

Notes

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1 Application d'assemblage quantitative

1.1 Longueur, propagation et suite exacte à six termes.

Dans la suite, \mathcal{G} dénote un groupoïde localement compact de base $\mathcal{G}^{(0)} = X$ et muni d'un système de Haar $(\lambda^x)_{x \in X}$. On rappelle qu'une suite exacte courte est dite semi-scindée si elle admet une section complètement positive. Par équivalence de Morita, on entend les isomorphismes de groupes

$$\mathcal{M}_A^{\epsilon, r} : K_*^{\epsilon, r}(A) \rightarrow K_*^{\epsilon, r}(A \otimes \mathbb{K}) \quad , \forall r \geq 0, \forall \epsilon \in (0, \frac{1}{4})$$

induit par $A \rightarrow A \otimes \mathbb{K}; x \mapsto x \otimes e$, où e est n'importe quel projecteur de rang 1, A une C^* -algèbre, et \mathbb{K} l'idéal des opérateurs compacts sur l'espace de Hilbert séparable.

On suppose que le groupoïde est muni d'un système de longueur l , i.e. d'une famille d'applications $(l^x)_{x \in X}$ définies sur les fibres \mathcal{G}^x à valeurs dans \mathbb{R}_+ , telles que

$$\begin{aligned} l^x(e_x) &= 0 \\ l^{r(\gamma)}(\gamma) &= l^{s(\gamma)}(\gamma^{-1}) \\ l^x(\gamma_1^{-1}\gamma_2) &\leq l^x(\gamma_1) + l^x(\gamma_2) \end{aligned}$$

ce qui permet de définir une filtration sur les algèbres produits croisés associées au groupoïde par :

$$(A \rtimes \mathcal{G})_r = \{f \in C_c(\mathcal{G}, A) : \text{supp } f \subset \cup_{x \in X} B_x(e_x, r)\}$$

où $B_x(e_x, r)$ dénote la boule $\{\gamma \in \mathcal{G} : l^{r(\gamma)} \leq r\}$. Ici, \rtimes peut être à loisir \rtimes_{red} ou \rtimes_{max} . On rappelle que $A \rtimes \mathcal{G}$ est fonctoriel en A , de la catégorie des \mathcal{G} - C^* -algèbres avec $*$ -morphisme équivariants, dans celle des C^* -algèbres avec $*$ -morphisme. Pour $\phi : A \rightarrow B$ un $*$ -morphisme, on note $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \rightarrow B \rtimes \mathcal{G}$ le morphisme induit.

Lemme 1. On suppose le groupoïde \mathcal{G} localement compact, muni d'un système de Haar et d'une longueur l .

Soit $0 \rightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \rightarrow 0$ une suite exacte semi-scindée de \mathcal{G} - C^* -algèbres.

Alors la suite $0 \rightarrow J \rtimes \mathcal{G} \xrightarrow{\phi_{\mathcal{G}}} A \rtimes \mathcal{G} \xrightarrow{\psi_{\mathcal{G}}} A/J \rtimes \mathcal{G} \rightarrow 0$ est une suite exacte semi-scindée et filtrée.

Preuve 1. Par fonctorialité, $\psi_{\mathcal{G}} \circ \phi_{\mathcal{G}} = 0$. Soit $f \in C_c(\mathcal{G}, A)$ telle que $\phi_{\mathcal{G}}(f) = 0$, i.e. $\phi(f(\gamma)) = 0, \forall \gamma \in \mathcal{G}$. Alors, pour chaque γ , il existe un unique $g(\gamma) \in A'$ telle que $\psi(g(\gamma)) = f(\gamma)$. De plus, $\gamma \mapsto g(\gamma)$ est continue, de support contenu dans celui de f , donc $g \in C_c(\mathcal{G}, A')$ et $\phi_{\mathcal{G}}(g) = f$.

Si maintenant $a \in A \rtimes \mathcal{G}$, il existe une suite de fonctions $f_n \in C_c(\mathcal{G}, A)$ telle que $\|a - f_n\| \rightarrow 0$. Mais $f_n - s_{\mathcal{G}} \circ \psi_{\mathcal{G}}(f_n)$ est annulée par $\psi_{\mathcal{G}}$, donc il existe $g_n \in C_c(\mathcal{G}, J)$ telle que $\phi_{\mathcal{G}}(g_n) = f_n - s_{\mathcal{G}} \circ \psi_{\mathcal{G}}(f_n)$. Alors

$$\lim_n \|a - \phi_{\mathcal{G}}(g_n)\| = 0$$

et comme $\text{Im } \phi_{\mathcal{G}}$ est fermé, $\text{Ker } \psi_G = \text{Im } \phi_{\mathcal{G}}$.

La suite est bien scindée par $\sigma_{\mathcal{G}}$, et les morphismes respecte la propagation, elle est donc filtrée. \square

La proposition suivante découle du lemme.

