Questions from Münster

Clément Dell'Aiera

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1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in K-theory, like determining generators of K-group of some known C^*-algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for $\mathbb{R}^n$. (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space $\mathcal{G} = X \rtimes G$. The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [3], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}.$$

Indeed, suppose Y is a $\mathcal{G}$-proper cocompact space, then $Y \rtimes \mathcal{G}$ is a proper groupoid, so there exists a cutt-off function $c : Y \to [0,1]$ such that :

$$\sum_{g \in \mathcal{G}^{p(y)}} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^{p(y)}} c(yg) \delta_g$$

which is a $\mathcal{G}$-equivariant continuous map. Moreover Z_d is proper and cocompact, and there exists a d s.t. the map takes its values in it.

Now if $\mathcal{G} = X \rtimes G$, $Z_d \simeq X \times Z'_d$ where $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$, so that $KK^{\mathcal{G}}(\Delta, A) \simeq KK^G(\Delta', A)$, where Δ and Δ' are respectively the 0-dimensional part of the equivariant complexes Z_d and Z'_d. This is true because the action of G on Z'_d is proper and cocompact, see lemma 3.6 of [3]. Now a standard Mayer-Vietoris argument (theorem 3.8 [3]) concludes to show that $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$.

As $C_r^* \mathcal{G} = C_0(X) \rtimes_r G$, we see that the Baum-Connes assembly map for $\mathcal{G}$ with coefficients in A is equivalent to

$$K_*^{top}(G, A) \to K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle $U(2) \to U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$. You can foliate the fibers by an irrational rotation θ, so that you have an action of $\mathbb{R}$ on $C(U(2))$. Reducing to a complete transversal (take $SU(2)$), the algebra $C(U(2)) \rtimes \mathbb{R}$ turns out to be Morita equivalent to $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$ (a general result of foliation groupoids I think). $\underline{A}$ can be reduced to $C(\overline{D}) \otimes A_\theta$ and to $Ind_{\mathbb{T}^2}^{U(2)}\ A_\theta$.

Question : Compute the generators of the K-theory of $\underline{A}$.

1.3 Foliations

1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus $M = \mathbb{T}^2$. Its fundamental group $\Gamma = \mathbb{Z}^2$ acts on its universal cover $\tilde{M} = \mathbb{R}^2$ by isometries, and the electromagnetic field A gives a two-form w (its curvature) on $\tilde{M}$, so a 2-cocycle on the fundamental groupoid of $\tilde{M}$:

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_\Delta \tilde{w}}$$

where Δ a geodesic triangle between the 3 points. It turns out that $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$, so that $\tilde{w}$ determines a number $\theta \in [0, 1)$, and the twisted reduced algebra of the fundamental groupoid w.r.t. $\tilde{w}$ is equal to $A_\theta = C(\mathbb{T}^2) \rtimes_{r,\theta} \mathbb{Z}^2$. This situation generalizes to general manifold whose fundamental cover are equiped with a line bundle and a conection. We can then associate a 2-cocycle on the fundamental groupoid of $\tilde{M}$ to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let G be a real reductive group. For all parabolic subgroup P, there is only one nilpotent normal subgroup N, and the Levi is defined as $P = LN$. The idea of Pierre Julg is to fix first a Levi susgroup L of G. Now there is only a finite numbers of choices for N, so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group $W_L = N_G(L)/L$ acts on it by $w.N = wNw^{-1}$. Pierre Clare defined a C_r^*L-module $C_r^*(G/N)$, equipped with and action of C_r^*G by compacts operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$, where $L = MA$, $\hat{M}_d$ is the discrete dual of M, and $\hat{A} = \mathfrak{a}^*$. Then $\sigma \otimes \tau$ is a représentation of $MA = L$, which we can trivially extend to N to induce it on G. Pierre Clare showed the following fact :

$$Ind_P^G\ H_{\sigma\otimes\tau\otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma\otimes\tau}.$$

For every $\tilde{w} \in N_G(L)$, the operator $\rho(\tilde{w}) : C_r^*(G/N) \to C_r^*(G/w.N)$ is well defined and gives a morphism

$$Ad\ \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \to \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because C_r^*G is acting on $C^*(G/N)$ by compact operators. This gives a morphism

$$C_r^*G \to \bigoplus_{[L]} \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard C^*-algebraic tools).

