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1 Groupoïds

1.1 Definitions

Définition 1. A groupoid is a small category whose arrows are all invertible. More concretly, it is the data of a set G together with a set of units $G^{(0)}$ and two maps $r, s : G \to G^{(0)}$. We can compose two arrows when the range of the first agrees with the source of the second. If we denote, for $x \in G^{(0)}$, $G_x = \{\gamma \in G : s(\gamma) = x\}$ and $G^x = \{\gamma \in G : r(\gamma) = x\}$, this can be rephrase as the existence of a family of maps

$$\left\{ \begin{array}{ccc} G_x \times G^x & \to & G \\ (\gamma, \gamma') & \mapsto & \gamma\gamma' \end{array} \right. , \forall x \in G^{(0)}.$$

An automorphism of a groupoid is just an endofunctor which is invertible.

Depending on the situation, we will require these to be topological spaces with continuous maps, manifolds with smooth functions, etc. In these cases, we will talk about topological or smooth groupoïds. For now on, L_γ denotes the left translation $G^{s(\gamma)} \to G^{r(\gamma)}; \gamma' \mapsto \gamma\gamma'$, and $X = G^{(0)}$ is the set of units.

Définition 2. A Haar system $\lambda = (\lambda^x)_{x \in G^{(0)}}$ is a family of borelian measures λ^x with support G^x such that :

1. for all continuous function with compact support $f \in C_c(G)$, the map $x \mapsto \int_{G^x} f d\lambda^x$ is continuous.
2. λ is left-invariant w.r.t G, i.e. $L_{\gamma,*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)} \forall \gamma \in G$ or

$$\int_{G^{s(\gamma)}} f(\gamma\gamma') d^{s(\gamma)}\gamma' = \int_{G^{r(\gamma)}} f(\gamma') d^{r(\gamma)}\gamma'.$$

From $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$, we deduce

$$\int_{G^{s(\alpha^{-1}(\gamma))}} f(\gamma\alpha(\gamma')) d\gamma' = \int_{G^{r(\gamma)}} f(\gamma') \frac{1}{\rho(\alpha^{-1}(\gamma^{-1}\gamma'))} d\gamma'$$

$$\int_{G^{s(\alpha^{-1}(\gamma))}} f(\alpha(\alpha^{-1}(\gamma)\gamma')) = \int_{G^{r(\gamma)}} f(\gamma') \frac{1}{\rho(\alpha^{-1}(\gamma'))} d\gamma'$$

and $\rho(\gamma^{-1}\gamma') = \rho(\gamma')$. In particular, ρ is constant on G_x, for all $x \in X$.

Définition 3. An automorphism α of G preserves a Haar system λ if, for each $x \in X$, $\alpha_*\lambda^x$ is absolutely continuous w.r.t $\lambda^{\alpha(x)}$ and there exists a continuous function $\rho_\alpha : G \to \mathbb{R}^+$ such that ρ_α restricted to $G^{\alpha(x)}$ is the Radon-Nikodym derivative $\frac{d\alpha_*\lambda^x}{d\lambda^{\alpha(x)}}$.

Définition 4. Given an automorphism α of a groupoid G, we can form the suspension groupoid relative to α as follow. It is the groupoid with arrows

$$G_\alpha = G \times \mathbb{R}/\sim \quad \text{where } (\gamma, t) \sim (\alpha(\gamma), t-1)$$

and units

$$X_\alpha = X \times \mathbb{R}/\sim \quad \text{where } (x, t) \sim (\alpha_X(x), t-1).$$

If $[\gamma, t]$ and $[x, t]$ denote the equivalence classes in G_α and X_α respectively, then the source and the range map are given by

$$s([\gamma, t])] = [s(\gamma), t] \quad \text{and} \quad r([\gamma, t])] = [r(\gamma), t].$$

The composition is $[\gamma, t][\gamma', t] = [\gamma\gamma', t]$.

