynamics (De la Harpe, Bridson, Osin).

Let us write a few words about the technique Powers used. For $\mathbb{F}_2 = \langle a, b \rangle$, let

$$\tau : C_r^*(\mathbb{F}_2) \to \mathbb{C}; a \mapsto \langle \delta_e, a\delta_e \rangle$$

be the canonical tracial state.

Theorem 1.4.4 (Powers [?])**.** For every $a \in C_r^*(\Gamma)$,

$$\tau(x) = \lim \frac{1}{mn} \sum_{i=1,nj=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}$$

Corollary 1.4.5. $\mathbb{F}_2$ is C^*-simple.

Proof. Let $J \lhd C_r^*(\mathbb{F}_2)$ be an ideal. For $x \in C_r^*(\mathbb{F}_2)$ let $x_{mn} = \sum_{i=1,nj=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}$. If $x \in J$ then $(x^*x)_{mn} \in J$ so $\tau(x^*x)1_{C_r^*(\mathbb{F}_2)} \in \overline{J}^{\|\ \|}$. If J is not trivial, it contains a non zero element x, which forces $1_{C_r^*(\mathbb{F}_2)} \in J$ as $\tau(x^*x) > 0$. This ensures that $J = C_r^*(\mathbb{F}_2)$ and we are done. □

Corollary 1.4.6. $C_r^*(\mathbb{F}_2)$ has a unique tracial state.

Proof. Let τ' be a tracial state on $C_r^*(\mathbb{F}_2)$. Then for $x \in C_r^*(\mathbb{F}_2)$,

$$\tau'(x) = \tau'(x_{mn}) \to \tau'(\tau(x)1) = \tau(x)\tau'(1) = \tau(x).$$

□

1.4.2 Definitions

We only consider discrete countable groups, usually denoted by Γ.

Definition 1.4.7. A group is said to be C^*-simple if its reduced C^*-algebra is simple, i.e. has no proper closed two sided ideals.

A motivation for the interest toward such a notion can be the following result of Murray and Von Neumman: the Von Neumman algebra $L(\Gamma)$ is simple (no proper weakly closed two sided ideals) iff it is a factor iff Γ is ICC (infinite conjugacy classes, i.e. all non trivial conjugacy classes are infinite). Another one is that simplicity is one out of the 5 criteria (unital simple separable UCT with finite nuclear dimension) needed in the classification theorem obtained by Winter et. al.

Recall that, given two unitary representations of Γ, we say that π is weakly contained in σ and write

$$\pi < \sigma$$

if every positive type function associated to π can be approximated uniformly on compact sets by finite sums of such things associated to σ. In other words, if for every $\xi \in H_\pi$, $F \subseteq \Gamma$ finite and every $\varepsilon > 0$, there exists $\eta_1, \eta_2, ..., \eta_k$ such that

$$|\langle \pi(s)\xi, \xi \rangle - \sum_i \langle \sigma(s)\eta_i, \eta_i \rangle| < \varepsilon \quad \forall s \in F.$$

Remark: one can restricts to convex combinations of normalized positive type functions.

If $\pi < \sigma$, then the identity $\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ extends to a surjective $*$-morphisms

$$C^*_\sigma(\Gamma) \to C^*_\pi(\Gamma).$$

Indeed, it suffices to show that for every $a \in \mathbb{C}[\Gamma]$,

$$\|\pi(a)\| \leq \|\sigma(a)\|.$$

As $\|\pi(a)\|^2 = \|\pi(a^*a)\|$, we can suppose a positive. Then

$$\begin{aligned}\langle \pi(s)\xi, \xi\rangle &\leq \sum_i t_i \langle \sigma(s)\eta_i, \eta_i\rangle + \varepsilon \\ &\leq \|\sigma(a)\| + \varepsilon\end{aligned}$$

hence $\|\pi(a)\| \leq \|\sigma(a)\| + \varepsilon$, and let just ε go to zero.

Definition 1.4.8. A group Γ is C^*-simple if its reduced C^*-algebra is simple (i.e. has no proper closed two sided ideal).

Theorem 1.4.9. If Γ has a non trivial amenable normal subgroup, then it is not C^*-simple.

Proof. Let N be a normal amenable subgroup of Γ. Let (F_k) be a sequence of Folner sets for N, and

$$\xi_k = \frac{1}{|F_k|}\chi_{F_k} \in l^2(\Gamma)$$

Then

$$\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle = 2 - \frac{|F_k \Delta s F_k|}{|F_k|}$$

wich is 0 if $s \notin N$, and goes to 1 as n goes to infinity if $s \in N$. In other words

$$\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle \to \langle \lambda_{\Gamma/N}(s)\delta_{eN}, \delta_{eN}\rangle,$$

which shows that $\lambda_{\Gamma/N} < \lambda_\Gamma$. This gives us a surjective $*$-morphism

$$\phi : C^*_r(\Gamma) \to C^*_{\Gamma/N}(\Gamma).$$

A faster way which still works out when the ambient group is only locally compact is to point out that, N being amenable,

$$1_N < \lambda_N,$$

ensures by induction

$$Ind_N^\Gamma 1_N = \lambda_{\Gamma/N} < Ind_N^\Gamma \lambda_N = \lambda_\Gamma.$$

But if $n \in N$ is non trivial, $\lambda_\Gamma(n)$ is non trivial and sent to $\lambda_{\Gamma/N}(n) = 1$ via ϕ, so that $Ker\ \phi$ is a proper ideal in $C^*_r(\Gamma)$.

□

After the talk, Erik Guentner suggested the following proof. It is even shorter and doesn't assume any knowledge about weak containment or induction of representations. It is a weakening of the following fact: when Γ is amenable, the trivial representation $1_\Gamma : C^*_{max}(\Gamma) \to \mathbb{C}$ extends to the reduced C^*-algebra.

Indeed let $a \in \mathbb{C}[\Gamma]$ and (F_n) be a sequence of Folner sets for the support of a. Define $\xi_n = \frac{1}{|F_n|^{\frac{1}{2}}} \chi_{F_n} \in l^2(\Gamma)$. Then, suppose a is positive, and compute

$$\begin{aligned} \langle a\xi_n, \xi_n \rangle &= \sum_{s \in \text{ supp } a} a_\gamma \frac{|F_n \cap sF_n|}{|F_n|} \\ &\to \|a\|_{1_\Gamma} \end{aligned}$$

so that $\|a\|_r \leq \|a\|_{1_\Gamma}$.

Now if N is a normal amenable subgroup of Γ...

We saw that $\mathbb{F}_2$ is C^*-simple, yet it has a copy of $\mathbb{Z}$ as an amenable subgroup (non normal), and a normal (non amenable) subgroup: the commutator subgroup, which is an infinite rank free group, $\langle [x, y] : x, y \in \mathbb{F}_2 \rangle = \mathbb{F}([a^n, b^m]; n, m)$. Both conditions are necessary.

This result led to following (false) conjecture: a group is C^*-simple iff it has no non trivial amenable normal subgroups.

1.4.3 Completely positive maps

If A and B are C^*-algebra, then a linear map $\phi : A \to B$ is called completely positive if

$$\phi^{(n)}(a) = (\phi(a_{ij}))_{ij} \geq 0 \quad \forall a \in M_n(A)_+.$$

Denote $CP(A, B)$ the normed vector space of completely positive maps form A to B. $S(A)$ denotes the state space of A, endowed with the weak-$*$ topology (it's then a convex subspace of A^*, compact when A is unital).

Then:

- $CP(C(X), C(Y)) \cong C(Y, P(X))$ via $\mu_y(f) = \Phi(f)(y)$;

- $CP(A, C(Y)) \cong C(Y, S(A))$ via $\omega_y(f) = \Phi(a)(y)$;

- What about $CP(C(X), B)$? Continuous sections on the continuous field of C^*-algebras $\bigoplus_{\omega \in S(B)} B(H_\omega)$.

1.4.4 Injective C^*-algebras

Recall that an abelian group M is injective if, given any injective homomorphism of abelian group $A \hookrightarrow B$, any homomorphism $A \to M$ extends to a homomorphism $B \to M$. In words: any homomorphism into M extends to super-objects. We will often use the following commutative diagram

$$\begin{tikzcd} B \arrow[dr, dotted, "\exists"] & \\ A \arrow[u, hook] \arrow[r] & M \end{tikzcd}$$

to represent this situation. We will now turn to a analog notion in the C^*-algebraic world.

Definition 1.4.10. A C^*-algebra M is *injective* if, given an inclusion of C^*-algebra $A \subset B$, any injective $*$-homomorphism $A \to M$ extends to B by a contractive completely positive (CCP) map.

$$\begin{tikzcd} B \arrow[dr, dotted, "\exists\ ccp"] & \\ A \arrow[u, hook] \arrow[r] & M \end{tikzcd}$$

Even if the straight arrow are here supposed to be $*$-homomorphism, Stinespring's dilation theorem ensures that we can suppose all the arrows to be only CCP maps. We will say that M is Γ-injective if Γ acts by automorphisms on all the C^*-algebras in the diagram, and all the arrows are Γ-equivariant.

We will define a particular class of compact spaces acted upon by Γ, called Γ-boundaries, and show that there exists a maximal Γ-boundary $\partial_F\Gamma$, called the *Thurston boundary.*

The first **major goal** of this presentation is to show that $C(\partial_F\Gamma)$ is Γ-injective.

Description of commutative injective algebras

Lemma 1.4.11. If M is injective and $S \subset M$, define

$$Ann_M(S) = \{m \in M \mid sm = 0 \forall s \in S.$$

Then there exists a projection $p \in M$ satisfying $Ann_M(S) = pM$.

Proof. This is true if $M = B(H)$ for some Hilbert space. In the general case, embed M unitally in some $B(H)$. By injectivity of M, there exists a CCP map $E : B(H) \to M$ such that $E(m) = m, \forall m \in M$ so $M \subset dom(E)$ (multiplicative domain). There exists a projection $p \in B(H)$ with $Ann_{B(H)}(S) = pB(H)$ (take the projection on $\cap_{s\in S} Ker(s)$). If $s \in S$,

$$sE(p) = E(sp) = 0 \text{ hence } E(p) \in Ann_M(S).$$

Moreover if $m \in Ann_M(S) \subset pB(H)$, $pm = m$ and

$$E(p)m = E(pm) = E(m) = m$$

so that for $m = E(p)$, we get $E(p)$ is a projection. This also proves that $E(p)Ann_M(S) = Ann_M(S)$. A slight fiddling ensures then that $Ann_M(S) = E(p)M$. $\square$

Corollary 1.4.12. Let X be a compact Hausdorff space. If $C(X)$ is injective then X is Stonean, i.e. $\overline{U}$ is open for every open subset $U \subset X$.

Proof. Let $U \subset X$ be open, and $S = C_0(U)$. By the previous lemma, there exists a projection $p \in C(X)$ such that $Ann_{C(X)}(S) = pC(X)$. But p cannot be anyone else than the characteristic function of $\overline{U}^c$ so that $1 - p = \chi_{\overline{U}}$ is continuous and $\overline{U}$ is open. □

Note: Infinite compact Stonean spaces are not metrizable (not even second countable). Suppose the contrary and get a sequence $x_i \to x$ in X and open sets $U_n = B(x_n, \varepsilon_n)$, with ε_n such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ for every $n \neq m$. Set $U = \cup_n U_{2n}$, then $x \in \overline{U}$ ($\overline{U}$ is open) so $x_n \in \overline{U}$ for large n but $x_n \notin \overline{U}$ for n odd.

1.4.5 Furstenburg boundary

If Γ is a discrete group acting on a compact Hausdorff space X (we will just say that X is a Γ-space), the space of probability measures $Prob(X)$ endowed with the weak-$*$ topology is homeomorphic to the state space $S(C(X))$ with the topology of simple convergence. We identify X with a closed subspace of $Prob(X)$ with the help of the Dirac masses ($X \hookrightarrow Prob(X); x \mapsto \delta_x$ is an embedding). Recall that the action can be extended to $Prob(X)$, which is then a Γ-space by Banach-Alaoglu's theorem.

Definition 1.4.13. A Γ-space X is:

- *minimal* if the only Γ-invariant closed subset of X are itself and $\emptyset$;
- *strongly proximal* if $\overline{\Gamma.\mu}^{weak-*}$ contains δ_x for some $x \in X$;
- a Γ-boundary if it is minimal and strongly proximal

$$X \subset \overline{\Gamma.\mu}^{weak-*} \quad \forall \mu \in Prob(X).$$

Example: Let $SL(2, \mathbb{Z})$ act on the projective line $\mathbb{R}P^1$ (the quotient of $\mathbb{R}^2 \backslash \{0\}$ by the group of dilations) given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Most $g \in Sl(2, \mathbb{Z})$ are acting hyperbolically (two distinct eigenspaces, one expansive one contractive). Take $\mu \in Prob(\mathbb{R}P^1)$, and a generic element $g \in SL(2, \mathbb{Z})$. As n goes to ∞,

$$g^n.\mu \to_{weak-*} \delta_{\text{Expanding eigenspace}}$$

unless $\mu(\{\text{contractive eigenspace}\}) > 0$, hence

$$\{\delta_{\text{Expanding eigenspace}}\}_{g \in SL(2,\mathbb{Z})} \subset \overline{\Gamma.\mu}^{wk-*}.$$

Exercise: the set of these is dense in $\mathbb{R}P^1 \subset Prob(\mathbb{R}P^1)$.

Theorem 1.4.14 (Furstenburg)**.** There exists a Γ-boundary $\partial_F \Gamma$ (now called the Furstenburg boundary) such that for any Γ-boundary X there exists a continuous Γ-equivariant surjection $\partial_F \Gamma \twoheadrightarrow X$.

Proof. Let $\mathcal{B}$ be the class of all Γ-boundaries. It is non empty as it contains the point space. Take

$$Z = \prod_{Y \in \mathcal{B}} Y$$

which is compact by Tychonoff's theorem. Equip Z by the diagonal Γ-action.

- It is strongly proximal: for any $\mu \in Prob(Z)$, a diagonal argument gives a weak-$*$ convergent net $g_i.\mu \to \delta_z$ for some $z \in Z$.
- It is not minimal, but Zorn's lemma ensures the existence of a minimal closed Γ-invariant subset $\partial_F\Gamma$ of Z.

We obtain the desired map as the composition of the inclusion $\partial_F\Gamma \hookrightarrow Z$ with the projection on the X-factor $Z \twoheadrightarrow X$. □

Theorem 1.4.15 (Kalantar-Kennedy)**.** $C(\partial_F\Gamma)$ is Γ-injective.

Lemma 1.4.16. There exists a bijective correspondence between the completely positive maps from $C(X)$ to $C(Y)$ and the continous maps from Y to $Prob(X)$. The statement remains true if one asks for equivariance. **send to a previous section on CP maps**

Lemma 1.4.17 (Furstenburg)**.** Let X and Y be two Γ-boundaries. Then any Γ-equivariant map $X \to Prob(Y)$ has image in Y, i.e. any UCP map $C(X) \to C(Y)$ is a $*$-homomorphism! Moreover there is at most one such map.

Proof. Take $\mu : X \to Prob(Y)$. The image $\mu(X) \subset Prob(Y)$ is a closed Γ-invariant subspace: by strong proximality of Y, there exists $y \in Y$ such that

$$\delta_y \in \overline{\Gamma.\mu_x}^{wk-*} \subset \mu(X).$$

By minimality of Y, $\overline{\Gamma.\mu_x}^{wk-*} \cap Y = Y$, By minimality of X, $\mu^{-1}(Y) = X$ i.e. $\mu(X) \subset Y$.

Let $\mu, \eta : X \to Prob(Y)$ be two such maps. Then $\frac{1}{2}\mu + \frac{1}{2}\eta$, μ and η all take values in Y so that they are all equal. □

Corollary 1.4.18. Any equivariant UCP map $C(\partial_F\Gamma) \to C(\partial_F\Gamma)$ is the identity.

Recall that if A is a unital Γ-algebra, its state space $S(A)$ is convex compact Γ-space.

Proposition 1.4.19 (Gleason)**.** Let $Z \subset S(A)$ be a Γ-invariant closed convex subspace, which is minimal w.r.t. these properties. (Such a thing exists by Zorn's lemma.) Then

$$\partial_{ex} Z = \{\phi \in Z \mid \phi \text{ is not a non trivial convex combination of anything in } Z\}$$

is a Γ-boundary.

Proof. There is a barycenter map $\beta : Prob(Z) \to Z$ such that

$$\int_Z f d\mu = f(\beta(\mu)) \quad \forall f \in C(Z) \text{ affine}.$$

Indeed, if $\mu = \delta_z$, $\beta(\mu) = z$ and if $\mu = \sum \alpha_i \delta_{z_i}$ with $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1$, then $\beta(\mu) = \sum \alpha_i z_i$. Finite convex combinations are weak-$*$ dense in $Prob(Z)$ by the Hahn-Banach separation theorem. As β is weak-$*$ continuous, and affine so uniformly weak-$*$

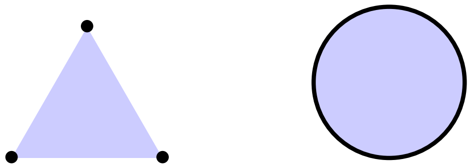


Figure 1.1: Two examples with Z in blue and $\partial_{ex}Z$ in black.

continuous, it extends to the whole space $Prob(Z)$.

Note: β is Γ-equivariant continuous and satisfies $\beta(\mu) = z \in \partial_{ex}Z$ iff $\mu = \delta_z$.

Then, for any $\mu \in Prob(Z)$,

$$\beta(\overline{conv(\Gamma\mu)}) = \overline{conv(\Gamma\beta(\mu))} = Z,$$

the first equality coming from continuity, Γ-equivariance and affinity. Now, $\partial_{ex}Z$ is minimal, and if $\mu \in \partial_{ex}Z$, then

$$AFINIR$$

□

We are now ready for the main result of this section.

Theorem 1.4.20 (Kalantar-Kennedy)**.** $C(\partial_F\Gamma)$ is Γ-injective.

Proof. First, observe that $l^\infty(\Gamma)$ is Γ-injective. Let indeed $A \subset B$ be an inclusion of C^*-algebras and $\phi : A \to l^\infty(\Gamma)$ a $*$-homomorphism. Then $ev_e \circ \phi$ is a state on A, so it extends to a state Ψ on B. Define $\tilde{\phi} : B \to l^\infty(\Gamma)$ by

$$\tilde{\phi}(b)(\gamma) = \Psi(\gamma^{-1}.b).$$

Then Ψ is a UCP Γ-equivariant map that extends ϕ.

Now, producing ucp equivariant maps

$$C(\partial_F\Gamma) \xrightarrow{\alpha} l^\infty(\Gamma) \xrightarrow{\beta} C(\partial_F\Gamma)$$

is sufficient to conclude, as their composition must be the identity by corollary 1.4.18.

Define $\alpha : C(\partial_F\Gamma) \to l^\infty(\Gamma)$ by fixing $\mu \in Prob(\partial_F\Gamma)$ and set

$$\alpha(f)(\gamma) = \mu(\gamma^{-1}.f).$$

By Gleason's theorem 1.4.19, there is a Γ-boundary $X \subset S(l^\infty(\Gamma))$. By universal property of $\partial_F\Gamma$, we have an equivariant surjection $\partial_F\Gamma \twoheadrightarrow X \subset S(l^\infty(\Gamma))$. By duality, we get a Γ-equivariant ucp map

$$\Psi : l^\infty(\Gamma) \to C(\partial_F\Gamma)$$

and we are done. □

As a final remark, one can point out that this last proof used the following useful fact: if B is injective and $\phi : A \to B$ is a split injective Γ-ucp map, then A is injective. We use this with $A = C(\partial_F\Gamma)$ and $B = l^\infty(\Gamma)$.

1.4.6 Dynamical characterization of C^*-simplicity

(Facts we are using:
$C(\partial_F\Gamma)$ is Γ-injective, in particular any Γ-equivariant u.c.p. $C(\partial_f\Gamma) \to A$ is split, so is an isometric embedding,
$\partial_F\Gamma$ is totally disconnected,)

The goal of this section is to prove the following theorem.

Theorem 1.4.21. Γ is C^*-simple iff the action of Γ on $\partial_F\Gamma$ is free.

Let's do first the forward direction.

Suppose the action is free. First, to show $C^*_r(\Gamma)$ is simple, it is enough to show that any representation

$$\pi : C^*_r(\Gamma) \to B(H)$$

is injective.

By Arveson's extension theorem, π extends to a u.c.p. map

$$\phi : C(\partial_F\Gamma) \rtimes_r \Gamma \to B(H).$$

Its restriction ϕ_0 to $C(\partial_F\Gamma)$ is Γ-equivariant, because $C(\partial_F\Gamma)$ is in the multiplicative domain of ϕ_0, and thus must be an isometric embedding, by Γ-injectivity of $C(\partial_F\Gamma)$ (it is split because $\mathbb{C} \subseteq B(H)$). The equivariant u.c.p. map ϕ_0 is an isomorphism onto its image: extend its inverse form $im\ \phi_0$ to $im\ \phi$ and denote the resulting u.c.p map by τ.

Claim: $\Psi = \tau \circ \phi$ is the canonical expectation $E : C(\partial_F\Gamma) \rtimes_r \Gamma \to C(\partial_F\Gamma)$ which is faithful. This implies π is injective.

Let's end up with the claim.

- $\Psi_{|C(\partial_F\Gamma)} = id_{C(\partial_F\Gamma)}$. Indeed, τ is the inverse of $\phi_0 = \phi_{C(\partial_F\Gamma)}$.
- If $\gamma \neq e_\Gamma$, the action being free, for every x there exists a function $f \in C(\partial_F\Gamma)$ such that
$$f(x) \neq 0 \quad \text{and} \quad f(s^{-1}x) = 0.$$
Now $C(\partial_F\Gamma)$ is in the multiplicative domain of Ψ, so
$$\Psi(\lambda_s)f = \Psi(\lambda_s f) = \Psi((sf)\lambda_s) = (sf)\Psi(\lambda_s)$$
which evaluated at x gives $\Psi(\lambda_s)(x) = 0$, for all x, so $\Psi(\lambda_s) = 0$.

The other direction is more intricated. It consists in two steps:

1. if $x \in \partial_F\Gamma$, then the stabilizer Γ_x is amenable, which implies that $\lambda_{\Gamma/\Gamma_x} < \lambda_\Gamma$,

2. if X is a X is a Γ-boundary, and $\gamma \neq 0$ such that $int(X_s) \neq \emptyset$, then $\lambda_\Gamma \not< \lambda_{\Gamma/\Gamma_x}$, so that the kernel of $C^*_r(\Gamma) \to C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma)$ is a non trivial two sided closed ideal.

This, together with the fact that $\partial_F\Gamma$ is topologically free iff it is free, concludes the proof.

First bullet:

- there exists a Γ_x-equivariant injective $*$-homomorphism

$$\rho : l^\infty(\Gamma_x) \to l^\infty(\Gamma)$$

  defined by $\rho(f)(ts_i) = f(t)$ for every $t \in \Gamma_x$, $\{s_i\}_i$ being a system of representatives of the right cosets $\Gamma_x \backslash \Gamma$.

- there exists a Γ_x-equivariant u.c.p. map

$$\psi : l^\infty \to C(\partial_F\Gamma),$$

  by universal property of $\partial_F\Gamma$, and the fact that the spectrum of $l^\infty(\Gamma)$ is $\beta\Gamma$. (for any compact Γ-space, there exists a Γ-map $\partial_F\Gamma \to P(X)$. take the dual of this map for $X = \beta\Gamma$).

- The composition $\phi = ev_x \circ \psi \circ \rho$ defines a Γ_x-invariant state on $l^\infty(\Gamma_x)$, which concludes the proof.

Second bullet:

This needs a lemma:

Lemma 1.4.22. Let X be a Γ-boundary. For every non empty subset of X, every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma \backslash \{e_\Gamma\}$ such that

$$\min_{t \in F} \mu(tU^c) < \varepsilon \quad \forall \mu \in P(X).$$

Proof. Let $x \in U$. By strong proximality, there exists $t_\mu \neq e_\Gamma$ such that

$$\delta_x(U) - \mu(t_\mu U) = \mu(t_\mu U^c) < \varepsilon,$$

and by continuity of the action

$$V_\mu = \{\nu \in P(X) \mid \nu(t_\mu U^c) < \varepsilon\}$$

is a neighboorhood of μ. By compactness of $P(X)$ in the weak-$*$ topology, we can extract a finite cover such that

$$P(X) = \cup_{i=1,m} V_{\mu_i}.$$

Then $F = \{t_{\mu_1}, ..., t_{\mu_m}\}$ fills the requirements of the lemma.

□

Suppose the action is not topologically free and let $s \neq e_\Gamma$ such that the interior U of X_s is not empty. Let F the finite subset given by the lemma for U and $\varepsilon = \frac{1}{3}$. Suppose

$$\lambda_\Gamma < \lambda_{\Gamma/\Gamma_x}.$$

We will show this is absurd by looking at the coefficient $c_\gamma = \langle \lambda_\Gamma(\gamma)\delta_e, \delta_e \rangle$, which is 0 unless $\gamma = e_\Gamma$.

On the finite subset $K = \{tst^{-1}\}_{t\in F}$, approximate c_γ up to ε by a convex combination

$$\sum_{j=1,n} \alpha_j \langle \lambda_{\Gamma/\Gamma_x}(\gamma)\xi_j, \xi_j \rangle$$

of coefficients of the quasi regular representation. Set

$$\mu_j = \sum_{y\in\Gamma.x} |\xi_j(y)|^2 \delta_y \in P(X) \quad \text{and} \quad \mu = \sum \alpha_j \mu_j,$$

where we identify $\Gamma.x$ with Γ/Γ_x. **A FINIR**

Questions:

- Can we get a more direct proof for the last implication? (without representation theory)

- It is not known in general wether the action of Γ on $\partial_F\Gamma$ is amenable. If X is a Γ-space such that one of the stabilizer is not amenble, the action cannot be amenable. Is it true that, if Γ is exact, this is the only obstruction for the amenability of the action?

1.4.7 Another proof

The last subsection uses representation theory (induction) which makes one wonder if this could be avoided. While the implication

$$\partial_F\Gamma \text{ is free } \Rightarrow \Gamma \text{ is } C^*\text{-simple}$$

is still good enough if one wants to stay clear of representation theoretic lingo, the other direction can be proven in another way.

This proof is taken from a set of notes that Ozawa wrote after giving lectures for the "Annual Meeting of Operator Theory and Operator Algebras" at Tokyo university, 24–26 December 2014.

For X a compact Γ-space and H a subgroup of Γ, we denote by:

- $E_x : C(X) \rtimes_r \Gamma \to C^*_r(\Gamma)$ the canonical conditional expectation onto $C^*_r(\Gamma)$ given by extending the evaluation at x,

- $E_H : C_r^*(\Gamma) \to C_r^*(H)$ the canonical conditional expectation given by $E(\lambda_s) = \delta_{s\in H}$,

- τ_H the canoncical trace $C_r^*(H) \to \mathbb{C}$.

The first thing one can show is the following.

Proposition 1.4.23. Let X be a Γ-boundary, then

$$C(X) \rtimes_r \Gamma$$

is simple.

Proof. It is enough to show that any quotient map

$$\pi : C(X) \rtimes_r \Gamma \to B$$

is injective. By C^*-simplicity, π restricts to an isomorphism on $C_r^*(\Gamma)$ so that the canoncial trace τ is well defined on $\pi(C_r^*(\Gamma)$. Seeing $\mathbb{C}$ as the sub-C^*-algebra of constant functions in $C(\partial_F\Gamma)$, we can extend τ to B.

$$\begin{tikzcd}
C(X) \rtimes_r \Gamma \arrow[r, "\pi"] & B \arrow[dr, dotted, "\phi"] & \\
C_r^*(\Gamma) \arrow[u, hook] \arrow[r, "\cong"] & \pi(C_r^*(\Gamma)) \arrow[u, hook] \arrow[r, "\tau"] & \mathbb{C} \subseteq C(\partial_F\Gamma)
\end{tikzcd}$$

Now $\phi \circ \pi$ restricts to a Γ-u.c.p. map $C(X) \to C(\partial_F\Gamma)$ which can only be the inclusion. This ensures that

$$C(X) \subseteq Dom(\phi \circ \pi).$$

As ϕ extends τ, $\phi \circ \pi$ is the canonical conditional expectation $C(X) \rtimes_r \Gamma \to C(X)$ which is faithful. In particular, π is faithful, and is injective. □

Applying this to $X = \partial_F\Gamma$, we get that $C(\partial_F\Gamma) \rtimes_r \Gamma$ is simple. In that case, every stabilizer

$$\Gamma_x = \{s \in \Gamma \mid sx = x\} \quad \forall x \in \partial_F\Gamma$$

is amenable. Moreover, the strong stabilizer

$$\Gamma_x^0 = \{s \in \Gamma \mid \exists U \text{ neighborhood of } x \text{ s.t. } s_U = id_U\}$$

is a normal subgroup of Γ_x.(In particular, is is amenable.) In that case, we will apply the following proposition.

Proposition 1.4.24. Let X be a minimal compact Γ-space. If

$$C(X) \rtimes_r \Gamma$$

is simple and there exists $x \in X$ such that Γ_x^0 is amenable, then X is topologically free.

Proof. By minimality, topological freeness is equivalent to $\Gamma_x^0 = 1$ for some x.

Indeed, if $\Gamma_x^0 = 1$ for some x, every non trivial group element cannot fix any neighborhood of x hence for every $s \neq e_\Gamma$, we get a sequence of points that converge to x which are not fixed by s. By minimality,
$$X_s = \{y \in X \mid sy \neq y\}$$
is a non empty dense open set of X for every $s \neq e_\Gamma$. By Baire category's theorem,

$$\cap_{s\in\Gamma\setminus\{e\}} X_s$$

is dense in X so that X is topologically free.

Let us show that $\Gamma_x^0 = 1$. Define a representation

$$\rho : C(X) \rtimes_r \Gamma \to B(l^2(\Gamma/\Gamma_x^0))$$

by $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(s\gamma.x)\delta_{s\gamma\Gamma_x^0}$. It is clearly covariant on the algebraic crossed-product.

To prove ρ extends to the whole crossed-product, i.e. $\|\rho(a)\| \leq \|a\|_{C(X)\rtimes_r\Gamma}$, it is enough to show that
$$\langle\ \rho(a)\delta_{\Gamma_x^0}, \delta_{\Gamma_x^0}\ \rangle \leq \|a\|_{C(X)\rtimes_r\Gamma}$$
because $\delta_{\Gamma_x^0}$ is cyclic. This follows from the fact that the latter is the composition $\tau \circ E_{\Gamma_x^0} \circ E_x$ of 3 u.c.p maps (so contractive).

Pick up x such that Γ_x^0 is amenable and $s \in \Gamma$ arbitrary that fixes some neighborhood of x: there exists a neighborhood U of x such that $s_{|U} = id_U$. Let $f \in C(X)$ be nonzero and supported in U. Let us compute
$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0}.$$

- If $\gamma.x \in U$, then $s\gamma.x = \gamma.x$ and

$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(\gamma.x)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}.$$

- If $\gamma.x \notin U$, $f(\gamma.x) = 0 = f(s\gamma.x)$, so that $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}$.

This shows that $\rho(f(\lambda_s - 1)) = 0$. By injectivity, $\lambda_s = 1$ and $s = e_\Gamma$ hence $\Gamma_x^0 = 1$ and we are done. □

1.4.8 Thompson's group V is C^*-simple

In this section, we prove that Thompson's group V is C^*-simple. Recall that V is defined as the group of piecewise linear bijections of $[0, 1)$ with finitely many points of non differentiability, all of which are dyadic rational numbers. Such a function f is entirely determined by two partitions

$$[0, 1) = \coprod_{i=1}^{n} I_i = \coprod_{i=1}^{n} J_i$$

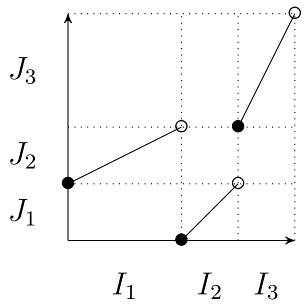


Figure 1.2: The graph of $\begin{pmatrix} I_1 & I_2 & I_3 \\ J_2 & J_1 & J_3 \end{pmatrix}$

and a bijection $\begin{pmatrix} I_1 & \dots & I_n \\ J_{\sigma(1)} & \dots & J_{\sigma(n)} \end{pmatrix}$. The intervals I_i and J_i are of the type $[a, a+2^{-n})$, with a dyadic rational in $[0,1)$. Then f is defined on I_i as the only linear increasing function applying I_i to $J_{\sigma(i)}$.