Proposition 1. Il existe une paire de contrôle (λ, h) telle que pour toute extension semi-scindée de \mathcal{G} - C^* -algèbres

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad ,$$

les diagrammes suivant soient commutatifs et exacts :

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_r \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_r \mathcal{G}) \end{array} \quad ,$$

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_{max} \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_{max} \mathcal{G}) \end{array} \quad .$$

1.2 Transformée de Kasparov

Soient A et B deux \mathcal{G} - C^* -algèbres. On notera H un espace de Hilbert séparable, $l^2(\mathbb{Z})$ par exemple, et $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. Le module hilbertien standard pour B est noté $H_B = H_{\mathcal{G}} \otimes B$, et la C^* -algèbre des opérateurs compacts pour H_B est noté $K_B = K \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Tout K -cycle $z \in KK^{\mathcal{G}}(A, B)$ peut être représenté comme la classe d'un triplet (H_B, π, T) où :

- $\pi : A \rightarrow \mathcal{L}_B(H_B)$ est une $*$ -représentation de A sur H_B .
- $T \in \mathcal{L}_B(H_B)$ est un opérateur autoadjoint
- π et T vérifient les conditions de K -cycles, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ et $\pi(a)(\gamma.T - T)$ sont des opérateurs compacts, pour tout $a \in A$, $\gamma \in \mathcal{G}$.

On peut définir $T_{\mathcal{G}} = T \otimes id_{B \rtimes_r \mathcal{G}} \in \mathcal{L}_B(H_B \otimes (B \rtimes_r \mathcal{G})) \simeq \mathcal{L}_B(H_{B \rtimes_r \mathcal{G}})$, et $\pi_{\mathcal{G}} : A \rtimes_r \mathcal{G} \rightarrow \mathcal{L}_B(H_{B \rtimes_r \mathcal{G}})$. Le K -cycle $(H_{B \rtimes_r \mathcal{G}}, \pi_{\mathcal{G}}, T_{\mathcal{G}})$ représente l'élément $j_{red, \mathcal{G}}(z) \in KK(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$. Nous allons construire un morphisme contrôlé associé à z ,

$$J_{red, \mathcal{G}}(z) : \hat{K}(A \rtimes_r \mathcal{G}) \rightarrow \hat{K}(B \rtimes_r \mathcal{G}),$$

qui induit la multiplication à droite par $j_{red, \mathcal{G}}(z)$ en K -théorie.

1.2.1 Cas impair

Considérons tout d'abord le cas où $z \in KK_1(A, B)$. Soit (H_B, π, T) un K -cycle représentant z . On pose $P = \frac{1+T}{2}$ et $P_{\mathcal{G}} = P \otimes id_{B \rtimes_r \mathcal{G}}$. Définissons

$$E_r^{(\pi, T)} := \{(x, P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}} + y) : x \in (A \rtimes_r \mathcal{G})_r, y \in K \otimes (B \rtimes_r \mathcal{G})_r\}$$

ce qui nous donne une extension de C^* -algèbres filtrées

$$0 \longrightarrow K_{B \rtimes_r \mathcal{G}} \longrightarrow E^{(\pi, T)} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

$$\text{et semi-scindée par } s : \begin{cases} A \rtimes_r \mathcal{G} & \rightarrow & E^{(\pi, T)} \\ x & \mapsto & (x, P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}}) \end{cases}.$$

Montrons que le bord de cette extension ne dépend que de la classe z du K -cycle. Soit donc $(H_B, \pi_j, T_j), j = 0, 1$ deux K -cycles homotopes via $(H_{B[0,1]}, \pi, T)$. On note e_t l'évaluation en $t \in [0, 1]$ pour un élément de $B[0, 1]$. L'application

$$\phi : \begin{cases} E^{(\pi, T)} & \rightarrow & E^{(\pi_t, T_t)} \\ (x, y) & \mapsto & (x, y_t) \end{cases}$$

vérifie $\phi(K_{B[0,1] \rtimes_r \mathcal{G}}) \subset K_{B \rtimes_r \mathcal{G}}$ et fait commuter le diagramme

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{B[0,1] \rtimes_r \mathcal{G}} & \longrightarrow & E^{(\pi, T)} & \longrightarrow & A \rtimes_r \mathcal{G} \longrightarrow 0 \\ & & \downarrow \phi|_{K_{B[0,1] \rtimes_r \mathcal{G}}} & & \downarrow \phi & & \downarrow = \\ 0 & \longrightarrow & K_{B \rtimes_r \mathcal{G}} & \longrightarrow & E^{(\pi_t, T_t)} & \longrightarrow & A \rtimes_r \mathcal{G} \longrightarrow 0 \end{array}.$$

D'après la remarque 3.7 de [6], on a donc la relation suivante sur les bords contrôlés

$$D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_t, T_t)}} = \phi_* \circ D_{K_{B[0,1] \rtimes_r \mathcal{G}}, E^{(\pi, T)}}.$$

Comme $id \otimes e_t$ établit une homotopie entre $id \otimes e_0$ et $id \otimes e_1$, et que si 2 $*$ -morphisms sont homotopes, alors ils sont égaux en K -théorie contrôlée, on obtient que

$$D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_0, T_0)}} = D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_1, T_1)}}$$

et le bord de l'extension $E^{(\pi, T)}$ ne dépend que de z .

