Questions from Münster

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1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in K-theory, like determining generators of K-group of some known C^*-algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for $\mathbb{R}^n$. (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space $\mathcal{G} = X \rtimes G$. The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [?], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}\,.$$

Indeed, suppose Y is a $\mathcal{G}$-proper cocompact space, then $Y \rtimes \mathcal{G}$ is a proper groupoid, so there exists a cutt-off function $c : Y \to [0,1]$ such that :

$$\sum_{g \in \mathcal{G}^{p(y)}} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^{p(y)}} c(yg)\delta_g$$

which is a $\mathcal{G}$-equivariant continuous map. Moreover Z_d is proper and cocompact, and there exists a d s.t. the map takes its values in it.

Now if $\mathcal{G} = X \rtimes G$, $Z_d \simeq X \times Z'_d$ where $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$, so that $KK^{\mathcal{G}}(\Delta, A) \simeq KK^{G}(\Delta', A)$, where Δ and Δ' are respectively the 0-dimensional part of the equivariant complexes Z_d and Z'_d. This is true because the action of G on Z'_d is proper and cocompact, see lemma 3.6 of [?]. Now a standard Mayer-Vietoris argument (theorem 3.8 [?]) concludes to show that $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$.

As $C^*_r\mathcal{G} = C_0(X) \rtimes_r G$, we see that the Baum-Connes assembly map for $\mathcal{G}$ with coefficients in A is equivalent to

$$K^{top}_*(G, A) \to K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle $U(2) \to U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$. You can foliate the fibers by an irrational rotation θ, so that you have an action of $\mathbb{R}$ on $C(U(2))$. Reducing to a complete transversal (take $SU(2)$), the algebra $C(U(2)) \rtimes \mathbb{R}$ turns out to be Morita equivalent to $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$ (a general result of foliation groupoids I think). $\underline{A}$ can be reduced to $C(\overline{D}) \otimes A_\theta$ and to $Ind_{\mathbb{T}^2}^{U(2)}\ A_\theta$.

Question : Compute the generators of the K-theory of $\underline{A}$.

1.3 Foliations

1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus $M = \mathbb{T}^2$. Its fundamental group $\Gamma = \mathbb{Z}^2$ acts on its universal cover $\tilde{M} = \mathbb{R}^2$ by isometries, and the electromagnetic field A gives a two-form w (its curvature) on $\tilde{M}$, so a 2-cocycle on the fundamental groupoid of $\tilde{M}$:

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_\Delta \tilde{w}}$$

where Δ a geodesic triangle between the 3 points. It turns out that $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$, so that $\tilde{w}$ determines a number $\theta \in [0, 1)$, and the twisted reduced algebra of the fundamental groupoid w.r.t. $\tilde{w}$ is equal to $A_\theta = C(\mathbb{T}^2) \rtimes_{r,\theta} \mathbb{Z}^2$. This situation generalizes to general manifold whose fundamental cover are equiped with a line bundle and a conection. We can then associate a 2-cocycle on the fundamental groupoid of $\tilde{M}$ to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let G be a real reductive group. For all parabolic subgroup P, there is only one nilpotent normal subgroup N, and the Levi is defined as $P = LN$. The idea of Pierre Julg is to fix first a Levi susgroup L of G. Now there is only a finite numbers of choices for N, so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group $W_L = N_G(L)/L$ acts on it by $w.N = wNw^{-1}$. Pierre Clare defined a C_r^*L-module $C_r^*(G/N)$, equipped with and action of C_r^*G by compacts operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$, where $L = MA$, $\hat{M}_d$ is the discrete dual of M, and $\hat{A} = \mathfrak{a}^*$. Then $\sigma \otimes \tau$ is a représentation of $MA = L$, which we can trivially extend to N to induce it on G. Pierre Clare showed the following fact :

$$Ind_P^G \; H_{\sigma\otimes\tau\otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma\otimes\tau}.$$

For every $\tilde{w} \in N_G(L)$, the operator $\rho(\tilde{w}) : C_r^*(G/N) \to C_r^*(G/w.N)$ is well defined and gives a morphism

$$Ad\; \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \to \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because C_r^*G is acting on $C^*(G/N)$ by compact operators. This gives a morphism

$$C_r^*G \to \bigoplus_{[L]} \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard C^*-algebraic tools).

The first step would be to prove that

$$\begin{array}{ccc} C_r^*G & \to & \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N))\right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for $F_N^{-1}(g) = \mathrm{Tr}_{C_d^*L}\; (T\pi_N(g^{-1}))$.

2.1 In $SL(2,\mathbb{R})$

In this case, G acts on the Poincaré disc by homographies, and P can be taken as the stabilizer of a point at infinity, and L stabilizes a geodesic, that is to say two points at infinity, so that

$$P_{1,1} \simeq \{\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}\}, \quad L \simeq \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\}, \quad N \simeq \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing P amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and L. Now fix two points at infinity, which gives you L. You now only have two choices for P, and the two are exchanges under the action of W_L on the nilpotent groups.

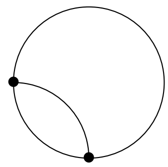

FIGURE 1 – Choices for the Levi subgroup

3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let $\mathcal{N}$ be the smallest class of C^*-algebras containing $\mathbb{C}$, closed under countable inductive limits, stable by KK-equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in $\mathcal{N}$, so is the third). Here is the classical theorem :

Théorème 1 (Universal Coefficient Theorem)**.** Let A and B be two separable C^*-algebras, where A is in $\mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow KK_*(A, B) \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ? ?
- The second map is given by the boundary element associated to any impair K-cycle. Namely, if $z \in KK^1(A, B)$, let (H_B, π, T) be a K-cycle representing z, and P the associated projector $P = \frac{1+T}{2}$. Define the pull-back
$$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$
Then the boundary of the folowing extension
$$0 \longrightarrow \mathfrak{K}_B \longrightarrow E^{(\pi,T)} \longrightarrow A \longrightarrow 0$$
is given by $\quad \partial = - \otimes z : K_*(A) \longrightarrow K_*(B) \quad$ which depends only on z.
The map is just $z \mapsto \partial$
- If $\partial = 0$, then the sequence associated to z splits and we have exact sequences
$$0 \longrightarrow K_*(B) \longrightarrow K_*\left(E^{(\pi,T)}\right) \longrightarrow K_*(A) \longrightarrow 0$$
which gives an element of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Miscenko and Kasparov.

Proposition 1. Let G be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold M. Then

$$K^{top}(G) \xrightarrow{\mu} K(C_r^*G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for G.

The original point being that G. Yu can prove this (how ?) without usig the heavy machinery of the Dirac Dual-Dirac method, nor anything related to KK^G-theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

Proposition 2. Let G be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture hold for G.

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^* P_d(X_{2k})) \xrightarrow{\beta} K\left(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k})\right))$$

is an isomorphism.

Here are some details : first decompose $G = G_0 \cup G_1$ into two subspaces, which are not necesseraly subgroups, such that each is a R-disjoint union of bounded subsets (in fact finite since G is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define $\prod^R C^*(P_d(X_{2k}) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), prop(T_{2k}) \leq R\}$, so that $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$, and each X_{2k} corasely embedds into some $\mathbb{R}^{m_k}$. The isomorphism of $\beta \circ \mu$ implies the injectivity of μ, and by cutting and pasting, μ can be shown to be injective for G so that Novikov is satisfied.