In order to prove that V is C^*-simple, we will:

- realize V as a countable group of homeomorphisms of the Cantor set;
- use the following result of Le Boudec and Matte-Bon ([**?**], thm 3.7):

  Theorem 1.4.25. Let X be a Hausdorff locally compact space and Γ be a countable subgroup of $Homeo(X)$. Suppose that for every non empty open subset $U \subset X$, the rigid stabilizer
  $$\Gamma_U = \{\gamma \in \Gamma \mid \gamma x = x \ \forall x \notin U\}$$
  is non amenable. Then Γ is C^*-simple.

Let G be an ample groupoid with compact base space. We also always suppsose that groupoids are second countable, Hausdorff and locally compact. Recall that a bisection $U \subset G$ is a set such that s and r are homoeomorphisms when restricted to U. In particular, any open bisection U induces a partial homeomorphism

$$\alpha_U \left\{ \begin{array}{rcl} s(U) & \to & r(U) \\ x & \mapsto & r \circ s_{|U}^{-1}(x) \end{array} \right.$$

The topological full group $[\![G]\!]$ is defined as the set of bisections U of G such that $s(U) = r(U) = G^0$. The operations are defined by

$$e = G^0, \quad UV = \{gg' \mid g \in U, g' \in V \text{ s.t. } s(g) = r(g')\}, \quad U^{-1} = \{g^{-1} \mid g \in U\}.$$

Recall that a Cantor space is any compact metrizable totally disconnected space without any isolated points. It is a standard fact that they are all homeomorphic. A model for Ω is the countable product A^X, where

- A is a finite set, often reffered to as the *alphabet*;
- X is a countable set.

Then elements of Ω are infinite words indexed by X. Denote by Ω_f the set of finite words

$$\Omega_f = \coprod_{\text{finite } F \subset X} A^F,$$

then the topology on Ω is the one generated by the *cylinders*

$$C_a = \{w \in \Omega \mid w(x) = a(x)\ \forall x \in F = supp(a)\}.$$

For finite words $a \in \Omega_f$, $l(a)$ denotes their length, and if $F = \mathbb{N}$, $x \in \Omega$, ax denotes the concatenation of a and x, i.e. the word obtained by first saying a and then x.

Examples:

1. Let Γ a countable discrete group acting on a Hausdorff compact space X by homeomorphisms. Then $[\![X \rtimes \Gamma]\!]$ consists of the bisections of the type

   $$S = \coprod U_i \times \{\gamma_i\}$$

   where $X = \coprod_{i=1}^n U_i = \coprod_{i=1}^n \gamma_i U_i$.

2. Let $\mathbb{Z}$ act on the Cantor space $\Omega = \{0,1\}^{\mathbb{Z}}$ by Bernoulli shift

   $$n(a_i)_i = (a_{i+n})_i \quad \forall n \in \mathbb{Z}, a \in \Omega.$$

   Then $[\![\Omega \rtimes \mathbb{Z}]\!]$ consists of homeomorphisms $\phi : \Omega \to \Omega$ such that there exists a continuous function $n : \Omega \to \mathbb{Z}$ such that

   $$\phi(x) = n(x).x \quad \forall x \in \Omega.$$

3. Let $\Omega = \{0,1\}^{\mathbb{N}}$ be another model for the Cantor space. Define $T : \Omega \to \Omega$ continuous to be the shift

   $$T(a_0, a_1, ...) = (a_1, a_2, ...).$$

   Let G_2 be the so-called *Cuntz* or *Renault-Deaconu* groupoid defined by

   $$\{(x, m-n, y) \mid x, y \in \Omega, m, n \in \mathbb{N} \text{ s.t. } T^m x = T^n y\}.$$

   Exercise: The reduced C^*-algebra of G_2 is isomorphic to the Cuntz algebra

   $$O_2 = C^*\langle s_1, s_2 \mid s_1 s_1^* + s_2 s_2^* = 1, s_1^* s_1 = s_2^* s_2 = 1\rangle.$$

   The open sets

   $$U_{a,b} = \{(ax, l(a) - l(b), bx) \mid x \in \Omega\}$$

   define compact open bisections which cover G_2 when a, b run across Ω_f.

   Then $[\![G_2]\!]$ consists of the bisections of the type

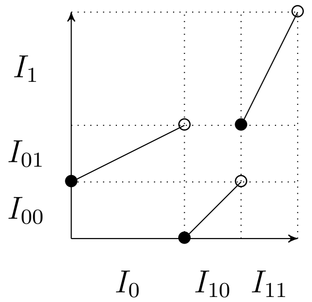
   $$S = \coprod_{i=1}^n U_{a_i, b_i}$$

where $\Omega = \coprod_{i=1,n} C_{a_i} = \coprod_{i=1,n} C_{b_i}$.

If for $a \in \Omega_f$, $I_a = [\overline{a}, \overline{a} + 2^{-l(a)}) \subset [0,1)$, then

$$\begin{cases} \llbracket G_2 \rrbracket & \to & V \\ \coprod_{i=1}^{n} U_{a_i,b_i} & \mapsto & \begin{pmatrix} I_{a_1} & \dots & I_{a_n} \\ I_{b_1} & \dots & I_{b_n} \end{pmatrix} \end{cases}$$

is an isomorphism of groups.



$$S = U_{0,01} \coprod U_{10,00} \coprod U_{11,1} \text{ corresponds to } \begin{pmatrix} I_0 & I_{10} & I_{11} \\ I_{00} & I_{01} & I_1 \end{pmatrix}$$

Figure 1.3: The isomorphism $\llbracket G_2 \rrbracket \cong V$

The last example realizes V as a countable subgroup of homeomorphsims of Ω. If $U = C_a$ is a cylinder for $a \in \Omega_f$, then the rigid stabilizer V_U is isomorphic to V. But V contains a nonabelian free groups, hence is nonamenable. The above theorem ensures that V is thus C^*-simple.

1.5 Weakly and non-weakly band dominated operators

1.5.1 Approximation of band dominated operators

In the following, X denotes a discrete metric space (e.g. $\mathbb{Z}$) with bounded geometry. This last requirement means that, for each positive number r, the cardinality of the r-balls is uniformly bounded, i.e. the number $N_r = \sup_{x\in X} |B(x,r)|$ is finite. For $T \in B(l^2X)$, define the *matrix coefficients* of T by

$$T_{xy} = \langle \delta_x, T\delta_y \rangle \quad \forall x, y \in X.$$

Think of T as a matrix (T_{xy}) indexed by X. The propagation of such a T will then be the (possibly infinite) number

$$prop(T) = \inf\{d(x,y) \mid T_{xy} \neq 0\}.$$

If the propagation of T is finite, we will say that T is *bounded* or has *finite propagation. Band dominated operators* are the norm limits of bounded operators. They form a C^*-algebra $C^*_u(X)$, called the *uniform Roe algebra* of X.

Questions:

1. If T is band dominated, how can we approximate it by bounded operators?
2. How can we recognize when T is band dominated?

The two next numbers will give partial answers to these two questions. Or at least try to explain why they are not trivial.

1.5.2 Approximation by bounded operators

For $T \in B(l^2X)$ band dominated, define $T^{(n)}$ to be the operator with matrix coefficients

$$T^{(n)}_{xy} = \begin{cases} T_{xy} & \text{if } d(x,y) \leq n \\ 0 & \text{otherwise.} \end{cases}$$

We hope that T^n converges to T in norm as n goes to ∞.

As an example, take $X = \mathbb{Z}$ with its canonical metric (given by the absolute value). Each $f \in C(\mathbb{S}^1)$ gives rise to a multiplication operator $M_f \in B(L^2(\mathbb{S}^1))$, and by Fourier transform to a convolution operator $T_f \in B(l^2\mathbb{Z})$. It is the operator of norm $\|f\|_\infty$ with matrix coefficients $(T_f)_{xy}$ proportional to $\hat{f}(x-y)$.
In particular, if $f = \sum_{n=-N}^{N} \lambda_n z^n$ is a trigonometric polynomial, then T_f is bounded as $\hat{f}(n) = 0$ for $|n| > N$. This ensures that every T_f is band dominated, as every continuous functions is a uniform limit of trigonometric polynomials. For such operators, our naive guess

$$\text{“ } T^{(n)}_f \to_{\|\|} T_f \text{ ”}$$

is equivalent to

$$\text{“ } \sum_{n=-N}^{N} \hat{f}(n) z^n \to_{\|\|_\infty} f \text{ ”}$$

which is false. It is even worse: one can have $\|T_f\| = 1$ while $\|T_f^{(n)}\|$ goes to ∞ (Baire category argument, see [?] p. 167) and this implies (by uniform boundedness theorem) that $(T_f^{(n)})_n$ does not even converges to T_f in the strong operator topology.

1.5.3 Weakly band dominated operators

Definition 1.5.1. An operator $T \in B(l^2X)$ has (r, ε)-propagation if for every subsets $A, B \subset X$ such that $d(A, B) > r$,

$$\|\chi_A T \chi_B\| < \varepsilon.$$

T is weakly band dominated if, for every $\varepsilon > 0$, there is $r > 0$ such that T has (r, ε)-propagation.

Note: Bounded implies weakly band dominated, therefore, weakly band dominated being a closed condition, band dominated implies weakly band dominated, as the intuition suggests.

Question: Does weakly bounded implies bounded?

This was claimed without proof for spaces with finite asymptotic dimension by J. Roe ca '97, and actually proved

- by Rabinovich-Roch-Silbermann in '00 for $X = \mathbb{Z}^n$ [?];
- by Špakula-Tikuisis in '16 for finite asymptotic dimension (and a bit more, finite decomposition complexity spaces for the curious reader) [?];
- by Špakula-Zhang in '18 for spaces with property A [?].

We have no counterexamples to this date (25 jan. 2019).

Theorem 1.5.2 (Folklore)**.** The following are equivalent:

1. T is weakly band dominated;
2. for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $f \in l^\infty(X)_1$ and $Lip(f) \leq \delta$ then $\|[T, f]\| < \varepsilon$.

Proof. Let us start with the reverse implication. Say $d(A, B) > r$, then there exists $f \in l^\infty(X)_1$ satisfying $0 \leq f \leq 1$, $f_{|A} = 1$, $f_{|B} = 0$ and $Lip(f) \leq \frac{1}{r}$. Then $f\chi_A = \chi_A$ and $f\chi_B = 0$ so that

$$\chi_A T \chi_B = \chi_A [f, T] \chi_B$$

and $\|\chi_A T \chi_B\| \leq \frac{1}{r}$.

Remark: the function

$$f(x) = \max\{0, 1 - \frac{d(x, A)}{r}\}$$

does the job. The Lipschitz constant is smaller than $\frac{1}{r}$ because of the easy inequality

$$|d(x, A) - d(y, A)| \leq d(x, y) \quad \forall x, y \in X, A \subset X.$$

□

1.5.4 Characterizing membership in the Roe algebra

The main goal of this section is to prove the following result, after the work of Spakula and Tikuisis.

Theorem 1.5.3. Consider the following properties of an operator $b \in B(l^2X)$.

1. $\lim \|[b, f_n]\| = 0$ for every very lipschitz sequence $(f_n) \subset C_b(X)$;
2. b is quasi local;
3. $[b, g] \in \mathfrak{K}(l^2X)$ for every Higson function $g \in C_h(X)$;
4. $b \in C_u^*(X)$.

Then (4) $\implies$ (1) $\iff$ (2) $\iff$ (3). Moreover if X has finite asymptotic dimension, then (4) is equivalent to all of these.

Some remarks are in order.

- These results grew out of a question of John Roe, who asked about the implication (2) $\implies$ (4) when X has *finite asymptotic dimension* (FAD).
- The theorem in [**?**] is better: (2) $\implies$ (4) when X has straight *finite decomposition complexity* (FDC), which is much weaker than FAD. For instance, $\mathbb{Z} \wr \mathbb{Z}$ has FDC but not FAD, while FAD always implies FDC.
- There is a follow up paper which shows (2) $\implies$ (4) when X has property A, an even weaker condition. This last result will be treated in a following number.

Let us understand the conditions better.

Very Lipschitz condition

Recall that a function f is Lipschitz if its Lipschitz modulus

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

is finite. More precisely, a function f is L-Lipschitz if $Lip(f) \leq L \iff |f(x) - f(y)| \leq Ld(x, y), \quad \forall x \neq y$.

A sequence $(f_n) \subset l^\infty(X)$ is *very Lipschitz* if

- the sequence is uniformly bounded: $\exists C > 0$ such that $\|f_n\| \leq C$;
- $\lim Lip(f_n) = 0$.

With this notation, the condition (1) is equivalent to

$$\forall \varepsilon > 0, \exists L > 0 \text{ s.t. if } \|f\| \leq 1 \text{ and } Lip(f) \leq L \text{ then } \|[b, f]\| < \varepsilon.$$

Indeed, one direction is obvious, and suppose there exists $\varepsilon > 0$ such that for every $L > 0$ there is a $f \in l^\infty(X)$ with $\|f\| \leq 1$, $Lip(f) \leq L$ and $\|[b, f]\| \geq \varepsilon$. Take $L = \frac{1}{n}$ to get a very Lipschitz sequence (f_n) with $\|[b, f]\| \geq \varepsilon > 0$, which contradicts (1).

Quasi-locality

Recall that $b \in B(l^2X)$ is quasi-local iff $\forall \varepsilon > 0$, b has finite ε-propagation, iff $\forall \varepsilon > 0, \exists r > 0$ such that $\forall f, g \in l^\infty(X)$, if $\|f\|, \|g\| \leq 1$ and $d(supp(f), supp(g)) \geq r$ then $\|fbg\| < \varepsilon$.

Let us introduce the space

$$C_{L,\varepsilon} = \{a \in B(l^2X) \ : \ \|[a,f]\| < \varepsilon \quad \forall f \in l^\infty(X)_1 \text{ s.t. } Lip(f) \leq L\}.$$

What was said above reduces to the fact that the algebra of quasi-local operators is exactly

$$\bigcap_{\varepsilon>0} \bigcup_{L>0} C_{L,\varepsilon}.$$

Higson functions

A function $g \in l^\infty(X)$ is said to be a Higson function, algebra denoted by $C_h(X)$ iff $\forall \varepsilon > 0, \forall r > 0$, there exists a finite subset $F \subset X$ such that if $x, y \notin F$ and $d(x,y) \leq r$, then $|g(x) - g(y)| \leq \varepsilon$.

Roe's question on conditions (2) and (4)

(4) $\implies$ (2) is not difficult. In short, quasi-locality is a closed condition, which is obviously satisfied by finite propagation bounded operators.

Closed condition If $\forall \delta > 0$, there is a quasi-local operator b' such that $\|b - b'\| < \delta$, then b is quasi-local.

Finite propagation operators are quasi-local If $\xi \in l^2(X)$ is finitely supported and $prop(b) \leq r$, then $supp(b\xi) \subset N_r(supp(\xi))$, and so if $d(supp(f), supp(g)) > r$, then $gbf = 0$.

(4) $\implies$ (1) is again not too hard.
Condition (1) is closed and is satisfied by finite propagation operators. This follows from elementary estimates and a calculation of the kernel of the commutator.

Lemma 1.5.4. If $b \in B(l^2X)$ such that $|b(x,y)| \leq C$ and $prop(b) \leq r$, then $\|b\| \leq CN_r$

Lemma 1.5.5. Let b as above and $f \in l^\infty(X)$. The kernel of $[b,f]$ is

$$[b,f](x,y) = b(x,y)(f(x) - f(y))).$$

Now (4) $\implies$ (1) follows: if $prop(b) \leq r$, then

$$prop([b,f]) \leq r \text{ and } |[b,f](x,y)| \leq CLip(f)r$$

so that also $\|[b,f]\| \leq CrN_r Lip(f)$. As for the lemmas, the first point reduces to:

$$\begin{aligned}
|b\xi(x)| &\leq \sum_{y\in B_r(x)} |b(x,y)|\ |\xi(y)| \\
&\leq CN_r^{\frac{1}{2}} \|\xi_{|B_r(x)}\|_2 \quad \text{by CBC.} \\
\implies \quad \|b\xi\|_2^2 &= \sum_x |b\xi(x)|^2 \\
&\leq \sum_x C^2 N_r \|\xi_{|B_r(x)}\|_2^2 \\
&\leq \sum_x \sum_{y\in B_r(y)} C^2 N_r |\xi(y)|^2 \\
&\leq C^2 N_r^2 \|\xi\|_2^2
\end{aligned}$$

The second point is a direct calculation.

(1) $\implies$ (2)

The key point is the following.

Lemma 1.5.6. If $A, B \subset X$ such that $d(A,B) \geq r$, then there exists a function $\phi : X \to [0,1]$ such that

- $\phi = 1$ on A,
- $\phi = 0$ on B,
- $Lip(\phi) \leq \frac{1}{r}$.

Proof. Let us show that it gives the claimed implication. Let $\varepsilon > 0$, condition (1) gives a constant L. Put $r > L^{-1}$. If then $f, g \in l^\infty(X)_1$ such that $d(supp(f), supp(g)) \geq r$ we have $f\phi = f$ and $g\phi = 0$ so that

$$\|fbg\| = \|f[\phi,b]g\| \leq \|[\phi,b]\| < \varepsilon.$$

As for the lemma, just take

$$\phi(x) = \max\{0, 1 - \frac{d(x,A)}{r}\}.$$

$\square$

(2) $\implies$ (1)

The key here idea is: if f has a small Lipschitz constant, then it varies slowly so that its level sets are well separated.

Let $f \in l^\infty(X)$ such that $0 \leq f \leq 1$ and $Lip(f) \leq L$, and put

$$\begin{aligned}
A_i &= \{x \mid \frac{i-1}{N} < f(x) \leq \frac{i}{N}\} \quad i = 2, N \\
A_1 &= \{x \mid 0 \leq f(x) \leq \frac{1}{N}\}
\end{aligned}$$

Then $f \sim \sum_{i=1}^{N} \frac{i}{N} A_i := g$ (actually $\|f-g\| \leq \frac{1}{N}$) and also

$$d(A_i, A_j) \geq \frac{1}{NL} \quad \text{if } |i-j| \geq 2.$$

Also the A_i's are disjoint and cover X. we will now estimate $\|[b,g]\|$. Let $\varepsilon > 0$,

$$\begin{aligned}
\|[g,b]\| &= \|[\sum_i \frac{i}{N} A_i, b]\| \\
&= \| \left(\sum_i \frac{i}{N} A_i \right) \; b \; \left(\sum_i A_i \right) - \left(\sum_i A_i \right) \; b \; \left(\sum_i \frac{i}{N} A_i \right) \| \\
&= \| \sum_{i,j} (\frac{i}{N} - \frac{j}{N}) A_i b A_j \| \\
&\leq \| \sum_{|i-j|=1} \frac{1}{N} A_i b A_j \| + \| \sum_{|i-j|\geq 2} (\frac{i}{N} - \frac{j}{N}) A_i b A_j \|
\end{aligned}$$

Let us label the summands of this last line by I and II. By quasi-locality of b, we get a $r = r(\varepsilon) > 0$, then for any choice of N, put $L = L(N, \varepsilon)$ such that $L < (rN)^{-1}$. Any such f with $Lip(f) \leq L$ satisfies $d(A_i, A_j) > r$ so that $\|A_i f A_j\| < \varepsilon$ for each term in the second summand, so that

$$(II) \leq N^2 \varepsilon.$$

For (I), the pairs (i,j) can be split up into 4 classes: (i odd, $j = i+1$), (i even, $j = i+1$) and the two symmetric cases. For each of these families, the sum is a block sum with orthogonal domain and range, hence the norm of the sum is less than the sup of the norm of the terms, so that

$$(I) \leq \frac{4}{N}.$$

Let us wrap all of this up: if ε is given, choose N such that $\frac{4}{N} < \varepsilon$, choose $L = L(N, \frac{\varepsilon}{N^2})$. This gives:

$$\begin{aligned}
\|[b,g]\| &\leq (I) + (II) \\
&\leq \frac{4}{N} + \frac{\varepsilon}{N^2} N^2 \\
&\leq 2\varepsilon.
\end{aligned}$$

1.5.5 Heart of the paper

Let us turn the attention to the most important result of the paper:

$$\text{If } X \text{ has FAD, then } (1) \implies (4).$$

Theorem 1.5.7. Let X be a bounded geometry uniformly discrete metric space. If X has finite asymptotic dimension, then

$$\forall \varepsilon > 0, \exists L > 0 \text{ s.t. } a \in Commut(L, \varepsilon) \implies a \in C_u^*(X)$$

and

$$a \in Commut(L, \varepsilon) \iff \|[a,f]\| < \varepsilon \quad \forall f \in l^\infty(X)_1, Lip(f) < L.$$

Review of asymptotic dimension

Recall that X has *asymptotic dimension* less than d if for every $r > 0$, there exists a bounded cover X which is (d, r)-separated. The typical example is the group $\mathbb{Z}$ with the metric induced by the absolute value, which has asymptotic dimension bounded by 1. As an exercise, prove that $asdim(\mathbb{Z}^n) \leq n$.

In the context of $asdim \leq 1$, conditional expectations into block subalgebras are very natural. Consider subsets $\{U_i\}$ of X which are pairwise disjoint and u_i the correponding multiplication operators. Define

$$\theta(a) = \sum_i u_i a u_i \quad \forall a \in B(l^2 X).$$

- $\theta(a)$ is SOT convergent;
- θ is lower continuous;
  Both of these follow essentially from

$$\begin{aligned} \|\theta(a)\xi\|^2 &= \sum_i \|u_i a u_i \xi\|^2 \quad \text{by orthogonality of the support,} \\ &\leq \|a\| \sum_i \|u_i \xi\|^2 \\ &\leq \|a\| \; \|\xi\|^2 \end{aligned}$$

  Take the directed systems of all the sums over finite subsets of I, in which case the sum is finite.

- θ is a conditional expectation. (Meaning it is CP, $\theta(xay) = x\theta(a)y$ when x, y are block diagonals wrt $\bigoplus_i l^2 U_i$, and $\theta(a)$ is block diagonal.)

Write $u = \sum_i u_i$.

Fact: if $prop(a) \leq r$ and $\mathcal{U}$ is $2r$-separated, then $uau = \theta(a)$.

Proof. If $\xi \in l^2 X$, $supp(u_i \xi) \subset U_i$, so that $supp(a u_i \xi) \subset N_r(U_i)$ which is disjoint from U_j, $j \neq i$. Hence the cross terms $u_j a u_i \xi$ vanish. We get for finite sums

$$\sum_{i,j \in F} u_i a u_j \xi = \sum_{i \in F} u_i a u_i \xi,$$

and the result follows by continuity. □

Consequence: If $b \in C^*_u(X)$, for every $\varepsilon > 0$, there exists $r > 0$ such that if $\mathcal{U}$ is r-separated, then

$$\|ubu - \theta(b)\| < \varepsilon.$$

This renders the next proposition natural.

Proposition 1.5.8. (Cor 4.3) If $a \in Commut(L, \varepsilon)$, with the notations above, if $\mathcal{U}$ is $\frac{2}{L}$-separated, then

$$\|uau - \theta(a)\| < \varepsilon.$$

Remark: if the theorem is true, then the above discussion shows that the result above must be true.

Proof. (of the theorem, assuming the above proposition) If $asdim(X) \leq 1$, fix a big $r > 0$: we get a bounded cover $\mathcal{Y}$ which is $(1, r)$-separated, meaning

$$\mathcal{Y} = \mathcal{U} \cup \mathcal{V}$$

with $\mathcal{U}$ and $\mathcal{V}$ r-separated families. Then, if $a \in B(l^2 X)$,

$$a = uau + uav + vau + vav.$$

we want to show that if $a \in Commut(L, \varepsilon)$ and $r > 4L^{-1}$, then each term on the right is near a finite propagation operator.

Claim: this is true for uau and vav.

This follows from the proposition: $\mathcal{U}$ is $r > 4L^{-1} > 2L^{-1}$-separated so that

$$\|uau - \theta(a)\| \leq \varepsilon$$

and $\theta(a)$ is block diagonal w.r.t. a bounded family, it is thus of finite propagation (less than $\sup_{\mathcal{U}} diam(U)$).

Claim: this is true for uav and vau.

Let $\mathcal{U}' = N_{L^{-1}}(\mathcal{U})$, same for $\mathcal{V}'$. Both are at least $2L^{-1}$-separated. We thus obtain $f = \sum f_i$ with f_i $[0,1]$-valued, with value 1 on U_i, 0 on $N_{L^{-1}}(U_i)^c$ and $Lip(f_i) \leq L$. Similarly for $\mathcal{V}$, we get $g = \sum_j g_j$. Then $uf = u$ and $vg = v$. Put

$$W_{ij} = N_{L^{-1}}(U_i) \cap N_{L^{-1}}(V_j) \quad , W = \coprod_{i,j} W_{ij}$$

which is at least $2L^{-1}$-separated. Similarly, build w and w_{ij}. we then calculate:

$$\begin{aligned} uav &= ufagv \\ &= ugafv + u[f,a]gv + u[a,g]fv \\ \text{So } \|uav - ugafv\| &\leq 2\varepsilon \end{aligned}$$

but now, $ugafv \in Commut(L, \varepsilon)$, so that the proposition applies using the $\{W_{ij}\}$ which are $2L^{-1}$-separated:

$$\|wugafvw - \theta_w(ugafv)\| \leq \varepsilon.$$

But $wugafvw = ugafv$ since $wugug$ and $fvw = fv$ ($supp(fv) \subset W$ and $supp(ug) \subset W$). And $\theta_w(ugafv)$ is block diagonal with finite propagation. □

It remains to prove the proposition.

Block diagonal symmetries

Lemma 1.5.9. If $a \in C_{L,\varepsilon}$ and $\mathcal{U}$ is a $\frac{2}{L}$-separated family of X then

$$\|[uau, \overline{u}]\| < \varepsilon$$

where $u = \sum u_i$ is our usual notation for the characteristic function of $\cup U_i$, and $\overline{u}$ is a *block diagonal symmetry*, i.e. an operator of the type $\sum \epsilon_i u_i$, $\epsilon_i \in \{-1, 1\}$.

Proof. Extend each u_i to a $[0,1]$-valued L-Lipschitz function, which is 1 on U_i and zero outside of $N_{L^{-1}}(U_i)$. Then the f_i have disjoint support so that

$$\overline{f} = \sum_i \epsilon_i f_i$$

satisfies $Lip(\overline{f}) \leq L$, $\|f\| \leq 1$ and $\overline{f}u = \overline{u}$. But

$$[uau, \overline{u}] = u[a, \overline{f}]u$$

which has norm smaller than ε. □

The block diagonal symmetries form a topological group (with the SOT topology), isomorphic to $\prod_{\mathcal{U}} \mathbb{Z}_2$ endowed with the product topology. It is thus a totally disconnected compact group, and has a unique Haar probability measure $d\overline{u}$.

Lemma 1.5.10. Let $Z \subset X$ and $b \in B(l^2 Z)$. If

$$\|[b, \overline{u}]\| < \varepsilon \quad \forall \overline{u} \text{ block diagonal symmetry}$$

then $\|b - E(b)\| < \varepsilon$, where $E : B(l^2 Z) \to \bigoplus l^\infty(U_i)$ is the canonical expectation onto the block diagonal. Furthermore,

$$E(b) = \int_G \overline{u} b \overline{u} \; d\overline{u}.$$

Remark: the example of the two point space is helpful to understand what is happening. Let say

$$\overline{u} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

then a simple calculation shows

$$\begin{aligned} \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} &= \frac{1}{4} \sum_{\overline{u} \in G} \overline{u} b \overline{u} \\ &= \frac{1}{4} \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} x & -y \\ -z & w \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} x & -y \\ -z & w \end{pmatrix} \right). \end{aligned}$$

The estimate is easy:

$$\|E(b) - b\| \leq \frac{1}{4} \sum \|b\overline{u} - \overline{u}b\| < \varepsilon.$$

Proof. First check that on the group G, the $*$-SOT, SOT and pointwise convergence coincide. Since the norms are all smaller than 1, we can consider finitely supported vectors, or even basis vectors.

Next the integral is understood in the weak sense, meaning that the assertion is

$$\langle E(b)\xi, \eta\rangle = \int_G \langle\ \overline{u}b\overline{u}\ \xi, \eta\rangle\ d\overline{u} \quad \forall \xi, \eta \in l^2X.$$

Ignoring existence, check the following matrix coefficients

$$\langle E(b)\delta_i, \delta_j\rangle = \begin{cases} b_{ii} & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

and

$$\begin{aligned} \int_G \langle \overline{u}b\overline{u}\ \delta_i, \delta_j\rangle\ d\overline{u} &= \int_G \langle b\overline{u}\ \delta_i, \overline{u}\ \delta_j\rangle d\overline{u} \\ &= (\int_G \epsilon_i\epsilon_j\ d\overline{u})\ b_{ij}. \end{aligned}$$

But

$$\int_G \epsilon_i\epsilon_j\ d\overline{u} = \mathbb{P}(\epsilon_i = \epsilon_j) - \mathbb{P}(\epsilon_i \neq \epsilon_j)$$

which is $\frac{1}{2} - \frac{1}{2} = 0$ if $i \neq j$, 1 otherwise. □

Finally let's put all the lemmas together to get the proposition.

Proof. Let $\mathcal{U}$ be a $\frac{2}{L}$-separated family and $a \in C_{L,\varepsilon}$.
The first lemma gives

$$\|[uau\ , \overline{u}]\| < \varepsilon \quad \forall \overline{u} \in G,$$

and now $uau \in B(l^2Z)$ for $Z = \cup U_i$, so by the second lemma,

$$\|uau - E(uau)\| < \varepsilon.$$

The canonical expectation $E(uau)$ is $\theta_{\mathcal{U}}(a)$, and this concludes the proof. □

1.5.6 Property (A)

The last part of the section is devoted to prove the assertion (quasi-locality implies locality) when X has property (A).

Property (A) and its friends

Motivation: let G be a countable discrete group with a bounded geometry left-invariant metric d. For each $A \subset X$ and every $r > 0$, define the r-corona of A to be the set

$$\partial_r A = \{x \in X \mid 0 < d(x, A) \leq R\}.$$

Here is a possible definition of amenability.

Definition 1.5.11. The group G is *amenable* if for all $r, \varepsilon > 0$, there exists a finite subset $A \subset X$ satisfying

$$|\partial_r A| < \varepsilon|A|.$$

Remark: we don't suppose the group to be finitely generated. For instance $G = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ with the metric $l(n) = \sum_i i|n_i|$ is not finitely generated, yet is of bounded geometry and amenable. If G is finitely generated, one does not need to quantify over r in the definition and can use ∂A instead.

This definition of amenability makes perfect sense for any bounded geometry metric space. However, it is a bit silly, since for any bounded geometry space X, the space $X \cup \mathbb{N}$ is amenable. Indeed, given $r > 0$, take $A_r = [r, r+N] \subset \mathbb{N}$. Then $\frac{|\partial_r A|}{|A|} = \frac{2r}{N+1}$ is very small for N large. This definition of amenabilty is thus local (" something nice happens somewhere") when we actually want to say something about the global structure of X.

Definition 1.5.12. The metric space X is *uniformly locally amenable*, abreviated $(ULA)_\mu$ after on, if for all $r, \varepsilon > 0$, there exists $s > 0$ such that for all probability measure $\mu \in Prob(X)$, there is a finite subset $A \subset X$ satisfying

$$diam(A) \leq s \quad \text{and} \quad \mu(\partial_r A) < \varepsilon \mu(A).$$

Remarks:

- The strict inequality is important, otherwise take $A = \emptyset$.
- The condition would be vacuous without the condition $diam(A) \leq s$, with s uniform on all probability measures. Otherwise just take the uniform probability on A for all A: the measure of the r-corona is 0.
- $(ULA)_\mu$ is equivalent to property (A), see [**?**].
- If G is a group, then if G is amenable, G is $(ULA)_\mu$. The proof is left as an exercise for the reader.