Définition 1. La transformée de Kasparov contrôlée d'un élément $z \in KK_1^{\mathcal{G}}(A, B)$ est définie comme la composition

$$J_{red, \mathcal{G}}(z) = \mathcal{M}_{B \rtimes_r \mathcal{G}}^{-1} \circ D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi, T)}}.$$

Proposition 2. Soient A et B deux \mathcal{G} - C^* -algèbres. Pour tout élément $z \in KK_1^{\mathcal{G}}(A, B)$, il existe un morphisme contrôlé

$$J_{red, \mathcal{G}}(z) : \hat{K}_*(A \rtimes_r \mathcal{G}) \rightarrow \hat{K}_{*+1}(B \rtimes_r \mathcal{G})$$

tel que :

- (i) $J_{red, \mathcal{G}}(z)$ induit en K -théorie la multiplication à droite par $j_{red, \mathcal{G}}(z)$;
- (ii) $J_{red, \mathcal{G}}$ est additif, i.e.

$$J_{red, \mathcal{G}}(z + z') = J_{red, \mathcal{G}}(z) + J_{red, \mathcal{G}}(z').$$

- (iii) Pour tout \mathcal{G} -morphisme $f : A_1 \rightarrow A_2$,

$$J_{red, \mathcal{G}}(f^*(z)) = J_{red, \mathcal{G}}(z) \circ f_{\mathcal{G}, red, *}$$

pour tout $z \in KK_1^G(A_2, B)$.

Preuve 2. (i) Le bord $[\delta_{K_{B \rtimes_r \mathcal{G}, E(p^i, T)}}] \in KK_1(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ associé à l'extension $E^{(\pi, T)}$ induit par définition, modulo équivalence de Morita, l'application $j_{red, \mathcal{G}}$, ce qui assure directement ce point.

- (ii)

1.3 Transformée de Kasparov ancien

Si l'élément $z \in KK^G(A, B)$ est représenté par un cycle (H, π, T) , nous allons définir sa transformée de Kasparov $J_{\mathcal{G}}(z) \in KK(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$.

Tout d'abord, le cas pair. Notons $P_{\mathcal{G}} = P \otimes_B id_{B \times_r \mathcal{G}}$ l'opérateur sur $H \otimes B \times_r \mathcal{G}$ induit par $P = \frac{T + id_{H \otimes B}}{2}$. Si l'on pose

$$\mathcal{E} := \{(x, y) \in A \times_r \mathcal{G} \oplus \mathcal{L}(H \otimes B \times_r \mathcal{G}) : P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}} = y \text{ mod } \mathbb{K} \otimes B \times_r \mathcal{G}\},$$

observons l'extension

$$(E) : 0 \rightarrow \mathbb{K} \otimes B \times_r \mathcal{G} \rightarrow \mathcal{E} \rightarrow A \times_r \mathcal{G} \rightarrow 0.$$

Toute extension $(Ext) : 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ induit une application contrôlée

$$D_{Ext} = D_{A'}^A : \hat{K}(A'') \rightarrow \hat{K}(A').$$

Montrons que D_E ne dépend que de la classe de z , et pas de π et T .

A FAIRE

Définition 2. La transformée de Kasparov d'un élément z de $KK^G(A, B)$ est le morphisme contrôlé

$$J_{\mathcal{G}}(z) = M_{B \times_r \mathcal{G}}^{-1} \circ D_E : \hat{K}_*(A \times_r \mathcal{G}) \rightarrow \hat{K}_*(B \times_r \mathcal{G}),$$

où (E) est l'extension précédemment décrite.

Ce morphisme $J_{\mathcal{G}} : KK^G(A, B) \rightarrow \text{Hom}_0(\hat{K}(A \times_r \mathcal{G}), \hat{K}(B \times_r \mathcal{G}))$ nous permet de définir l'application d'assemblage associée à n'importe quel élément de $\hat{K}(A \times_r \mathcal{G})$ par simple évaluation :

$$Ind_x(z) = J_{\mathcal{G}}(z)(x).$$

La conjecture de Baum-Connes s'intéresse à l'application d'assemblage associée à un certain élément. Dans le cas des groupoïdes, il existe une fonction continue à support compact $h : P_d(\mathcal{G}) \rightarrow [0, 1]$ telle que

$$\sum_{\gamma \in \mathcal{G}} \gamma(h^2) = 1.$$

Alors $\gamma \rightarrow \sum_{h \in \mathcal{G}} h\gamma(h)$ définit un projecteur de $A = C_0(P_d(\mathcal{G})) \rtimes_r \mathcal{G}$ de propagation finie, majorée par une certaine constante s . Comme les fonctions h admissibles forment un ensemble convexe, la classe de $[e_h, 0] \in K_0^{\epsilon, r(A)}$ ne dépend pas de la fonction h choisie, et l'application d'assemblage de Baum-Connes est définie par l'évaluation en cette classe.

2 Géométrie asymptotique

Pour tout $z \in KK^\Gamma(A, B)$, on peut construire une application de descente

$$- \otimes \sigma_X(z) : K_*(C^*(X, A)) \rightarrow K_*(C^*(X, B)),$$

et l'application d'assemblage asymptotique est simplement l'application de descente prise en un certain élément. On va montrer que l'on peut en fait construire un morphisme contrôlé $\tau_X(z) : \dot{K}_*(C^*(X, A)) \rightarrow \dot{K}_*(C^*(X, B))$ qui induit la multiplication à droite par $\sigma_X(z)$.

