# Notes

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# Table des matières

During the past years, there has been a growing interest on the links between several conjectures involving assembly maps. This report will focus on the link between the coarse Baum-Connes conjecture and the Novikov conjecture. If $\Gamma$ is a finitely generated group, the descent principle assures that if the coarse Baum-Connes map for $\Gamma$ as a metric space with the word length is an isomorphism, then the Baum-Connes assembly map for $\Gamma$ is injective, thus the Novikov conjecture holds for $\Gamma$.

Following ideas of M. Gromov, G. Yu introduced new coarse concepts in the study of these assembly maps. He was able to prove the coarse Baum-Connes conjecture for proper metric spaces with finite asymptotic dimension, which is a coarse analogue of the topological covering dimension. Later on, in a paper with Guenter and R. Tessera [4], they defined decomposition complexity for metric spaces, which is a broad generalization of asymptotic dimension. In particular, proper metric spaces with finite asymptotic dimension are of finite decomposition complexity. At the end of [4], as concluding remarks, the authors point out that one should be able to derive a new proof of the coarse Baum-Connes conjecture for spaces with finite decomposition complexity. We should emphasize that this is already known : a space which is finitely decomposable has property (A), hence verifies the coarse Baum-Connes conjecture by the work of G. Yu. [27] But the techniques of this proof is highly analytical, it uses a Dirac-Dual Dirac type construction, which involves infinite dimensional analysis. The suggestion of [4] is to give a geometrical proof, using a coarse Mayer-Vietoris argument in the spirit of the proof of the Baum-Connes conjecture for spaces with finite asymptotic dimension.

Such a proof was given in the setting of algebraic $K$-theory in a paper of D. A. Ramras, R. Tessera and G. Yu where they established the integral Novikov conjecture for algebraic $K$-theory of group rings $R[\Gamma]$ when the group $\Gamma$ has FDC (finite decomposition complexity). Their proof uses the continuously controlled algebraic $K$-theory groups very intensively : their key lemma is a vanishing theorem of these groups. In a series of papers [7][8], H. Oyono-Oyono and G. Yu developed an analogue of this controlled $K$-theory for operator algebras, which they named quantitative $K$-theory. It consists of a family of groups $\hat{K}(A) = (K^{\epsilon,r}(A))$ for $r \geq 0, 0 < \epsilon < \frac{1}{4}$ and $A$ a filtered $C^*$-algebra, which we shall describe later. They were able to define quantitative assembly maps that factorize the usual ones, and to give equivalence between isomorphisms of the assembly map and quantitative statements.

Following the route of these articles [7][8], we will define quantitative assembly maps for étale groupoids with a proper length. These assembly maps are equivalent to the coarse quantitative assembly maps for proper metric spaces $X$ defined in [8] if one takes $G = G(X)$, the coarse groupoid of $X$. We give also quantitative statements equivalent to a certain isomorphism. **(rerédiger ce paragraphe de façon plus précise une fois les résultats écrits)**

# 1 Review of quantitative $K$-theory

This section presents basic constructions of quantitative $K$-theory for operator algebras that we shall use. For more details, see the original article of H. Oyono-Oyono and G. Yu.[7] We will refer either to quantitative or controlled $K$-theory for the same object, namely a family of abelian groups $\hat{K}(A) = (K^{\epsilon,R})$ where $R > 0, 0 < \epsilon < \frac{1}{4}$, defined for a filtered $C^*$-algebra $A$. The motivating idea is to keep track of propagation of an operator while taking his (possibly higher) index. The main example is that of Roe algebras.

## 1.1 Roe algebras and filtration

Let $(X, d)$ be a discrete proper metric space, i.e. its closed ball are compact, that is uniformly bounded, so that for every $R > 0$, there exists an integer $N \geq 0$ such that every ball of radius $R$ contains less than $N$ elements. A $X$-module is a hilbert space $H$ equiped with a $C^*$-morphism $\rho : C_0(X) \to \mathcal{L}(H)$. To lighten notations, we write $fx$ instead of $\rho(f)x$ if $f \in C_0(X)$ and $x \in X$. All these definitions can be found in [**?**]

**Définition 1.** Let $H$ be a $X$-module.

- An operator $T \in \mathcal{L}(H)$ is locally compact if for every $f \in C_0(X)$, $fT$ and $Tf$ are compact operators, where $f$ is understood as a multiplication operator.
- An operator $T \in \mathcal{L}(H)$ is of finite propagation bounded by $R > 0$ if for every pair of functions $f, g \in C_0(X)$ such that $d(\text{supp } f, \text{supp } g) > R$, $fTg = 0$.
- We denote by $C_R[X]$ the set of locally compact operators with finite propagation bounded by $R$. The Roe algebra of $X$ is $C^*(X)$, the closure of $\cup_{R>0} C_R[X]$ in the operator topology of $\mathcal{L}(H)$.