More definitions.

Definition 1.5.13 ([**?**])**.** The metric space X is *exact* if for all $r, \varepsilon > 0$, there exists $s > 0$ and a partition of unity $\{\phi_i\}_i$ on X such that

- if $d(x, y) < r$, then
$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \varepsilon,$$
- $diam(supp(\phi_i)) \leq s$ for every $i \in I$.

Definition 1.5.14 ([**?**])**.** The metric space X has the *metric sparsification property*, abreviated MSP after on, if for all $r, \varepsilon > 0$, there exists $s > 0$ such that for all $\mu \in Prob(X)$, there exists $\Omega \subset X$ such that

- $\mu(\Omega) \geq 1 - \varepsilon$,
- Ω is a r-disjoint union of s-bounded sets.

Theorem 1.5.15 (by everyone above)**.** Exact $\Longrightarrow_{(1)}$ $(ULA)_\mu$ $\Longrightarrow_{(2)}$ MSP $\Longrightarrow_{(3)}$ *Exact*.

The implication (3) is harder, see Sako [**?**]. The proof is C^*-algebraic: can we find a direct proof?

Proof. (1) Given $r, \varepsilon > 0$, let $\mu \in Prob(X)$, and $\{\phi_i\}$ be as in the definition with

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \frac{\varepsilon}{N_r}.$$

Hence for each fixed x,

$$\sum_{y:d(x,y)\leq r} \sum_{i\in I} |\phi_i(x) - \phi_i(y)| < \varepsilon = \varepsilon \sum_i \phi_i(x).$$

As μ is a probability measure,

$$\sum_x \mu(x) \sum_{y:d(x,y)\leq r} \sum_{i\in I} |\phi_i(x) - \phi_i(y)| < \varepsilon \sum_x \mu(x) \sum_i \phi_i(x).$$

hence there exists an index i_0 such that

$$\sum_x \mu(x) \sum_{y:d(x,y)\leq r} |\phi(x) - \phi(y)| < \varepsilon \sum_x \mu(x)\phi(x).$$

with $\phi = \phi_{i_0}$. now write $\phi = \sum a_i \chi_{F_i}$ where $a_i > 0$ and $F_{i+1} \subset F_i$. All the F_i's are in $supp(\phi)$ so their diameter is bounded above by s.

$$\sum_x \mu(x) \sum_{y:d(x,y)\leq r} |\sum_k a_k(\chi_{F_k}(x) - \chi_{F_k}(y))| < \varepsilon \sum_x \mu(x) \sum_k a_k \chi_{F_k}(x)$$
$$\sum_x \mu(x) \sum_{y:d(x,y)\leq r} \sum_k a_k |\chi_{F_k}(x) - \chi_{F_k}(y)| < \varepsilon \sum_x \mu(x) \sum_k a_k \chi_{F_k}(x).$$

Hence
$$\sum_x \mu(x) \sum_{y:d(x,y)\leq r} |\chi_{F_k}(x) - \chi_{F_k}(y)| < \varepsilon \sum_x \mu(x)\chi_{F_k}(x) = \varepsilon\mu(F_k)$$

for some $k = k_0$, and for $x \in \partial_r F_{k_0}$,

$$\sum_{y:d(x,y)\leq r} |\chi_{F_k}(x) - \chi_{F_k}(y)| \leq 1 \leq \sum_{x\in\partial_r F_{k_0}} \mu(x) = \mu(\partial_r F_{k_0}).$$

Set $A = F_{k_0}$, then
$$\mu(\partial_r F_{k_0}) < \varepsilon\mu(A).$$

□

Quasi-locality and property (A)

The main goal of this section is to provide a proof of (1) $\implies$ (4) in the case where X has property (A). Let us fix some notations.

For (X, d) a metric space, a partition of unity will be given by a pair $(\phi, \mathcal{U})$ where $\mathcal{U}$ is a cover of X and ϕ is a map

$$\phi : X \to l^2(\mathcal{U})_{1,+} \quad ,$$

such that $x \mapsto \phi_U(x)$ is supported in U for every $U \in \mathcal{U}$. (The notation $l^2(\mathcal{U})_{1,+}$ means positive elements of norm 1.) If $\mathcal{U} = \{U_i\}_{i \in I}$, we will identify $l^2(\mathcal{U})$ with $l^2(I)$.

The characterization of property (A) which we use is the following, obtained by Dadarlat and Guentner in [**?**].

Theorem 1.5.16. A metric space X is called *exact* if, for every $r, \varepsilon > 0$, there exists a partition of unity $\phi : X \to l^2(\mathcal{U})$ such that $\mathcal{U}$ is uniformly bounded with finite multiplicity and

$$d(x, y) \leq r \implies \|\phi(x) - \phi(y)\|_2 \leq \varepsilon.$$

If X is discrete and of bounded geometry, exactness and property (A) are equivalent.

We will also need to know that property (A) implies the metric sparsification property, which was proven in the last section.

The key idea of the proof relies on an approximation property of quasi-local operators: their norm can be approximated by finitely supported vectors. This means that if $b \in \bigcap_\varepsilon \bigcup_L C_{L,\varepsilon}$,

$$\|b\| = \sup_{\|v\|=1 \ , \ diam(supp(v))<\infty} \|bv\|.$$

This relies on the following lemma.

Lemma 1.5.17 ([**?**], lemma 5.2)**.** For every M, L, ε, there exists $s > 0$ such that, for every $b \in C_{L,\varepsilon}$ with $\|b\| \leq M$, there exists $v \in l^2(X)$ satisfying $\|v\| = 1$, $diam(supp(v)) < s$ and

$$\|bv\| \geq \|b\| - \varepsilon.$$

Proof. (of the result, using the lemma) Let X discrete with bounded geometry and property (A), and say $b \in B(l^2X)$ is quasi-local and fix $\varepsilon > 0$. Then there is $L > 0$ such that $b \in C_{L,\varepsilon}$ and, by the lemma, a $s > 0$ such that $\|T\|$ can be approximated up to ε by s-supported vectors for every $T \in C_{2\varepsilon,L}$ with $\|T\| \leq M$.

Choose a partition of unity ϕ with uniformly bounded support and

$$d(x, y) \leq s + \frac{1}{L} \implies \|\phi(x) - \phi(y)\| \leq \varepsilon.$$

Let us show that the norm of

$$a = b - \sum_i \phi_i b \phi_i = \sum_i \phi_i [\phi_i, b]$$

is small enough.

The following computation shows that $a \in C_{2\varepsilon,L}$:

$$\begin{aligned}
\|[a,f]\| &= \|[\sum_i \phi_i[\phi_i,b],f]\| \\
&\leq \|\sum_i \phi_i[b,f]\phi_i\| + \|[b,f]\| \\
&\leq 2\varepsilon
\end{aligned}$$

where we used $\|\sum_i \phi_i[b,f]\phi_i - \phi[b,f]\phi\| < \varepsilon$. This follows from the fact that, if e_j are positive contractions with $\frac{2}{L}$-separated support, and $T \in C_{\varepsilon,L}$, then $\|eTe - \sum_i e_iTe_i\| < \varepsilon$. This is not a trivial statement, and was proven in the last section (Cor 5.3 of [?]).

Of course, $\|a\| \leq 2M$, so we can apply the statement of the first paragraph to a: there exists a unit vector $v \in l^2X$ with support F satisfying $diam(F) < s$ and $\|av\| \geq \|a\| - \varepsilon$, and

$$\begin{aligned}
|\sum_i \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}| &\leq M(\sum_i \phi_i^2(x))^{\frac{1}{2}} \ (\sum_i |\phi_i(x) - \phi_i(y)|^2)^{\frac{1}{2}} \\
&\leq M\|\phi(x) - \phi(y)\|_2
\end{aligned}$$

so that if $x \in N_{L^{-1}}(F)$, $\|\phi(x) - \phi(y)\|_2 \leq \varepsilon$, and

$$\begin{aligned}
|av|_x &= |\sum_{i,y\in F} \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}v_y| \\
&\leq \sum_{y\in F} |\sum_i \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}| \ |v_y| \\
&\leq \varepsilon M \sum_{y\in F} |v_y| \\
&\leq \varepsilon M N_s^{\frac{1}{2}} \ \|v\|.
\end{aligned}$$

Now $\|av\|^2 = \sum_x |av|_x^2 \leq M^2N_s^2\|v\|^2\varepsilon^2 + \sum_{x\in N_{L-1}} |av|_x^2$, but a being in $C_{2\varepsilon,L}$,

$$\|\chi_F a \chi_{N_{L-1}(F)}\| < \varepsilon$$

hence $\|av\|^2 \leq (M^2N_s^2 + 1)^{\frac{1}{2}}\|v\|\varepsilon$. □

It remains to prove the lemma.

Proof. Let $b \in C_{\varepsilon,L}$ and $M = \|b\|$. Let $v \in l^2(X)$ be a unit vector such that $\|bv\| \leq \|b\| - \frac{\varepsilon}{2M}$ (so that $\|bw\| \geq \|bv\| - \varepsilon$). Denote by μ the probablity measure on X defined by

$$\mu(\{x\}) = |v_x|^2.$$

The MSP implies that there is a subset $\Omega \subset X$ with $\mu(\Omega^c) < \varepsilon$ and Ω is a $\frac{4}{L}$-separated disjoint union

$$\Omega = \coprod_{\frac{4}{L}} \Omega_i$$

of uniformly bounded subsets, i.e. $diam(\Omega_i) < s$ for all i. Denote by $w_i = P_{\Omega_i}v$, and $w = \sum_i w_i$. Then the condition above says that $\|v - w\|^2 < \varepsilon$ and $diam(supp(w_i)) < s$

so if we could approximate $\|b\|$ using one of the w_i's, that would end the proof.

There exists $f_i \in l^\infty(X)_1$ such that

- $Lip(f_i) \leq L$,
- $supp(f_i) \subset N_{L^{-1}}(\Omega_i)$,
- $f_i = 1$ on Ω_i and 0 outside of $N_{L^{-1}}(\Omega_i)$.

Then $f = \sum_i f_i$ and $1 - f$ are also L-lipschitz functions and $fw = w$. But $bw = [b, f]w + fbw$ so

$$\begin{aligned}\|bw\| &\leq \varepsilon\|w\| + \|fbfw\|\\ &\leq 2\varepsilon\|w\| + \|\sum_i f_i b f_i w\|\end{aligned}$$

In the last line, we used that $\|fbf - \sum f_i b f_i\| \leq \varepsilon$:

Now, the same trick $f_i b = [f_i, b] + bf_i$ entails that

$$\begin{aligned}\|\sum_i f_i b f_i w\|^2 &= \sum_i \|f_i b w\|^2\\ &\leq \varepsilon \sum_i \|w_i\|^2 + \sum_i \|bw_i\|^2\\ &\leq \varepsilon\|w\|^2 + \sum_i \|bw_i\|^2\end{aligned}$$

so that

$$(\|bw\| - 3\varepsilon\|w\|)^2 \leq \sum_i \|bw_i\|^2 \leq \sum_i \frac{\|bw_i\|^2}{\|w_i\|^2}\|w_i\|^2 \leq \sup_i(\frac{\|bw_i\|^2}{\|w_i\|^2})\|w\|^2$$

from which follows that

$$\frac{\|bw\|}{\|w\|} \leq \sup_i \frac{\|bw_i\|}{\|w_i\|} + 3\varepsilon.$$

and

$$\sup_i \frac{\|bw_i\|}{\|w_i\|} + 3\varepsilon \geq \frac{\|bv\| - \varepsilon\|v\|}{\|w\|} \geq \|bv\| - \varepsilon \geq \|b\| - 2\varepsilon$$

so that there exists i_0 such that $\frac{\|bw_{i_0}\|}{\|w_{i_0}\|} \geq 6\varepsilon$, and $diam(supp(w_{i_0})) < s$. □

1.6 Haagerup

1.6.1 An example of a nonuclear C^*-algebra which has the MAP

In his 1979 paper [], Haagerup showed that $C_r^*(\mathbb{F}_n)$ has the Metric Approximation property, this settling that CPAP and MAP are not equivalent. This paper is considered a landmark of the field, for several reasons. It contains several ideas that would bloom as properties of their own: Haagerup's property, Rapid Decay, etc.

The following statement can be extracted from the paper.

Theorem 1.6.1. Let G be a discrete group which has length l of conditionally negative type. If G has $(RD)_l$, then $C_r^*(G)$ has the MAP.

Actually what is proven is that if G has $(RD)_l$ w.r.t. a (CNT) length, then there exists a net of contrative bounded linear maps

$$T_\alpha : C_r^*(G) \to C_r^*(G)$$

which converges poinwise to the identity, i.e.

$$\lim_\alpha \|T_\alpha(x) - x\| = 0 \quad \forall x \in C_r^*(G).$$

You need to know how to build multipliers on the group C^*-algebra.

Lemma 1.6.2. We say that $\phi : G \to \mathbb{C}$ is a *multiplier* if there exists a unique bounded linear map $M_\phi : C_r^*(G) \to C_r^*(G)$ such that

$$M_\phi(\lambda_s) = \phi(s)\lambda_s \quad \forall s \in G.$$

If either one of these conditions is satisfied, then it is the case:

- ϕ is completely positive, and then $\|M_\phi\| = \phi(e)$;
- G has (RD) with constants C, α, and $K = \sup_g |\phi(g)|(1+|g|)^\alpha < \infty$. Then $\|M_\phi\| \leq CK$.

In particular, finitely supported functions induce multipliers.

Let's prove the statement.

Proof. Since free groups are metric trees, the length function is of (CNT) and, by Schonberg's theroem,

$$\phi_t(g) = e^{-t|g|} \quad \forall t > 0$$

are of (PT).

Define their restriction to the ball of radius $n > 0$ and the error

$$\phi_{n,t}(g) = e^{-t|g|}1_{|g|\leq n} \text{ and } r_t(g) = e^{-t|g|}$$

Both satisfy the second point of the lemma, $\phi_{n,t}$ is finitely supported, and $e_{t,n}$ has

$$K_n = \sup \sup_{|g|>n} e^{-t|g|}(1 + |g|)^\alpha \to 0.$$

From this, it is standard to prove that up to renormalization, one can extract from the net $M_{\phi_{t,n}}$, which are of finite rank, a net which gives the MAP. $\square$

The proof that $\mathbb{F}_n$ has (RD) is based on the estimate

$$\|\lambda(a)\| \leq (k+1)\|a\|_2$$

if $a \in \mathbb{C}_{S_k}[G]$, then by Cauchy-Schwartz

$$\begin{aligned}
\|\lambda(a)\| &\leq \sum (k+1)\|a_k\|_2 \\
&\leq (\sum \frac{1}{(1+k)^2})^{\frac{1}{2}} \cdot (\sum_k (1+k)^2 \|a_k\|_2^2)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{\pi}{6}} \cdot \|a\|_{l,1}
\end{aligned}$$

Erik told me that a better proof exists in the paper of Ozawa, Weak amneability of hyperbolic groups, based on an idea of Bozejko and Picardello, in Weakly amenable groups and amalgamated products.

1.7 Right-angled Artin groups

1.7.1 Subgroups of RAAGS generated by 2 elements

Let us start this section by presenting some classical notions from geometric group theory.

Definition 1.7.1. A group G is called residually finite if all its finite index normal subgroups intersect trivially, i.e.

$$\bigcap_{HG \ [G:H]<\infty} H = 1.$$

RF groups are in a sense well approximated by their quotients. Indeed, for any non trivial element $g \in G$, there is a normal finite index subgroup which does not contain g, so that being RF is equivalent to the property: for every $g \neq e$, there exists an epimorphism onto a finite group $\Psi : G \to Q$ such that $\Psi(g) \neq e$.

Examples of RF groups include finite groups, $\mathbb{Z}^n$, $\mathbb{F}_n$, $SL(n, \mathbb{Z})$, etc.

Definition 1.7.2. A group G is called Hofpian if any epimorphism from G to itself is in fact an isomorphsim.

Hopfian groups are groups which cannot be isomorphic to any of their proper quotients. Examples include virtually polycyclic groups, torsion free word hyperbolic groups, and all our previous examples of RF groups using the following result.

Proposition 1.7.3. A finitely generated RF group is Hopfian.

Proof. Let $\phi : G \to G$ be an epimorphism, and let $g \in Ker\ \phi$. Suppose $g \neq e$. Firstly, notice that ϕ^m is still surjective, and by residual finiteness pick a finite group Q and an epimorphsim $\Psi : G \to Q$ such that $\Psi(g) \neq e$. Let $m > n$, and let $h \in G$ such that $\phi^n(h) = g$. Then

$$\Psi \circ \phi^m(h) = \Psi \circ \phi^{m-n}(g) = e \neq \Psi(g) = \Psi \circ \phi^n(h),$$

hence $\Psi \circ \phi^m \neq \Psi \circ \phi^n$ for every $n \neq m$. But this is impossible since $Hom(G, Q)$ is a finite set: G is finitely generated so that a morphism is entirely determined by a finite set of choice for the images of the generators.

□

Proposition 1.7.4. Finitely generated free groups are residually-p, for every prime p.

The last part of the section is devoted to prove this result of Baudisch.

Theorem 1.7.5. If $G = \langle x, y \rangle$ is a 2-generated subgroup of a RAAG A_Γ, then either G is free or it is abelian.

The following lemma contains the key idea for the proof.

Lemma 1.7.6. Let X be a finite regular cover of the Salvetti complex S_Γ and T, T' be two connected preimages of hyperplanes of S_Γ in X. Suppose $T \cup T'$ does not separate X and $T \cap T' = \emptyset$. Then the oriented intersection morphism defines an epimorpshim $G \to \mathbb{F}_2$.

1.8 Noncommutative geometry

1.8.1 Basic objects and constructions

Mainly, I'm interested in $*$-algebras A (and their completions) which are k-algebras equipped with an involution $*$. Usually, $k = \mathbb{C}$ is the field of complex numbers. A very famous example of $*$-algebra is the algebra of the quantum harmonic oscillator,

$$\mathcal{H} = k\langle x, y\rangle/(xy - yx = 1).$$

When $k = \mathbb{C}$, one often represent A as a sub-$*$-algebra of the bounded operators on aHilbert space $\mathcal{L}(H)$,and complete w.r.t. to the norm. Note that not all complex $*$-algebras admit such a representation.

For instance, for$\mathcal{H}$, one easily get that

$$[x, P(y)] = P'(y) \quad \forall P \in \mathbb{C}[t]$$

Then if $|| \ ||$ is a multiplicative norm on$\mathcal{H}$, it satisfies

$$2||x|| \ ||y|| \geq n \quad \forall n > 0.$$

Basic construction:

- separation-completion: in our sense, a norm can be degenerate. Being multiplicative, the annhiliator of any norm is a closed ideal in A, so that there is an induced (classical/ nondegenerate) norm on the quotient algebra. The separation-completion is defined to be the completion of the quotient w.r.t. the induced norm. Let us say that if α is such a norm, wedenote by A_α the associated separation-completion. Any inequality

  $$\alpha(x) \leq \beta(x) \quad \forall x \in A$$

  induces an inclusion of annilihator $N_\beta \subset N_\alpha$, and gives a canonical quotient map

  $$A_\beta \to A_\alpha.$$

  The basic class of examples comes from completion of the complex group ring $\mathbb{C}[\Gamma]$. For any family of unitary representations $\mathcal{F}$, one can define the $*$-norm

  $$||x||_{\mathcal{F}} = \sup\{||\pi(x)|| : \pi \in \mathcal{F}\}$$

  on $\mathbb{C}[\Gamma]$. The separation-completion is a C^*-algebra denoted $C^*_{\mathcal{F}}(\Gamma)$. For instance, if $\mathcal{F}$ consists of all unitary representations of Γ, then one gets the maximal C^*-algebra $C^*_{max}(\Gamma)$, while if the family is reduced to the left regular representation λ_Γ, one gets the reduced C^*-algebra $C^*_r(\Gamma)$. By inclusion, one gets the canonical quotient map

  $$\lambda_\Gamma : C^*_{max}(\Gamma) \to C^*_r(\Gamma).$$

Crossed-product: the basic ingredients are a $*$-algebra H endowed with a coassociative coproduct

$$\Delta : H \to H \otimes H,$$

and a C^*-algebra A on which H acts via a $*$-homomorphism

$$\alpha : A \to A \otimes H$$

such that $(1 \otimes \Delta)\alpha = (\alpha \otimes 1)\alpha$. The crossed-product is a twisted version of the tensor product.

$$(a \otimes x)(a' \otimes y) := (a \otimes 1_{M(H)})\alpha(a')(1_{M(A)} \otimes xy)$$

1.8.2 Quantum groups

A C^*-bialgebra is a pair (H, Δ) where H is a C^*-algebra and

$$\Delta : H \to M(\tilde{H} \otimes_{min} H + H \otimes_{min} \tilde{H}, H \otimes_{min} H)$$

is a non-degenerate $*$-homomorphism such that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

A H-algebra is a pair (A, α) where A is a C^*-algebra and

$$\alpha : A \to M(\tilde{A} \otimes_{min} H, A \otimes_{min} H)$$

such that $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha)$. Its principal map is

$$\Psi : \begin{cases} A \otimes_{alg} A & \to & M(A \otimes_{min} H) \\ x \otimes y & \mapsto & (x \otimes 1_{M(H)})\alpha(y) \end{cases}$$

Let (H, Δ) be a C^*-bialgebra and (A, α) a H-algebra, with principal map

$$\Psi : A \otimes A \to M(A \otimes_{min} H).$$

- free if the range of Ψ is strictly dense in $M(A \otimes_{min} H)$
- proper if the range of Ψ is contained in $A \otimes_{min} H$
- principal if $\Psi(A \otimes_{alg} A)$ is a norm dense subset of $A \otimes_{min} H$

principal = free and proper

1.8.3 Why $SU_q(2)$?

Apparently, some people are interested in deformation of classical Lie groups such as $SU_q(2)$, which is the Hopf algebra generated by 3 generators E, F, K satisfying the relations

$$R.$$

I wanted to understand where these relations are coming from, which led me to interesting ideas developed by several people, including Yuri Manin. The idea is to define $SU_q(2)$ as a special group like object of the automorphism group of some noncommutative space, the quantum plane.

Let k be a field. The free (noncommutative) k-algebra on n generators is denoted by $k\langle x_1, ..., x_n\rangle$.

Definition 1.8.1. A quadratic algebra

$$A = \oplus_{i \geq 0} A_i$$

is a $\mathbb{N}$-graded finitely generated algebra such that:

- $A_0 = k$, and A_1 generates A,
- the relations on generators are in $A_1 \otimes A_1$.

The quadratic algebra A is said to be a Frobenius algebra of dimension d if moreover

- $A_d = k$ and $A_i = 0$ for all $i > d$,
- the multiplication map
$$m : A_i \otimes A_{d-i} \to A_d$$
is a perfect duality.

The main example is the quantum plane

$$\mathbb{A}_q^2 = k\langle x, y\rangle/(xy - qyx)$$

where $q \in k^\times$. More generally, the quantum space of dimension $n|m$ is

$$\mathbb{A}_q^{n|m} = k\langle x_1, .., x_n, \eta_1, ..., \eta_m\rangle/(x_i x_j - q x_j x_i, q\eta_i\eta_j + \eta_j\eta_i).$$

This example is suppose to come from physics. In quantum field theories, physicists deal with two kind of particles, bosons and fermions, and use commuting variables for one type, and anticommuting for the other. One object they appeal to are called supermanifolds, which are manifolds enriched with anticommuting variables. Formally, it means they look at ringed spaces $(X, \mathcal{O})$ locally isomorphic to $(\mathbb{R}^n, C^\infty[\eta_1, ..., \eta_m])$, where $C^\infty[\eta_1, ..., \eta_m]$ is the free sheaf of rings generated by anticommuting variables η_i over the smooth complex valued functions $C^\infty(\mathbb{R}^n)$.

Remark that a quadratic algebra A is a quotient of $k\langle x_1, ..., x_n\rangle$ by elements $r_\alpha \in A_1 \otimes A_1$, which we will denote as

$$A = k\langle x_1, ..., x_n\rangle/(r_\alpha)$$

or

$$A = \langle A_1, R_A\rangle$$

with $R_A \subseteq A_1 \otimes A_1$.

Manin defines the quantum dual of a quadratic algebra as

$$A^! = k\langle x^i\rangle/(r^\beta)$$

where $r_{ij}^\beta r_\alpha^{ij} = 0$, i.e. $R_{A^!} = R_A^\perp$. Then, the quantum endormorphisms between two quadratic algebra is

$$Hom(A, B) = k\langle z_i^j\rangle/(r_\alpha^\beta)$$

where $r_\alpha^\beta = r_\alpha^{ij} r_{kl}^\beta z_i^k z_j^l$. If $End(A) = Hom(A, A)$, then $End(A)$ satisfies the universal property to be intial in the category of k-algebras (B, β) endowed with an algebra homomorphism $\beta : A \to A \otimes B$.

If one does that to the quantum plane $\mathbb{A}_q^2$, one stil doesn't find quite $M_q(2)$: half of the relations are missing. Also

$$(\mathbb{A}_q^{2|0})^! = \mathbb{A}_q^{0|2}?$$

Exercise.

1.8.4 TQFT

Motivations

This section is aimed at being an introduction to *Topological Field Theories*. One of the difficulties of this particular topic is that it comes from different areas and can be attacked in different ways. The following is my attempt to make sense out of the large amount of information available on the subject. In particular, I do not claim exhaustivity or expertise.

The starting point are probably *path integral* formulations in Physics. In Statistical Mechanics and in Quantum Physics, the values predicted by the theory can often be written as expectation of the type

$$\mathbb{E}[\exp(-\int V(q(t))) \quad \text{or} \quad \mathbb{E}[\exp \frac{i}{h} S(q)].]$$

Physically, one tries to define a positive function on the phase space M (such as an energy (Hamiltonian) H or an action S, the integral of a Lagrangian). The probability distribution of the system should then be

$$\frac{1}{Z_M} e^{-\beta H(\omega)} D\omega \quad \text{or} \quad \frac{1}{Z_M} e^{\frac{i}{h} S(\omega)} D\omega.$$

The first case is the one known as Gibbs measures, and describes the behaviour of a system in contact with a thermostat at inverse temperature β. The second case is the so called Feynman integral of quantum mechanics. The reason these formulas are used is that systems should satisfy some *minimization principle*. In the classical case, the observed trajectories are the minima of the energy function, whereas in the quantum case the observe deviations from the classical trajectories up to the amplitude $e^{iS/h}$.

That point is exactly where it starts to be complicated. Physicists want to define something propoptional to these exponential, and the measure $D\omega$ is supposed to be a reference measure with nice invariance properties. In the finite dimensional case, the natural measure would be the Lebesgue measure. But no such thing exists on a general functional space, which makes the definition above useless.

It turns out that these integral have very interesting invariance properties, notably in topology. More precisely, the *partition function* Z_M gives topological invariant when M is a closed manifolds. This fact gave motivation to mathematicians to study more attentively these functional integrals. While you can try to define integrals analytically (see **REFERENCES**), there exists an algebraic approach which proposes intuitively to define the partition function on simple manifolds (possibly with border) and in coherent manner so that the partition function of a closed manifold can be computed by cutting it in simple pieces, computing the corresponding values, and reassembling these to get the final result. Mathematically, this requires:

- describing an algebraic structure on the family of *n-dimensional manifolds*, and giving generators;
- setting up the value of the partition function;

- showing that all of this makes sense.

This is accomplished by considering the n-category of bordisms and defining the (algebraic) partition function Z to be a nice functor between the latter and the n-category of finite dimensional vector spaces over some fields. In dimension 1 and 2, it will be understood that only one value needs to be fixed (respectively on the point space and on the circle), while this still holds in higher dimension under more hypothesis.

Summary of the talks

We recalled the definitions of a monoidal category, a braided category, and a symmetric monoidal category. The two main examples are the category of bordisms $Bord^d$ in dimension d, and the category of vector spaces over a field k. The first talk focused on topological quantum fields theories in dimension 1 and 2.

Definition 1.8.2. A TQFT in dimension d is a monoidal symmetric functor

$$Z : Bord_d \to Vect_k.$$

The two main results we showed are:

- there is an equivalence of categories
$$TQFT_1 \cong Vect_k$$
obtained as $Z \mapsto Z(pt)$.
- there is an equivalence of categories
$$TQFT_2 \cong Frob_k$$
obtained as $Z \mapsto Z(\mathbb{S}^1)$.

A nice example in dimension 2: $Z(\mathbb{S}^1) = \mathbb{C}[t]/(t^2 - 1)$ is the Frobenius algebra given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad \epsilon(1) = 0 \quad \epsilon(t) = 1.$$

Then the handle element is $h = 2t$ and

$$Z(\Sigma_g) = \begin{cases} 2^g & \text{if } g \text{ is odd} \\ 0 & \text{if } g \text{ is even.} \end{cases}$$

The second talk was directed towards extended field theories. First recall some higher category theory: n-categories, etc... And an extented TFT is a symmetric monoidal functor between symmetric monoidal n-categories

$$Z : Cob_n \to \mathcal{C}.$$

Then the following theorem was proved in [**?**].

Theorem 1.8.3. The evaluation functor

$$Z \mapsto Z(*)$$

establishes a bijective correspondance between extended n-dimensional TFT and fully dualizable objects of $\mathcal{C}$.

We now give an application of this result to the Jones polynomial. In [?], Witten gives an interpretation of the Jones polynomial, an isotopy invariant of links, as induced from a 3-dimensional TFT. The drawback of this article (for us) is that Witten uses Physical TFT's, i.e. gauge theories. The Jones polynmial is then shown to be the value of the partition function of a gauge field theory on $\mathbb{S}^3$ with gauge group $SU(2)$. I propose to rewrite this result in our setting as an exercise.

A link is a disjoint union of embedding of the circle into $\mathbb{S}^3$

$$\mathcal{L} = \{\text{embeddings } \coprod_{i=1}^{k} \mathbb{S}^1 \hookrightarrow \mathbb{S}^3\}.$$

we will often make no distinction between the embedding and its image in the 3-sphere, which we will denote by L. The Jones polynomial of a link L is defined as an isotopy invariant polynomial $V : \mathcal{L} \to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ satisfying the Skein relations

$$-t^{\frac{1}{2}} V_+ + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_0 + t^{-\frac{1}{2}} V_- = 0.$$

To a link L one can associated the 3-manifold $M_L = \mathbb{S}^3 - L$. Consider the extended TFT

$$Z^{(n)} : Cob_3 \to \mathcal{C}$$

given by $Z() = V_n$ where is the fundamental representation of $\mathfrak{su}(n)$. By the cobordism theorem, it is enough to define the TFT on all of Cob_3. Then

$$\phi(V_L) = Z^{(2)}(M_L),$$

where $\phi : \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \to \mathbb{C}$ is the evaluation at a root of unity $q \in \mathbb{C}^\times$. This can be proved by showing that $Z^{(n)}(M_L)$ satisfies the skein relation

$$-q^{\frac{n}{2}} V_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) V_0 + q^{-\frac{n}{2}} V_- = 0$$

1.8.5 Reminder

A locally ringed space is a topological space X together with a sheaf or ring $\mathcal{O}_X$ over X such that all stalks are local rings, ie have a unique maximal ideal.

For R a ring, $X = Spec(R)$ denotes the topological space obtained as the set of prime ideals of R endowed with the Zariski topology, i.e. the topology generated by the closed subsets

$$V_I = \{J \text{ ideals in } R \text{ s.t. } I \subset J\}.$$

Equivalently, a basis of open subsets is given by

$$D_f = \{J \text{ ideals in } R \text{ s.t. } f \notin J\}$$

for every $f \in R$. Let S_f be the multiplicative domain given by the powers of f. Then define a sheaf of ring over X by

$$\mathcal{O}_X(D_f) = S_f^{-1}R.$$

It is called the structural sheaf of $Spec(R)$. Any locally ringed space isomorphic to

$$(Spec(R), \mathcal{O}_{Spec(R)})$$

with R commutative is called an affine variety.

Note: the functor *Spec* gives an antiequivalence of categories between the categories of commutative rings and the category of affine varieties.