The first step would be to prove that

$$\begin{array}{ccc} C_r^*G & \to & \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N))\right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for $F_N^{-1}(g) = \mathrm{Tr}_{C_d^*L}\ (T\pi_N(g^{-1}))$.

2.1 In $SL(2, \mathbb{R})$

In this case, G acts on the Poincaré disc by homographies, and P can be taken as the stabilizer of a point at infinity, and L stabilizes a geodesic, that is to say two points at infinity, so that

$$P_{1,1} \simeq \{\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}\}, \quad L \simeq \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\}, \quad N \simeq \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing P amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and L. Now fix two points at infinity, which gives you L. You now only have two choices for P, and the two are exchanges under the action of W_L on the nilpotent groups.

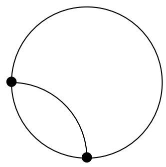

FIGURE 1 – Choices for the Levi subgroup

3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let $\mathcal{N}$ be the smallest class of C^*-algebras containing $\mathbb{C}$, closed under countable inductive limits, stable by KK-equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in $\mathcal{N}$, so is the third). Here is the classical theorem :

Théorème 1 (Universal Coefficient Theorem)**.** Let A and B be two separable C^*-algebras, where A is in $\mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow KK_*(A,B) \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ? ?
- The second map is given by the boundary element associated to any impair K-cycle. Namely, if $z \in KK^1(A,B)$, let (H_B, π, T) be a K-cycle representing z, and P the associated projector $P = \frac{1+T}{2}$. Define the pull-back

  $$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$

  Then the boundary of the folowing extension

  $$0 \longrightarrow \mathfrak{K}_B \longrightarrow E^{(\pi,T)} \longrightarrow A \longrightarrow 0$$

  is given by $\quad \partial = - \otimes z : K_*(A) \longrightarrow K_*(B) \quad$ which depends only on z. The map is just $z \mapsto \partial$
- If $\partial = 0$, then the sequence associated to z splits and we have exact sequences

  $$0 \longrightarrow K_*(B) \longrightarrow K_*\left(E^{(\pi,T)}\right) \longrightarrow K_*(A) \longrightarrow 0$$

  which gives an element of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Miscenko and Kasparov.

Proposition 1. Let G be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold M. Then

$$K^{top}(G) \xrightarrow{\mu} K(C^*_r G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for G.

The original point being that G. Yu can prove this (how ?) without usig the heavy machinery of the Dirac Dual-Dirac method, nor anything related to KK^G-theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

Proposition 2. Let G be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture hold for G.

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^* P_d(X_{2k})) \xrightarrow{\beta} K\left(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k})\right))$$

is an isomorphism.

Here are some details : first decompose $G = G_0 \cup G_1$ into two subspaces, which are not necesseraly subgroups, such that each is a R-disjoint union of bounded subsets (in fact finite since G is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define $\prod^R C^*(P_d(X_{2k}) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), prop(T_{2k}) \leq R\}$, so that $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$, and each X_{2k} corasely embedds into some $\mathbb{R}^{m_k}$. The isomorphism of $\beta \circ \mu$ implies the injectivity of μ, and by cutting and pasting, μ can be shown to be injective for G so that Novikov is satisfied.

4 Funky questions, ideas of talks

4.1 Expanders

Here are some interesting questions I had after a talk on expanders.

4.1.1 Plan of the talk

I first gave a motivation for considering expanders. Namely, we are interested in the following network theory problem : can we construct a network as big as we want, such that the cost is controlled and which is not subject to easy failure ?

Building a network as big as we want means we want to consider a family of graphs $X_j = (V_j, E_j)$ such that $|V_j| \to +\infty$, and controlling the cost means that $deg(X_j) < k$ for all j. But what does "not easily subject to failure" means ? For this, I want to explain why we should ask our family to stay well connected and why the second value of the discrete Laplacian is a good way to measure that.