An simple example is given by $l^2(X) \otimes H$ with $H$ a separable Hilbert space, in which $C_R[X]$ is the algebra of operators $(T_{xy})_{x,y \in X}$ such that $T_{x,y} \in K(H)$ for every $x, y \in X$, and $T_{xy} = 0$ as soon as $d(x, y) > R$.
Remark : one coulde replace Hilbert spaces by Hilbert modules $E$ over a $C^*$-algebra $B$ in this definition, $\mathcal{L}(H)$ by adjoinable operators $\mathcal{L}_B(E)$ and $K(H)$ by compact operators $K_B(E)$, to obtain $C^*(X, B)$, the Roe algebra with coefficient in $B$. The Roe algebra $C^*(X, B)$ enjoys functorial properties in $B$.

This example motivates the following definition.

**Définition 2.** A $C^*$-algebra $A$ is said to be filtered if there are closed $*$-stable linear subspaces $A_R$ for every $R > 0$ such that

- $A_s \subset A_r$ when $s \leq r$,
- $\cup_{R>0} A_R$ is dense in $A$,
- $A_s.A_r \subset A_{s+r}$ for every $r, s \geq 0$,
- $\forall r > 0, 1 \in A_r$ when $A$ is unital.

A $C^*$-morphism between filtered $C^*$-algebras $\phi : A \to B$ is filtered if $\phi(A_R) \subset B_R$ for every $R > 0$.

If $A$ is a non-unital $C^*$-algebra, let $A^+$ be the unital $C^*$-algebra containing $A$ as a two-sided ideal, defined as :

$$A^+ = \{(a, \lambda) \in A \times \mathbb{C}\}$$
$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a, \lambda)^* = (a^*, \overline{\lambda})$$

with the norm operator

$$||(a, \lambda)|| = \sup\{||ax + \lambda x|| : x \in A, ||x|| = 1\}.$$

When $A$ is not unital and filtered by $(A_R)_{R>0}$, $A^+$ is filtered by $A_R^+ = \{(x, \lambda) : x \in A_R, \lambda \in \mathbb{C}\}$.

### 1.2 Definition of quantitative $K$-theory

### 1.3 Morita equivalence

As in classical $K$-theory, we have an isomorphism which we call the (controlled) Morita equivalence.

**Proposition 1.** Let $A$ be a filtered $C^*$-algebra and $H$ a separable Hilbert space. We denote by $K_A$ the $C^*$-algebra of compact operators of the standard Hilbert module $H_A$, which is $C^*$-isomorphic to $A \otimes K(H)$. Let $e$ be any rank-one projection in $K(H)$. Then the $C^*$-morphism

$$\begin{array}{ccc} A & \to & K_A \\ a & \mapsto & a \otimes e \end{array}$$

induces an $\mathbb{Z}_2$-graded isomorphism

$$M_A^{\epsilon,R} : K^{\epsilon,R}(A) \to K^{\epsilon,R}(K_A)$$

for every $R > 0$ and $0 < \epsilon < \frac{1}{4}$.