Definition 1.8.4. A scheme is a locally ringed space locally isomorphic to an affine variety.

Chapter 2

Zoology of groups and C^*-algebras, and other wild creatures

2.1 A list of books

A list of books I like about general knowledge in science:

- L'aventure des nombres, Godefroy
- L'autobigraphie de Paul Levy, Laurent Schwartz, et Yuri Manin.
- Recoltes et semailles, Grothendieck.
- Lee Smolin, The trouble with physics, the rise of String theory, the fall of a Science, and what comes next,
- Julian Barbour, The End of Time, The next revolution in Physics,
- Carlo Rovelli, Et si le temps n'existait pas, un peu de science subversive,
- Mandlebrot, The (Mis)Behaviour of markets, Fractals and Chaos, the Mandelbrot set and beyond, The fractal geometry of nature.
- Manjit Kumar:
- Amir Alexander, Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World
- Ian Stewart, Does God play dice?
- History of Statistics, Stielger
- Logicomix

Overview and more specialized books:

- Moonshine beyond the Monster, Terry Gannon
- Le theoreme d'uniformisation, Saint-Gervais
- Invitation aux mathematiques de Fermat, Hellgouarch
- Rached Mneime, tous ses livres!
- Hubbard West pour les equa diff
- Noether's theorem, Yvette K
- Nother's wonderful theorem
- The annus mirabellus of Einstein
- The Road to Reality, Sir Roger Penrose

Books about Einstein:

- Subtle is the Lord, Abraham Pais [**?**]; biography of Einstein by someone who knew him;

- Einstein's miraculous year: Five papers that changed the face of physics, Penrose & Einstein [?]; English translations of the five papers Einstein published in 1905 while working at the patent office in Bern.
- Quantum: Einstein, Bohr, and the great debate about the nature of reality, Kumar [?]; history of quantum theory from Planck's blackbody radiation to the EPR paradox.

2.2 Groups

- Amenable, a-T-menable, property T, with a diagram
- Mapping class groups
- Profinite groups, locally profinite groups, $Aut(\overline{\mathbb{Q}}/\mathbb{Q})$
- Automorphism of a regular tree, the Grigorchuk group,
- Lamplighter groups, usually

$$\mathbb{Z}_2 \wr \mathbb{Z} = \oplus \mathbb{Z}_2 \rtimes \mathbb{Z}.$$

More generally, wreath products $H \wr \Gamma = H^\Gamma \rtimes \Gamma$.

- $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ generate a free group of finite index in $SL(2, \mathbb{Z})$. The corresponding semi-direct product $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{F}_2$ does not have Haagerup's property ($(\Gamma, \mathbb{Z}^2)$ has relative property (T)).

- Cayley graphs: finite groups, symmetric groups, $\mathbb{Z}^2$, $\mathbb{F}_2$, $\mathbb{Z}$ with original generating sets. $B(1, 2)$. Lamplighter groups: meta-abelian without finite presentation.

$SL(2, \mathbb{Z})$ has presentation

$$< x, y \mid x^4 = 1, x^2 = y^3 > \quad p, q \geq 1,$$

and in this presentation, the quotient by $< x^2 >$ is isomorphic to

$$PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

This gives a way to draw their Cayley graph easily.

- Baumslag-Solitar monster

$$BS_{p,q} =< a, b \mid ab^p a^{-1} = b^q >$$

are nonhopfian when p and q are coprime and at least 2. $BS_{p,q}$ is the Higman-Neumann-Neumann extension $HNN(\mathbb{Z}, p\mathbb{Z}, \theta)$ where $\theta(p) = q$:

$$BS_{p,q} < Aut(T_{p+q}) \quad \text{where } T_{p+q} \text{ homogeneous tree of degree } p + q.$$

On the other hand, one has a non injective morphism $BS_{p,q} \to Aff(\mathbb{R}); a \mapsto \frac{qx}{p}; b \mapsto x+1$. The diagonal morphism $BS_{p,pq} \to Aut(T_{p+q}) \times Aff(\mathbb{R})$ has discrete image and $BS_{p,q}$ has Haagerup's property (because both $Aut(T_{p+q})$ and $Aff(\mathbb{R})$ have it.

- Infinite torsion questions: subgroups of $GL(n, \mathbb{Z}[\frac{1}{p}])$.

- Tarski monsters: p a prime, then every x generates a cyclic subgroup of order p, and the set of x together with any element not contained in this cyclic subgroup subgroup generates Γ.

Tessera and Arhantseva showed that there exists a group which is a split extension of two groups that are coarsely embeddable into Hilberrt space, and that does not admit such an embedding.

- **Amenability** Abelian, Compact, extension of such (Elementary amenable), Grigorchuk group: amenable but not elementary amenable (first example of finitely generated group with intermediate growth, i.e. faster than polynomial but subexponential). Every group with subexponential growth (equivalent to virtually nilpotent by Gromov's Polynomial growth theorem). When discrete, Γ is amenable iff $C_r^*(\Gamma)$ is nuclear.

  In terms of CP functions? Γ is amenable iff there exists a net of commpactly supported continuous positive definite functions converging pointwise to 1.

- **Haagerup's property** Introduced by Haagerup on his work on the Free groups. Incidentatly, $\mathbb{F}_2$ is not amenable but has Haagerup's property. Stability by closed subgroups so $\mathbb{F}_n$ and the free group with countably many generators. Equivalent to Gromov's a-T-menability and property FH in the locally compact case. Every such group satisfies the Baum-Connes conjecture with coefficients, and is K-amenable, i.e.
$$\lambda \in KK_0(C_{max}(\Gamma), C_r^*(\Gamma))$$
  is invertible. Same for $SL(2,\mathbb{Z})$. Amenable groups, Coxeter groups, Groups acting metrically properly on trees or spaces with walls. $SU(n,1)$ and $SO(n,1)$: $g \mapsto d(gx_0, x_0)$ is conditionally negative and definite, where d is the hyperbolic distance and x_0 any point in real or complex projective space. Baumslag-Solitar 's groups $BS_{p,q}$.

  In terms of CP functions? G has Haagerup's property iff there exists a continuous proper conditionally negative definite function $G \to \mathbb{R}_+$, iff there exists a sequence of continuous normalized positive definite functions converging uniformly on compact subsets of G.

- **Property T** Any compact group. $SL(n,\mathbb{Z})$ for $n \geq 3$. Simple real Lie groups with real rank ≥ 2 and their lattices: $SL(n,\mathbb{R})$, $n \geq 3$; $SO(p,q)$, $p > q \geq 2$; $SO(p,p)$,

$p \geq 3$. Simple algebraic groups of rank ≥ 2 over a local field. $Sp(n,1)$, $n \geq 2$ which is a simple real Lie group of real rank 1, and its lattices, which are dicrete countable hyperbolic groups. $Aut(\mathbb{F}_5)$. Mapping class groups are supposed to have property (T), but the proof is still not clear and contains gaps.

$$\text{Property (T)} \ + \ \text{Haagerup} \ = \ \text{Compact}.$$

In terms of CP functions? Γ has (T) iff every sequence of continuous normalized positive definite functions that converges uniformly on compact subsets to 1 converges uniformly to 1. Or iff every continuous conditionally negative definite function on Γ is bounded.

- **Asymptotic dimension** asdim $|\Gamma| = \dim_{nuc}(C_u^*(\Gamma))$. $\mathbb{Z}^n$ of asymptotic dimension n. Asymptotic dimension of a tree is one: $asdim(\mathbb{F}_n) = 1$. Hyperbolic groups (trees from far away) are of finite asymptotic dimension (which can be arbitrarily large). Finitely generated solvable goups such that the abelian quotients are finitely generated have finite asymptotic dimension. Example of such: the group

  $$Sol = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \quad \text{where} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

  $$\{e\} < \mathbb{Z} < \mathbb{Z}^2 < Sol \quad \text{ with } Sol/\mathbb{Z}^2 \cong \mathbb{Z}.$$

  Every almost connected Lie group has finite asymptotic dimension, and any of their discrete subgroup. For instance $SL(n,\mathbb{Z})$ for every n. Mapping class groups have finite asymptotic dimension.

  All finite asymtotic dimension groups satisfy the Novikov conjecture.

  The groups $\mathbb{Z}^{(\infty)} = \bigoplus_{j=0}^{\infty} \mathbb{Z}$ with $d(x,y) = \sum j|x_j - y_j|$ and $\mathbb{Z} \wr \mathbb{Z}$ have infinite asymptotic dimension.

- **FDC** $\mathbb{Z}^{(\infty)}$ has FDC and infinite asymptotic dimension, but is not finitely generated. The following subgroup of $SL(2,\mathbb{R})$ has FDC, infinite asymptotic dimension and is finitely generated:

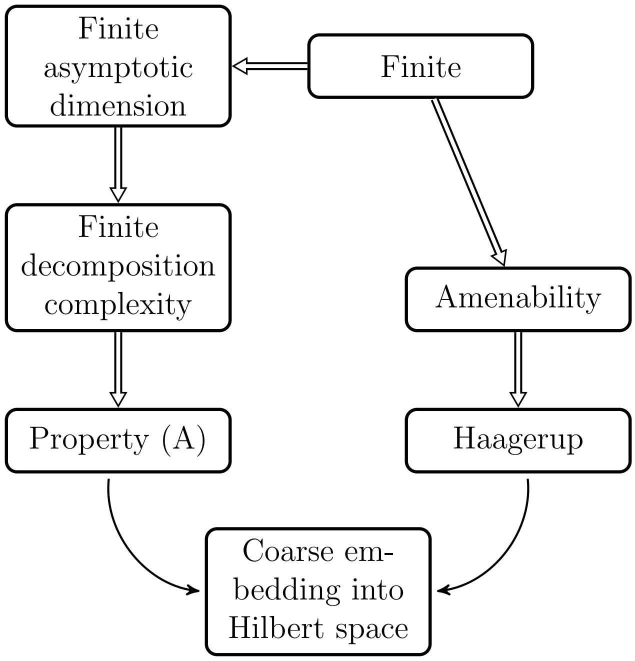
  $$G = \left\{ \begin{pmatrix} \pi^n & P(\pi) \\ 0 & \pi^{-n} \end{pmatrix} | n \in \mathbb{Z}, P \text{ Laurent polynomial with integer coefficients} \right\},$$

  with $\left\{ \begin{pmatrix} 1 & P(\pi) \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z} \wr \mathbb{Z}$ as a subgroup (so infinite dimension). Any countable subgroup of $GL(n,R)$ for R a commutative ring has FDC, countable subgroups of almost connected Lie groups, elementary amenable, finite asymptotic dimension and hyperbolic, all have FDC.

- **Property A** $|\Gamma|$ has property (A) iff $C_u^*(\Gamma)$ is nuclear (Ozawa, but Guentner-Kaminker...) iff $\beta\Gamma \rtimes \Gamma$ is amenable. Non-equivariant version of Haagerup's property. All FDC groups have (A).

- **Coarsely embeddable into Hilbert space** $|\Gamma|$ coarsely embeds iff $\beta\Gamma \rtimes \Gamma$ is a-T-menable.

Other properties: hyperbolicity in Gromov's sense, K-amenability, polya-$\mathcal{P}$ (polyabelian = solvable?, polycyclic,..), virtually abelian or nilpotent, Rapid decay property,... Exactness: Gromov's monsters are the only groups known not to be exact. C^*-simplicity: nonabelian Free groups,



Stability:

	Amenablitiy	**Haagerup**	**(T)**	**Baum-Connes**
Product	Yes			
Subgroups	Yes	No Closed yes	No, $\mathbb{Z} < SL(3,\mathbb{Z})$ Finite index yes	
Quotients	Yes		Yes	
Extensions	Yes			
Direct limits	Yes			
Free products	No, $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$			
HNN extensions				
Free amalgamated products				

	Finite Asymptotic dimension	**FDC**	**(A)**
Product	Yes	Yes	
Subgroups	Yes	Yes	
Quotients			By amenable subgroups
Extensions	Yes	Yes	Yes
Direct limits			Yes
Direct unions	No, $\mathbb{Z}^{(\infty)}$	Yes	
Free products			Yes
HNN extensions	Yes	Yes	
Free amalgamated products	Yes	Yes	

Other group like objects, but with less properties.

2.2.1 Groupoids

- The *coarse groupoid* $G(X)$: étale (even ample) with totally disconnected basis βX. Dynamical asymptotic dimension of asymptotic dimension of X. Amenable iff X has property A. A-T-menable iff X coarsely embeds into Hilbert space.
- HLS groupoid associated to a sequence of finite metric spaces X_n equipped with maps $X_n \to \Gamma$ to a finitely generated group $\Gamma = \langle S \rangle$.
- Groupoids of germs of semigroup of partial homeomorphisms acting on a topological space
- Full topological groups of an ample second-countable groupoid with compact base space
- Tillings groupoids ($\Omega \rtimes G$, usually amenable)
- Holonomy groupoids of a foliation
- Action groupoids $X \rtimes \Gamma$, principal bundles groupoids $P \times_G P$, where $P \to X$ is a G-bundle
- Equivalence relation groupoids

If G is étale, G is amenable iff $C_r^*(G)$ is nuclear. Amenability implies that the full C^*-algebra and the reduced coincides, but the converse is false by a result of Willett [?].

2.2.2 Quantum groups

One of the most useful ideas used by quantum groups theorists is to try and adapt concepts from geometric group theory in their setting. We could think of it as a way to algebraize notions like amenability, a-T-menability, etc. In the case of a discrete group Γ, it is known [?] that

$$asdim(\Gamma) = dim_{nuc}(l^\infty(\Gamma) \rtimes_r \Gamma).$$

By analogy, define

$$asdim(\hat{G}) = dim_{nuc}(l^\infty(\hat{G}) \rtimes_r \hat{G})$$

for a discrete quantum group $(\hat{G}, \hat{\Delta})$. Here

$$l^\infty(\hat{G}) = \prod_{x \in Irr(G)} B(H_x)$$

is naturally a $\hat{G}$-algebra. (Describe explicitely the action and give the example of $\hat{SU}(2)$.)

Remark post-discussion with Mehrdad: In general, $l^\infty(\hat{G}) \rtimes_r \hat{G}$ is not exact so its nuclear dimension is not finite. Such a definition is thus hopeless.

The natural filtration of any crossed-product of a $\hat{G}$-algebra by $\hat{G}$ is given by the *coarse structure* $\mathcal{E}_G$ of the finite dimensional symmetric representations of the compact dual G.

This suggests that the *coarse geometry* of the discrete quantum group $\hat{G}$ is encoded in $\mathcal{E}_G$. Indeed, the first thing one can do is to define a notion of S-separation for $x, y \in Irr(G)$ and $S \subseteq Irr(G)$:

$$(x, y) \in \Delta_S \text{ iff } \Delta(p_x)(p_y \otimes p_S) \neq 0.$$

- is it true that $asdim(\hat{G}) = d$ iff for every $R \in \mathcal{E}_G$, there exists a partition

  $$Irr(G) = U_0 \coprod U_1 \coprod \ ... \ \coprod U_d$$

  such that each U_i is a disjoint union $\coprod_j U_{ij}$ of uniformly bounded subsets R-separated:

  1. there exists $S \in \mathcal{E}_G$ such that $(x, y) \in \Delta_S$ for every $x, y \in U_{ij}$,
  2. if $x \in U_{ij}$ and $y \in U_{ik}$, $j \neq k$, then $(x, y) \neq \Delta_R$.

- in the presence of finite asymptotic dimension, do we get a controlled Mayer-Vietoris pair?

  $$\prod_{x \in U_i} B(H_x) \rtimes_r \hat{G}$$

- Define an assembly map for $l^\infty(\hat{G}) \rtimes_r \hat{G}$, and try to prove it is an isomorphism with *controlled cutting and pasting* techniques.

2.3 C^*-algebras

How to construct C^*-algebras?

- Finitely generated/presented C^*-algebras? You have to get *bounded relations.* (See Loring's book, *Lifiting solutions to perturbing problems in C^*-algebras* [?].)

- Basic blocks: commutative $C_0(X)$, finite dimensional or matrix blocks $\oplus M_{d_k}(\mathbb{C})$, $B(H)$ and its essential ideal $\mathfrak{K}(H)$, the Calkin quotient $Q(H)$,

- Sum and tensor products.

- Convolution algebras.

- Crossed product by an action by automorphisms by a group-like object: groups, groupoids, semi-groups, quantum groups. Stress that it is a kind of semi-direct product in the category of C^*-algebras: for instance, $A \rtimes_r \Gamma$ can be defined as a particular completion of the algebraic tensor product
$$A \otimes_{alg} C_r^*(\Gamma)$$
where the product is not the usual one, but twisted by the action of Γ on A, or, it is the same, the coaction of $C_r^*(\Gamma)$ on A.

Example of C^*-algebras:

- CAR algebra $C^*\langle a_i, a_i a_j + a_j a_i = \delta_{ij}\rangle$ or $\bigotimes M_2(\mathbb{C})$ or
$$\varinjlim \left\{ \begin{array}{rcl} M_{2^n}(\mathbb{C}) & \to & M_{2^{n+1}}(\mathbb{C}) \\ a & \mapsto & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{array} \right.$$

Class of C^*-algebras.

- Nuclear C^*-algebras. Finite dimensional, commutative, AF-algebras are nuclear. $C_r^*(\Gamma)$ is nuclear iff Γ is amenable (true for a discrete group, and more generally an étale groupoid). $C_u^*(X)$ is nuclear iff X has Yu's property A. So $C_r^*(\mathbb{F}_2)$ is not nuclear but $l^\infty(\mathbb{F}_2) \rtimes_r \mathbb{F}_2$ is, giving an instance where nuclarity fails to pass to sub-algebras.

- The bootstrap class $\mathcal{B}$. If Γ has Haagerup's property, then $C_r^*(\Gamma)$ is Bootstrap. It is a specialization of a famous result of Tu ([?], lemma 10.6) that if G is a-T-menable, there exists a G-proper C^*-algebra A, KK^G-equivalent to $\mathbb{C}$, such that $A \rtimes_r G$ is Bootstrap.

- The class $\mathcal{N}$ of C^*-algebras A such that the map
$$\alpha_{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$
is an isomorphism for every C^*-algebra B such that $K_*(B)$ is a free abelian group. In [?], it is shown that $\mathcal{N}$ contains all of the bootstrap class.

- Exact C^*-algebras. We say that Γ is exact if $C^*_r(\Gamma)$ is exact. It is shown by Ozawa (completing work of Guentner and Kaminker [?]) in [?] that Γ is exact iff $\beta\Gamma \rtimes \Gamma$ is amenable iff $C^*_u(|\Gamma|)$ is nuclear.

- Non exact C^*-algebras:

  - For any integer sequence k_n which tends to ∞ as n goes to ∞,
  $$\prod_{n \geq 0} M_{k_n}$$
  is not exact. As a result, for any discrete quantum group $\hat{G}$ which is truely noncommutative, $l^\infty(\hat{G})$ is not exact. So is any of its crossed-product, so that the naive definition of the uniform-Roe algebra
  $$l^\infty(\hat{G}) \rtimes_r \hat{G}$$
  is not exact, hence not nuclear.

  - The reduced C^*-algebra $C^*_r(\Gamma)$ of a finitely generated group whose Cayley graph contains expander. Using Ozawa's result [?], one can construct finite dimensional C^*-algebras M_{X_n} such that
  $$0 \to C^*_r(\Gamma) \otimes \bigoplus M_{X_n} \to C^*_r(\Gamma) \otimes \prod M_{X_n} \to C^*_r(\Gamma) \otimes \Big(\prod M_{X_n} / \bigoplus M_{X_n} \Big) \to 0$$
  is not exact in the middle.
  The problem of the existence of such a group is an interesting question, which was stated by Gromov and proved rigorously by several people in the wake of this.

  - If Γ is a discrete group with Kirchberg's approximation property, then Γ is amenable is equivalent to the maximal C^*-algebra $C^*(\Gamma)$ is exact ([?], prop 3.7.11).

  Any residually finite group satisfies Kirchberg's approximation property, hence *any nonamenable residually finite group has a non-exact maximal C^*-algebra.* For instance, $C^*(\mathbb{F}_n)$ and $C^*(SL(n, \mathbb{Z}))$, $n \geq 2$, are not-exact, for the reason that
  $$0 \longrightarrow J \otimes_{min} C^*(\Gamma) \longrightarrow C^*(\Gamma) \otimes_{min} C^*(\Gamma) \longrightarrow C^*_r(\Gamma) \otimes_{min} C^*(\Gamma) \longrightarrow 0$$

is not exact, where $J =$ ker $C^*(\Gamma) \to C_r^*(\Gamma)$.

Recall that Γ has Kirchberg's approximation property if

$$\lambda \times \rho : C^*(\Gamma) \otimes_{alg} C^*(\Gamma) \to B(l^2\Gamma)$$

is min-continuous, i.e. extends to $C^*(\Gamma) \otimes_{min} C^*(\Gamma)$.

One can define analog of approximation properties in the setting of K-theory.

- A is K-nuclear if the class of the natural map
$$p_{A,B} : A \otimes_{max} B \to A \otimes_{min} B$$
is invertible as an element of $KK(A \otimes_{max} B, A \otimes_{min} B)$.

- G is K-amenable if the class of the regular representation
$$\lambda_G : C^*_{max}(G) \to C^*_r(G)$$
is invertible as an element of $KK(C^*_{max}(G), C^*_r(G))$.

For instance, Skandalis proves in [?] that, if Λ is an infinite hyperbolic property T group, then $C_r^*(\Lambda)$ is not K-nuclear. In particular, it is not KK-equivalent to a nuclear C^*-algebra, and cannot be Bootstrap. This completely renders proving the Baum-Connes conjecture by mean of Dirac-Dual-Dirac method hopeless. An example of such a group is given by any lattice in $Sp(n, 1)$ for instance. (higher rank algebraic semisimple groups?)

After developing a restriction principle for groupoids, a natural question was to find a C^*-algebra coming from a groupoid crossed-product that we were able to prove that it satisfied the Künneth formula, while still not being a consequence of previous results. One could have started with the so called HLS groupoid $G_{\mathcal{N}}(\Gamma)$ associated to a residually finite finitely generated group Γ and a nested sequence of decreasing normal sugroups of finite index $\mathcal{N}$.

One always has the following exact sequence of $*$-algebras

$$0 \to \oplus\mathbb{C}[\Gamma_n] \to C_c(G) \to \mathbb{C}[\Gamma] \to 0$$

which induces the following exact sequence of C^*-algebras

$$0 \to \oplus\mathbb{C}[\Gamma_n] \to C_r^*(G) \to C_{\mathcal{N}}^*(\Gamma) \to 0$$

where $C_{\mathcal{N}}^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ w.r.t. to the norm

$$||x||_{\mathcal{N}} = \sup_{N \in \mathcal{N}} ||\lambda_N(x)|| \quad x \in \mathbb{C}[\Gamma]$$

induced by the quasi-regular representations $\lambda_N : C^*_{max}(\Gamma) \to \mathcal{L}(l^2(\Gamma/N))$.

Now this exact sequence intertwines the Baum-Connes assembly maps, and the Baum-Connes conjecture for $G_{\mathcal{N}}(\Gamma)$ is equivalent to $\mu_{\Gamma,\mathcal{N}}$ being an isomorphism.

- If $\Gamma = \mathbb{F}_2$ and
$$N_n = \cap ker\phi$$
for ϕ running accross all group homomorphisms from Γ to a finite group of cardinality less than n, then $C^*_{\mathcal{N}}(\Gamma) \cong C^*_{max}(\Gamma)$ and G satifies the Baum-Connes conjecture, is ample and satisfies the restriction condition. So we get that $C^*_r(G)$ satisfies the Künneth formula. It is still a result that one can get using the fact that Γ being a-T-menable, it is K-amenable. Hence $C^*_{max}(\Gamma)$ and $C^*_r(\Gamma)$ are KK-equivalent and bootstrap, so that $C^*_r(G)$ also is by extension stability of bootstrapness. A remark of R. Willett is worth mentioning: $\mathbb{F}_2$ being the fundamental group of the wedge of two circles, it is KK-equivalent to $C(\mathbb{S}^1 \wedge \mathbb{S}^1)$.

- One can artificially try to get rid of bootstrapiness by spatially tensoring this exact sequence by $C^*_r(\Lambda)$ for a infinite hyperbolic property T group. One then get the extension
$$0 \to \oplus \mathbb{C}[\Gamma_n] \otimes_{min} C^*_r(\Lambda) \to C^*_r(G \times \Lambda) \to C^*_{\mathcal{N}}(\Gamma) \otimes_{min} C^*_r(\Lambda) \to 0.$$
The restriction principle applies for the groupoid $G_{\mathcal{N}}(\Gamma) \times \Lambda$, and induces that its reduced C^*-algebra satisfies the Künneth formula. But then again, one can deduce this from a previous result, namely the restriction principle for groups. Indeed, apply it to Λ with coefficient on the trivial bootstrap Λ-algebra $C^*_r(G)$.

- Bekka shows that ??

2.4 Useful constructions in KK-theory

This section tries to compile interesting constructions in bivariant KK-theory that can be applied for the study of apporximation properties of C^*-algebras.

Extensions, boundaries and KK_1

Kasparov-Stinespring and $E^{(\pi,T)}$.
Toeplitz and suspension..

Mapping cone and double cone constructions

The mapping cone of a $*$-homomorphism $\phi : A \to B$ is the C^*-algebra
$$C_\phi = \{(a, f) \in A \oplus B[0,1] \mid f(0) = 0 \text{ and } f(1) = \phi(a)\}.$$
The mapping cone naturally fits in the short exact sequence
$$s: \ 0 \longrightarrow SB \longrightarrow C_\phi \longrightarrow A \longrightarrow 0$$
and the boundary map of this sequence coincides with ϕ_* modulo suspension, i.e.
$$\partial_{SB,CB} \otimes \partial_s = \phi_*.$$

(This remains true for any homology or cohomology theory.)

Given a sequence

$$s: \ 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

with zero composition, consider the natural inclusion $\gamma : A \to C_\phi$. We call C_γ the *double cone* of s and denote it by $C(s)$. Notice that $C(A \otimes C(s)) = A \otimes C(s)$.

Proposition 2.4.1 (see [?] rk 4.3 and [?] section 1)**.** The sequence $K(A) \to K(B) \to K(C)$ is exact iff $K(C(s)) = 0$.

For the proof, use that $K(C(s)) = 0$ iff γ_* is an isomorphism. This property is useful to show failure of K-exactness.

Corollary 2.4.2. If A is in $\mathcal{N}$ (satisfies the Künneth formula and separable...), then A is K-exact.

Proof. If A is not K-exact, there exists a short exact sequence s such that $A \otimes s$ is not exact in K-theory. Then $K(A \otimes C(s)) \neq 0$ while $K(C(s)) = 0$, which prevents A from satisfying the Künneth formula. □

Geometric injective and projective resolutions

2.5 Baum-Connes

- Compact groups, or better: proper groupoids: Green-Julg.

- Proof for the integer group $\mathbb{Z}$:

  A model for $\underline{E}\Gamma$ is the space of finitely supported probabilty measures. For $\mathbb{Z}$, the barycenter map

  $$\begin{cases} \underline{E}\mathbb{Z} & \to & \mathbb{R} \\ \sum_n p_n \delta_n & \mapsto & \sum_n p_n n \end{cases}$$

  is an equivariant continuous map homotopic to the identity.

  Prove the isomorphism

  $$RK_*^{\mathbb{Z}}(\mathbb{R}, B) \cong RK_*(\mathbb{S}^1, B),$$

  under which the assembly map sends the Toeplitz extension, which is a generator of the right side, to a generator of the K-theory group of $C^*(\mathbb{Z})$.

- For the free group on two elements $\mathbb{F}_2$, take $\mathbb{F}_2$'s Cayley graph T as a model for $\underline{E}\mathbb{F}_2 = E\mathbb{F}_2$, and $B\mathbb{F}_2$ is the wedge of two circles,

  $$RK_*^{\mathbb{F}_2}(T, B) \cong RK_*(\mathbb{S}^1 \wedge \mathbb{S}^1, B),$$

  and then?

- Connes-Kasparov: proof by representation theory (Wasserman, etc)

- Kasparov's Conspectus: towards Higson-Kasparov paper and the proof for Haagerup (J-L. Tu's general version in KK-theory, plus the beautiful result that aTmenability implies bootstrap)

- Ideas from Coarse geometry, and Yu and Roe's work, SkandalisTuYu etc.

- **Naturality of the conjecture** The question of the morphisms of groups which preserve the conjecture is quite an interesting one. It is looked at in Valette's book, *Proper actions and the Baum-Connes conjecture.* He states and proves that the assembly map is natural under monomorphisms, and under epimorphisms with amenable kernels. By natural, I mean that there exist intertwiners of the assembly maps, not that isomorphism is preserved. (The intertwiners are not always isomorphisms)

  The most complete result, to my knowledge, is Hervé's result on extensions. After Chabert, Echterhoff, tackle the conjecture for semi-direct product, Hervé gives a clean conceptual proof which can be stated as follows.

  Let us assume that one has an epimorphism with torsion free codomain that satisfies the Baum-Connes conjecture. (This simplifies the statement) If the kernel satisfies the Baum-Connes conjecture, then the domain also satisfies the Baum-Connes conjecture. In this proof, you see the commutative diagram, but it involves partial assembly maps.

  The maximal version is natural in an easier way, see Valette's book.

- A γ-element is a class $\gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ such that there exists a $\underline{E}\Gamma \rtimes \Gamma$-algebra A and elements
$$\eta \in KK^\Gamma(\mathbb{C}, A) \quad \text{and} \quad D \in KK^\Gamma(A, \mathbb{C})$$
  such that $\gamma = \eta \otimes_A D$ and $p^*(\gamma) = 1$ in $KK^\Gamma(C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma))$. See [**?**].

  Then γ and $D \otimes \eta$ are projections, and γ is unique.

  Theorem 2.5.1. If Γ has a γ-element, then $K^{top}(\underline{E}\Gamma, B)$ identifies with
$$K((A \otimes B) \rtimes_r \Gamma)p$$
  where $p = j_\Gamma(\Sigma_{\underline{E}\Gamma,B}(\gamma))$ and the assembly maps $\mu_{r,\Gamma}$ and $\mu_{max,\Gamma}$ are injective. Moreover if
$$j_\Gamma(\gamma)_* : K(B \rtimes \Gamma) \to K(B \rtimes \Gamma)$$
  is the identity, $\mu_{max,\Gamma}$ is an isomorphism. If $\gamma = 1$, then $\lambda_* \in KK(C_{max}(\Gamma), C^*_r(\Gamma))$ is invertible and $\mu_{r,\Gamma}$ and $\mu_{max,\Gamma}$ are isomorphisms.

- Rubén asked:

  Do you know a group satisfying Baum-Connes but which doesn't have a γ-element equal to 1? Do you know a group which is not K-amenable?

  Answer: Any non compact group having property T cannot have $\gamma = 1$, because the class of the Kazhdan projection is not zero in $K_0(C^*_{max}(\Gamma))$ but is in $K_0(C^*_r(\Gamma))$. For any infinite hyperbolic group Γ having T, its reduced C^*-algebra $C^*_r(\Gamma)$ is not K-nuclear ([**?**]), so any lattice in $Sp(n,1)$ works out. For instance: $Sp_{n,1}(\mathbb{Z})$.

  Do you know when Γ is amenable is equivalent to its maximal C^*-algebra being exact? When does $C^*_r(\Gamma)$ is exact implies $C^*(Gamma)$ is exact.

  Answer: When Γ has Kirchberg's approximation property, i.e. $\lambda \otimes \rho$ extends to $C^*(\Gamma) \otimes_{min} C^*(\Gamma)$, then Γ is amenable iff $C^*(\Gamma)$ is exact.