Pour tout $z \in KK^\Gamma(A, B)$, il existe un morphisme contrôlé

$$\tau_X(z) : K_*(C^*(X, A)) \rightarrow K_*(C^*(X, B)).$$

Rappelons que le morphisme de groupoïdes $\iota : \{e_x\} \hookrightarrow \Gamma$ induit un isomorphisme de groupes abéliens

$$\iota^* : KK_*^\Gamma(\tilde{A}, l^\infty(X, B \otimes \mathbb{K})) \xrightarrow{\sim} KK_*(A, B) \quad ,$$

où \tilde{A} est la $C(\Gamma)$ -algèbre $C_0(P_d(\Gamma))$, de fibre $A = \tilde{A}_x = C_0(P_d(X))$. On dispose de plus d'un $*$ -isomorphisme $\Phi_B : C^*(X, B) \rightarrow l^\infty(X, B \otimes \mathbb{K}) \rtimes_r \Gamma$ pour toute C^* -algèbre B , qui préserve la filtration, et donc induit un morphisme contrôlé. On peut alors définir, pour $z \in KK(A, B)$, le morphisme contrôlé $\tau_X(z)$ par

$$J_\Gamma(\iota^{*-1}(z)) = \Phi_{B*} \circ \tau_X(z) \circ \Phi_{\tilde{A}*}^{-1},$$

où $\tilde{B} := l^\infty(X, B \otimes \mathbb{K}) \rtimes_r \Gamma$. Montrons que $\tau_X(z)$ induit la multiplication par σ_X en KK -théorie. **A FAIRE**

3 English Notes

In a serie of papers, H. Oyono-Oyono and G. Yu have defined a controlled version of operator K -theory, [8] [6] [7], that allows them to define a quantitative and local assembly map, and a related Baum-Connes Conjecture. The aim of this work is to extend their work to the realm of groupoids. In a second part, we shall see how this setting can help us understand better the relation between the coarse-Baum-Connes conjecture and the Baum-Connes Conjecture with coefficient for groupoids. Indeed, in [20] and [18], G. Skandalis, J-L.Tu and G. Yu proved that there is a commutative diagramm

$$\begin{array}{ccc} K_*(X) & \xrightarrow{A} & K_*(C^*X) \\ \downarrow \simeq & & \downarrow \simeq \\ K_*(\mathcal{E}\Gamma) & \xrightarrow{\mu_r} & K_*(C_r\Gamma) \end{array}$$

where :

- the left sides are K -homology of certain spaces, the left sides are K -theory of certain C^* -algebras,
- the first line is the coarse assembly map associated to a coarse space X , the second being the assembly map for groupoids associated to the coarse groupoid of X , namely $\Gamma = G(X)$,
- the vertical arrows are isomorphisms on the level of KK -groups. The left one will be studied later, the right one derives from an isomorphism on the level of the C^* -algebras. Indeed, it has been shown that the Roe algebra is $*$ -isomorphic to the crossed product of $l^\infty(X)$ by Γ : $C^*(X) \simeq l^\infty(X) \rtimes \Gamma$.

We shall see that this relation is already true "locally" and factorises through quantitative K -theory. Locally here means that we can factorize the assembly map through $KK(C_0(P_d(X)), B)$. Indeed, the K -homology can be expressed as an inductive limit

$$K_*(X, B) = \varinjlim_{d \rightarrow \infty} KK(C_0(P_d(X)), B) \quad \text{and}$$

hence the local.

The local quantitative assembly maps defined in [6] are of the form

$$KK^F(C_0(P_d(F)), B) \rightarrow K_*^{\epsilon, r}(B \rtimes_r \Gamma)$$

where F is a finite group.

We will be using the bivariant functor KK^Γ introduced by P-Y. Le Gall in his thesis [4], which is a generalization of Kasparov's bifunctor for the case where Γ is a Hausdorff locally compact groupoid with Haar system. As for the KK -theory for Banach algebras introduced by V. Lafforgue [10], there is no (not yet ?) a Kasparov product in quantitative K -theory. The crucial point to define an assembly map is the existence of a morphism

$$\hat{J} : KK^\Gamma(A, B) \rightarrow Hom^*(\hat{K}_*(A), \hat{K}_*(B))$$

which allows us, for every element $x \in K_*(A)$, to consider the related index

$$\text{Ind}_x \begin{cases} KK_*(A, B) & \rightarrow & K_*(B) \\ z & \mapsto & \hat{J}(z)(x) \end{cases}$$

to construct \hat{J} , we will mimic the construction for \mathcal{J} in [6], which gives the right morphism when considering finite groups. The starting point is the following

Lemme 2. Let Γ be a Hausdorff locally compact groupoid with Haar system. Then, if A is a $\Gamma - C^*$ -algebra, forming the reduced (and maximal) crossed product $A \times_r \Gamma$ is functorial in A . Moreover, if Γ is *étale*, then it preserves short semi-split filtered exact sequences of $\Gamma - C^*$ -algebras, meaning that if

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is a semi-split filtered exact sequence of $\Gamma - C^*$ -algebras, then

$$0 \rightarrow A' \times_r \Gamma \rightarrow A \times_r \Gamma \rightarrow A'' \times_r \Gamma \rightarrow 0$$

is too.