## 2 Quantitative statements

The more general setting of the Baum-Connes conjecture [21] is that of a locally compact $\sigma$-compact Hausdorff groupoid $\mathcal{G}$ endowed with a Haar system, together with a coefficient $C^*$-algebra $B$ acted upon by $\mathcal{G}$, which give rise to an assembly map

$$\mu_r : K_*^{top}(\mathcal{G}, B) \to K_*(B \rtimes_r \mathcal{G}).$$

The left hand side $K_*^{top}(\mathcal{G}, B)$ is the $K$-homology of the classifying space $\mathcal{E}\mathcal{G}$ for proper actions of $\mathcal{G}$ in coefficient in $B$. We give a sketch of the construction when $\mathcal{G}$ is étale. Let $d \geq 0$ and $P_d(\mathcal{G})$ be the Rips complex of $\mathcal{G}$, i.e. the space of probabilities supported on a fiber $\mathcal{G}^x$ for a $x \in \mathcal{G}^{(0)}$

$$P_d(\mathcal{G}) = \{p \in \mathcal{P}(\mathcal{G}) : \exists x \in \mathcal{G}^{(0)}, r^*p = \delta_x, \text{supp } p \subset B(e_x, d)\}.$$

Then $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B)$ is defined to be the inductive limite of $KK^{\mathcal{G}}(C_0(X), B)$ for $X$ $\mathcal{G}$-proper $\mathcal{G}$-spaces (such that $X/G$ is compact). If $d \leq d'$, we have a morphism $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B) \to KK^{\mathcal{G}}(C_0(P_{d'}(\mathcal{G})), B)$ naturally induced by the inclusion $P_d(\mathcal{G}) \subset P_{d'}(\mathcal{G})$, and the $K$-homology of $\mathcal{G}$ is defined as

$$K_*^{top}(\mathcal{G}, B) = \lim_{d \to \infty} KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B).$$

In his thesis [5], P.-Y. Le Gall constructed the Kasparov transform for the action of a groupoid

$$j_{\mathcal{G}} : KK^{\mathcal{G}}(A,B) \to KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$$

for any $\mathcal{G}$-$C^*$-algebras $A$ and $B$. It is also in this paper that equivariant $KK$-theory for groupoids and the corresponding Kasparov product are defined. One can then give an formula for the assembly map, namely if $z \in KK^{G}(C_0(X), B)$ for a $\mathcal{G}$-proper $\mathcal{G}$-space $X$ of $P_d(\mathcal{G})$, then

$$\mu_r(z) = [\mathcal{L}_X] \otimes_{C_0(X) \rtimes_r \mathcal{G}} j_{\mathcal{G}}(z) \in K_*(B \rtimes_r \mathcal{G})$$

holds, where $[\mathcal{L}_X]$ is the class of a canonical element associated to $X$ which is to be thought of as a Miscenko bundle over $C_0(X) \rtimes_r \mathcal{G}$.

The remaining of this section will be devoted to the construction of a controlled Kasparov transformation for every $z \in KK^{\mathcal{G}}(A,B)$ :

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G})$$

which is of course a controlled morphism which induces right multiplication by $j_{\mathcal{G}}(z)$ in $K$-theory. This will allow us to define a bunch of quantitative assembly maps

$$\mu_{\mathcal{G}}^{\epsilon,R} : K^{top}(\mathcal{G}, B) \to K^{\epsilon,R}(B \rtimes \mathcal{G})$$

inducing the assembly map in $K$-theory, and to study the relation between the quantitative Baum-Connes conjecture and the classical one for $\mathcal{G}$.

## 2.1 Length, propagation and controlled six-terms exact sequence

Let $\mathcal{G}$ be a locally compact groupoid with base $\mathcal{G}^{(0)} = X$, a compact space, endowed with a Haar system $\lambda = (\lambda^x)_{x \in X}$. We suppose that $\mathcal{G}$ comes with a proper length $l$, that is a family of application $(l^x)_{x \in X}$ defined on the fibers $\mathcal{G}^x$ with values in $\mathbb{R}_+$, such that

$$\begin{aligned} &l^x(e_x) = 0 \\ &l^{r(\gamma)}(\gamma) = l^{s(\gamma)}(\gamma^{-1}) \\ &l^x(\gamma_1^{-1}\gamma_2) \leq l^x(\gamma_1) + l^x(\gamma_2). \end{aligned}$$

That length allows us to define a filtration on crossed-product algebras of $\mathcal{G}$ by

$$(A \rtimes \mathcal{G})_r = \{ f \in C_c(\mathcal{G}, A) : \text{ supp } f \subset \cup_{x \in X} B_x(e_x, r) \}$$

for any $\mathcal{G}$-algebra $A$. Here, $B_x(e_x, r)$ is the ball $\{\gamma \in \mathcal{G} : l^{r(\gamma)} \leq r\}$, and $\rtimes$ can be either the reduced cross-product $\rtimes_r$ or the maximal one $\rtimes_{max}$. Recall that $A \rtimes \mathcal{G}$ is functorial in $A$, from the category of $\mathcal{G}$-$C^*$-algebras with $\mathcal{G}$-equivariant $C^*$-morphisms to the category of $C^*$-algebras with $C^*$-morphisms. For $\phi : A \to B$ a $\mathcal{G}$-equivariant $C^*$-morphism, we denote by $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$ the induced $C^*$-morphism.