- Direct splitting method (Nishikawa 2018 [**?**]):

  Definition 2.5.2. A Kasparov cycle $(H,T) \in E^\Gamma(\mathbb{C},\mathbb{C})$ has property (γ) if there exists a non-degenerate representation of the Γ-algebra $(C_0(\underline{E}\Gamma),\alpha)$,

  $$\pi : C_0(\underline{E}\Gamma) \to \mathcal{L}(H),$$

  such that

  $$\gamma \mapsto [\alpha_\gamma(\phi), T] \in C_0(\Gamma, \mathfrak{K}(H)) \quad \forall \phi \in C_0(\underline{E}\Gamma)$$

  and

  $$\int_\Gamma \alpha_\gamma(c^{\frac{1}{2}}) T \alpha_\gamma(c^{\frac{1}{2}}) d\mu_\Gamma - T \in \mathfrak{K}(H),$$

  for some cutoff function c on Γ and Haar measure μ_Γ. (integral in the strong topology)

  If such a pair (H,T) and π is given, define:

  - the Γ-equivariant Hilbert A-module $\tilde{H} = H \otimes l^2(\Gamma) \otimes A$,
  - the Fredholm operator $(\tilde{T})_{\gamma\gamma} = \gamma T \gamma^*$,
  - the representation $\tilde{\pi} = \pi \otimes \rho_{\Gamma,A}$, where $\rho_{\Gamma,A}$ is the right regular representation on $l^2(\Gamma) \otimes A$.

  Then $(\tilde{H}, \tilde{\pi}, \tilde{T})$ defines a class in

  $$\gamma \in KK_0(C_0(\underline{E}\Gamma) \otimes (A \rtimes_r \Gamma), A)$$

and the splitting map is defined as

$$\nu_{\Gamma,A} : \begin{cases} K_*(A \rtimes_r \Gamma) & \to & KK^{\Gamma}_*(C_0(\underline{E}\Gamma), A) \\ z & \mapsto & \tau_{C_0(\underline{E}\Gamma)}(z) \otimes \gamma \end{cases}$$

It is functorial in A w.r.t. Γ-equivariant $*$-homomorphisms. The main result is the following:

Theorem 2.5.3. The composition $\mu_{\Gamma,A} \circ \nu_{\Gamma,A}$ coincides the endomorphism of $K_*(A \rtimes_r \Gamma)$ induced by (H, T).

2.6 GPOTS & NCGOA 2018

2.6.1 Arnaud Brothier: some representations of the Thompson group

2.6.2 Piotr Nowak: Property T for $Out(\mathbb{F}_n)$

2.6.3 Wilhem Winter: Relative nuclear dimension

2.6.4 Rufus Willett: Exactness and exotic crossed-product

2.7 Coarse geometry & dynamics

2.8 Langlands

A modular form of weight k is a section of

$$\Lambda^{k+2} T^* M.$$

The projective space of the $\mathbb{N}$-graded algebra

$$A = \bigoplus \Lambda^{k+2} T^* M$$

is the compactification of the modular curve

$$\mathbb{P}(A) \cong \tilde{\mathcal{C}}.$$

If $F = \mathbb{Q}$ and $G = GL_2$, the finite part of the adele

$$\mathbb{A}_f = \prod_{\text{finite places}} F_\nu = \prod_{p \in \mathcal{P}} \mathbb{Q}_p$$

is ?? and $G(\hat{\mathbb{Z}})$ is the maximal compact of $G(\mathbb{A}_f)$ with $G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$ being two copies of the upper half plane $\mathbb{H}$, and $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ is the modular curve.

Is the right part $G(\mathbb{A}_f)/G(K)$ is isomorphic to the inductive limit $G(\mathbb{Z}/p^k\mathbb{Z})$?

Yes if $G = GL_1$:

$$\varinjlim \mathbb{Z}/p^k\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p.$$

2.9 Haagerup property, cocycles and the mapping class group

If Σ is a closed oriented connected surface (with marked points), we denote by $Mod(\Sigma)$ its so-called mapping class group.

In [?] are used bounded representations of the mapping class group parametrized by a complex number $z \in \mathbb{D}$:

$$\pi_z : \Gamma \to \mathcal{L}(H).$$

Here, H is the Hilbert space obtained as the free Hilbert space generated by multicurves having a finite number of intersections with a fixed triangulation τ of Σ.

2.10 Representations of groupoids

Mettre les references et des rappels sur les champs continus et mesurables d'espaces de Hilbert.

This section is a reminder on the different notions of representations for groupoids that exists. Let us first begin by a reminder on continuous fields of C^*-algebras and Hilbert spaces. All this material can be found in Diximier's book[**?**].

A continuous field of Banach spaces over a topological space X is a pair

$$E = (\{E_x\}_{x\in X}, \Gamma_E)$$

where:

- E_x is a Banach space, with norm denoted $||\ .\ ||_x$,
- Γ_E is a linear subspace of $\prod_{x\in X} E_x$ such that $x \mapsto ||\gamma(x)||_x$ is continuous (or upper semi-continuous according to Lafforgue) for every $\gamma \in \Gamma_E$, and if every $\sigma \in \prod_{x\in X} A_x$ is locally uniformly approximable by sections, i.e. if for every $x \in X$ and $\varepsilon > 0$, there exists $\gamma \in \Gamma_A$ and a neighborhood U of x such that $\sup_{y\in U} ||\sigma - \gamma||_y < \varepsilon$, then $\sigma \in \Gamma_A$,
- $\{\gamma(x)\}_{x\in X}$ is dense in E_x.

The elements of Γ_E are called the continuous sections of E. A continuous section is said to be bounded if

$$\sup_x ||\gamma(x)||_x < \infty.$$

The space of contiuous bounded sections with the sup-norm is a Banach space.

A continuous field of C^*-algebras over a locally compact space X is a pair

$$(\{A_x\}_{x\in X}, \Gamma_A)$$

where:

- A_x is a C^*-algebra,
- $\Gamma_A \subset \prod_{x\in X} A_x$ is a $*$-algebra such that $x \mapsto ||\gamma(x)||$ is continuous for every $\gamma \in \Gamma_A$, and if every $\sigma \in \prod_{x\in X} A_x$ is locally uniformly approximable by sections, i.e. if for every $x \in X$ and $\varepsilon > 0$, there exists $\gamma \in \Gamma_A$ such that $||\sigma - \gamma|| < \varepsilon$, then $\sigma \in \Gamma_A$,
- $\{\gamma(x)\}_{x\in X}$ is dense in A_x.

On the other hand a $C_0(X)$-algebra is a C^*-algebra A endowed with a nondegenerate $*$-homomorphism

$$\phi : C_0(X) \to M(Z(A)).$$

Any field $(\{A_x\}_{x\in X}, \Gamma_A)$ over X defines a $C_0(X)$-algebra

$$C^*(\Gamma_A) := \{\gamma \in \Gamma_A \text{ s.t. } x \mapsto ||\gamma(x)|| \in C_0(X)\}.$$

A $C_0(X)$-algebra A is continuous if $x \mapsto ||a_x||$ is continuous for each $a \in A$. Here, a_x denotes the image of a under the map $A \to A/\phi(I_x)A$,with I_x the ideal of functions vanishing at x.

There is a correspondence between these two notions.

Let G be a locally compact groupoid. Renault defines a representation of G as the following data:

- a measure μ on G^0,
- a measurable field of Hilbert spaces $(\mathcal{H}, \mu)$ over G^0,
- a family of bounded operators $L_g : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ for each $g \in G$ satisfying $L_{g_1}L_{g_2} = L_{g_1g_2}$ for every $(g_1, g_2) \in G^2$, $L_{e_x} = id_{\mathcal{H}_x}$, and

  $$g \mapsto \langle L_g(\xi_{s(g)}), \eta_{r(g)}\rangle$$

  is measurable for every pair of measurable sections.

The main examples are the left and right regular representations, and the trivial one. The left regular representation λ is defined as the family of operators

$$\lambda_g : \begin{cases} L^2(G^x, \lambda^x) & \to & L^2(G^x, \lambda^x) \\ \xi & \mapsto & [\gamma \mapsto \xi_{g^{-1}\gamma}] \end{cases}$$

whereas the trivial representation is defined as

$$\tau_g : \begin{cases} H_{s(g)} = \mathbb{C} & \to & H_{r(g)} = \mathbb{C} \\ \xi & \mapsto & \xi \end{cases}$$

Recall the following: given a $C_0(G^0)$-Hilbert module E, a unitary representation G is a unitary

$$V \in \mathcal{L}_{s^*C_0(G^0)}(s^*E, r^*E)$$

which satisfies $V_1V_2 = \Delta V$, where:

- V_i is the $C_0(G^2)$-operator induced from V by the projection $p_i : G^2 \to G$,

- Δ is the (comultiplication) map $C_0(G) \to M(C_0(G^2))$-operator induced by the multiplication $\Delta : G^2 \to G$.

Fiberwise this gives you a more restrictive class than the representations in the sense of Renault. Indeed, in the case of a trivial groupoid over a locally compact space X, the spectral theorem ensures that any $*$-representation on a Hilbert space H

$$\pi : C_0(X) \to \mathcal{L}(H)$$

disintegrates into a representation in the sense of Renault on a field of Hilbert space. However, our $\{V_g\}$ gives a continuous field of representation over a continuous field of Hilbert space, which is a priori stronger.

As in the case of groups, one can try to define a integrated representation, by

$$(V(f)\xi)_x = \int_{g \in G^x} f(g)V_g(\xi_{s(g)})d\lambda^x(g) \quad f \in C_c(G), \xi \in E.$$

This defines a map $C_c(G) \to \mathcal{B}(E)$, where $\mathcal{B}(E)$ denotes the bounded operator of E seen as a Banach space. But this map is not even multiplicative!

Instead, consider G^0 to be discrete, and

$$V(f)_{xy} = \sum_{g \in G^x_y} f(g)V_g,$$

so that

$$(V(f)\xi)_x = \sum_{y \in X} V(f)_{xy}\xi_y = \sum_{g \in G^x} f(g)V_g(\xi_{s(g)}),$$

but this time

$$V(f * g)_{xz} = \sum_{y \in X} V(f)_{xy}V(g)_{yz}.$$

If G^0 is not discrete, suppose there is a measure μ on G^0. Then

$$V(f * g)_{xz} = \int_X V(f)_{xy}V(g)_{yz}d\mu(y).$$

Some facts.

Let G be proper and $V \in \mathcal{L}_{s^*C_0(G^0)}(s^*E, r^*E)$ be a unitary represntations. If $c : G^0 \to [0,1]$ is a cutoff function, any vector $\xi \in E$ can be averaged as to get an invariant one:

$$(\overline{\xi})_x = \int_{G^x} c(x\gamma)V_\gamma(\xi_{s(\gamma)})d\lambda^x(\gamma).$$

A simple computation and $(L_g)^*\lambda^{s(g)} = \lambda^{r(g)}$ gives indeed the $\overline{\xi}$ is G-invariant.
This ensures that any proper groupoid has property T.

2.11 Grothendieck and tensor products, the origin of nuclearity

This section is based on a talk given by Gilles Pisier, and his (exceptionally good) survey article.

Grothendieck started his work in functional analysis. While this is well known, I wanted to write a little post about how his work is important in my field.

Grothendieck did his Licence (his "undergrad") in the south of France, in the city of Montpellier.
If $x = \sum_j \alpha_j \otimes \beta_j$,

$$||x||_\wedge = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_\wedge = \sup\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_H = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

14 fundamental norms.

Chapter 3

Research projects

3.1 Hawaii

3.1.1 HLS groupoids

Let $(\Gamma, \mathcal{N})$ be an *approximated group* and $G_{\mathcal{N}}$ its associated HLS groupoid. Then:

- $G_{\mathcal{N}}$ is amenable iff Γ is amenable,

- if $G_{\mathcal{N}}$ is a-T-menable, then Γ is a-T-menable. The converse doesn't hold: in [**?**], the authors construct an approximated pair $(\mathbb{F}_2, \mathbb{N})$ such that the assembly map $\mu_{G_{\mathcal{N}},r}$ is not surjective, even if $\mathbb{F}_2$ is a-t-menable.

- $G_{\mathcal{N}}$ has T iff Γ has T,

- the algebraic exact sequence
$$0 \longrightarrow \bigoplus_n \mathbb{C}[\Gamma_n] \longrightarrow C_c(G_{\mathcal{N}}) \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$
extends to
$$0 \longrightarrow \bigoplus_n C_r^*(\Gamma_n) \longrightarrow C_r^*(G_{\mathcal{N}}) \longrightarrow C_{r,\infty}^*(\Gamma) \longrightarrow 0 \ ,$$
where the right side algebra is the completion of $\mathbb{C}[\Gamma]$ w.r.t. the norm
$$||x||_{r,\infty} = \sup\{||y||_r : q(y) = x\} \quad \forall x \in \mathbb{C}[\Gamma].$$
This is not an exotic crossed product functor, but one can still define an assembly map $\mu_{\Gamma,r,\infty}$ as the composition of $\mu_{\Gamma,max}$ with the induced at the level of K-theory of the quotient map $C_{max}^*(\Gamma) \to C_{r,\infty}^*(\Gamma)$. This exact sequence and the one induced by the decomposition of $G^0 = \overline{\mathbb{N}}$ is $\mathbb{N}$ and ∞ intertwines the assembly maps so that the next point follows:

- $G_{\mathcal{N}}$ satisfies BC iff Γ satifies BC for $\mu_{\Gamma,r,\infty}$.

- If Γ has T, then if μ_Γ is injective (which is the case for all closed subgroups of connected Lie groups), then $\mu_{\mathcal{G}_{\mathcal{N}}}$ fails to be surjective.

- **Congruence subgroup property.** If Γ has c.s.p. , then the assembly map fails to be surjective for any HLS groupoid $G_{\mathcal{N}}(\Gamma)$. If one can find such a groupoid which is a-T-menable for $SO(n,1)$, then this would imply Serre's c.s.p. conjecture: Any lattice in $SO(n,1)$ does not have c.s.p.

A useful fact from [**?**]:
$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$
is exact implies that the cone C_γ of the natural inclusion $\gamma : J \to C_\beta$ has vanishin K-groups:
$$K_*(C_\gamma) = 0.$$

3.1.2 Visit to PennState, September 18th to 21st 2018

Michael Francis:

- Dixmier-Malliavin theorem: for every Lie group G,

  $$C_c^\infty(G) * C_c^\infty(G) = C_c^\infty(G).$$

  The idea is to decompose, in the real case, the Dirac mass at 0 as a derivative of $\delta_0 = g^{(n)}$ for some $g \in C^{n-2}(\mathbb{R})$, but this doesn't quite do the job, so that they show that there exists $g \in C_c^\infty(G)$ and a_n going to 0 as fast as needed so that

  $$\delta_0 = \sum a_k g^{(k)}.$$

  The result follows from $f = f * \delta = (\sum (-1)^k a_k f^{(k)}) * g$. Michael extended this result to Lie groupoids.

- a remak of John Roe in his lectures on Coarse Geometry, that there exists a Svarc-Milnor theorem for foliations.

Sarah Browne:

- Advice to read a book Nate Brown gave her, *Lifting solutions to perturbing problems in C^*-algebras*by Terry A. Loring (Fields Institute Monographs). In here canbefind te definition o semi projective C^*-algebras.

3.1.3 Visit to Texas A&M, February 12th to 14th 2019, and University of Houston February 15th

Künneth formula, useless stability and extensions

Proposition 3.1.1. If for every r, X is 2-decomposable w.r.t. a family of uniformly embeddable spaces (into Hilbert space), then X is itself CEH.

This contradicts the (false) example I gave in the preprint with Christian of a group whose uniform Roe algebra satisfies the Künneth formula. For this I used a split extension of two CEH groups, which is not itself CEH built by Arzhantseva and Tessera ([?]). A misuse of the fibering theorem made me believe that this group was 2-decomposable w.r.t. a uniformly CEH family.

The main ingredients for this result are the following, and are all contained in a paper of Dadarlat and Guentner (see [?]).

Proof. Let us fix some notation: X is a discrete metric space, and $\mathcal{U}$ is a cover, by which we mean a collection of subset of X whose union form X. We say that $\mathcal{U}$

- has Lebesgue number at least L ($Leb(\mathcal{U}) \geq L$) if any ball of radius L is completely contained in some $U \in \mathcal{U}$,
- has R-multiplicity less than k ($R - mult(\mathcal{U}) \geq R$) if any ball of radius R interesect at most k elements of $\mathcal{U}$. If ignored, R is zero.
- is R-separated if any two elements of $\mathcal{U}$ a at least R-apart,
- is (k, R)-separated if $\mathcal{U}$ admits a partition into $k+1$ families which are R-separated.

By a partition of unity ϕ subordinated to $\mathcal{U}$, we mean a collection of function $\{\phi_U\}_{U \in \mathcal{U}}$, each $\phi_U : X \to [0,1]$ being zero outside of U, and such that $\sum_U \phi_U(x) = 1$ for every $x \in X$. We say $Lip(\phi_U) \leq C$ is there exist $\delta > 0$ such that if $d(x,y) < \delta$, $|f(x) - f(y)| < C$. And $Lip_{l^1(\mathcal{U})}(\phi) < C$ if there exists δ such that if $d(x,y) < \delta$, $\|\phi(x) - \phi(y)\|_{l^1(\mathcal{U})} = \sum_U |\phi_U(x) - \phi_U(y)| < C$.

1. if $\mathcal{U}$ is a cover of X with $Leb(\mathcal{U}) \geq L$ and $mult(\mathcal{U}) \leq k$ then
$$\phi_U(x) = \frac{d(x, X - U)}{\sum_{V \in \mathcal{U}} d(x, X - V)}$$
defines a PDU such that
$$Lip(\phi_U) \leq \frac{2k+3}{L} \text{ and } Lip_{l^1(\mathcal{U})} \leq \frac{(2k+2)(2k+3)}{L}.$$
2. if $\mathcal{U}$ is (k, L)-separated, then $mult(\mathcal{U}) \leq k+1$;
3. if $\mathcal{U}$ is $(k, 2R)$-separated, then $R - mult(\mathcal{U}) \leq k+1$;
4. if $L - mult(\mathcal{U}) \leq k+1$, $Leb(\mathcal{U}_L) \geq L$;
5. *Summary:* If $\mathcal{U}$ is $(k, 2L)$-separated, then $mult(\mathcal{U}_L) \leq k+1$ and $Leb(\mathcal{U}_L) \geq L$. Also $L - mult(\mathcal{U}) \leq k+1$.

Using [?], thm 3.2, the result follows.

□

Can we show that under suitable conditions, if $C_u^*|N|$ and $C_u^*|H|$ satisfy the Künneth formula, then so does $C_u^*|G|$?

Proposition 3.1.2. Let N be a coarsely embeddable discrete group, H a discrete group with Haagerup's property and $G = N \rtimes H$ be a semi-direct product. Then the uniform Roe algebra $C_u^*(G)$ satisfies the Künneth formula. In particular, there exists a group which is not coarsely embeddable and satisfies the result.

Proof. By classical properties of semi-direct products (see [?], prop. 3.11),

$$C_u^*(G) \cong (l^\infty(G) \rtimes_r N) \rtimes_r H.$$

The first projection $N \rtimes H \to N$ induces a proper surjective N-equivariant map

$$\beta(N \rtimes H) \twoheadrightarrow \beta N,$$

inducing a proper surjective groupoid morphism

$$\beta(N \rtimes H) \rtimes N \twoheadrightarrow \beta N \rtimes N.$$

Now, by a construction similar to the proof of theorem 7.1, the ample groupoid

$$\beta(N \rtimes H) \rtimes N,$$

where the action of N on H is trivial, is an inductive limit $Y_i \rtimes N$ of a-T-menable second-countable ample groupoids, so that its reduced C^*-algebra $A = l^\infty(N \times H) \rtimes_r N$ satisfies the Künneth formula. Now if H has Haagerup's property, it satisfies the Baum-Connes conjecture with coefficients so that we can apply the Going-Down principle:

$$A \rtimes_r H \cong C_u^*(G)$$

satisfies the Künneth formula.

Indeed, one has to show that $A_{|K} \rtimes_r K$ satisfies the Künneth formula for every finite subgroup K of H. In that case,

$$A_{|K} \rtimes_r K \cong (l^\infty(N \times H) \rtimes_r N) \rtimes F.$$

K being a finite group, the action map $(\beta(N \rtimes H) \rtimes N) \rtimes K \to \beta(N \rtimes H) \rtimes N$ is proper. Any proper negative type function on $\beta N \rtimes N$ can then be pulled back to $(\beta(N \rtimes H) \rtimes N) \rtimes K$, which is thus a-T-menable.

Taking the example of [?] we get an example of a group which does not coarsely embed, such that its uniform Roe algebra satisfies the Künneth formula.

□

Another interesting question was the following. The rule that associate to a coarse space X its coarse groupoid is a functor, where the domain category is the one of coarse spaces and coarse maps, and the target category is the one of groupoid and generalized morphisms.

Does any generalized morphism between coarse groupoids arises from a coarse map?

The answer should be no. Take an extension of groups

$$1 \to H \to G \to C \to 1,$$

maybe split. Then the quotient arrow should give a generalized morphism between the coarse groupoids, but it is not a coarse map as soon as H is unbounded.

Let $\phi : G \to H$ be a group homomorphism (they are discrete). It induces $l^\infty(H) \to l^\infty(G)$ and a continuous map $\beta\phi : \beta G \to \beta H$, which is G-equivariant (the action of G on H being given by composition by ϕ and multiplication). This gives a strict groupoid morphism

$$\beta G \rtimes G \to \beta H \rtimes H$$

(Take $(x, g) \mapsto (\beta\phi(x), \phi(g))$. If ϕ is surjective, this map is surjective.

Wreath product and counterexample

In [?], thm 8.3, Willett and Yu constructed a counterexample for the Baum-Connes conjecture *with coefficients*. Let G be a group whose Cayley graph contains an expander X. G does not act on X, but we can enlarge X and set

$$N_\infty(X) = \cup_{R>0} N_R(X),$$

so that G acts on $N_\infty(X)$ and on

$$A = l^\infty(N_\infty(X)) = \overline{\cup_{R>0} l^\infty(N_R(X))}.$$

Then the Baum-Connes assembly map for G with coefficients in A fails to be surjective.

Now can we force this C^*-algebraic coefficients into a group so as to build a counterexample for Baum-Connes without coefficients? For instance, take the unrestricted wreath product

$$\Gamma = \mathbb{Z}_2 \wr_X G = \prod_X \mathbb{Z}_2 \rtimes G.$$

One sees that the reduced C^*-algebra $C^*_r(\Gamma)$ should really look like $A \rtimes_r G$. Indeed, $N = \prod_X \mathbb{Z}_2$ is normal abelian in Γ, so that $C^*_r(\Gamma) \cong C^*_r(N) \rtimes_r G \cong C(\hat{N}) \rtimes_r G$.

Matui's conjecture

See the section below.

Failure of K-exactness

Show that if X is an expander, then $C^*_u(X)$ is not K-exact.

Automata groups and exotic construction

Can one use automata groups to build an example of a discrete group which has A but is not FDC?

Andrzej Zuk build a example of an amenable automata group which is not elementary amenable. The idea of FDC is that it should be the "elementary property A" groups.

3.1.4 Visit to SCMS (Fudan University, Shanghai), 21 July to 15 August 2019

The C^*-algebra associated to a Hilbert-Hadamard space

The best hope for computing the operator K-theory of crossed-product C^*-algebras is generally to establish the Baum-Connes conjecture. In that case, for a nice group (more generally a nice enough groupoid) G and a C^*-algebra on which G acts by automorphisms, the K-theory groups $K_*(A \rtimes_r G)$ is isomorphic to Kasparov's K-homology $RK_*^G(\underline{E}G, A)$. The first argument of the latter is the so called *classifying space for proper actions* of G, a locally finite (nope) CW-complex. Classical computational methods allow in principle to compute the K-homology of such a space. Still, in order to actually carry these computations, one should also have a concrete model for $\underline{E}G$, which is not an easy task.

The Baum-Connes conjecture has been proven for a very large class of groups, including almost connected locally compact Hausdorff groups, a-T-menable groups, and hyperbolic groups. The main idea behind the vast majority of these proofs is the *Dirac-Dual-Dirac* method:

- Find a (possibly infinite dimensional) space M endowed with an action of G,
- Build a C^*-algebra $A(M)$ from M such that G-acts on it,
- Prove that $A(M)$ is KK^G-equivalent to $\mathbb{C}$.

The last step implies the conjecture by nonsensical diagram chase and natural properties of the assembly map. The first appearance of this method was in a paper of Kasparov where the Novikov conjecture is proven for groups acting properly by isometries on non-positively curved spaces. The peak of its career was nonetheless attained in the celebrated work of Higson-Kasparov for a-T-menalble groups. In the first case, the space M is the non-positively curved manifold acted upon by G whereas in the a-T-menable case, it is a Hilbert space on which G has an isometric metrically proper action.

The presentation of the algebra $A(H)$ is done by inductive limits on the finite dimensional spaces of the Hilbert space, and is quite cumbersome. In their recent paper, Gong, Wu and Yu give an alternative construction for $A(M)$, which avoids inductive limits in the infinite dimensional case, and has the flexibility of applying to a infinite diemsnional version of non-positively curved spaces that the authors call *Hilbert-Hadamard spaces.*

Definition 3.1.3. A Hilbert-Hadamard space (M, d) is a metric space which is CAT(0) and such that each of its tangent cone embed isometrically in a Hilbert space.

The last condition is equivalent, by Schoenberg's theorem, to the metric of the tangent cone to be a negative definite kernel. Any Hadamard manifold or Hilbert space is of course a Hilbert-Hadamard space. The prototypical example will be the space of square integrable loops of a Hadamard manifold M, i.e. $L^2(\mathbb{S}^1, w, M)$.

Let us first recall the construction of $A(M)$.

Let (X, d) be a Hilbert-Hadamard space. We will denote by $\mathbb{R}_t$ the real vector space $\mathbb{R}$ for positive t, 0 otherwise. Let $T = X \times \mathbb{R}_+$, and for each $(x, t) \in T$, let $C_{x,t}$ be the Clifford algebra associated to the real vector space $T_xX \oplus \mathbb{R}_t$ endowed with the nondegenerate bilinear symmetric form $q(u_x, v) = \|u_x\|_x^2 + v^2$, and form the infinite $*$-algebraic product

$$\mathcal{A}(X) = \prod_{(x,t)\in T} C_{x,t}.$$

For every $x_0 \in X$, let $\sigma(x,t)$ be the image of the tangent vector $\dot\gamma(1) \in T_xX \oplus \mathbb{R}_t$ in $C_{x,t}$, where γ is the unique geodesic $[0,1] \to X$ from $(x_0, 0)$ to (x,t). Any complex valued function $f \in C_0(\mathbb{R})$ can be written uniquely as the sum of an even and odd functions, that is $f(s) = f_0(s^2) + sf_1(s^2)$. For $f \in C_0(\mathbb{R})$ and $x_0 \in X$, a_{f,x_0} denotes the element $\mathcal{A}(X)$ given by

$$(x,t) \mapsto f_0(d(x,x_0)^2 + t^2) + f_1(d(x,x_0)^2 + t^2)\sigma_{x_0}(x,t).$$

Let Γ_X be the complex subspace generated by $\{a_{f,x_0}\}_{f\in C_0(\mathbb{R}), x_0\in X}$. Each $C_{x,t}$ acts by bounded operators on $\Lambda_{\mathbb{C}}(T_xX \oplus \mathbb{R}_t)$, and one can check that the $a_{f,x}$ are bounded sections of $\mathcal{A}(X)$, whereas the σ_x are not. That is one of the reason for considering functional calculus.

Definition 3.1.4. The C^*-algebra $A(X)$ is obtained as the sections of the continuous field

$$(\mathcal{A}(X), \Gamma_X).$$

This presentation of $A(X)$ as the C^*-algebra associated to a continuous field gives a simple criterium to decide when two Hilbert-Hadamard spaces give isomorphic algebras (at least up to Morita equivalence). It will also allow us to compute its K-theory. For details on continuous fields of C^*-algebras, we refer to chapter 10 of Dixmier's book[**?**].

Suppose that there exist:

(A) an isometry $\phi : M \to N$

(B) and invertible isometries (with isometric inverses) $u_m : T_mM \to T_{\phi(m)}N$ for every $m \in M$.

The latter define $*$-isomorphisms $C_{m,t} \cong C_{\phi(m),t}$ and thus an ismorphism $u : \mathcal{A}(M) \to \mathcal{A}(N)$. Hence if $u(\Gamma_M) \subset \Gamma_N$, we have an ismorphism $A(M) \cong A(N)$. Note: a stronger condition would be $u_m(\beta^M_{m_0}(m)) = \beta^N_{\phi(m_0)}(\phi(m))$.

We will see that can happen for:

- finite dimensional Riemanian manifolds with scalar curvature bounded below, only locally, but that should be enough for computing the K-theory;
- L^2 product of finite dimensional Hadamard manifold M, i.e. $L^2(N, w, M)$.

In the first case, the injectivity radius is uniformly bounded below, let us say by ε. We thus know that the exponential map gives a diffeomorphism when restricted to balls of radius ε in T_mM. The details are given in the following example.

Let (M, g) be a finite dimensional Riemanian manifold, and let $\varepsilon > 0$ such that $exp_m : T_m M \to M$ is a diffeomorphism when restricted to the ball centered at 0 of radius ε. We have a local isomorphism of vector bundles

$$T(T_m M) \to TM$$

over the aformentioned ball, given by the linear isomorphism

$$Dexp_m : (v, w) \mapsto (exp_m(v), D_{m'} exp_m(w))$$

for every $v \in B(0, \varepsilon)$ (here $m' = exp_m(v)$). Now the bounded linear operator $a_v : T_m M \to T'_m M; w \to D_{m'} exp_m(w - v)$ (composition of the derivative of the exponential with the identification $T_v V$ with V given by translation) is inversible and thus we have an invertible isometry $u_v : T_m M \to T_{exp_m(v)}$ given by functional calculus (I know this is using a hammer to kill a fly, but I want something that works in infinite dimension)

$$u_v = (a_v a_v^*)^{-\frac{1}{2}} a_v.$$

In the finite dimensional case, this coincides with the parallel transport from m to m' along the unique geodesic joining them. The relation $u_v C_w(v) = -C_p(x)$ where $x = exp_m(v)$ and $p = exp_x(w)$ ensures that u_v induces an isomorphism $u_v : C_m \to C_{exp_m(v)}$, $\forall v \in B(0, \varepsilon)$. All these isomorphisms are homotopic to $u_0 = Id$ by $s \mapsto u_{sv}$.

Now proposition 10.1.13 in Dixmier's book [?] allow us to know how $A(M)$ looks like. Indeed, we have an open cover (U_i, m_i) of M with isomorphisms $\phi_i : A(T_{m_i} M) \to A(M)_{U_i}$ such that $g_{ij} = \phi_i^{-1} \phi_j$ satisfies $g_{ij} g_{jk} = g_{ik}$: $A(M)$ is isomorphisc to the sections of the con-itnous field given by the cocycle $g = \{g_{ij}\}$. As each $A(T_{m_i} M) \cong C_0(U_i, Cliff_{\mathbb{C}}(T_{m_i} M))$ has K-theory $\mathbb{Z}$, the K-theory of $A(M)$ is the K-theory of the bundle determined by the cocycle g.

In the case of $X = L^2(N, \omega, M)$ with M finite dimensional Hadamard, the same proof as in the finite dimensional case work: we can define the exponential map as the L^2-product of the exponential map on M. It satisfies the properties (A) and (B) follows form the fact that geodesics in X are in correspondance with L^2-products of geodesics in M. Notice that in the finite dimensional case, $\mathcal{A}(M) \cong C_0(M, Cliff(M))$.

The rest of the proof does not go through the infinite dimensional barrier. Indeed, the correct base space for $A(M)$ is the Gelfand spectrum of $A(M)_0$, which does not coincides with $M \times \mathbb{R}_+$ anymore. Proposition 1.10.13 of Dixmier's book applies in the case of a totally regular base space. Here I am confused, because any metric space is so, hence $M \times R_+$ should be, but maybe $A(M)$ is not a $C_0(M \times \mathbb{R}_+)$-algebra, and that is the point. Another remark: in the finite dimensional case, it seems the unique geodesic property is not needed since the Jacobi vector fields generates locally the whole Clifford algebra. Thus we can just define localy the trivial Clifford bundle, and clutch it via a cocycle (which could not be a possibly non existing Clifford bundle in the case the cocyle is not trivial). The problem is that in the infinite dimensional case, the Clifford algebra is not a C^*-algebra anymore, and we do not have a nice descritpion, even locally, of what the sections $a_{f,x}$ generates.