Preuve 3. Let

$$0 \rightarrow A' \xrightarrow{\Psi} A \xrightarrow{\Phi} A'' \rightarrow 0$$

be a semi-split filtered exact sequence of $\Gamma - C^*$ -algebras.

Let $f \in C_c(\Gamma, A)$ such that $\Phi_\Gamma(f) = 0$. For every $\gamma \in \Gamma$, there exists a unique $g(\gamma) \in A'$ such that $\Psi(g(\gamma)) = f(\gamma)$. But, as Ψ and Φ commute with the action of Γ , we have $g(\gamma'^{-1}\gamma) = \gamma'.g(\gamma)$, so that g is continuous.

Let $a \in A \times \Gamma$ such that $\Phi_\Gamma(a) = 0$. We can approximate a by a sequence of $f_n \in C_c(\Gamma, A)$. But, with the preceeding work on compactly supported continuous functions, there exist a sequence $g_n \in C_c(\Gamma, A')$ such that :

$$f_n = \Psi(g_n) + \sigma \circ \Phi(f_n).$$

Moreover, $\text{Im } \Psi$ is a closed ideal in A , and $\lim \|f_n - \Psi(g_n)\| = 0$ so that $a \in \text{Im } \Psi$: the sequence is exact in the middle. \square

3.1

Let X be a discrete metric space with bounded geometry and Γ its coarse groupoid. $\mathbb{C}[X]$ is the algebra of locally compacts operators with finite propagation on $l^2(X) \otimes H$. We denote by \tilde{A} the $C(\Gamma)$ -algebra $C_0(P_d(\Gamma))$, and by A the C^* -algebra $C_0(P_d(X))$. As $P_d(\Gamma)$ is equivariantly homeomorphic to $\beta X \times P_d(X)$, the fiber \tilde{A}_x is isomorphic to A as a C^* -algebra.

Lemme 3. Let B be a C^* -algebra and $x \in X$. The morphism of groupoids $\iota : \{e_x\} \rightarrow \Gamma$ induces an isomorphism in KK -theory

$$KK_*^\Gamma(\tilde{A}, l^\infty(X, B)) \xrightarrow{\cong} KK_*(A, B) \quad .$$

Preuve 4. Let us first check that the fiber over $x \in X$ of $\tilde{B} := l^\infty(X, B)$ is B : easily, if I_x denotes the kernel of the evaluation map $\tilde{B} \rightarrow B$ at x , $\tilde{B}_x = \tilde{B}/I_x \tilde{B} \simeq B$. Moreover, as $\tilde{A}_x \simeq A$, functoriality assures that ι induces $\iota^* : KK_*^\Gamma(\tilde{A}, l^\infty(X, B)) \rightarrow KK_*(A, B)$.

We can now construct an explicit R inverse for ι^* . Let $z = (\hat{H}_{\tilde{B}}, \Phi, T) \in \mathbb{E}^\Gamma(\tilde{A}, \tilde{B})$ be a K -cycle. By stabilization theorem, we can suppose that the Hilbert \tilde{B} module is in canonical form, that is the standard separable Hilbert module over \tilde{B} with usual grading. We will also suppose that T is Γ -equivariant, which is always true up to compact perturbation. The image of z under R is defined to be its restriction to x , (\hat{H}_B, Φ_x, T_x) . This definition gives immediatly $R \circ \iota^* = 1$. Now let $V : s^* \hat{H}_{\tilde{B}} \rightarrow \hat{H}_{\tilde{B}}$ be the unitary implementing the action of Γ on $\hat{H}_{\tilde{B}}$. The equivariance of T can be written as

$$V_\gamma T_{s(\gamma)} V_\gamma^* = T_{r(\gamma)} \quad , \forall \gamma \in \Gamma.$$

If $T' = \iota^* \circ R(T)$, it is the constant operator with fiber $T_x \in \mathcal{L}_B(\hat{H}_B)$ on the trivial Hilbert module $\bigoplus_{y \in X} (H_{\tilde{B}})_x$. Define $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$. It is unitary and intertwines T and T' , i.e. the cycles are homotopic and $\iota^* \circ R = 1$. \square

Another result of interest for our study is the expression of the Roe algebras of our discrete space X as crossed products. It was first proven by G. Yu in the case of X being a finitely generated group with the word metric, the general version can be found in the article of G. Skandalis, J-L. Tu and G. Yu. [18]

Lemme 4. Let B be a coefficient C^* -algebra. The uniform Roe algebra of X and the Roe algebra of X are $*$ -isomorphic to crossed-product algebras by Γ , respectively :

$$\begin{aligned} C_u^*(X, B) &\simeq l^\infty(X, B) \rtimes_r \Gamma \\ C^*(X, B) &\simeq l^\infty(X, \mathbb{K} \otimes B) \rtimes_r \Gamma \end{aligned}$$

Preuve 5. If $T \in \mathbb{C}[X]$. For $x, y \in X$, $T_{xy} \in \mathbb{K}$ is the compact operator defined by Riesz representation theorem as

$$\langle T(\delta_x \otimes \xi), \delta_y \otimes \eta \rangle = \langle T_{xy} \xi, \eta \rangle \quad , \xi, \eta \in H.$$