The idea is to relate the Laplacian to the uniform random walk on the graph, and to show that $\lambda_1(X)$ controlls the speed of convergence of the uniform random walk to the stationary measure which is the uniform probability on the graph, given by $\nu(x) = C.deg(x)$.

A family of graphs satifying the previous conditions and such that $\lambda_1(X_j) > c > 0$ is called an expander. If time allows, one can then elaborate on metric properties of this type of graphs. The impossibility to embed them coarsely into any separable Hilbert space, and the relations to the Baum-Connes conjecture are close to my work.

4.1.2 Questions

- Paolo Pigato : What is the dynamic at the limit ?
- Anne Briquet : Is $\lambda_1(X)$ such a good way to measure the connectedness of a graph, if you consider the phenomenon of cuttoff for finite Markov Chains.

4.2 Ideas of funky talks

- What is the relation between the Fourier transform and quantum groups ?
- What is the relation between the Runge Kutta methode and renormalization in QFT ?
- What is the relation between Brownian motion and second quantization ?

5 A list of books

A list of books I like about general knowledge in science :
- L'aventure des nombres, Godefroy
- L'autobigraphie de Paul Levy, Laurent Schwartz, et Yuri Manin.
- Recoltes et semailles, Grothendieck.
- Lee Smolin, The trouble with physics, the rise of String theory, the fall of a Science, and what comes next,
- Julian Barbour, The End of Time, The next revolution in Physics,
- Carlo Rovelli, Et si le temps n'existait pas, un peu de science subversive,
- Mandlebrot, The (Mis)Behaviour of markets, Fractals and Chaos, the Mandelbrot set and beyond, The fractal geometry of nature.
- Manjit Kumar :
- Amir Alexander, Infinitesimal : How a Dangerous Mathematical Theory Shaped the Modern World
- Ian Stewart, Does God play dice ?
- History of Statistics, Stielger
- Logicomix

Overview and more specialized books :
- Moonshine beyond the Monster, Terry Gannon
- Le theoreme d'uniformisation, Saint-Gervais
- Invitation aux mathematiques de Fermat, Hellgouarch
- Rached Mneime, tous ses livres !
- Hubbard West pour les equa diff
- Noether's theorem, Yvette K
- Nother's wonderful theorem
- The annus mirabellus of Einstein
- The Road to Reality, Sir Roger Penrose

6 Seminar

6.1 Cartan subalgebras

Out of any inclusion of C^*-algebras $A \subseteq B$ with A unital commutative, we construct an action of the normalizer of A in B by partial homeomorphism on X the spectrum of A, i.e. a homomorphism of semigroup

$$\alpha : N_B(A) \to SHomeo(X).$$

If $n \in N_B(A)$ and $x \in Spec(A)$, set

$$\langle \alpha_n(x), a \rangle = \langle x, n^* a n \rangle.$$

This defines a homeomorphism

$$\alpha_n : U_n \to U_{n^*},$$

where $U_n = \{x \in Spec(A), n^* n(x) > 0\}$ such that $\alpha_{nm} = \alpha_n \circ \alpha_m$.

If A is maximal abelian in B, and other conditions, then B is shown to be isomorphic to the twisted reduced C^*-algebra of the groupoid of stalks of $N_B(A)$. This can be seen as an extension of the Gelfand transform

$$\left\{ \begin{array}{ccc} B & \to & C_r^*(G) \\ b & \mapsto & \end{array} \right.$$

6.2 Classification and the UCT

For A a simple unital C^*-algebra, the Elliot invariant is :

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \to S(K_0(A)))\,,$$

here $T(A)$ is the trace space and r_A the paring $r_A(\tau)([p]) = [\tau(p)]$.

Elliot's conjecture : Separable, simple, nuclear are classifiable by Elliot's invariants.

Théorème 2. Separable, simple, unital, nuclear, $\mathcal{Z}$-stable, UCT algebras are classifiable by Elliot's invariants.