Removing the non positive curvature assumption

Say that X has a uniformly bounded injectivity radius r. As every ball of radius r is uniquely geodesic, the vector field $\sigma_{x_0}(x,t)$ is still well defined when $d(x,x_0) < r$, and so is a_{f,x_0} for $f \in C_0(\mathbb{R})$ with $supp(f) \subset (-r,r)$. Denote the subspace of such functions by S_r and Γ_r to be the subspace of $\mathcal{A}(X)$ generated by $\{a_{f,x_0} \mid x_0 \in X, f \in S_r\}$.

Definition 3.1.5. The C^*-algebra $A(X)_r$ is defined to be the sections of the continuous field $(\mathcal{A}(X), \Gamma_r)$.

Questions:

- How changing r affects the definition of $A(X)$?
- Compute $K(A(X))$ in the case of $L^2(\mathbb{S}^1, \mathbb{S}^1)$ and $L^2(\mathbb{S}^1, \mathbb{R})$.

The first question can be solved quickly in finite dimension. Here is a nice averaging argument. The normal vector to the sphere of radius r can be obtained as the average of the family of vector fields parametrized by the sphere of radius $r-\varepsilon$, i.e. if $n_{p,r}$ denotes the normal vector field to the sphere of center p and radius r, denoted by $S(p,r)$,

$$n_{x_0,r}(p) = \int_{S(x_0,r-\varepsilon)} n_{x,\varepsilon}(p) d\sigma(x) \quad \forall p \in S(x_0,r).$$

This gives

$$\beta_{x_0}(f)(p,t) = \int_{S(x_0,r-\varepsilon)} \beta_x(f^\tau)(p,t) d\sigma(x) \quad \forall p \in S(x_0,r)$$

where $f^\tau(s) = \frac{1}{\tau} f(\tau s)$ and $\tau = \frac{d(x_0,p)}{d(x,p)} = \frac{r}{\varepsilon}$. So if f is supported in $(0,r)$, one can decompose it as a finite sum of continuous f_i supported in small bands (r_i, r_{i+1}) of length less than ϵ and f is in the closure of $A(X)_\epsilon$.

In infinite dimension, the integral only converges weakly so I don't know if the argument can be extended. (Details: use $C_{x_0} = \tau C_x$.)

To do

Another description in term of $C_0(Z)$-algebras? Later. Also: is it a groupoid algebra with $G^0 = A(X)_0$? It could give insights in how to build interesting states on $A(M)$ when M is infinite dimensional.

Groupoid picture: $B = A(X)_0$ is maximal abelian and contains the unit of $A(M)$. The map defined by linear extension of $\beta_x(f) \mapsto \beta_x(f_0)$ is continuous, and so define a linear map $E : A(X) \to A(X)_0$ which is a conditional expectation, and it is faithful. The normaliser of B in $A(X)$ is $A(X)_1$, and is generating so that B is regular and B is a Cartan subalgebra of $A(X)$.

Let us determine what twisted groupoid (G, Σ) model can be given to $A(X)_0 \subset A(X)$. G is the trivial groupoid over the Gelfand spectrum of $A(X)_0$. This just means that there is no dynamic: we have a field of algebras, and nothing gets moved around.

Random unrelated facts about groupoids

We will prove a lazy generalization of a result of Tikuisis and ... that the conditional expectation onto a Cartan algebra can in some case be obtained as an average over an abelian totally disconnected group. Their result applies to $l^\infty(X)$ in $C_u^*(X)$ for a bounded geometry discrete countable metric space X. We will do it for an étale groupoid (so the general case by a result of Renault).

Let G be an étale groupoid, with a second countable base space G^0. Let $\mathcal{U}$ be a (countable) Borel cover of G^0, and let $\mathbb{G}_\mathcal{U} = \prod_\mathcal{U} \mathbb{Z}/2\mathbb{Z}$ endowed with the product topology. Such covers of G^0 form a directed system, and define $\mathbb{G} = \varinjlim \mathbb{G}_\mathcal{U}$. Now, $\mathbb{G}_\mathcal{U}$ can be realized as a subgroup of the unitary group of the GNS Hilbert space for G. Indeed, for $\varepsilon \in \mathbb{G}_\mathcal{U}$, denote

$$u = \sum_{U\in\mathcal{U}} (-1)^{\varepsilon_U} \chi_U \in U(L^2G).$$

The product topology on $\mathbb{G}_\mathcal{U}$ coincides with the weak-$*$ topology on $U(L^2G)$, and as long as none of the Borel sets in $\mathcal{U}$ have empty interior, the representation is faithful. We thus identify $\mathbb{G}_\mathcal{U}$ with its image in $U(L^2G)$.

Then, $\mathbb{G}_\mathcal{U}$ is locally compact, denote the Haar measure by du. Define

$$E_\mathcal{U}(f) = \int_{\mathbb{G}_\mathcal{U}} u^* f u du$$

for $f \in C_c(G)$. A priori this belongs to $L^\infty(G^0)$. Define E to be the limit of the $E_\mathcal{U}$'s.

Proposition 3.1.6. The map above is actually the conditional expectation $C_r^*(G) \to C_0(G^0)$.

Proof. A simple computation gives that if $u = \sum_i (-1)^{\varepsilon_i} \chi_{U_i}$,

$$\langle u^* f u \delta_g, \delta'_g\rangle = \varepsilon_i\varepsilon_j f(g^{-1}g') \quad \text{if } g \in G_{U_i}^{U_i}, g' \in G_{U_j}^{U_j},$$

so that

$$\langle \int u^* f u du.\delta_g, \delta'_g\rangle = \sum_{ij} E(\varepsilon_i\varepsilon_j) f_{G_{U_i}^{U_j}} = f_{\coprod G_{U_i}^{U_i}}.$$

If f is supported in a compact subset of G, since G is étale, there is a Borel cover such that $\coprod G_U^U \cap K = G^0 \cap K$, so that the average of f over this cover is f restricted to G^0, which is the conditional expectation on $C_0(G^0)$. In particular, $C_c(G)$ is stable by E.

□

3.1.5 Visit to Texas (A&M and University of Houston), September 30 to October 5 August 2019

Exhaustive representations of Roe algebras

Motivated by characterizations of Fredholm and spectral questions of N-body Hamiltonians, Victor Nistor and Nicolas Prudhon introduced in [?] the notion of exhaustivity of a family of $*$-representation of a C^*-algebra.

Definition 3.1.7. A family of representations

$$\mathcal{F} = \{\phi : A \to B(H_\phi)\}$$

is exhaustive if, for every irreducible representation $\sigma : A \to B(H_\sigma)$, there exists $\phi \in \mathcal{F}$ such that $ker\phi \subset ker\sigma$

Let G be a locally compact groupoid with Haar system, and $C_c(G)$ the $*$-algebra of compactly supported complex valued functions with the convolution product. Define the *maximal C^*-algebra* $C^*(G)$ to be its envelopping C^*-algebra, and the *reduced C^*-algebra* $C^*_r(G)$ to be its completion under the norm

$$\|a\|_r = \sup_{x \in G^0} \|\lambda_x(a)\|$$

where $\lambda_x \in B(L^2(G_x, \mu_x))$ is the left regular representation based at $x \in G^0$.

Let $U \subset G^0$ be an invariant open subset, and F its complementary. From [?], the exact sequence of $*$-algebras

$$0 \to C_c(G_{|U}) \to C_c(G) \to C_c(G_{|F}) \to 0$$

extends to an exact sequence of C^*-algebra,

$$0 \to C^*(G_{|U}) \to C^*(G) \to C^*(G_{|F}) \to 0$$

while at the level of the reduced norm, the sequence

$$0 \to C^*_r(G_{|U}) \to C^*_r(G) \to C^*_r(G_{|F}) \to 0$$

might fail to be exact in the middle. We still have a injective morphism on the left and a surjective morphism on the right. Let us call the *ghost ideal* I_G the kernel of the map $C^*_r(G) \to C^*_r(G_{|F})$. Then $C^*_r(G_{|U}) \subset I_G$, and we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{|U}) \arrow[r, hook] \arrow[d, two heads] & C^*(G) \arrow[r, two heads] \arrow[d, two heads] & C^*(G_{|F}) \arrow[r] \arrow[d, two heads, "\lambda_F"] & 0 \\
0 \arrow[r] & C^*_r(G_{|U}) \arrow[r, hook] \arrow[d, hook] & C^*_r(G) \arrow[r, two heads] \arrow[d, equal] & C^*_r(G_{|F}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_G \arrow[r, hook] & C^*_r(G) \arrow[r, two heads] & C^*_r(G_{|F}) \arrow[r] & 0
\end{tikzcd}$$

with exact first and third rows. A diagram chase ensures that if λ_F is an isomorphism, then $I_G = C^*_r(G_U)$, or *all ghosts are compacts.* But exhaustivity of the boundary regular representations $R_F = \{\lambda_x\}_{x \in F}$ always implies that λ_F is an isomorphism. We thus get the following.

Proposition 3.1.8. If G_F is metrically amenable, then $I_G = C_r^*(G_U)$. Equivalently

$$0 \to C_r^*(G_{|U}) \to C_r^*(G) \to C_r^*(G_{|F}) \to 0$$

is exact. In particular, this is satisfied if R_F is exhaustive.

Two cases of interests are will be the coarse setting, and so called HLS groupoids. In these cases $C^*(G) \cong C_r^*(G_U) \cong \mathfrak{K}$ (explaining why we call the exactitude of the sequence " all ghosts are compacts").

We will recall now a construction of limit spaces due to Špakula and Willett (see [?]). Let (X, d) be a discrete metric space with bounded geometry, i.e. $\forall R > 0, \sup_{x \in X} |B(x, R)| < \infty$. The distance function $d : X \times X \to \mathbb{R}$ can be extended to $\tilde{d} : \beta X \times \beta X \to \mathbb{R} \cup \{\infty\}$. If $\omega \in \beta X$, define

$$X(\omega) = \{\alpha \in \beta X / \tilde{d}(\omega, \alpha) < \infty\}.$$

Then, it is shown in [?] that $(X(\omega), \tilde{d})$ is a discrete metric space with bounded geometry. (For instance, if $X = |\Gamma|$, $X(\omega) = |\Gamma|$, and if X is a box space, then $X(\omega) = X$ when $\omega \in X$, and $|\Gamma|$ if $\omega \in \partial\beta X$.)

Given $\omega \in \beta X$, and $T \in C_m^*(X)$, one can look at the coefficients $T_{xy} = \langle \delta_y, T\delta_x \rangle$, which are bounded on $X \times X$, hence

$$T^{(\omega)} = \lim_{\omega} T_{xy}$$

defines a bounded operator $B(l^2(X(\omega)))$, and $T \mapsto T^{(\omega)}$ defines what is called a representation at infinity $\phi_\omega : C_m^*(X) \to B(l^2(X(\omega))))$. The family $F_\infty = \{\phi_\omega\}_{\omega \in \partial\beta X}$ is called the family or representations at infinity of X. Note: they coincide with the regular representations of $C_c(G(X))$.

The coarse groupoid admits a description in that setting. Recall that a partial translation of (X, d) is a partial bijection with controlled graph, that is a bijection $t : D \to R$ between subsets $D, R \subset X$, such $\sup_{x \in D} d(x, t(x)) < \infty$.

It is proven in [?] that property Yu's (A) (see [?] for a survey) is equivalent to $I_G \cong \mathfrak{K}(l^2X)$. So in that case, we get

Corollary 3.1.9. A space X has property (A) if and only if the representations at infinity $\mathcal{F}_\infty$ is an exhaustive family for the uniform Roe algebra $C_u^*(X)$.

Proof. If $\mathcal{F}_\infty$ is exhaustive then $G(X)$ is metrically amenable, hence all ghosts are compact by the previous proposition, which gives property (A).
If X has property (A), then $G(X)$ is topologically amenable, and so metrically amenable. Any irreducible representations $\phi : C_m^*(X) \to B(H)$ is either the induction of the unique representation of $\mathfrak{K}(l^2X)$, either factorizes through $C_m^*(X)/\mathfrak{K}(l^2X)$. In the second case,

$$C_m^*(X)/\mathfrak{K}(l^2X) \cong C_u^*(X)/\mathfrak{K}(l^2X) \cong C_u^*(X)/I_G \cong l^\infty(X)/c_0(X) \rtimes_r G(X),$$

so any representation factorizes through $l^\infty(X)/c_0(X) \rtimes_r G(X)$ which implies F_∞ is exhaustive.

□

In the case of the HLS groupoid, we have the following corollary.

Corollary 3.1.10. Let $(\Gamma, \mathcal{N})$ be an approximated group, and $G_{\mathcal{N}}(\Gamma)$ the associated HLS groupoid. The following are equivalent:

(i) the family of regular representations $\{\lambda_x\}_{x\in\overline{N}}$ is exhaustive for $C^*G_{\mathcal{N}}$,

(ii) the regular representation at infinity is exhaustive for $C^*G_{\mathcal{N}}$ is exhaustive for $C^G_{\mathcal{N}}$,

(iii) the group Γ is amenable,

(iv) the groupoid $G_{\mathcal{N}}(\Gamma)$ is amenable.

Proof. The equivalence of (iii) and (iv) is done in [**?**].

The trivial representation $1_\Gamma : C^*\Gamma \to \mathbb{C}$ always extends $C^*G_{\mathcal{N}}(\Gamma)$, let us call it τ. If $\{\lambda_\Gamma\}$ is exhaustive, τ extends to $C^*_r(\Gamma) = C^*G_{\mathcal{N}}(\Gamma)/ker(\lambda_\Gamma)$, so that Γ is amenable.

□

We can compare this to the example built in [**?**]: Willett shows that a certain HLS groupoid associated to $\Gamma = \mathbb{F}_2$ is metrically amenable while not being itself amenable. What we have here is an example of a groupoid which is metrically amenable while the regular representations are not exhaustive.

In both cases, it turns out that topological amenability and exhaustivity of the boundary regular representations are equivalent. This result can be generalized to the following case.

Suppose there exists an invariant open dense subset $U \subset G^0$ such that $G_{|U} \cong U \times U$. Then we can identify, for any point $x \in U$, the fiber and the open subset, i.e. $G_x \cong U$. The representation that corresponds to any λ_x under that isomorphism is called the vector representation, and denoted $\pi_0 : C^*G \to B(L^2(U,\mu))$. Notice that μ is the pull-back of any of the Haar measure under the previous isomorphism.

The diagram reduces to:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(L^2(U,\mu)) \arrow[r, hook] \arrow[d, two heads] & C^*(G) \arrow[r, two heads] \arrow[d, two heads] & C^*(G_{|F}) \arrow[r] \arrow[d, two heads, "\lambda_F"] & 0 \\
0 \arrow[r] & C^*_r(G_{|U}) \arrow[r, hook] \arrow[d, hook] & C^*_r(G) \arrow[r, two heads] \arrow[d, equal] & C^*_r(G_{|F}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_G \arrow[r, hook] & C^*_r(G) \arrow[r, two heads] & C^*_r(G_{|F}) \arrow[r] & 0
\end{tikzcd}$$

so that if $\{\lambda_x\}_{x\in F}$ is exhaustive, then λ_F is an isomorphism and $\mathfrak{K}(L^2(U,\mu)) \cong C^*_r(G) \cong I_G$. What about the reverse?

3.2 Property T and non K-exactness

Definition 3.2.1. A C^*-algebra A is K-exact if, for every exact sequence

$$0 \to J \to B \to B/J \to 0$$

of C^*-algebras, the induced sequence

$$K(J) \to K(B) \to K(B/J)$$

is exact in the middle.

In the separable case, C^*-algebras in the Bootstrap class (equivalent to K-commutative) are K-nuclear, which are themselves K-exact. This gives a lot of examples, such as the reduced C^*-algebras of amenable groups (more generally a-T-menable groupoids).

On the counterexample side, Ozawa proved bot that $\prod M_n$ and $C_r^*(G)$ are not K-exact, if the Cayley graph of G contains isometrically an expander. Spakula showed that if X is a box space obtained from a residually finite group G with property (T), then its uniform Roe algebra $C_u^*(X)$ is not K-exact. The proof of these results relies on building a Kazdhan-type projection which witnesses the failure of the sequence $(*)$. We will provide a general procedure to do so with the help of what we call a *twisted Laplacian*. A natural condition on its spectrum, which would be a analog of property (T) in that setting, will allow to produce a Kazdhan-type projection that will lead to failure of K-exactness.

3.2.1 Twisted Laplacians

We first recall some facts about representation theory of C^*-algebras. We will denote $\phi : A \to B(H)$ and $\sigma : A \to B(K)$ two representations, i.e. $*$-homomorphisms. Then one can define the space of intertwiners

$$Hom_A(K, H) = \{T \in B(K, H)/T\sigma(a) = \phi(a)T \forall a \in A\}.$$

If H and K are finite dimensional, it is itself a finite dimensional representation of A, isomorphic to $\Lambda_{\sigma,\phi} = \{\eta \in K^* \otimes H/(\phi(a) \otimes 1)\eta = (1 \otimes \sigma(a^*)\eta, \forall a \in A\}$.

In the cas where only K is finite dimensional, one can restrict to the Hilbert-Schmidt operators to get a Hilbert space, i.e. one consider the Hilbert space $K^* \otimes_2 H$ of bounded operators from K to H endowed with the inner-product

$$\langle a, b\rangle = Tr(a^*b)$$

where Tr is the normalized trace on $B(K) \cong M_d$. Then A acts on $K^* \otimes_2 H$ via

$$a.T = \phi(a)T\sigma(a)^*$$

and invariant vectors ($a.T = T$) correspond to intertwiners.

We suppose σ is (topologically) *irreducible*, meaning that any vector is cyclic. There is a A-morphism

$$B(K, H) \otimes K \to H \ ; \ T \otimes v \mapsto T(v),$$

whose image we denote by H^σ or H^K. It is the subspace of *σ-isotypical components* of H. Then H_σ will denote the orthogonal complement of H^σ. If σ and σ' are two non-equivalent irreducible representations, then H^σ and $H^{\sigma'}$ are in direct sum. An important consequence for us is that, if H is finite dimensional, only finitely many H^σ are non zero for distinct σ's.

Fix a finite self-adjoint set $S = \{a_i\}_{i=1,N}$ in A.

Definition 3.2.2. The twisted Laplacian is defined as

$$\Delta_{\sigma,\phi} = \sum_{s\in S} x_s^* x_s$$

with $x_s = \phi(s) \otimes 1 - 1 \otimes \sigma(s)$.

Lemma 3.2.3. The kernel of $\Delta_{\sigma,\phi}$ is isomorphic to the set of σ-isotypical components of H, seen as representations of $C^*(S)$, the sub-C^*-algebra of A generated by S, i.e.

$$Ker(\Delta_{\sigma,\phi}) \cong Hom_{C^*(S)}(K, H).$$

Proof. Indeed,

$$\langle \xi, \Delta\xi\rangle = \sum_{s\in S} \|x_s\xi\|^2 \quad \forall \xi \in H \otimes K,$$

ensuring that the kernel of Δ is comprised of vectors $\xi \in H \otimes K$ such that $(s \otimes 1)\xi = (1 \otimes s^*)\xi$ for all $s \in S$. But then

$$(ss') \otimes 1\xi = (s' \otimes 1)(s \otimes 1)\xi = (s' \otimes 1)(1 \otimes s^*)\xi = (1 \otimes s^*)(s' \otimes 1)\xi = 1 \otimes (ss')^*\xi,$$

proving that

$$Ker\Delta = \{\xi \in H \otimes K | a \otimes 1\xi = 1 \otimes a^*\xi \quad \forall a \in C^*(S)\}.$$

□

A fun but useless remark: Denote by $\mathcal{O}_S$ the complexified Clifford algebra over $\mathbb{R}^S$, with that canonical inner product associated to the basis $\{e_s\}$, and $c_s = c(e_s)$. Then

$$\partial = \sum_{s\in S} x_s \otimes c_s$$

satisfies $\partial^*\partial = \Delta \otimes 1_{\mathcal{O}_S}$, thus ∂ can be thought of as a twisted Dirac operator.

The projection onto the kernel of $\Delta_{\sigma,\phi}$ always exists in the bidual of A, by Borel functional calculus.

$$p_{\sigma,\phi} = \chi(\Delta_{\sigma,\phi})$$

where χ is the characteristic function of the singleton $\{0\}$.

3.2.2 Kazdhan projections

Let $A \subset B(H)$ be a unital finitely generated C^*-algebra, S a finite self-adjoint generating set and $l(a)$ the associated length. We denote by A_r the closed self-adjoint subspace of elements of length lesser than r. Then we have a filtration

- $A = \overline{\cup_{r>0} A_r}$,
- $A_r A_s \subset A_{s+t}$,
- $1_A \in A_r$ for every $r > 0$.

Diagonal case

Suppose that there exists

- a decomposition $H = \bigoplus H_m$, with $dim(H_m)$ finite, diverging to ∞ such that
$$A \subset \prod_m B_m$$
with $B_m = B(H_m)$,
- pairwise inequivalent finite dimensional irreducible representations
$$\sigma_n : A \to B_n = B(K_n)$$
such that $K_n < H_n$.

This is Ozawa's example $\prod M_n$. Is it a different proof? The first condition applies for instance when $X = \coprod X_n$ is an expander with infinite distance between the finite components.

Consider $\sigma = \oplus \sigma_i$ and $\Delta = \Delta_{\sigma,\phi}$, where ϕ is the representation $A \to \prod B_m$. Then Δ stabilizes $H_m \otimes K_n$: denote by $\Delta_{m,n}$ the associated coefficients. If the family $\{\Delta_{m,n}\}$ has uniform spectral gap, i.e. there exists $\varepsilon > 0$ such that
$$sp(\Delta_{m,n}) \subset \{0\} \cup (\varepsilon, \infty)$$
then the spectral projection $p = \chi(\Delta)$ belongs, by continuous functional calculus, in $A \otimes \prod_n B_n$. By naturality of continuous functional calculus, p stabilizes $H_n \otimes K_m$, and $p_{m,n}$ is the projection onto the isotypical component $H_m^{\sigma_n}$. H_n being finite dimensional, at n fixed, p is finitely supported in m, so that it goes to zero in $A \otimes \oplus B_m$.

Moreover $K_n < H_n$ so $Tr(p_{nn}) > 0$ (normalized trace on $B(H_n \otimes K_n)$. The map $p \mapsto (Tr(p_{nn})$ induces a morphism
$$\tau : K(A \otimes \prod_n B_n) \to \prod_n \mathbb{Z} / \oplus_n \mathbb{Z}.$$
This morphism kills anything coming from $K(A \otimes \oplus B_n)$, but we just showed that $\tau(p) \neq 0$.

Asymptotically diagonal case

Suppose that:

- there exists a decomposition $H = \bigoplus H_m$, with $dim(H_m)$ finite
- for every $r > 0$, there exists $M > 0$ such that
$$A_r \subset \mathcal{B}_M \oplus \prod_{m \geq M+1} B_m$$
with $\mathcal{B}_M = B(\oplus_{m=1}^M H_M)$ and $B_m = B(H_m)$.

The C^*-algebras $\{\mathcal{B}_M \oplus \prod_{m\geq M+1} B_m\}_M$ naturally form an inductive system, denote by $\mathcal{B}_\infty$ the inductive limit. Then the second condition ensures the existence of an embedding

$$\phi : A \hookrightarrow \mathcal{B}_\infty.$$

Suppose that there exists a family of finite dimensional irreducible representations $\sigma_n : A \to B(K_n)$ such that

$$sp(\Delta_n) \subset \{0\} \cup (\varepsilon, \infty)$$

with $\Delta_n = \Delta_{\sigma_n,\phi}$. In particular, if $K \oplus K_n$,

$$\Delta = (\Delta_n)_n \in A \otimes \prod_n B(K_n) \subset B(H \otimes K)$$

Denote $\prod_n B(K_n)$ by C, and its ideal $\bigoplus_n B(K_n)$ by J. We then have the exact sequence

$$0 \to J \to C \to C/J \to 0$$

of C^*-algebras, and the assumptions on our representations allow us to define the Kazdhan type projection $p = \chi(\Delta) \in A \otimes C$.

Proposition 3.2.4. The class

$$[p] \in K_0(A \otimes C)$$

is not in the image of $K_0(A \otimes J)$.

Proof. The completely positive map $\mathcal{B}_N \oplus \prod_{n\geq N+1} B_n \to \prod_{n\geq 1} B_n$ induces a morphism

$$\tau : K_0(A \otimes C) \to \prod \mathbb{Z}/ \oplus \mathbb{Z}.$$

Notice that τ is zero on $A \otimes J$, but $\tau([p]) \neq 0$.

□

Proposition 3.2.5. The projection p goes to zero in $A \otimes C/J$.

Proof. □

Note: I need an asymptotic version of property (T) with asymptotic intertwiners

$$T : K \to H \quad \|[T_{nn}, a]\| \to 0$$

and asymptotic invariant vectors (almost invariant vectors?)

$$\eta \in K \otimes H \quad a \otimes 1\eta_n - 1 \otimes a^*\eta \to 0$$

3.2.3 First draft

Let $A \subset B(H)$ be a unital concrete filtered C^*-algebra, and

$$\mathcal{N} = \{\sigma : A \to B(H_\sigma) = M_{k(\sigma)}\}$$

a family of cyclic (irreducible?) representations.

Definition 3.2.6. We say that A has property $(\tau)_{\mathcal{N}}$ if there exist a controlled set E and a positive constant $\varepsilon > 0$ such that

$$\forall \eta \in \Lambda_i, \exists a \in (A_E)_{+,1}, \quad \|a \otimes 1\xi - 1 \otimes \sigma_i(a)\eta\| > \varepsilon.$$

Examples:

- Γ has property T iff $C^*\Gamma$ has (τ) with respect to the trivial representation.
- Γ has property T with respect to $\mathcal{F}$ iff $C^*_{\mathcal{F}}\Gamma$ has (τ) with respect to the trivial representation.
- X has geometric property T iff $C^*_u(X)$ has (τ) with respect to the trivial representation.
- G has topological property T with respect to F iff $C^*_F(G)$ has (τ) with respect to the trivial representation.

Suppose we have an abelian sub-C^*-algebra $1_A \in B \subset A$ such that A_E is a finitely generated B-module which normalizes B, and that we can find generators a_i, $i = 1, N$ such that

$$\sum a_i^* a_i = 1.$$

Define $\Delta = \sum_i x_i^* x_i$ with $x_i = a \otimes 1 - 1 \otimes \sigma(a)$.

Fix two representations $\phi : A \to B(H_\phi)$ and $\sigma : A \to B(H_\sigma)$ and define the subspace of $H_\phi \otimes H_\sigma$:

$$\Lambda_{\phi,\sigma} = \{\eta \in H_\phi \otimes H_\sigma \mid (\pi(a) \otimes 1)\eta = (1 \otimes \sigma(a)^*)\eta \ \forall a \in A\}.$$

Then $\Lambda_{\phi,\sigma}$ naturally comes equipped with a representation of A defined by either $\pi \otimes 1$ or $1 \otimes \sigma$.

Lemma 3.2.7. $\Lambda_{\phi,\sigma}$ is nonzero iff there exists a nonzero intertwiner $T : H_\sigma \to H_\phi$.

Proof. The correspondance between the intertwiners is given by

$$T = \sum_i e_i^* \otimes x_i \longleftrightarrow \eta = \sum_i e_i \otimes x_i.$$

□

Let G be an étale locally compact groupoid, compactly generated. Fix a compact generating subset $K \subset G$ that is symmetric, $K^{-1} = K$, and cover it by bisections U_i, $i = 1, N$. There exists $\{a_i\}_{i=1,N}$, $a_i \in C_K(G)_{+,1}$ such that $\sum a_i = 1$. Suppose

- there are a countable family of finite dimensional representations

  $$\phi_i : C_c(G) \to B(H_i)$$

  such that $\bigoplus_i \phi_i$ is faithful on a quotient of $A = C^*_r(G)$.
- we can find finite dimensional representations $\sigma_j : A \to B(K_j) = M_{d_j}$ satisfying

  (i) $d_j = dim(K_j) \to \infty$,

(ii) there exists a Kazhdan pair (K, ε) such that $\{\phi_i\}$ and $\{\sigma_j\}$ have (K, ε)-spectral gap.

Define

$$\Delta = \sum_{k=1}^{N} (a_k \otimes 1 - 1 \otimes (\sigma_j(a_k)^*)_j)^*(a_k \otimes 1 - 1 \otimes (\sigma_j(a_k)^*)_j) \in C_c(G) \otimes_{alg} \prod M_{d_j}.$$

Then the spectral projection of Δ belongs to $C_r^*(G) \otimes \prod M_{d_j}$ and goes to zero in $C_r^*(G) \otimes \prod M_{d_j} / \bigoplus M_{d_j}$.

Indeed, $\Delta = (\Delta_{ij})$. The first thing to realize is that $ker(\Delta_{ij}) = \Lambda_{ij}$. The spectral gap condition ensures that the characteristic function of the singleton $\{0\}$ is continous on the spectrum of Δ in

$$\bigoplus_{ij} H_i \otimes K_j = \bigoplus_{ij} \Lambda_{ij} \oplus \Lambda_{ij}^{\perp},$$

so that by continous functional calculus $p \in C_r^*(G)$. Decompose $p = (p_{ij})$, then p_{ij} is nonzero iff Λ_{ij} is nonzero iff there is an nonzero intertwiner $K_j \to H_i$. H_i being finite dimensional and K_j being irreducible, that can only happen for finitely many j so that $p_i = (p_{ij})_j$ is finitely supported in j and the second claims follow.

Suppose that:

- there exists a decomposition $H = \bigoplus H_n$ with finite dimensional H_n such that $\lim dimH_n = +\infty$ and $A \subset \bigoplus B(H_n)$;
- there exists a sequence of finite dimensional irreducible representations $\sigma_m : A \to B(K_m)$ such that $\lim dimK_m = +\infty$ and $Hom_A(K_m, H_m) \neq 0$;
- there exist $r > 0$ and $\varepsilon > 0$ such that for all $\eta \in \bigoplus_m (H_m)_{\sigma_m}$, there exists $a \in A_{+,1}$ such that $\|(a_m \otimes 1)\eta - (1 \otimes \sigma(a_m^*)\eta\| \geq \varepsilon \|\eta\|$.

Denote by $\phi_n : A \to B(H_n)$ the restriction to the n^{th}-block, and consider $\oplus_{n,m} \Delta_{n,m} \in B(H \otimes K)$, where $\Delta_{n,m}$ is the operator $\Delta_{\phi_n, \sigma_m}$ associated to the finite generating set of A. By the hypothesis, for a fixed n, $(\Delta_{n,m})_m$ is finitely supported in m. Moreover, $Spec(\Delta_{n,m}) \subset \{0\} \cup (\varepsilon, \infty)$, so that the spectral projection p onto its kernel belongs to $A \otimes \prod_m B(K_m)$. Indeed, the characteristic function χ of the singleton $\{0\}$ is continuous on the spectrum of Δ, and $p = \chi(\Delta)$ is defined by continuous functional calculus. Moreover p has the same block decomposition that Δ, hence its image is 0 in $A \otimes (\prod_m B(K_m) / \bigoplus B(K_m))$.

As $Hom_A(K_m, H_m) \neq 0$, $tr(p_{nn}) \geq \frac{dimK_n}{dimH_n}$. Define

$$K_0(A \otimes \prod_m B(K_m)) \to \prod \mathbb{R} / \oplus \mathbb{R}$$

by $\tau([p]) = (tr(p_{nn}))_n$.
This trace-like map kills $A \otimes \bigoplus B(K_m)$, and $\tau([p]) \neq 0$.

3.3 Coarse decompositions for groupoids, stability of the Baum-Connes conjecture

Let us define the localization algebra for groupoids.

Let G be a locally compact étale groupoid with base space G^0, and $C_c(G)$ its $*$-algebra of complex compactly supported continuous functions.

Let $K \subset G$ be a compact subset of G. Recall the definition of the Rips complex $P_K(G)$. Blablabla. It is a G-CW-complex, locally finite,...