Let us recall how one construct the left regular representation of Γ . For $x \in \Gamma^{(0)} = \beta X$, Γ^x has a natural action on $l^2(\Gamma^x)$:

$$(\lambda_x(\gamma)\eta)(\gamma') = \eta(\gamma^{-1}\gamma') \quad , \gamma, \gamma' \in \Gamma^x, \eta \in l^2(\Gamma^x).$$

To any couple $(x, y) \in \beta X \times \beta X$, there exists a unique $\gamma_x^y \in \Gamma$ such that $s(\gamma) = x$ and $r(\gamma) = y$. The map

$$T \rightarrow \sum_{x, y} T_{xy} \lambda_x(\gamma_x^y)$$

establishes an $*$ -isomorphism between $\mathbb{C}[X]$ and the algebra of finitely supported functions from Γ to $l^\infty(X, \mathbb{K})$, its inverse being the $*$ -morphism that associates to a function a the convolution operator :

$$(a(\gamma)(\eta \otimes \xi))(x) = (a(\gamma) * \eta)(x) \xi \quad , x \in X, \eta \in l^2(X), \xi \in H.$$

That is it of finite propagation and locally compact follows from the finiteness of the support (it is of propagation the diameter of the support of a !). As this two maps are continuous and their domain are respectively dense in the Roe algebra and in the crossed product, the morphism extends to the $*$ -isomorphism we were looking for. \square

This $*$ -isomorphism gives an isomorphism at the level of K -theory

$$K_*(C^*(X, B)) \rightarrow K_*(l^\infty(X, \mathbb{K} \otimes B) \rtimes_r \Gamma).$$

These lemmas allows us to state a refinement of a result of G. Skandalis, J-L.Tu and G. Yu [18], namely that the two isomorphisms described above intertwine the (local quantitative?) assembly maps for the groupoid Γ with coefficients in $l^\infty(X)$ with the coarse assembly map of X . (If local and quantitative :) This result at the level of controlled K -theory on the right side entails that of [18]

Théorème 1. The following diagram

$$\begin{array}{ccc} A : KK_*(A, B) & \longrightarrow & K_*(C^*(X, B)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mu_r : KK^\Gamma & \longrightarrow & K_*(l^\infty(X, \mathbb{K} \otimes B) \rtimes_r \Gamma) \end{array}$$

commutes, and the vertical arrows are isomorphisms.

4 Assembly maps for groupoids and for coarse spaces

4.1 The case of a finitely generated group

Let Γ be a discrete finitely generated group. The word length provides a structure of metric space, of which the class up to coarse equivalence is independent of the set of generators.

Denoting $C^*(\Gamma)$ the Roe algebra, i.e. the C^* -algebra generated by locally compact operators on $l^2(\Gamma) \otimes H$ with finite propagation, we can show that

$$C_u^*(\Gamma) = l^\infty(\Gamma) \times_\alpha \Gamma$$

$$C^*(\Gamma) = l^\infty(\Gamma, \mathfrak{K}(H)) \times_\alpha \Gamma.$$

Here $\alpha \in \text{Aut}(A)$ is the automorphism encoding the left action of Γ on A :

$$\alpha_\gamma(a) = s \rightarrow a_{s\gamma^{-1}}.$$

Let S_γ be the operator acting on $l^2(\Gamma)$ as

$$(S_\gamma \eta)_s = \eta_{s\gamma^{-1}}.$$

We see $l^\infty(\Gamma)$ as an algebra of operator, acting by left multiplication on $l^2(\Gamma)$. Then

$$S_\gamma a S_\gamma^* = \alpha_\gamma(a),$$

for any $a \in l^\infty(\Gamma)$ and $\gamma \in \Gamma$. The algebra $C^*(\Gamma)$ is generated by finite sums of the form

$$\sum_\gamma a_\gamma S_\gamma$$

which are of finite propagation $\max_\gamma \{l(\gamma) : a_\gamma \neq 0\}$ and locally compact.

4.2 Relation between the coarse and the groupoid assembly maps

We have to show that there is an isomorphism

$$KX_*(X) \rightarrow KK_*^{top}(G(X), l^\infty(X, \mathfrak{K})).$$

Let us recall that the Stone-Cech compactification of our coarse groupoid $\Gamma = G(X)$ identified itself to the spectrum of the bounded continuous functions over X , which is discrete. We have

$$C(\beta X) \simeq l^\infty(X)$$

and we can think of $C(\beta X)$ -algebras as $l^\infty(X)$ -algebras.

The left handside $KX_*(X)$ is defined as the limit of the directed groups

$$KK_*(C_0(P_E(X), \mathbb{C}))$$

when E is an entourage of X . Here $P_E(X)$ denotes the Rips complex defined by the entourage E , which is the set of simplexes $[x_0, \dots, x_n]$ such that $(x_i, x_j) \in E$.