An example of a classification theorem : Elliot's theorem,

Théorème 3. Let A and B unital AF-algebras and

$$\alpha : K_0(A) \to K_1(A)$$

a unital order isomorphism, i.e.

$$\alpha(K_0(A)_+) \subseteq K_0(B)_+ \quad \text{and} \quad \alpha([1_A]) = [1_B].$$

Then there exists a unital $*$-isomorphism $\phi; A \to B$ such that $\phi_* = \alpha$.

7 Groups

- Amenable, a-T-menable, property T, with a diagram
- Mapping class groups
- Profinite groups, locally profinite groups, $Aut(\overline{\mathbb{Q}}/\mathbb{Q})$
- Automorphism of a regular tree, the Grigorchuk group,
- Lamplighter groups $H^\Gamma \rtimes \Gamma$, usually

$$\oplus\mathbb{Z}_2 \rtimes \mathbb{Z}.$$

And groupoids :
- the coarse groupoid $G(X)$: étale (even ample) with totally disconnected basis βX. Dynamical asymptotic dimension of asymptotic dimension of X. A-T-menable iff X has property A
- HLS groupoid associated to a sequence of finite metric spaces X_n equipped with maps $X_n \to \Gamma$ to a finitely generated group $\Gamma = \langle S \rangle$.
- groupoids of germs of semigroup of partial homeomorphisms acting on a topological space
- holonomy groupoids of a foliation
- action groupoids $X \rtimes \Gamma$, principal bundles groupoids $P \times_G P$, where $P \to X$ is a G-bundle
- equivalence relation groupoids

8 C^*-algebras

- CAR algebra $C^*\langle a_i, a_i a_j + a_j a_i = \delta_{ij}\rangle$ or $\bigotimes M_2(\mathbb{C})$ or

$$\varinjlim \left\{ \begin{array}{rcl} M_{2^n}(\mathbb{C}) & \to & M_{2^{n+1}}(\mathbb{C}) \\ a & \mapsto & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{array} \right.$$

Class of C^*-algebras.

- The bootstrap class $\mathcal{B}$

- The class $\mathcal{N}$ of C^*-algebras A such that the map

$$\alpha_{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

  is an isomorphism for every C^*-algebra B such that $K_*(B)$ is a free abelian group. In [**?**], it is shown that $\mathcal{N}$ contains all of the bootstrap class.

- Non exact C^*-algebra : to my knowledge only one example is known : the reduced C^*-algebra $C^*_r(\Gamma)$ of a finitely generated group whose Cayley graph contains expander. Using Ozawa's result [**?**], one can construct finite dimensional C^*-algebras M_{X_n} such that

$$0 \to C^*_r(\Gamma) \otimes \bigoplus M_{X_n} \to C^*_r(\Gamma) \otimes \prod M_{X_n} \to C^*_r(\Gamma) \otimes \Big(\prod M_{X_n} / \bigoplus M_{X_n} \Big) \to 0$$

  is not exact in the middle.
  The problem of the existence of such a group is an interesting question, which was stated by Gromov and proved rigorously by several people in the wake of this.

One can define analog of approximation properties in the setting of K-theory.

- A is K-nuclear if the class of the natural map

$$p_{A,B} : A \otimes_{max} B \to A \otimes_{min} B$$

  is invertible as an element of $KK(A \otimes_{max} B, A \otimes_{min} B)$.

- G is K-amenable if the class of the regular representation

$$\lambda_G : C^*_{max}(G) \to C^*_r(G)$$

  is invertible as an element of $KK(C^*_{max}(G), C^*_r(G))$.

For instance, Skandalis proves in [**?**] that, if Λ is an infinite hyperbolic property T group, then $C^*_r(\Lambda)$ is not K-nuclear. In particular, it is not KK-equivalent to a nuclear C^*-algebra, and cannot be Bootstrap. This completely renders proving the Baum-Connes conjecture by mean of Dirac-Dual-Dirac method hopeless. An example of such a group is given by any lattice in $Sp(n,1)$ for instance. (higher rank algebraic semisimple groups ?)