Let Z be a G-space and E be a Hilbert G-module equipped with a non-degenerate G-equivariant representation $\phi : C_0(G^0) \to \mathcal{L}(E)$. For $K \in \mathcal{E}_G$, let A_K be the self-adjoint subspace of G-invariant operators $T \in \mathcal{L}(E)$ with $prop(T) \subset K$ and $[T, f] \in \mathfrak{K}(E)$.

Define J_K to be the ideal

$$J_K = \{T \in A_K | \forall f \in C_0(Z),\ Tf \in \mathfrak{K}(E)\}.$$

Definition 3.3.1. The Roe algebra $C^*_G(Z, E)$ is defined to be the completion of $\cup_{K\in\mathcal{E}_G} J_K$ in the operator norm of $\mathcal{L}(E)$, and the *localization algebra* $C^*_L(Z, E)$ is the completion of

$$\{a : [0, \infty) \to \mathbb{C}[Z, E] | a(t) \in J_{K_t} \text{ s.t. } \cap K_t \subset G^0\}$$

under the norm

$$\|a\|_L = \sup_{t\in[0,\infty)} \|a(t)\|_{C^*(Z,E)}.$$

Prove that when Z is a free proper space, the localization map is equivalent to the Baum-Connes assembly map.

Recall the definition of absorbing representations. Let A a G-algebra and $\phi : A \to \mathcal{L}_B(H)$ a faithful representation of A on a B-G-Hilbert module, considered as an inclusion $A \subset \mathcal{L}_B(H)$. Then ϕ is called *absorbing* if for every Hilbert G-B-module K and representation $\sigma : A \to \mathcal{L}_B(K)$, there exists a sequence of G-invariant isometries $v_n : K \to H$ such that $v_n^* a v_n - \sigma(a) \in \mathfrak{K}(K)$ and $\lim \|v_n^* a v_n - \sigma(a)\| = 0$, $\forall a \in A$.

If H is absorbing,

$$RK^G_*(Z, B) \cong K_{*+1}(Q^*_G(Z, H)).$$

3.4 Paschke duality for groupoids

Let X be a proper right G-space, with $p : X \to G^0$ the anchor map. It is a local homeomorphism, and the fiber $X_s = p^{-1}(s)$ is discrete.

Given a G-algebra A and a G-Hilbert A-module H, we will build two Hilbert modules.

Define $L^2(X, H)$ to be the completion of the sections $\Gamma_c(G, r^*H)$ with the A-valued inner product

$$\langle \eta, \xi \rangle_s = \sum_{x\in X_s} \langle \eta(x), \xi(x) \rangle_H \quad \forall \eta, \xi \in \Gamma_c(G, r^*H).$$

The A-module structure is given by $(\eta a)(x) = \eta(x)a(p(x))$, and multiplication $(f\eta)(x) = f(x)\eta(x)$ defines a non-degenerate G-equivariant representation $\phi : C_0(X) \to \mathcal{L}_A(L^2(X,E))$.

The pull-back $s^*L^2(X,H) := C_0(G) \otimes_s L^2(X,H)$ is naturally ismorphic to the s^*A-Hilbert module, obtained by completion of the sections $\Gamma_c(X \times_{p,s} G, H)$ with $C_0(G)$-valued inner product

$$\langle \eta, \xi \rangle_g = \sum_{x \in X_{s(g)}} \langle \eta(x,g), \xi(x,g) \rangle_H \quad \forall \eta, \xi \in \Gamma_c(X \times_{p,s} G, H)$$

and s^*A-module structure given by $(\eta f)(x,g) = \eta(x,g)f(g)$.
The same is true when r replaces s, and the map $(U\eta)(x,g) = \eta(xg,g)$ defines a adjoinable operator

$$U : s^*L^2(X,H) \to r^*L^2(X,H)$$

which is unitary, such that $U_gU_h = U_{gh}$ for all $(g,h) \in G^2$.

Define $L^2_G(X,H)$ to be the $A \rtimes_r G$ Hilbert module obtained by completion of the sections $\Gamma_c(X, r^*H)$ with respect to the inner-product

$$\langle \eta, \xi \rangle_g = \sum_{x \in X_{r(g)}} \langle \eta(x), \xi(xg) \rangle_H$$

and $C_r^*(G)$-module structure given by $(\eta f)(x) = \sum_{g \in G_{p(x)}} \eta(xg^{-1})f(g)$ for $f \in C_c(G)$, $\eta \in \Gamma_c(X, r^*H)$.

The case of $X = G$ and $H = A$ is particularly interesting, as it allows a definition for the left regular representation. The left regular representation $\lambda_{G,A}$ is given by

$$\lambda(f)\eta = f * \eta \quad \forall f, \eta \in \Gamma_c(G, r^*A).$$

This induces a injective $*$-homomorphism

$$\lambda_{G,A} : A \rtimes_r G \to \mathcal{L}_A(L^2(G,A)).$$

Using the internal tensor product, we can thus define

$$E = L^2_G(X,H) \otimes_\lambda L^2(G,A).$$

We have $s^*E \cong L^2_G(X,H) \otimes_{s^*\lambda} s^*L^2(G,A)$ (and the same for r).

Proposition 3.4.1. For $\eta \otimes \xi \in L^2_G(X,H) \otimes_\lambda L^2(G,A)$, the map

$$(V\eta \otimes \xi)(x) = \sum_{g \in G_{p(x)}} \eta(xg^{-1})\xi(g)$$

induces an isomorphism of G-Hilbert $C_0(G^0)$-module

$$V : L^2_G(X,H) \otimes_\lambda L^2(G,A) \to L^2(X,H).$$

Corollary 3.4.2. The isomorphism V induces a $*$-isomorphism

$$\mathfrak{K}_{A \rtimes_r G}(L^2(G,A)) \cong C^*_G(X, L^2(X,H))$$

by $T \mapsto U^*(1 \otimes T)U$.

3.5 Matui's conjecture

Matui's conjecture states that

$$K_i(C_r^*(G)) \cong \bigoplus_k H^{2k+i}(G)$$

for every second countable étale essentially principal minimal groupoid with base space homeomorphic to a Cantor space. See [?] for a survey, by Matui himself.

Basically, there are two directions that one can take: proving it, for instance for groupoids with FAD, of even for groupoids satisfying the Baum-Connes conjecture? on the other direction: try to find a counterexample.

A counterexample was given by Scarparo in [?]. The groupoid is obtained as $G = \Omega \rtimes \Gamma$, where Γ is the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ and the action is given by shift on $\varinjlim \Gamma/\Gamma_i$, $\Gamma_j = n_j\mathbb{Z} \rtimes \mathbb{Z}$. Is it FAD? A-T-menable?

Pick a strictly increasing sequence of integers (n_i) such that $n_i | n_{i+1}$ and look at the following box-space

$$X = \coprod_{i \geq 1} |\mathbb{Z}_{n_i}|,$$

which has property A.

Does Matui's conjecture hold for the coarse groupoid $G(X)$?

Recall that $C_r^*(G) \cong C_u^*(X)$ and let us compute its K-theory.

Here are some ideas on how to do it. K_0 is big. Show that

$$K_1(C_u^*(X)) \cong \{(k_j)_{j\in\mathbb{Z}} \mid k \text{ is bounded }\}/\{k = (k_j)_{j\in\mathbb{Z}} \mid \lim_j k = 0\}.$$

- Let $u_j \in M_{n_j}$ be the shift on $\mathbb{Z}_{n_i}$. Then any unitary in $U_N(C_u^*(X))$ is stably homotopic to a product $(u_j^{k_j})_j$.
- Such a product is not trivial in K-theory. To show that a unitary is not trivial in K_1, one can for instance compute the trace of

  $$D(u)u^{-1}$$

  where D is a derivation. For instance $D(u) = [N, u]$ where N is the propagation operator (multiplication by the length).
  Remark: if D is the derivation on the circle, then $\int D(u)u^{-1}$ is the winding number.
- Take the exact sequence associated to the decompostion $\beta X = X \cup \partial\beta X$, which because X has property A (ghost operators are compact) is

  $$0 \to \mathfrak{K} \to C_u^*(X) \to l^\infty(X)/c_0(X) \rtimes_r \mathbb{Z} \to 0$$

  then Pimsner-Voiculescu should conclude.

- Mayer-Vietoris argument: X splits into two copies of U, coarse unions of lines of length $\frac{n_j}{2}$, with intersection W the coarse union of two-points spaces that are further and further apart, so that the K-theory is

$$K(C_u^*(\coprod_{n \geq N} W_n)) \cong K(l^\infty \mathbb{N}).$$

  Compute $K(C_u^* U)$.

To compute the homology, first read [?]. Also, if X is a box-space, $G(X)_{|\partial\beta X} = \partial\beta X \rtimes \Gamma$. We can use the Mayer-Vietoris exact sequence to compute $H(G(X))$ from $H(X \times X)$ and $H(\partial\beta X \rtimes \Gamma) \cong H(\Gamma, C(\partial\beta X))$. To compute the latter, start with a projective resolution of $\mathbb{Z}[\Gamma]$:

$$0 \longrightarrow \mathbb{Z}[\Gamma] \xrightarrow{\partial} \mathbb{Z}[\Gamma] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

and tensor it by $C(\partial\beta X)$. The homology of this complex is $H(\Gamma, C(\partial\beta X))$.

Christian and Jamie did the following to build a map from Matui's homology to the K-theory groups of G. They use Putnam's paper [?].

$$\begin{tikzcd}
0 \arrow[r] & H_1(G) \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & H_0(G) \arrow[d] \\
0 \arrow[r] & K_1(C_r^*(G)) \arrow[r] & K_0(C_\iota) \arrow[r] & K_0(C_0(G^0)) \arrow[r, "\iota"] & K_0(C_r^*(G))
\end{tikzcd}$$

On our part, we tried to understand how to build a Chern like map. Here are some questions:

- Can Matui's homology be computed as a inductive limit $H_*(G) = \varinjlim_E H_*(P_E(G)$ on the Rips complexes of some topological homology theory? The motivating example is that the coarse homology of a coarse space X is obtained as $HX_*(X) = \varinjlim H_*(P_d(X))$.
- Show that if G si d-BLR (see []), then $H_n(G) = 0$ for $n > d$.

3.5.1 About G-rings and G-modules.

Let R be a ring and M be a R-module. Let G be an ample groupoid and X a right G-space with étale momentum map. We will denote by

- $R[G]$ the abelian group $C_c(G, R)$ (if R does not come up with a topology, we take the discrete one);
- $M[X]$ the abelian group $C_c(X, M)$ (same remark).

We will define a ring structure on $R[G]$, and a $R[G]$-module structure on $M[X]$. For that, we suppose that G acts on R, i.e. we have a morphism (of rings)

$$\alpha : \mathbb{Z}[G] \to R.$$

Define for $a, b \in R[G]$,

$$(ab)(g) = \sum_{h \in G^{r(g)}} a(h)\alpha_h b(h^{-1}g) \quad g \in G.$$

and $m \in M[X]$,

$$(ma)(x) = \sum_{g \in G^{p(x)}} m(xg^{-1})a(g) \quad g \in G.$$

Similarly,

$$(am)(x) = \sum_{g \in G^{p(x)}} a(g)m(g^{-1}x) \quad g \in G,$$

defines a structure of left $R[G]$-module in the case where X is a left G-space.

Recall that if $p : X \to G^0$ and $q : Y \to G^0$ are spaces over G^0, then their fibred product is

$$X \times_{p,q} Y = \{(x, y) \in X \times Y \mid p(x) = q(y)\}.$$

It is a closed subspace of $X \times Y$.

By a G-module, we mean a space $Y \to G^0$ over G^0 equipped with commuting left and right actions of G. If Y is G-module and N a R-bimodule, we then have a $R[G]$-bimodule structure on $N[Y]$. The left action restricts to a structure of $R[G^0]$-module. We can thus form the balanced tensor product

$$M[X] \otimes_{R[G^0]} N[Y],$$

which we declare a right $R[G]$-module by $(m \otimes n)a = m \otimes (na)$.

Proposition 3.5.1. There is an isomorphism of $R[G]$-modules

$$R[X] \otimes_{R[G^0]} M[Y] \cong M[X \times_{p,q} Y]$$

Proof. The first thing to see is that $R[X] \otimes M[Y] \cong M[X \times Y]$. Indeed, G being ample and p, q being étales ensure that any compact subset of $X \times Y$ is contained in a set of the form $\cup_{i=1}^N V_i \times K_i$, where the V_i's are disjoint a compact open subsets such that $p_{|V_i}$ is a homeomorphism, and K_i is a compact subset of Y. If $f \in M[X \times Y]$ is supported in such a set, then

$$f = \sum_{i=1}^N \chi_{V_i} \otimes f_i$$

Then the injection $R[X] \otimes M[Y] \to M[X \times Y]$ is surjective, so is an isomorphism. The canonical projection $R[X] \otimes_{\mathbb{Z}} M[Y] \to R[X] \otimes_{R[G^0]} M[Y]$ is thus $M[X \times Y] \to R[X] \otimes_{R[G^0]} M[Y]$.

Seeing $X \times_{p,q} Y$ as a closed subspace of $X \times Y$, it is immediate to see that if $m \otimes n \in M[X \times_{p,q} Y]$,

$$(ma \otimes n)(x, y) = m(x)a(p(x))n(y) = m(x)a(q(y))n(y) = (m \otimes an)(x, y)$$

which gives that the canonical projection $M[X \times Y] \to R[X] \otimes_{R[G^0]} M[Y]$ has cokernel $M[X \times_{p,q} Y]$, so $M[X] \otimes_{R[G^0]} M[Y] \cong M[X \times_{p,q} Y]$ as abelian groups, which concludes since the isomorphism is clearly $R[G]$-equivariant.

□

I guess it would not be harder to show $M[X] \otimes_{R[G^0]} N[Y] \cong (M \otimes_R N)[X \times_{p,q} Y]$.

Examples:
Let $R = M = \mathbb{Z}$. Then G^n is nothing else than the itereated fibred product $G \times_{s,r} G \times_{s,r}$ ... $\times_{s,r} G$, so that $G^{n+1} = G \times_{s,r} G^n$. The previous proposition gives then

$$\mathbb{Z}[G^{n+1}] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \mathbb{Z}[G^n].$$

To Do
I would like to look at the category of G-equivariant sheaves over proper G-compact spaces, i.e. contravariant G-equivariant functors

$$\mathcal{O}(X) \to C$$

where X is a proper G-compact space and C a category equipped with an action of G.

Is it interesting to do induction? Restriction principle? Let N be a $R[G]$-module. If X is a left G-space and a right G'-space, then $M[X]$ is a $R[G]$-$R[G']$-bimodule and

$$ind_X N := (N \otimes_{R[G^0]} M[X])^{G'}.$$

3.5.2 Vanishing theorem and another question

It is natural to go from the ordered complex $\{G^n\}$ to the unordered one $\Delta = \{G^n/S_n\}$. This corresponds to passing from free proper actions to proper actions (the latter can have some isotropy due to the fact that one can stabilize a simplex), and the Rips complex $\cup P_K(G)$ is a topological realization of Δ. So let us consider

$$\Delta_n = G^n/S_n$$

where the permutation group S_n acts by permutting (shocking) the coordinates.

It is probable that the counterexample of Scarparo does not survive this change of homology, meaning that the K-theory $K_*(C(\Omega) \rtimes \Gamma)$ is isomorphic to

$$\begin{aligned} H_*(G) &= H_*(\Gamma, \Omega) \\ &= \varinjlim H_*(\Gamma, \Gamma/\Gamma_i) \\ &= \varinjlim H_*(\Gamma_i) \end{aligned}$$

In that case, $\Gamma_i = n_i\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, and in our case, the homology coincides with the singular homology of the calssifying space for proper actions, i.e.

$$H(\Gamma) \cong H(\underline{B}\Gamma)$$

with $\underline{B}(\mathbb{Z} \rtimes \mathbb{Z}_2) \cong \mathbb{R}$, with action by translation and flipping. Thus $\mathbb{R}$ can be decomposed into two invariant subsets

$$X_0 = \cup_{k\in\mathbb{Z}}[k \pm \frac{1}{4}] \quad \text{and} \quad X_1 = \cup_{k\in\mathbb{Z}+\frac{1}{2}}[k \pm \frac{1}{4}].$$

Moreover $X_0 \cong D_\infty \times_{\mathbb{Z}/2} *$, and $H(X_0) \cong H^{\mathbb{Z}/2}(*) = \mathbb{Z} \oplus \mathbb{Z}$.

The other question is to show properly that some dynamical condition implies that G has finite projective dimension. This is well studied in the case of groups (for instance see Brown's book: if a group acts freely and properly on a finite dimensional CW-complex, its has finite projective dimension).

3.6 Non-proper actions, restriction principle and the Baum-Connes conjecture

A motivating example.

Let $\Gamma = SL(2,\mathbb{Z})$ acting on the hyperbolic plane with all rational points added at infinity, $\mathbb{H}\cup\mathbb{Q}$. The stabilizer of any of the infinite points is $\Lambda \cong \mathbb{Z}$. If X denotes the Cayley graph of Γ, with a new vertex added for each of these rational points q, and an edge connecting q to any element in Γ_q, then X is quasi-isometric to the subspace of the boundary points, which is the Farey graph, qi to a tree. We thus can use Oyono-Oyono's theorem to conclude that Γ acts on a tree with amenable stabilizers, hence satisfies the Baum-Connes conjecture.

The question is to see how far we can push that technique, here deducing the conjecture for $SL(2,\mathbb{Z})$ from $\mathbb{Z}$. One can also look at $\mathbb{Z}^2$ acting on $\mathbb{Z}$ by translation by the first component.

Let Γ acting on a real Hilbert space H by affine isometries. There are two ways for the action to fail to be proper: having huge stabilizers, or having a bad orbit. We will focus on the first case. Suppose that every orbit $X_v = \Gamma \cdot v$ is proper and that the family $\{X_v\}_v$ uniformly coarsely embeds into a Hilbert space. Can we deduce the Baum-Connes conjecture for Γ?

3.6.1 Breakdown of the argument of Hervé for extensions: Erik's take

Let $1 \to N \to G \to Q \to 1$ be a group extension. Suppose Q is torsion free. The goal is to deduce the Baum-Connes conjecture for G from the Baum-Connes conjecture for N and Q.

Consider the space $Z = Q$. It is endowed with two actions:

- $G \times Q$ acts on Z via $(g,q)z = gzq^{-1}$; the generic stabilizer is isomorphic to G, but not as $G \times 1 \subset G \times Q$.
- G acts on Z by left multiplication. The generic stabilizer is $stab(e) \cong N$.

Now, the restriction principle gives that

- $BC(G, C_0(Z))$ is equivalent to $BC(N,\mathbb{C})$.
- $BC(G,\mathbb{C})$ is equivalent to $BC(G\times Q, C_0(Z))$.

If we suppose $BC(N,\mathbb{C})$, it is enough to show

$$BC(G, C_0(Z)) + BC(Q, C_0(Z) \rtimes_r G) \Rightarrow BC(G \times Q, C_0(Z)),$$

which is where the partial assembly map comes in. The diagram

$$\begin{tikzcd}
K^{top}(G\times Q, C_0(Z)) \arrow[r, "\mu_{G\times Q, C_0(Z)}"] \arrow[d, "\mu^Q_{G,C_0(Z)}"'] & K(C_0(Z) \rtimes_r (G\times Q)) \\
K^{top}(Q, C_0(Z)\rtimes_r G) \arrow[ur, "\mu_{Q,C_0(Z)\rtimes_r G}"] &
\end{tikzcd}$$

commutes, and $\mu_{Q,C_0(Z)\rtimes_r G}$ is an isomorphism by hypothesis. The remaining point is to show that $BC(G, C_0(Z))$ implies that the partial assembly map is an isomorphism.

Here is a nice application. It follows from this result that an extension of two a-T-menable groups with Q being torsion free satisfies the Baum-Connes conjecture. Arzhantseva and Tessera build such an example that is not coarsely embeddable into Hilbert space.

To adapt the argument, we want to replace that Z G-coarsely embeds into a Hilbert space.

3.6.2 Breakdown of the argument of Hervé for extensions: my take

Hervé's proof has three ingredients:

- an extension of Green's isomorphism in K-homology. This is a result from his thesis, the study of Baum-Connes for groups acting on trees. It uses in a crucial way the restriction principle;
- a factorization of the assembly of a product via a partial assembly map;
- the restriction principle applied to the partial assembly map.

Green's isomorphism

A nice way to see where Green's isomorphism is coming from is to look at the case of a discrete groupoid G. Choose a transverse set $\Omega \subset G^0$, meaning a subset which contains only one representant of each orbit on G^0 by left translation of G. For each of the points $x \in G.s$, where $s \in \Omega$, choose one element $s_x \in G$ with $d(s_x) = s$ and $t(s_x) = x$. Then $g \mapsto s_y^{-1} g s_x$, (x, y) with $d(g) = x$ and $t(g) = y$ defines a groupoid isomorphism

$$G \to \coprod_{s\in\Omega} Stab(s) \times (X_s \times X_s)$$

where $X_s = G \cdot s$ is the orbit of s. The inverse of the map is given by

$$g, (x, y) \mapsto s_y g s_x^{-1}.$$

The right hand side is the disjoint union of the stabilizer subgroups by the pair groupoid on the corresponding orbits.

Hervé's statement is as follows (okay, I pumped it up a little): for any discrete groupoid G and any G-algebra A, there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{s\in\Omega} KK^{Stab(s)}(\underline{E}G, A_s) \arrow[r] \arrow[d, "\bigoplus \mu_{G_s,A_s}"] & KK^G(\underline{E}G, A) \arrow[d, "\mu_{G,A}"] \\ \bigoplus_{s\in\Omega} K(A_s \rtimes_G G_s) \arrow[r] & K(A \rtimes_r G) \end{tikzcd}$$

with horizontal isomorphisms given by:

- the induction map for the top row

$$Ind^{G}_{Iso(G)}$$

- Green isomorphism in K-theory for the bottom row.

A particular consequence that Hervé points out is the stability of Baum-Connes by taking subgroups, which was "oddly never proven before". A more interesting consequence for us is that for a group action G on X,

$$BC(G_s, A) \iff BC(G, A \otimes C_0(X_s))$$

Partial assembly maps

The nice trick used in the paper on extensions is the following: given a surjective group morphism $\phi : G \to Q$, one considers the action of $G \times Q$ on Q given by

$$(g, q) \cdot x = \phi(g) x q^{-1}.$$

The stabilizer of the neutral element is isomorphic to G:

$$Stab(e_Q) = \{(g, \phi(g))\} \cong G.$$

Moreover, when one only considers the restrcited action of $G \times F$, where $F < Q$ is a subgroup, then

$$Stab_{G\times F}(e_Q) = \{(g, \phi(g)) \ : \ \phi(g) \in F\} \cong \phi^{-1}(F).$$

Green's isomorphism allows to conclude, by the discussion above, that

$$BC(G, A) \iff BC(G \times Q, A \otimes c_0(Q))$$

and

$$BC(\phi^{-1}(F), A) \iff BC(G \times F, A \otimes c_0(Q)),$$

here the orbit is still Q since ϕ is surjective, so $(G \times F) \cdot e_Q = Q e_Q F = Q$.

Using a partial assembly map, one gets a commutative diagram

$$\begin{tikzcd}
KK^{G\times Q}(\underline{E}G \times \underline{E}Q, A) \arrow[r, "\mu_G^Q"] \arrow[dr, "\mu_{G\times Q}"] & KK^{Q}(\underline{E}Q, A \rtimes_r G) \arrow[d, "\mu_Q"] \\
 & K(A \rtimes_r (G \times Q))
\end{tikzcd}$$

Hence

$$BC(Q, A \rtimes_r G) + \mu^Q_{G,A} \implies BC(G \times Q, A).$$

Combined with the previous discussion, it reduces to

$$BC(Q, (A \rtimes_r G) \otimes C_0(Q)) + \mu^Q_{G, A\otimes C_0(Q)} \implies BC(G, A).$$

Supposing that Q satisfies the Baum-Connes conjecture, it remains to understand when is the partial assembly map an isomorphism.

Restriction principle

The partial assembly map has a classifying space on the left side argument:

$$\mu^Q_{G,A} : KK^{G\times Q}(\underline{E}G \times \underline{E}Q, A) \to KK^Q(\underline{E}Q, A \rtimes_r G),$$

so the restrcition machinery reduces the statement to finite subgroups $F < Q$, i.e. if

$$\mu^F_{G,A} : KK^{G\times F}(\underline{E}G \times U, A_{|F}) \to KK^F(U, A \rtimes_r G),$$

is an isomorphism for all finite subgroups $F < Q$, then $\mu^Q_{G,A}$ is an isomorphism. The second assembly is equivalent to $\mu_{G\times F,A}$, which by the discussion above, is equivalent to $\mu_{\phi^{-1}(F),A}$. In the end, we get a necessary condition

$$BC(\phi^{-1}(F), A) \quad \forall F < Q \text{ finite } \implies \mu^Q_{G,A}.$$

Trees

Say now that G acts on a tree T, so that we have a morphism

$$\phi : G \to Isom(T),$$

which we can restrict to its image $S = \phi(G)$ to reduce to the case of a surjective morphism.

- when is it true that
$$\underline{E}S = \cup_{d>0} P_d(T)?$$
In this case, the assembly map for $G \times S$ factorizes through the partial assembly and the assembly
$$\lim_d KK^S(P_d(T), A \rtimes_r G) \to K(A \rtimes_r G \times S)$$
which could be maybe rehrased in a nice property for the orbit, such as "The orbit satisfies the coarse Baum-Connes conjecture S-equivariantly".

- Is it true that having proper orbits and finite stabilizers is equivalent to a proper action? Can "proper orbits" be defined as a *slice property*: for every $p \in X$, there is a G-invariant neighborhood W which is induced by a subgroup H, i.e. $W \cong_G G \times_H U$. Here H is not necessarily compact.

New remarks:
Erik pointed out that, the result for trees is quite good since, in the case of an almalgated free product $G = G_1 * G_2$, G acts on the Bass-Serre tree, which is not of bounded geometry. But that does not matter. There is one orbit for edges and two for vertices. Maybe the answer lies in the topological dynamic at infinity (my suggestion).
In the hyperbolic case, one can look at the action of G on the rational points of the Gromov-boundary

$$\partial_Q G = \{g^+ \ : \ order(g) = \infty\},$$

which is dense in ∂G, and the set of pole pairs $\{(g^-, g^+) \ : \ order(g) = \infty\}$ is dense in $\partial G \times \partial G$. Let S be the image of G in the homeomorphism of the boundary

$$G \to S < Homeo(\partial G)$$

and $G \times S$ acts on the groupoid of germs of partial homeomorphisms on ∂G.

The case of $SL(2,\mathbb{Z})$ acting on the Farey tree is even harder: the tree is not of bounded geometry, quasi-isometric to the regular tree of $\mathbb{N}$-valency, the orbit are not proper and the stabilizers are big.

What properties do we know about $SL(n,\mathbb{Z})$?

For $n = 2$, it is a-T-menable, but has box spaces which are not CEH. For $n \geq 3$, it has property (T). It is not hyperbolic or reletively hyperbolic (Erik knows for a fact, i don't know why). It acts amenably on the space $SL(n,\mathbb{R})$ modulo the upper triangular matrices.

3.6.3 Argument for trees

Hervé Oyono-Oyono proves that, if G acts properly by isometries on a tree X, the Baum-Connes conjecture for the stabilizers implies it for the group G.

Question: why is the extension

$$0 \longrightarrow C_0(X^0) \rtimes \mathbb{Z} \longrightarrow C_+(X^1) \rtimes \mathbb{Z} \longrightarrow C_r^*(\mathbb{Z}) \longrightarrow 0$$

equivalent to the Topelitz extension in the case of $G = \mathbb{Z}$ and $X = \mathbb{Z}$?

The left slot is $c_0(\mathbb{Z}) \rtimes \mathbb{Z} \cong \mathfrak{K}(l^2\mathbb{Z})$, and the right slot is $C_r^*(\mathbb{Z}) \cong C(\mathbb{S}^1)$. As $C_+(X^1)$ is the function on $\mathbb{Z}$ that have limit in $+\infty$ and tend to zero at $-\infty$, it is generated by $c_0(\mathbb{Z})$ and χ_F where F is any neighboorhood of infinity, $F = [N, +\infty]$. The crossed product is thus generated by $c_0(\mathbb{Z}), \chi_F, \lambda_1$. But $c_0(\mathbb{Z}), S$ can be taken as well as a generating set since $SS^* = \chi_{[n+1,\infty]}$ and $S^*S = \chi_F$. Then S is the shift, and $C_+(X^1) \rtimes_r \mathbb{Z} \cong C^*(S)$.

It seems the important bit is to be able to give an orientation to path, and then, if G acts on X, take Y to be the set of points that you can obtained by following a positively oriented path from a fixed point x_0. Let $G = (X \rtimes G)_{|Y}$ and $C_+(X^1)$ corresponds to the functions that admit a limit along positively oriented paths going outward from x_0.

3.6.4 Relative hyperbolicity

Proposition 3.6.1. Let G be a discrete group which admits a polynomial growth subgroup P such that $P < G$ is relatively hyperbolic. Then G satisfies the Baum-Connes conjecture.

Let G be a discrete group, and P be a polynomial growth subgroup such that $P < G$ is relatively hyperbolic. We use the Groves-Manning picture of relative hyperbolicity.

Let (α, C, d) such that bounded geometry (α, C, d)-thinnings of P exists (it does by polynomial growth and [**?**]) and let $\mathbb{X}$ the space of such thinnings, endowed with the product topology: it is a compact space endowed with a G-invariant probability measure μ.

Define a new distance on G by

$$\tilde{d}(s,t) = \int_{\mathbb{X}} d_T(x,y)d\mu(T).$$

Then show that:

- $\tilde{d}$ is G-invariant,
- $\tilde{d}$ is quasi-isometric to the word metric,
- $(G, \tilde{d})$ is weakly geodesic and strongly bolic.

Form [?], we know that this concludes the proof.

3.6.5 Examples

Let K be the figure eight knot, and M is the complementary of K in $\mathbb{S}^3$. In [?], Long and Reid constructed a family of irreducible representations

$$\beta_n : \pi_1(M) \to SL(3, \mathbb{Z})$$

such that $N_n = ker\ \beta_n$ has finite index in $SL(3, \mathbb{Z})$ and $\cap N_n = 1$.

Already, this would prove that these representations are never injective, since otherwise $SL(3, \mathbb{Z})$ would have a proper action on a l^p. (I think I don't remember that part correctly. It relies on the extension of a proper cocycle from a finite index subgroup to the big ambient group, see the Bekka-Valette. If the representation is injective, $\pi_1(M)$ is such a subgroup of $SL(3, \mathbb{Z})$, and these group admit a proper affine action on a l^p I think (ask Erik), so that it's impossible since $SL(3, \mathbb{Z})$ has strong (T).)

3.6.6 A question

Jiantao Deng proves in [] that if $G = N \rtimes H$ is a semi-direct product of two CEH groups N and H, then G satisfies the Novikov conjecture.

Erik asks if this could not be done directly as a consequence of Hervé's result on extensions.

Maybe start as follows. We know there exist

- a compact metrizable Q-space Y such that $Y \rtimes Q$ is a-T-menable,
- a compact metrizable G-space X such that $X \rtimes N$ is a-T-menable,
- a locally compact G-space W which is N-proper, and a G-$C_0(W)$-algebra $A_X(H)$ which is KK^N-equivlent to $C(X)$.

By Tu, $BC(H, C(Y) \otimes \mathcal{A})$ holds for any C^*-algebra $\mathcal{A}$. By Chabert and Oyono-Oyono, we can factorize the assembly map for G as follows

$$\begin{tikzcd}
K^{top}(G, C(Y) \otimes A_X(H) \otimes B) \arrow[r, "\mu^H_{N,C(Y)\otimes A_X(H)\otimes B}"] \arrow[dr, "\mu_{G,C(Y)\otimes A_X(H)\otimes B}"'] & K^{top}(H, (C(Y) \otimes A_X(H) \otimes B) \rtimes_r N) \arrow[d, "\mu_{H,C(Y)\otimes\mathcal{A}}"] \\
 & K((C(Y) \otimes A_X(H) \otimes B) \rtimes_r G)
\end{tikzcd},$$

so that $BC(G, C(X\times Y)\otimes B)$ is equivalent to $B(G, C(Y)\otimes A_X(H)\otimes B)$ which is equivalent to the partial assembly map $\mu_{H,C(Y)\otimes A_X(H)\otimes B}$ being an isomorphism. If that is proven, injectivity of $\mu_{G,B}$ follows if one is aware of the commutative diagram

$$\begin{tikzcd} K^{top}(G,B) \arrow[r, "\mu_{G,B}"] \arrow[d] & K(B \rtimes_r G) \arrow[d] \\ K^{top}(G, C(X\times Y)\otimes B) \arrow[r, "\mu_{G,C(X\times Y)\otimes B}"] & K((C(X\times Y)\otimes B)\rtimes_r G)) \end{tikzcd}$$

(The left vertical arrow is an isomorphism, I think. At least it is necessary.)