Now the classifying space $\mathcal{E}\Gamma$ of the groupoid $G(X)$ is unique up to homotopy, and can be realised by the space of measures μ on $G(X)$ which satisfied $s^*\mu$ is a Dirac measure on $G^{(0)} = \beta X$, and $\frac{1}{2} < |\mu| \leq 1$. Saying that $s^*\mu$ is a Dirac measure is the same as demanding μ to be supported in a fiber Γ_x for some $x \in X$. The abelian group $KK_*^{top}(G(X), l^\infty(X, \mathfrak{K}))$ is defined as the inductive limit of

$$KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$$

when Y is a Γ -compact space of $\mathcal{E}\Gamma$.

Let E be an entourage of X . A Fredholm module (H, ϕ, F) in $E(C_0(P_E(X)), \mathbb{C})$ is defined by a Hilbert space H , a $*$ -homomorphism $\phi : C_0(P_E(X)) \rightarrow \mathcal{L}(H)$ and an operator F satisfying all definitions.

We can form the $l^\infty(X, \mathfrak{K})$ -module $\mathcal{E} = H \otimes_{\mathbb{C}} l^2(\Gamma, \mathfrak{K})$, and extend ϕ into $\phi \otimes id : C_0(P_E(X)) \rightarrow \mathcal{L}(H \otimes l^2(\Gamma, \mathfrak{K}))$. We do the same with $F : \hat{F} := F \otimes id$. Then, as $P_E(X)$ identifies itself as a G -compact of $\mathcal{E}G$, $(\mathcal{E}, \phi \otimes id, F \otimes id)$ defines an element of $KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$.

5 Correspondance between the coarse K -homology of a space and the one of its coarse groupoid

The aim of this section is to give a proof of a result of [20], in which it is stated that the following diagram commutes :

$$\begin{array}{ccc} KX_*(X, B) & \xrightarrow{A} & K_*(C^*X, B) \\ \downarrow \simeq & & \downarrow \simeq \\ K_*(G(X), l^\infty(X, B)) & \xrightarrow{\mu} & K_*(C_r(G(X)), B). \end{array}$$

The vertical arrow from the left comes from an isomorphism at the C^* -algebraic level, as

$$C^*(X) \simeq l^\infty(X) \times G(X).$$

The rest of this section is devoted to describe the vertical arrow from the right in the langage of Kasparov KK -theory, i.e.

$$\varinjlim_d KK(C_0(P_d(X)), B) \rightarrow \varinjlim_{Y \subset \mathcal{E}G(X)} KK(C_0(Y), B),$$

where the inductive limite on the right is taken among the proper $G(X)$ -compact subsets Y of the universal classifying space for proper actions of $G(X)$.

Recall from [19] that we can take for $\mathcal{E}G(X)$ the space \mathfrak{M} of positive measures μ on $G(X)$ satisfying :

- $\frac{1}{2} < \mu(G(X)) \leq 1$,
- $s^*\mu$ is a Dirac measure, i.e. its support consists of arrows of $G(X)$ that all source from the same base point of βX .

If \mathfrak{M}_d denotes the space of measures μ of \mathfrak{M} such that :

- μ is a probability measure
- for all γ and γ' in the support of μ , $\gamma'\gamma^{-1}$ is d -controlled, i.e. $d(r(\gamma), r(\gamma')) \leq d$,

then $\mathfrak{M} = \varinjlim \mathfrak{M}_d$.

The Rips complex of X , denoted $P_d(X)$, is the topological space of the complexes of diameter less than d , identified with probability measures on X with support of diameter less than d , with the weak topology coming from $C_c(C)$. We will write $[y, t]$ for a point of a simplex defined by barycentric coordinates of k points y_1, \dots, y_k , ie $\sum t_j \delta_{y_j}$. To such a point $[y, t]$ and an element of the Stone-Cech compactification $w \in \beta X$, we can associate a measure of \mathfrak{M}_d in the following way. As $G(X)$ is a principal and transitive groupoid, there exists only one arrow γ_j such that $s(\gamma_j) = x$ and $r(\gamma_j) = y_j$. To $z = ([y, t], w) = (z_w, w)$, we associate

$$\phi_d(z) = \sum_{j=1,k} t_j \delta_{\gamma_j} \in \mathfrak{M}_d.$$

Proposition 3. The map

$$\phi_d : P_d(X) \times \beta X \rightarrow \mathfrak{M}_d$$

is an homeomorphism.

Preuve 6. It is clearly bijective. The bicontinuity comes from the identity :

$$\langle z_w, f \rangle = \langle \phi_d(z), f \circ r \rangle$$

for all $z = (z_w, w) \in P_d(X) \times \beta X$, and $f \in C_c(X)$. □

This homeomorphism ϕ_d gives an $*$ -isomorphism at the level of C^* -algebras

$$\Psi_d : C_0(\mathfrak{M}_d) \rightarrow C_0(P_d(X) \times \beta X).$$

Let $(\mathcal{E}, \pi, F) \in \mathbb{E}(C_0(P_d(X)), B)$ be an elliptic operator. **A FINIR**

Let $X_0 \subset X_1 \subset \dots \subset X_j \subset \dots$ the n -skeleton decomposition associated to the simplicial structure of the Rips complex $P_d(X)$, and similarly $\tilde{X}_0 \subset \tilde{X}_1 \subset \dots \subset \tilde{X}_j \subset \dots$ for \mathfrak{M}_d , and

$$Z_j = C_0(X_j) \quad \text{and} \quad \tilde{Z}_j = C_0(\tilde{X}_j).$$

$$Z_{j-1}^j = C_0(X_j - X_{j-1}) \quad \text{and} \quad \tilde{Z}_{j-1}^j = C_0(\tilde{X}_j - \tilde{X}_{j-1}).$$