After developing a restriction principle for groupoids, a natural question was to find a C^*-algebra coming from a groupoid crossed-product that we were able to prove that it satisfied the Künneth formula, while still not being a consequence of previous results. One could have started with the so called HLS groupoid $G_{\mathcal{N}}(\Gamma)$ associated to a residually finite finitely generated group Γ and a nested sequence of decreasing normal sugroups of finite index $\mathcal{N}$.

One always has the following exact sequence of $*$-algebras

$$0 \to \oplus \mathbb{C}[\Gamma_n] \to C_c(G) \to \mathbb{C}[\Gamma] \to 0$$

which induces the following exact sequence of C^*-algebras

$$0 \to \oplus \mathbb{C}[\Gamma_n] \to C_r^*(G) \to C_{\mathcal{N}}^*(\Gamma) \to 0$$

where $C_{\mathcal{N}}^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ w.r.t. to the norm

$$||x||_{\mathcal{N}} = \sup_{N \in \mathcal{N}} ||\lambda_N(x)|| \quad x \in \mathbb{C}[\Gamma]$$

induced by the quasi-regular representations $\lambda_N : C_{max}^*(\Gamma) \to \mathcal{L}(l^2(\Gamma/N))$.

Now this exact sequence intertwines the Baum-Connes assembly maps, and the Baum-Connes conjecture for $G_{\mathcal{N}}(\Gamma)$ is equivalent to $\mu_{\Gamma,\mathcal{N}}$ being an isomorphism.

- If $\Gamma = \mathbb{F}_2$ and
$$N_n = \cap ker\phi$$
for ϕ running accross all group homomorphisms from Γ to a finite group of cardinality less than n, then $C_{\mathcal{N}}^*(\Gamma) \cong C_{max}^*(\Gamma)$ and G satifies the Baum-Connes conjecture, is ample and satisfies the restriction condition. So we get that $C_r^*(G)$ satisfies the Künneth formula. It is still a result that one can get using the fact that Γ being a-T-menable, it is K-amenable. Hence $C_{max}^*(\Gamma)$ and $C_r^*(\Gamma)$ are KK-equivalent and bootstrap, so that $C_r^*(G)$ also is by extension stability of bootstrapness. A remark of R. Willett is worth mentioning : $\mathbb{F}_2$ being the fundamental group of the wedge of two circles, it is KK-equivalent to $C(\mathbb{S}^1 \wedge \mathbb{S}^1)$.

- One can artificially try to get rid of bootstrapiness by spatially tensoring this exact sequence by $C_r^*(\Lambda)$ for a infinite hyperbolic property T group. One then get the extension
$$0 \to \oplus \mathbb{C}[\Gamma_n] \otimes_{min} C_r^*(\Lambda) \to C_r^*(G \times \Lambda) \to C_{\mathcal{N}}^*(\Gamma) \otimes_{min} C_r^*(\Lambda) \to 0.$$
The restriction principle applies for the groupoid $G_{\mathcal{N}}(\Gamma) \times \Lambda$, and induces that its reduced C^*-algebra satisfies the Künneth formula. But then again, one can deduce this from a previous result, namely the restriction principle for groups. Indeed, apply it to Λ with coefficient on the trivial bootstrap Λ-algebra $C_r^*(G)$.

- Bekka shows that ? ?

9 Baum-Connes

Compact groups, or better : proper groupoids : Green-Julg.
Connes-Kasparov : proof by representation theory (Wasserman, etc)
Kasparov's Conspectus : towards Higson-Kasparov paper and the proof for Haagerup (J-L. Tu's general version in KK-theory, plus the beautiful result that aTmenability implies bootstrap)
Ideas from Coarse geometry, and Yu and Roe's work, SkandalisTuYu etc.