Lemme 1. If $\rho_\alpha \circ \alpha = \rho_\alpha$, then the suspension groupoid G_α admits a Haar system λ_α, given by

$$\lambda^{[x,t]}(f) = \int_{G^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma).$$

Preuve 1. We shall first demonstrate that this definition does make sense, i.e. that it is independent of the representant of the class $[x, t]$.

$$\begin{aligned}
\lambda^{[x,t]}(f) &= \int_{G^x} \rho(\alpha(\gamma))^{-t} f([\gamma, t]) d^x\gamma \\
&= \int_{G^{\alpha(x)}} \rho(\gamma)^{-t} f([\alpha^{-1}(\gamma), t]) \frac{d^{\alpha(x)}\gamma}{\rho(\gamma)} \\
&= \int_{G^{\alpha(x)}} \rho(\gamma)^{-t+1} f([\gamma, t-1]) d^{\alpha(x)}\gamma \qquad = \lambda^{[\alpha(x), t-1]}(f).
\end{aligned}$$

As the continuity is clear, we can conclude by showing the left-invariance.

$$\begin{aligned}
\int_{G_\alpha^{[s(\gamma),t]}} f([\gamma\gamma', t]) d^{[s(\gamma),t]}[\gamma', t] &= \int_{G^{s(\gamma)}} \rho^{-t}(\gamma') f([\gamma\gamma', t]) d^{s(\gamma)}\gamma' \\
&= \int_{G^{r(\gamma)}} \rho^{-t}(\gamma^{-1}\gamma') f([\gamma', t]) d^{r(\gamma)}\gamma' \\
&= \int_{G^{r(\gamma)}} \rho^{-t}(\gamma') f([\gamma', t]) d^{r(\gamma)}\gamma'
\end{aligned}$$

The last equality follows from the fact that ρ is constant on G_x, for all $x \in X$, and then

$$\int_{G_\alpha^{[s(\gamma),t]}} f([\gamma\gamma', t]) d^{[s(\gamma),t]}[\gamma', t] = \int_{G_\alpha^{[r(\gamma),t]}} f([\gamma', t]) d^{[r(\gamma),t]}[\gamma', t].$$

□

1.2 Principal *étale* groupoids

In this section, we are interested in locally compact groupoids. The maps $r, s : G \to X$, the composition and inverse maps are continuous.

Définition 5. A groupoid is said to be *étale* if $r : G \to X$ is a local homeomorphism.
It is principal if the product map $s \times r : G \to X \times X$ is one-to-one.

Let $x \in X$ and $\gamma \in G^x$. If G is *étale*,there exists a neighborhood U of γ such that $r_{|U}$ is a homeomorphism. So $G^x \cap U = \{\gamma\}$ is open in G^x. That show that the fibers G^x are discrete for all $x \in X$.

Proposition 1. If G is a principal étale groupoid, the fibers G^x are discrete for all $x \in X$ and the only Haar systems are the multiple of the counting measure on the fibers.

Preuve 2. If λ is a non-zero Haar system and G is principal, λ^x is a measure on the discrete space G^x, which entails that there exists a $\gamma \in G^x$ such that $\lambda(\gamma) > 0$. By left-invariance,

$$\lambda^{r(\gamma')}\{\gamma'\gamma\} = \lambda\{\gamma\} > 0.$$

Replacing $\gamma' = \gamma^{-1}$ in this relation, we have $\lambda^x\{x\} > 0$, which we can suppose equal to 1. The left invariane assures then that

$$\lambda^x\{\gamma\} = 1 \quad \forall \gamma \in G^x.$$

$\square$

2 Asymptotic dimension

Définition 6. Let X be a metric space.
The multiplicity of a cover $\mathcal{U}$ of X is the largest number $n \in \mathbb{N}$ such that every point $x \in X$ is contained in at most n elements of $\mathcal{U}$.
If $R > 0$, the R-multiplicity is the defined as the multiplicity restricted on covers uniformly bounded by R.
The asymptotic dimension of x is the smallest natural integer $n \in \mathbb{N}$ such that, for all $R > 0$, there exists a uniformly bounded cover $\{U_j\}$ with R-multiplicity $n + 1$.
Such a space is said to have finite asymptotic dimension if this number, denoted $\dim_\infty X$, is bounded.