If $\quad 0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad$ is a semi-split exact sequence of $\mathcal{G}$-$C^*$-algebras,

then $\quad 0 \longrightarrow J \rtimes \mathcal{G} \xrightarrow{\phi_{\mathcal{G}}} A \rtimes \mathcal{G} \xrightarrow{\psi_{\mathcal{G}}} A/J \rtimes \mathcal{G} \longrightarrow 0 \quad$ is a flitered semi-split exact sequence. From this, we can state the following proposition.

**Proposition 2.** There exists a control pair $(\lambda, h)$ such that for every semi-split extension of $\mathcal{G}$-$C^*$-algebras

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad ,$$

the following diagrams commutes and are exact

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_r \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_r \mathcal{G}) \end{array} \quad ,$$

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_{max} \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_{max} \mathcal{G}) \end{array} \quad .$$

## 2.2 The Kasparov transform

Let $A$ and $B$ be two $\mathcal{G}$-$C^*$-algebras, and $H$ a separable Hilbert space, $l^2(\mathbb{Z})$ for instance, and $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. The standard Hilbert module over $B$ is denoted by $H_B = H_{\mathcal{G}} \otimes B$, and $K_B$ is the algebra of compact operators for $H_B$, i.e. $K(H) \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Every $K$-cycle $z \in KK^G(A, B)$ can be represented as a triplet $(H_B, \pi, T)$ where :

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of $A$ on $H_B$.
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator.
- $T$ and $\pi$ verify the $K$-cycle condition, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ and $\pi(a)(g.T - T)$ are compact operator over $H_B$ for all $a \in A, g \in \mathcal{G}$.

Set $T_{\mathcal{G}} = T \otimes id_{B \rtimes \mathcal{G}} \in \mathcal{L}_{B \rtimes \mathcal{G}}(H_B \otimes (B \rtimes \mathcal{G})) \simeq \mathcal{L}_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$, and $\pi_G : A \rtimes \mathcal{G}v \to L_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$. Then, according to Le Gall [5], $(H_{B \rtimes \mathcal{G}, \pi_{\mathcal{G}}, T_{\mathcal{G}}})$ represents the $K$-cycle $j_{\mathcal{G}}(z) \in KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$. Let us construct a controlled morphism associated to $z$,

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to K(B \rtimes \mathcal{G}),$$

which induces right multiplication by $j_{\mathcal{G}}(z)$ in $K$-theory.

### 2.2.1 Odd case

Let first do the for work for $z \in KK_1^{\mathcal{G}}$. Let $(H_B, \pi, T)$ be a $K$-cycle representing $z$. Set $P = \frac{1+T}{2}$ and $P_{\mathcal{G}} = P \otimes id_{B \rtimes \mathcal{G}}$. We define

$$E^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in A \rtimes \mathcal{G}, y_i n K_{B \rtimes \mathcal{G}}\}$$

a $C^*$-algebra which is filtered by

$$E_R^{(\pi,T)} = \{(x, P_G\pi_G(x)P_{\mathcal{G}} + y) : x \in (A \rtimes \mathcal{G})_R, y_i n K \otimes (B \rtimes \mathcal{G})_R\}$$

which gives us a filtered extension

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow E^{(\pi,T)} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

and semi split by $s : \begin{cases} A \rtimes_r \mathcal{G} & \to & E^{(\pi,T)} \\ x & \mapsto & (x, P_{\mathcal{G}}\pi_{\mathcal{G}}(x)P_{\mathcal{G}}) \end{cases}$.