10 GPOTS & NCGOA 2018

10.1 Arnaud Brothier : some representations of the Thompson group

10.2 Piotr Nowak : Property T for $Out(\mathbb{F}_n)$

10.3 Wilhem Winter : Relative nuclear dimension

10.4 Rufus Willett : Exactness and exotic crossed-product

11 Coarse geometry & dynamics

12 Noncommutative geometry

12.1 Basic objects and constructions

Mainly, I'm interested in $*$-algebras A (and their completions) which are k-algebras equipped with an involution $*$. Usually, $k = \mathbb{C}$ is the field of complex numbers. A very famous example of $*$-algebra is the algebra of the quantum harmonic oscillator,

$$\mathcal{H} = k\langle x, y\rangle/(xy - yx = 1).$$

When $k = \mathbb{C}$, one often represent A as a sub-$*$-algebra of the bounded operators on aHilbert space $\mathcal{L}(H)$,and complete w.r.t. to the norm. Note that not all complex $*$-algebras admit such a representation.

For instance, for$\mathcal{H}$, one easily get that

$$[x, P(y)] = P'(y) \quad \forall P \in \mathbb{C}[t]$$

Then if $|| \ ||$ is a multiplicative norm on$\mathcal{H}$, it satisfies

$$2||x|| \ ||y|| \geq n \quad \forall n > 0.$$

Basic construction :

- separation-completion : in our sense, a norm can be degenerate. Being multiplicative, the annhiliator of any norm is a closed ideal in A, so that there is an induced (classical/ nondegenerate) norm on the quotient algebra. The separation-completion is defined to be the completion of the quotient w.r.t.

the induced norm. Let us say that if α is such a norm, wedenote by A_α the associated separation-completion. Any inequality

$$\alpha(x) \leq \beta(x) \quad \forall x \in A$$

induces an inclusion of annilihator $N_\beta \subset N_\alpha$, and gives a canonical quotient map

$$A_\beta \to A_\alpha.$$

The basic class of examples comes from completion of the complex group ring $\mathbb{C}[\Gamma]$. For any family of unitary representations $\mathcal{F}$, one can define the $*$-norm

$$||x||_{\mathcal{F}} = \sup\{||\pi(x)|| : \pi \in \mathcal{F}\}$$

on $\mathbb{C}[\Gamma]$. The separation-completion is a C^*-algebra denoted $C^*_{\mathcal{F}}(\Gamma)$. For instance, if $\mathcal{F}$ consists of all unitary representations of Γ, then one gets the maximal C^*-algebra $C^*_{max}(\Gamma)$, while if the family is reduced to the left regular representation λ_Γ, one gets the reduced C^*-algebra $C^*_r(\Gamma)$. By inclusion, one gets the canonical quotient map

$$\lambda_\Gamma : C^*_{max}(\Gamma) \to C^*_r(\Gamma).$$

Crossed-product : the basic ingredients are a $*$-algebra H endowed with a coassociative coproduct

$$\Delta : H \to H \otimes H,$$

and a C^*-algebra A on which H acts via a $*$-homomorphism

$$\alpha : A \to A \otimes H$$

such that $(1 \otimes \Delta)\alpha = (\alpha \otimes 1)\alpha$. The crossed-product is a twisted version of the tensor product.

$$(a \otimes x)(a' \otimes y) := (a \otimes 1_{M(H)})\alpha(a')(1_{M(A)} \otimes xy)$$

12.2 Quantum groups

A C^*-bialgebra is a pair (H, Δ) where H is a C^*-algebra and

$$\Delta : H \to M(\tilde{H} \otimes_{min} H + H \otimes_{min} \tilde{H}, H \otimes_{min} H)$$

is a non-degenerate $*$-homomorphism such that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

A H-algebra is a pair (A, α) where A is a C^*-algebra and

$$\alpha : A \to M(\tilde{A} \otimes_{min} H, A \otimes_{min} H)$$

such that $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha)$. Its principal map is

$$\Psi : \begin{cases} A \otimes_{alg} A & \to \quad M(A \otimes_{min} H) \\ \quad x \otimes y & \mapsto \quad (x \otimes 1_{M(H)})\alpha(y) \end{cases}$$