3 Correspondance between the coarse K-homology of a space and the one of its coarse groupoid

The aim of this section is to give a proof of a result of [12], in which it is stated that the following diagramm commutes :

$$\begin{tikzcd} KX_*(X,B) \arrow[r, "A"] \arrow[d, "\simeq"] & K_*(C^*X,B) \arrow[d, "\simeq"] \\ K_*(G(X),l^\infty(X,B)) \arrow[r, "\mu"] & K_*(C_r(G(X)),B). \end{tikzcd}$$

The vertical arrow from the left comes from an isomorphism at the C^*-algebraic level, as

$$C^*(X) \simeq l^\infty(X) \times G(X).$$

The rest of this section is devoted to describe the vertical arrow from the right in the langage of Kasparov KK-theory, i.e.

$$\varinjlim_d KK(C_0(P_d(X)),B) \to \varinjlim_{Y\subset\mathcal{E}G(X)} KK(C_0(Y),B),$$

were the inductive limite on the right is taken among the proper $G(X)$-compact subsets Y of the universal classifying space for proper actions of $G(X)$.

Recall from [11] that we can take for $\mathcal{E}G(X)$ the space $\mathfrak{M}$ of positive measures μ on $G(X)$ satisfying :
- $\frac{1}{2} < \mu(G(X)) \leq 1$,
- $s^*\mu$ is a Dirac measure, i.e. its support consists of arrows of $G(X)$ that all source from the same base point of βX.

If $\mathfrak{M}_d$ denotes the space of measures μ of $\mathfrak{M}$ such that :
- μ is a probability measure
- for all γ and γ' in the support of μ, $\gamma'\gamma^{-1}$ is d-controlled, i.e. $d(r(\gamma),r(\gamma')) \leq d$,

then $\mathfrak{M} = \varinjlim \mathfrak{M}_d$.

The Rips complex of X, denoted $P_d(X)$, is the topological space of the complexes of diameter less than d, identified with probability measures on X with support of diameter less than d, with the weak topology coming from $C_c(C)$. We will write $[y,t]$ for a point of a simplex defined by barycentric coordinates of k points $y_1,...,y_k$, ie $\sum t_j\delta_{y_j}$. To such a point $[y,t]$ and an element of the Stone-Cech compactification $w \in \beta X$, we can associate a measure of $\mathfrak{M}_d$ in the following way. As $G(X)$ is a principal and transitive groupoid, there exists only one arrow γ_j such that $s(\gamma_j) = x$ and $r(\gamma_j) = y_j$. To $z = ([y,t],w) = (z_w,w)$, we associate

$$\phi_d(z) = \sum_{j=1,k} t_j\delta_{\gamma_j} \in \mathfrak{M}_d.$$

Proposition 2. The map

$$\phi_d : P_d(X) \times \beta X \to \mathfrak{M}_d$$

is an homeomorphism.

Preuve 3. It is clearly bijective. The bicontinuity comes from the identity :

$$\langle z_w, f\rangle = \langle \phi_d(z), f \circ r\rangle$$

for all $z = (z_w, w) \in P_d(X) \times \beta X$, and $f \in C_c(X)$. $\square$

This homeomorphism ϕ_d gives an $*$-isomorphism at the level of C^*-algebras

$$\Psi_d : C_0(\mathfrak{M}_d) \to C_0(P_d(X) \times \beta X).$$

Let $(\mathcal{E}, \pi, F) \in \mathbb{E}(C_0(P_d(X)), B)$ be an elliptic operator.

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