Haagerup's property and constructions

Recall some useful constructions: if G has a CEH subgroup N, there exists a compact metrizable G-space X such that $X \rtimes N$ is a-T-menable. Indeed, one takes the space of probability measures, with the weak-$*$ topology, on the spectrum of the (commutative) sub-C^*-algebra of $l^\infty(G)$ generated by χ_G, $c_0(G)$ and the functions $\{\phi_n\}_{n\in N}$, where

$$\phi_g : G \to \mathbb{C} \quad ; \quad x \mapsto \|\phi(gx) - \phi(x)\|^2.$$

Then

$$\eta(m,g) = \int \phi_g(x)d(gm)(x)$$

is a negative type function on $X \rtimes G$ which is proper on $X \rtimes N$.

Moreover if G acts (metrically) properly by affine isometries on a Hilbert space H, it acts properly on H with the weak topology. The space $Y = \mathbb{R}_+ \times H$ witht the weak topology on H is just an artifact to get an embedding $A(0) \hookrightarrow A(H)$. In the groupoid case, one take

$$\mathbb{R}_+ \times \coprod_{x\in G^0} H_x$$

together with the topology generated by the open subsets (basis of neighboorhood of (x_0, t_0, v_0))

$$\mathcal{V}_{\varepsilon,V_{x_0}} = \{(t-t_0)^2 + (\|v\|_x^2 - \|v_0\|_{x_0})^2 < \varepsilon, x \in V_{x_0}\}$$

In particular, the weak closed balls

$$\{(t-t_0)^2 + (\|v\|_x^2 - \|v_0\|_{x_0})^2 < R^2\}$$

are compact. From there is it easy to see that for every a-T-menable groupoid, there exists a proper G-space $W = \mathbb{R}_+ \times \mathcal{H}$ and $A(\mathcal{H})$ is a $C(X)$-algebra.

Valette has a good explanation for it in his first book on the Baum-Connes conjecture (example 4.1.8 part 2). In particular, he explains how when G acts by affine isometries metrically properly on a real affine Hilbert space X, then a model for $\underline{E}G$ is $X \times \mathbb{R}_+$, with the topology coming from the inverse image of

$$\begin{array}{rcl} X \times \mathbb{R}_+ & \to & X_\omega \times \mathbb{R}_+ \\ (x,t) & \mapsto & (x, \sqrt{\|x\|^2 + t^2}) \end{array}$$

X_ω is X endowed with the weak topology. The reason the topology has to be changed is that $X \times \mathbb{R}_+$ is not locally compact. What I don't understand though is why we take the factor $\mathbb{R}_+$ in account? Why not just X_ω?

Generalization

Remark that Deng's result could be strenghten and be a direct application of stability of Baum-Connes for extension of groupoids. Indeed, when $G = N \rtimes H$, the is an extension of coarse groupoids

$$\beta N \rtimes N \hookrightarrow \beta G \rtimes G \twoheadrightarrow \beta H \rtimes H.$$

It would imply that if G is in the smallest class of groups containing the coarsely embeddable ones and stable by extension, G statisfies the coarse Baum-Connes conjecture, and thus the Novikov conjecture.

This also generalisz quite nicely to a less equivariant setting. Say G acts by isometries on a bounded geometry metric space X, but we don't assume the action to be proper. Then there is a extension of groupoids

$$\beta X \times \{G_s\}_{s \in X} \hookrightarrow \beta X \rtimes G \twoheadrightarrow G(X).$$

(The first term is the isotropy subgroupoid)

Coarse Green isomorphism

Recall that in the case of a transitive action of a group discrete G on a discrete space X, choose a base point $s \in X$ and an element $g_x \in G$ such that $g_x.s = x$ for all $X \in X$. If G_s denotes the stabilizer, then the maps $(x, g) \mapsto (g_x^{-1}g, (x, gx))$ and $(g, (x, y)) \mapsto (g_x g, x)$ define a groupoid isomorphism

$$X \rtimes G \cong G_s \times (X \times X)$$

which gives Green's isomorphism

$$C_0(X) \rtimes_r G \cong C_r^*(G_s) \otimes \mathfrak{K}(l^2 X).$$

Show that this isomorphism extends to an isomorphism

$$G(X) \rtimes G \cong G_s \times G(X) \quad \text{and} \quad C_G^*(X) \cong C_r^*(G_s) \otimes C_u^*(X)$$

In the spirit of Oyono-Oyono, it should follow that if a group acts properly by isometries on a space X and such that there exists an orbit that is proper, and satisfies the coarse Baum-Connes conjecture, with a stabilizer that satisfies the Baum-Connes conjecture, then X satisfies the coarse Baum-Connes conjecture.

In the non transitive case, one has

$$C_G^*(X) \cong \bigoplus_{s \in \Omega} C_r^*(G_s) \otimes C_u^*(X_s)$$

with Ω a fundamental domain. When are $C_G^*(X)$ and $C_r^*(G)$ Morita equivalent?

3.6.7 A direct proof

Let G a countable discrete group acting on a metric space with bounded geometry X. Suppose

$$\sup_{x\in X} d(x,gx) < \infty \quad \forall g \in G.$$

Then we get an action of G on the coarse groupoid $G(X)$, given by $g\cdot[x,\alpha] = [x, g\circ\alpha]$, where we see an element of the group as a bisection with full support in the total groupoid. We also have maps

$$\phi_\omega : G \to G(X); g \mapsto [\omega, g]$$

for every $\omega \in \beta X$.
The groupoid $G \times G(X)$ (with base space βX) acts on $G(X)$ via

$$(g,\alpha)x = gx\alpha^{-1}.$$

The stabilizer of a point in the base space $\omega \in \beta X$ is

$$St(\omega) = \{(g,\alpha) \mid \phi_\omega(g) = \alpha\} = \{(g,[\omega,g])\} \cong G.$$

If $K < G(X)$ is a subgroupoid inclusion, we can restrict the action, and the stabilizers become

$$St^K(\omega) = \{(g,\alpha) \mid \phi_\omega(g) = \alpha \in K\} \cong \phi_\omega^{-1}(K).$$

Let $Z = \mathcal{O}(\omega)$ be the orbit for this action, and Z_K the one for the restricted action.
Now the Green-Julg isomorphism (extended to the transitive groupoid case by Muhly-Renault-Williams) gives that

$$A \rtimes_r (G \times G(X)) \cong (A_x \rtimes_r G) \otimes \mathfrak{K}(l^2X)$$

for any $C_0(Z)$-algebra A.

So for every $G\times G(X)$-algebra A, $BC(G\times G(X), A) \iff BC(G, A_x)$, in particular

$$BC(G,A) \iff BC(G\times G(X), C_0(Z)\otimes A).$$

$$BC(\phi_\omega^{-1}(K), A) \iff BC(G\times K, C(Z_K)\otimes A).$$

We can factorize the assembly map by a partial assembly, as to get the following commutative diagram

$$\begin{tikzcd}
KK^{G\times G(X)}(\underline{E}G \times \underline{G}(X), A) \arrow[r, "\mu_G^X"] \arrow[dr, "\mu_{G\times G(X)}"'] & KK^{G(X)}(\underline{E}G(X), A\rtimes_r G) \arrow[d, "\mu_X"] \\
 & K(A \rtimes_r (G\times G(X)))
\end{tikzcd}$$

If X satisfies the coarse Baum-Connes conjecture with coefficients in $A \rtimes_r G$ (say X is CEH), then the vertical map is an isomorphism. This reduces the statement to know when the partial assembly is an isomorphism.

First, if $\Lambda = \phi_\omega^{-1}(K)$, let us show that Λ contains G_ω a subgroup of finite index. Let W be a neighborhood of w, small enough that we can find a finite cover of K_W by bisections U_i, $i = 1, N$, such that $W \subset U_i$ for every $i = 1, N$. Let α_i be the corresponding partial homeomorphisms. Then

$$\Lambda = \coprod_{i=1,N} \{g \ : \ [\omega, g] = [\omega, \alpha_i]\},$$

each subset on the right hand side being a coset of $\Lambda_0 = \{g \ : \ [\omega, g] = [\omega, id]\}$. As $\Lambda_0 < G_\omega$ and $[\Lambda : \Lambda_0] = N$, it is clear that $[\Lambda : G_\omega]$ is finite.

Proposition 3.6.2. If all the subgroups of G which contain a stabilizer of the action of G on X as a subgroup of finite index satisfy the Baum-Connes conjecture with coeeficients, then the partial assembly map is an isomorphism.

Proof. Restriction principle

$$\begin{array}{ccc}
KK^{G\times G(X)}(\underline{E}G \times G(X) \times_K U, A) & \xrightarrow{\mu_G^X} & KK^{G(X)}(G(X) \times_K U, A \rtimes_r G) \\
\downarrow & & \downarrow \\
KK^{G\times K}(\underline{E}G \times U, A) & \xrightarrow{\mu_G^K} & KK^{K}(U, A \rtimes_r G) \\
\downarrow & & \downarrow{\scriptstyle \mu_X} \\
KK^{G\times G(X)}(\underline{E}G \times U, A) & \xrightarrow{\mu_G^X} & KK^{G(X)}(U, A \rtimes_r G)
\end{array}$$

□

We get that a group which acts on a metric space X with orbits that are CEH and stabilizers that satisfy BC satisfies itself BC (an improvment on Herve's result for tree, using his technique for extension for groupoids). Also this implies Deng's result: on can then show (being careful for the coefficients) that $BC(G, A \otimes C(X))$ holds where X is a G-space such that $X \rtimes N$ is a-T-menable. This gives that the injectivity of $\mu_{G,A}$. What more?

Use it when a group acts on a space by partial homeomorphism, then $\phi : G \to G(S)$. Also this is a particular case of, maybe, a invariance of the conjecture with coefficients for groupoids w.r.t. generalized morphism.

$$KK^G(A, B) \to KK^H(X^*(A), X^*(B))$$

gives

$$\begin{array}{ccc}
KK^G(\underline{E}G, A) & \xrightarrow{\mu_X} & KK^H(\underline{E}H, X^*(A) \rtimes_r G) \\
\downarrow{\scriptstyle \mu_G} & & \downarrow{\scriptstyle \mu_H} \\
K(A \rtimes_r G) & \longrightarrow & K(X^*(A) \rtimes_r G \rtimes_r H)
\end{array}$$

with a big ? Maybe: this diagram is always commutative, with vertical isomorphisms if X is a Morita equivalence.

There should be also a result of the type: if G acts transitively, which a metrically proper orbit which fiberly coarsely embeds and stabilizer satisfying the maximal Baum-Connes, then G satisfies the maximal Baum-Connes conjecture.

3.6.8 Generalized descent morphism

Let X be space endowed with commuting actions of two groups G and G', a left action of G and a right action of G'. Let (E, ϕ, T) be a $KK^{G'}$-cycle and define:

- $A_X = (A \otimes C_0(X))^G \cong \{a \in C_0(X, A) \mid s \cdot a(x) = a(s.x)\}$;
- E_X is the completion of $(E \otimes C_0(X))^G$ wrt the B_X-valued inner product
$$\langle \xi, \eta \rangle(x) = \sum_s s \cdot \langle \xi(x), \eta(s.x) \rangle_E$$
- $\phi_X = \phi \otimes id$ and $T_X = T \otimes 1$.

This defines a morphism:

$$j_X : KK^{G'}(A, B) \to KK^G(A_X, B_X)$$

Examples:

- If $X = G$ with G acting on the right and the trivial group acting on the left, j_G is Kasparov's descent morphism.
- If $X = G$ with G acting on the left and a subgroup H acting on the right, we get $j_X = ind_H^G$. It the actions are flipped, we get $j_X = res_H^G$.
- If $G = N \rtimes H$, and H acts on the left on $X = N$, and G acts on the right, then j_X is the partial descent morphism j_N^H constructed by Chabert in [].

The descent morphism has the usual properties.

Proposition 3.6.3. j_X is additive

$$j_X(x \otimes y) = j_X(x) \otimes j_X(y)$$

$$j_X \circ j_Y = j_{X \times_{G'} Y}$$

$j_X(\partial) = \partial_X$

Here is an application. In the case of a semi-direct product $G = N \rtimes H$, let K be a subgroup of H. Let $X = H$ be the K-H-bimodule associated to the inclusion $K < H$ and $Y = N$ is the $H - G$-bimodule associated to the semi-direct product. Let $X' = N$ be the K-$N \rtimes K$-bimodule associated to the semi-direct product, and $Y' = N$ is the $N \rtimes K - G$-bimodule associated to the inclusion $N \rtimes K < G$. Then

$$X \times_H Y \cong X' \times_{N \rtimes K} Y',$$

giving

$$res_K^H \circ J_N^H = j_N^{N \rtimes K} \circ res_{N \rtimes K}^G$$

directly, a result in Chabert.

If Z is a proper cocompact G'-space and if $A = C_0(Z)$, then

$$A_X \cong C_0(Z \times_G X),$$

so that we get a map

$$RK^{G'}(Z, A) \to RK^G(Z \times_G X, A_X).$$

If X is a G'-proper space, $Z \times_G X$ is a G'-proper space, and there is a continuous G-equivariant map $Z \times_G X \to \underline{E}G$, thus giving a morphism

$$K^{top}(G', A) \to K^{top}(G, A_X)$$

In the case of a transtive action of G on X, with stabilizers G_s. If G acts transitively and properly on X, we have the Green-Julg isomorphism

$$\Psi : (A \rtimes G_s) \otimes \mathfrak{K}(l^2(G/G_s)) \to (A \otimes C_0(X)) \rtimes_r G$$

and denote by $M_D : D \to D \otimes \mathfrak{K}$ the Morita equivalence. Then the following diagram commutes

$$\begin{array}{ccc} K^{top}(G_s, A) & \xrightarrow{\mu_{G_s,A}} & K(A \rtimes G_s) \\ \Big\downarrow{\scriptstyle j_X} & & \Big\downarrow{\scriptstyle \Psi_* \circ M_{A \rtimes G_s}} \\ K^{top}(G, A_X) & \xrightarrow{\mu_{G,A_X}} & K(A_X \rtimes G) \end{array}$$

with vertical isomorphisms. This shows that

$$BC(G_s, A) \iff BC(G, A_X)$$

This can be generalized a lot, at the expense of loosing clarity. Indeed, in the case a generalized morphism of groupoids $\phi : G \to G'$, LeGall has shown [] how to build a morphism

$$\phi^* : KK^{G'}(A, B) \to KK^G(\phi^* A, \phi^* B)$$

and j_X is ϕ^* for the generalized morphism induced by a bimodule.

There is actually a proof in Valette's book (proper actions...) showing the naturality of the maximal assembly map, and the naturality of the assembly map for either monomorphisms, of general morphisms where the domain is K-amenable. This follows from the previous construction, considering the graph of the morphism. Also, we can get a bit more: the assembly map is natural with respect to proper morphisms, which in the group case is equivalent to having a compact kernel.

3.6.9 Partial assembly map

One has to define a cutoff for G from a cuttof from G', so that the assembly maps are intertwined.

Example: In the case $G = N \rtimes H$, from a cuttof for G acting properly on X, one get a H-equivariant cuttoff for the action of N by

$$c_N(x) = \sum_{h \in H} c(hx),$$

and the corresponding projections satisfy $p_{N,X} \rtimes H = p_{G,X}$ so that

$$j_H([p_{N,X}]) = [p_{G,X}],$$

and

$$\begin{aligned} j_H \circ \mu_N^H(x) &= j_H([p_{N,X}]) \otimes j_H \circ j_N^H(X) \\ &= [p_{G,X}]) \otimes j_G(x) \\ &= \mu_G(x) \end{aligned}$$

3.7 Higher assembly maps

In all this piece, G is an étale groupoid, with base space G^0. The main goal is to prove a stability result of the Baum-Connes conjecture concerning extensions of groupoids. This generalizes previous results of the aurhor's advisor []. The new ingredient is the definition of an assembly map for every *generalized morphism* of groupoids, which is natural. The restriction principle (refered as Going-Down in [?]) then gives a criteria to transfer the Baum-Connes conjecture from one groupoid to the other using nice morphisms. This technique gives a conceptual way of understanding several existing proofs (Baum-Connes for trees, extensions) while giving new results (non-proper actions by affine isometries on a Hilbert space).

3.7.1 Equivariant setting for C^*-algebras and Hilbert modules

Recall that a G-algebra A is given by a

- a C^*-algebra A,
- a structure of $C_0(G^0)$-algebra on A, i.e. a non-degenerate $*$-morphism $\phi : C_0(G^0) \to Z(M(A))$,
- an isomorphism $\alpha : s^*A \to r^*A$ of $C_0(G)$-algebras such that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for every $(g, h) \in G^{(2)}$.

Note that a $C(G^0)$-algebra A defines a *continuous field of C^*-algebra $\mathcal{A}$* (see [], Dixmier section ...), with fibres

$$A_x = A/(\phi(I_x)A) \quad \forall x \in G^0,$$

where $I_x = \{f \in C_0(G^0) \mid f(x) = 0\}$. Denote by a_x the image of $a \in A$ under the quotient map $A \to A_x$. The $C_0(G^0)$-algebra A is called continnous if

$$x \mapsto \|a_x\|_{A_x}$$

is continuous for every $a \in A$. There is a correspondence between continuous $C_0(G^0)$-algebras and continuous fields. More precisely, one can rebuild A from its C^*-algebra of sections

$$A \cong C^*\{s \in \Gamma_{\mathcal{A}} \mid x \mapsto \|s_x\| \in C_0(G^0)\}.$$

3.7.2 Induced G-algebras and G-modules

Recall that LeGall defined in his thesis ([?]) induction for generalized morphisms of groupoids. A strict morphism betweem topological groupoids is a continuous functor.

Definition 3.7.1. A *generalised morphism* from H to G is a triple

$$(X, p, f)$$

where:

- X is a locally compact space,
- $p : X \to H^0$ is a continuous, open and surjective map,

- a strict morphism $f : H_X \to G$.

For convenience, we denote by X the generalized morphism (X, p, f). If A is a H-algebra, recall that $Ind_X(A)$ ([**?**][**?**]) is identified with the C_0-sections of

$$s \in \Gamma_b(X, p^*\mathcal{A}) \text{ s.t. } \alpha_g(s(x)) = f(xg^{-1}) \text{ and } xG \to \|s(x)\| \in C_0(X/G)$$

If the right G-action is proper, $B = Ind_X(A)$ is a $C_0(X/G)$-algebra, with fibers

$$B_{xG} = (Ind_X(A))_{xG} \cong A_{p(x)} \quad \forall x \in X.$$

The action of H is given by $\beta s^*B \to r^*B$

$$\beta_h(s(x)) = s(h^{-1}x)\forall(x,h) \in X \times_{r,q} H.$$

Lemma 3.7.2. If A is a proper G-algebra and X is proper, then $Ind_X(A)$ is a proper H-algebra.

Proof. Let Z be a proper right-G-space, then

$$Ind_X(C_0(Z)) \cong C_0(\tilde{Z})$$

with $\tilde{Z}$ being the quotient pull-back

$$\tilde{Z} = X \times_{G^0} Z/G$$

where the G-action is $g \cdot (x, z) = (xg^{-1}, gz)$.

I have a problem with the definitions: with the usual definitions of generalized morphisms, for each $(h, x) \in H \times_{H^0} X$, there exists a unique $\gamma(h, x) \in G^{p(x)}$ such that

$$h \cdot x = x \cdot \gamma(h, x).$$

Then if the action of G on Z is proper and the map $X \to H^0$ is proper, so is the induced space. □

For convenience, $Ind_X(A) = A_X$. Note: if G is a group, and X is the inclusion of the point, then Ind_X is the crossed product functor. The induction map induces a natural transformation

$$j_X : KK^G(A, B) \to KK^H(A_X, B_X)$$

which preserves properness and G-compactness, so we get a map

$$j_X : RK^G(\underline{E}G, A) \to RK^H(\underline{H}, A_X).$$

Lemma 3.7.3.

$$j_X \circ j_Y = j_{X \circ Y}$$

3.8 Topics class in Analysis

Title: Large scale geometry and operator algebras

Spring 2020, Clément Dell'Aiera

Description

Large scale geometry studies geometric properties of a metric space that emerge when you look at it from far away. The motivation behind this course is the use of this idea in operator algebras and index theory. More precisely, we will construct *exotic* operator algebras using the geometry at infinity of discrete groups. The main goal is to study some non exact sequences of C^*-algebras, which are of importance in Noncommutative Geometry and are related to the Novikov conjecture, an important problem in Algebraic Topology.

Organization of the class

The first part of the class will focus on finitely generated groups and their large scale geometry. We will introduce Cayley graphs and box spaces, two type of metric spaces associated to groups. We will cover a lot of examples, and discuss expanders (with maybe Margulis' proof of their existence).

The second part of the class will be devoted to building operator algebras associated to these objects. We will introduce $*$-algebras and their completions, with a gentle reminder on every notion. Metric amenabilty and its relationship to approximation properties of the so-called Roe algebra will be studied. We will introduce exact sequences of C^*-algebra and define a particular sequence associated to the uniform Roe algebra of a metric space, and its localization at infinity.

If time allows, we will end by a informal introduction to the problems in algebraic topology that these allow to solve.

Prerequisites

Knowing what are groups, rings, vector spaces, topological spaces and matrices. We will provide a reminder on all the notions that need to be covered, and no prior knowledge is assumed in functional analysis, algebraic topology or noncommutative geometry.

Looking at things from far away

What is the chronology of ideas concerning looking at infinity? Maybe it starts with projective geometry: embedding the affine plane $\mathbb{A}^n$ into the projective space $\mathbb{P}^n$. You also have the Gromov boundary of a hyperbolic group $\overline{\Gamma} = \Gamma \cup \partial\Gamma$. This idea generalizes to CAT(0) spaces (see Bridson-Haefliger). Formalization gives you the notion of compactification: a compact topological space $\overline{X}$ containing X as an open dense subset. The boundary ∂X is then the complement of X in $\overline{X}$. Of course, the compactification will always have more structure. Typically, X comes with a group of automorphisms which extends to homeomorphisms on the boundary: in projective space, any affine map induces a homography, and a quasi-isometry extends to an homeomorphism of the Gromov boundary of a hyperbolic group. The biggest compactification is the Stone-Čech compactification, a totally disconnected space whose points are ultrafilters. This time, the "group" of automorphisms is only a semigroup, that of *partial translations*, which extends to a pseudogroup of partial homeomorphisms acting on the compactification.

3.9 Mayer-Vietoris

3.10 Quantum groups

3.11 Property T

3.12 Number theory

3.13 Fock spaces, CuntzKrieger algebras, and second quantization

Chapter 4

Old notes

4.1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in K-theory, like determining generators of K-group of some known C^*-algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for $\mathbb{R}^n$. (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space $\mathcal{G} = X \rtimes G$. The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [?], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}.$$

Indeed, suppose Y is a $\mathcal{G}$-proper cocompact space, then $Y \rtimes \mathcal{G}$ is a proper groupoid, so there exists a cutt-off function $c : Y \to [0,1]$ such that :

$$\sum_{g \in \mathcal{G}^{p(y)}} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^{p(y)}} c(yg)\delta_g$$

which is a $\mathcal{G}$-equivariant continuous map. Moreover Z_d is proper and cocompact, and there exists a d s.t. the map takes its values in it.

Now if $\mathcal{G} = X \rtimes G$, $Z_d \simeq X \times Z'_d$ where $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$, so that $KK^{\mathcal{G}}(\Delta, A) \simeq KK^G(\Delta', A)$, where Δ and Δ' are respectively the 0-dimensional part of the equivariant complexes Z_d and Z'_d. This is true because the action of G on Z'_d is proper and cocompact, see lemma 3.6 of [?]. Now a standard Mayer-Vietoris argument (theorem 3.8 [?]) concludes to show that $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$.

As $C^*_r\mathcal{G} = C_0(X) \rtimes_r G$, we see that the Baum-Connes assembly map for $\mathcal{G}$ with coefficients in A is equivalent to

$$K^{top}_*(G, A) \to K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

4.1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

4.1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle $U(2) \to U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$. You can foliate the fibers by an irrational rotation θ, so that you have an action of $\mathbb{R}$ on $C(U(2))$. Reducing to a complete transversal (take $SU(2)$), the algebra $C(U(2)) \rtimes \mathbb{R}$ turns out to be Morita equivalent to $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$ (a general result of foliation groupoids I think). $\underline{A}$ can be reduced to $C(\overline{D}) \otimes A_\theta$ and to $Ind_{\mathbb{T}^2}^{U(2)}\ A_\theta$.

Question : Compute the generators of the K-theory of $\underline{A}$.

4.1.3 Foliations

4.1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus $M = \mathbb{T}^2$. Its fundamental group $\Gamma = \mathbb{Z}^2$ acts on its universal cover $\tilde{M} = \mathbb{R}^2$ by isometries, and the electromagnetic field A gives a two-form w (its curvature) on $\tilde{M}$, so a 2-cocycle on the fundamental groupoid of $\tilde{M}$:

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_\Delta \tilde{w}}$$

where Δ a geodesic triangle between the 3 points. It turns out that $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$, so that $\tilde{w}$ determines a number $\theta \in [0,1)$, and the twisted reduced algebra of the fundamental groupoid w.r.t. $\tilde{w}$ is equal to $A_\theta = C(\mathbb{T}^2) \rtimes_{r,\theta} \mathbb{Z}^2$. This situation generalizes to general manifold whose fundamental cover are equiped with a line bundle and a conection. We can then associate a 2-cocycle on the fundamental groupoid of $\tilde{M}$ to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

4.2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let G be a real reductive group. For all parabolic subgroup P, there is only one nilpotent normal subgroup N, and the Levi is defined as $P = LN$. The idea of Pierre Julg is to fix first a Levi susgroup L of G. Now there is only a finite numbers of choices for N, so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group $W_L = N_G(L)/L$ acts on it by $w.N = wNw^{-1}$.
Pierre Clare defined a C_r^*L-module $C_r^*(G/N)$, equipped with and action of C_r^*G by compacts operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$, where $L = MA$, $\hat{M}_d$ is the discrete dual of M, and $\hat{A} = \mathfrak{a}^*$. Then $\sigma \otimes \tau$ is a représentation of $MA = L$, which we can trivially extend to N to induce it on G. Pierre Clare showed the following fact :

$$Ind_P^G\ H_{\sigma\otimes\tau\otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma\otimes\tau}.$$

For every $\tilde{w} \in N_G(L)$, the operator $\rho(\tilde{w}) : C_r^*(G/N) \to C_r^*(G/w.N)$ is well defined and gives a morphism

$$Ad\ \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \to \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because C_r^*G is acting on $C^*(G/N)$ by compact operators. This gives a morphism

$$C_r^*G \to \bigoplus_{[L]} \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard C^*-algebraic tools).

The first step would be to prove that

$$\begin{array}{rcl} C_r^*G & \to & \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N))\right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for $F_N^{-1}(g) = \mathrm{Tr}_{C_d^*L}\ (T\pi_N(g^{-1}))$.

4.2.1 In $SL(2, \mathbb{R})$

In this case, G acts on the Poincaré disc by homographies, and P can be taken as the stabilizer of a point at infinity, and L stabilizes a geodesic, that is to say two points at infinity, so that

$$P_{1,1} \simeq \{\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}\}, \quad L \simeq \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\}, \quad N \simeq \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing P amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and L. Now fix two points at infinity, which gives you L. You now only have two choices for P, and the two are exchanges under the action of W_L on the nilpotent groups.

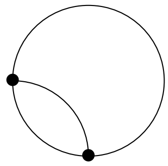


Figure 4.1: Choices for the Levi subgroup

4.3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let $\mathcal{N}$ be the smallest class of C^*-algebras containing $\mathbb{C}$, closed under countable inductive limits, stable by KK-equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in $\mathcal{N}$, so is the third). Here is the classical theorem :

Theorem 4.3.1 (Universal Coefficient Theorem)**.** Let A and B be two separable C^*-algebras, where A is in $\mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow KK_*(A,B) \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ??
- The second map is given by the boundary element associated to any impair K-cycle. Namely, if $z \in KK^1(A,B)$, let (H_B, π, T) be a K-cycle representing z, and P the associated projector $P = \frac{1+T}{2}$. Define the pull-back

  $$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$

  Then the boundary of the folowing extension

  $$0 \longrightarrow \mathfrak{K}_B \longrightarrow E^{(\pi,T)} \longrightarrow A \longrightarrow 0$$

  is given by $\partial = -\otimes z : K_*(A) \longrightarrow K_*(B)$ which depends only on z. The map is just $z \mapsto \partial$

- If $\partial = 0$, then the sequence associated to z splits and we have exact sequences

  $$0 \longrightarrow K_*(B) \longrightarrow K_*\left(E^{(\pi,T)}\right) \longrightarrow K_*(A) \longrightarrow 0$$

  which gives an element of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

4.3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Miscenko and Kasparov.

Proposition 4.3.2. Let G be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold M. Then

$$K^{top}(G) \xrightarrow{\mu} K(C_r^*G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for G.

The original point being that G. Yu can prove this (how ?) without usig the heavy machinery of the Dirac Dual-Dirac method, nor anything related to KK^G-theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

Proposition 4.3.3. Let G be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture hold for G.

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^* P_d(X_{2k})) \xrightarrow{\beta} K\left(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k}))\right)$$

is an isomorphism.

Here are some details : first decompose $G = G_0 \cup G_1$ into two subspaces, which are not necesseraly subgroups, such that each is a R-disjoint union of bounded subsets (in fact finite since G is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define $\prod^R C^*(P_d(X_{2k}) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), prop(T_{2k}) \leq R\}$, so that $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$, and each X_{2k} corasely embedds into some $\mathbb{R}^{m_k}$. The isomorphism of $\beta \circ \mu$ implies the injectivity of μ, and by cutting and pasting, μ can be shown to be injective for G so that Novikov is satisfied.

4.4 Funky questions, ideas of talks

4.4.1 Expanders

Here are some interesting questions I had after a talk on expanders.

Plan of the talk

I first gave a motivation for considering expanders. Namely, we are interested in the following network theory problem : can we construct a network as big as we want, such that the cost is controlled and which is not subject to easy failure ?

Building a network as big as we want means we want to consider a family of graphs $X_j = (V_j, E_j)$ such that $|V_j| \to +\infty$, and controlling the cost means that $deg(X_j) < k$ for all j. But what does "not easily subject to failure" means ? For this, I want to explain why we should ask our family to stay well connected and why the second value of the discrete Laplacian is a good way to measure that.

The idea is to relate the Laplacian to the uniform random walk on the graph, and to show that $\lambda_1(X)$ controlls the speed of convergence of the uniform random walk to the stationary measure which is the uniform probability on the graph, given by $\nu(x) = C.deg(x)$.

A family of graphs satifying the previous conditions and such that $\lambda_1(X_j) > c > 0$ is called an expander. If time allows, one can then elaborate on metric properties of this type of graphs. The impossibility to embed them coarsely into any separable Hilbert space, and the relations to the Baum-Connes conjecture are close to my work.

Questions

- Paolo Pigato : What is the dynamic at the limit ?
- Anne Briquet : Is $\lambda_1(X)$ such a good way to measure the connectedness of a graph, if you consider the phenomenon of cuttoff for finite Markov Chains.

4.4.2 Ideas of funky talks

- What is the relation between the Fourier transform and quantum groups ?
- What is the relation between the Runge Kutta methode and renormalization in QFT ?
- What is the relation between Brownian motion and second quantization ?