Let us show that the controlled boundary map of this extension does not depend on the representant chosen, but only on the class $z$.
Let $(H_B, \pi_j, T_j), j = 0, 1$ two $K$-cycles which are homotopic via $(H_{B[0,1]}, \pi, T)$. We denote $e_t$ the evaluation at $t \in [0, 1]$ for an element of $B[0, 1]$, and set $y_t = e_t(y)$ for such a $y$. The $*$-morphism

$$\phi : \begin{cases} E^{(\pi,T)} & \to & E^{(\pi_t,T_t)} \\ (x, y) & \mapsto & (x, y_t) \end{cases}$$

satisfies $\phi(K_{B[0,1]\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}})$ and makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & K_{B[0,1]\rtimes_r\mathcal{G}} \arrow[r] \arrow[d, "\phi_{|K_{B[0,1]\rtimes_r\mathcal{G}}}"] & E^{(\pi,T)} \arrow[r] \arrow[d, "\phi"] & A \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & K_{B\rtimes_r\mathcal{G}} \arrow[r] & E^{(\pi_t,T_t)} \arrow[r] & A \rtimes_r \mathcal{G} \arrow[r] & 0 \end{tikzcd}.$$

According to [7], remark 3.7., the following holds

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_t,T_t)}} = \phi_* \circ D_{K_{B[0,1]\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As $id\otimes e_t$ gives a homotopy between $id\otimes e_0$ and $id\otimes e_1$, and as if two $*$-morphisms are homotopic, then they are equal in controlled $K$-theory,

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_0,T_0)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_1,T_1)}}$$

holds, and the boundary of the extension $E^{(\pi,T)}$ depends only on $z$.

**Définition 3.** The controlled Kasparov transform of an element $z \in KK_1^{\mathcal{G}}(A, B)$ is defined as the compostion

$$J_{red,\mathcal{G}}(z) = \mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}} \circ D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

**Proposition 3.** Let $A$ and $B$ two $\mathcal{G}$-$C^*$-algebras. For every $z \in KK_1^{\mathcal{G}}(A, B)$, there exists a controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}_*(A \rtimes_r \mathcal{G}) \to \hat{K}_{*+1}(B \rtimes_r \mathcal{G})$$

such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in $K$-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z+z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_1^G(A_2, B)$.

**Preuve 1.** (i) Le bord $[\delta_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}] \in KK_1(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ associé à l'extension $E^{(\pi,T)}$ induit par définition, modulo équivalence de Morita, l'application $j_{red,\mathcal{G}}$, ce qui assure directement ce point.

(ii) Si $z, z'$ sont deux eléments de $KK_1^G(A, B)$, repésentés par des $K$-cycles $(H_B, \pi_j, T_j)$, et si l'on note $(H_B, \pi, T)$ un $K$-cycle représentant la somme $z+z'$, alors $E^{(\pi,T)}$ est naturellement isomorphe à l'extension somme des $E_j := E^{(\pi_j,T_j)}$

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow D \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

où

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j, E_i) \right\}.$$

Par naturalité du bord contrôlé [7], le bord de la somme de deux extensions est la somme des bords de chaque extension, d'où le résultat.

(iii) Soit $z \in KK_1^{\mathcal{G}}(A_2, B)$, représenté par un cycle $(H_B, \pi, T)$. Un représentant de $f^*(z)$ est $(H_B, f^*\pi, T)$ avec bien sûr $f^*\pi = \pi \circ f$. L'application

$$\phi : \begin{cases} E^{f^*(\pi,T)} & \to \quad E^{(\pi,T)} \\ (x, P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) & \to \quad (f_{\mathcal{G}}(x), P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) \end{cases}$$

vérifie

- $\phi(K_{B\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$, et s'insère dans le diagramme

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{f^*(\pi,T)} & \longrightarrow & A_1 \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\phi} & & \downarrow{f_{\mathcal{G}}} & & \\ 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A_r \rtimes_r \mathcal{G} & \longrightarrow & 0 \end{array} .$$

- Elle entrelace les sections de ces deux extensions.

La remarque 3.7 de [7] assure donc que

$$D_{K_{B\rtimes_r\mathcal{G}},E^{f^*(\pi,T)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}} \circ f_{\mathcal{G},*}$$

, et l'assertion est claire en composant par $\mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}}$.

## 2.3 Quantitative assembly maps

## 2.4 Quantitative statements

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