We will show the isomorphism by a Mayer-Vietoris type argument. By applying the KK -theory to the exact sequence :

$$0 \longrightarrow C_0(X_j - X_{j-1}) \longrightarrow C_0(X_j) \longrightarrow C_0(X_{j-1}) \longrightarrow 0$$

we have a commutative diagramm with exact lines :

$$\begin{array}{ccccccccc}
KK_*(Z_{j-1}^j, B) & \xrightarrow{\delta} & KK_*(Z_{j-1}, B) & \longrightarrow & KK_*(Z_j, B) & \longrightarrow & KK_*(Z_{j-1}^j, B) & \xrightarrow{\delta} & KK_*(Z_{j-1}, B) \\
\downarrow \theta_{j-1}^j & & \downarrow \theta_{j-1} & & \downarrow \theta_j & & \downarrow \theta_{j-1}^j & & \downarrow \theta_{j-1} \\
KK_*(\tilde{Z}_{j-1}^j, B) & \xrightarrow{\delta} & KK_*(\tilde{Z}_{j-1}, B) & \longrightarrow & KK_*(\tilde{Z}_j, B) & \longrightarrow & KK_*(\tilde{Z}_{j-1}^j, B) & \xrightarrow{\delta} & KK_*(\tilde{Z}_{j-1}, B)
\end{array}$$

The five lemma assures that if θ_{j-1} and θ_{j-1}^j are isomorphisms, then so is θ_j . Moreover, $X_j - X_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where $\mathring{\sigma}_j$ denotes the interior of the standard simplex, and Σ_j is the set of centers of j -simplices of X_j . Bott periodicity assures then that, if θ_{j-1} is an isomorphism, then so is θ_{j-1}^j . By induction, proving that θ_0 is an isomorphism concludes the proof.

Références

- [1] J.P. Serre A. Borel. Le théorème de riemann-roch. *Bulletin de la S.M.F.*, 86 :97–136, 1958.
- [2] J. Bellissard. Gap labelling theorems for schrödinger operators. *From Number theory to Physics, Les Houches*, 89 :538–630, 1993.
- [3] Joachim Cuntz. K-theory and c^* -algebras. *Algebraic K-theory, Number Theory, Geometry and Analysis*, pages 55–79, 1982.
- [4] Pierre-Yves Le Gall. Thesis.
- [5] Alexandre Grothendieck. Produits tensoriels d’espace topologiques et espaces nucléaires. *Séminaire N. Bourbaki*, 69 :193–200, 1951-1954.
- [6] G. Yu H. Oyono-Oyono. On quantitative operator k-theory. *Annales de l’Institut Fourier*.
- [7] G. Yu H. Oyono-Oyono. Persistance approximation property and controlled k-theory.
- [8] G. Yu H. Oyono-Oyono. Ktheory for the maximal roe algebra of certain expanders. *Journal of functionnal analysis*, 257 :3239–3292, 2009.
- [9] A. Hatcher. *Algebraic Topology*. 2001.
- [10] Vincent Lafforgue. Banach kk -theory and the baum-connes conjecture. *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002)(Higher Ed. Press, Beijing) :795–812, 2002.
- [11] D. Voiculescu M. Pimsner. Exact sequences for k -groups and ext -groups of certain cross-products of c^* -algebras. *Operator theory*, 4 :93–118, 1980.
- [12] Gerard J. Murphy. *C^* -algebras and operator theory*. Academic Press Inc., 1990.
- [13] Alain Connes Paul Baum. Geometric k -theory for lie groups and foliations. *Enseign. Math.*, 46 :3–42, 2000.
- [14] Nigel Higson Paul Baum, Alain Connes. Classifying space for proper actions and k -theory of group c^* -algebras. *Contemporary Mathematics*, 197 :241–291, 1994.
- [15] Michael V. Pimsner. A class of c^* -algebras generalizing both cuntz-krieger algebras and crossed products by F . *Fields Institute Communications*, 12 :189–212, 1997.
- [16] Jean Renault. *A groupoid approach to C^* -algebras*. Springer-Verlag, 1980.
- [17] Marc Rieffel. c^* -algebras associated with irrational rotations. *Pacific Journal of Mathematics*., 93(2) :415–429, 1981.
- [18] Tu J.L. Yu G. Skandalis, G. The coarse baum-connes conjecture and groupoids. *Topology*, 41(4) :807–834, 2002.
- [19] Jean-Louis Tu. a conjcture de novikov pour les feuilletages hyperboliques. *K-theory*, 16(2) :129–184, 1999.
- [20] Jean-Louis Tu. The baum-connes conjecture for groupoids. *C^* -algebras*, 18 :227–242, 1999.
- [21] Jean-Louis Tu. The coarse baum-connes conjecture and groupoids ii. *New York J. Math.*, 18 :1–27, 2012.
- [22] N.E. Wegge-Olsen. *K-theory and C^* -algebras, a friendly approach*. Oxford University Press, 1993.