Let (H, Δ) be a C^*-bialgebra and (A, α) a H-algebra, with principal map

$$\Psi : A \otimes A \to M(A \otimes_{min} H).$$

- free if the range of Ψ is strictly dense in $M(A \otimes_{min} H)$
- proper if the range of Ψ is contained in $A \otimes_{min} H$
- principal if $\Psi(A \otimes_{alg} A)$ is a norm dense subset of $A \otimes_{min} H$

principal = free and proper

12.3 Why $SU_q(2)$?

Apparently, some people are interested in deformation of classical Lie groups such as $SU_q(2)$, which is the Hopf algebra generated by 3 generators E, F, K satisfying the relations

$$R.$$

I wanted to understand where these relations are coming from, which led me to interesting ideas developed by several people, including Yuri Manin. The idea is to define $SU_q(2)$ as a special group like object of the automorphism group of some noncommutative space, the quantum plane.

Let k be a field. The free (noncommutative) k-algebra on n generators is denoted by $k\langle x_1, ..., x_n\rangle$.

Définition 1. A quadratic algebra

$$A = \oplus_{i\geq 0} A_i$$

is a $\mathbb{N}$-graded finitely generated algebra such that :
- $A_0 = k$, and A_1 generates A,
- the relations on generators are in $A_1 \otimes A_1$.

The quadratic algebra A is said to be a Frobenius algebra of dimension d if moreover
- $A_d = k$ and $A_i = 0$ for all $i > d$,
- the multiplication map
$$m : A_i \otimes A_{d-i} \to A_d$$
is a perfect duality.

The main example is the quantum plane

$$\mathbb{A}_q^2 = k\langle x, y\rangle/(xy - qyx)$$

where $q \in k^\times$. More generally, the quantum space of dimension $n|m$ is

$$\mathbb{A}_q^{n|m} = k\langle x_1, .., x_n, \eta_1, ..., \eta_m\rangle/(x_i x_j - q x_j x_i, q\eta_i\eta_j + \eta_j\eta_i).$$

This example is suppose to come from physics. In quantum field theories, physicists deal with two kind of particles, bosons and fermions, and use commuting variables for one type, and anticommuting for the other. One object they appeal to are called supermanifolds, which are manifolds enriched with anticommuting variables. Formally, it means they look at ringed spaces $(X, \mathcal{O})$ locally isomorphic to $(\mathbb{R}^n, C^\infty[\eta_1, ..., \eta_m])$, where $C^\infty[\eta_1, ..., \eta_m]$ is the free sheaf of rings generated by anticommuting variables η_i over the smooth complex valued functions $C^\infty(\mathbb{R}^n)$.

Remark that a quadratic algebra A is a quotient of $k\langle x_1, ..., x_n\rangle$ by elements $r_\alpha \in A_1 \otimes A_1$, which we will denote as

$$A = k\langle x_1, ..., x_n\rangle/(r_\alpha)$$

or

$$A = \langle A_1, R_A\rangle$$

with $R_A \subseteq A_1 \otimes A_1$.

Manin defines the quantum dual of a quadratic algebra as

$$A^! = k\langle x^i \rangle / (r^\beta)$$

where $r^\beta_{ij} r^{ij}_\alpha = 0$, i.e. $R_{A^!} = R_A^\perp$. Then, the quantum endormorphisms between two quadratic algebra is

$$Hom(A, B) = k\langle z_i^j \rangle / (r_\alpha^\beta)$$

where $r^\beta_\alpha = r^{ij}_\alpha r^\beta_{kl} z_i^k z_j^l$. If $End(A) = Hom(A, A)$, then $End(A)$ satisfies the universal property to be intial in the category of k-algebras (B, β) endowed with an algebra homomorphism $\beta : A \to A \otimes B$.

If one does that to the quantum plane $\mathbb{A}_q^2$, one stil doesn't find quite $M_q(2)$: half of the relations are missing. Also

$$(\mathbb{A}_q^{2|0})^! = \mathbb{A}_q^{0|2}?$$

Exercise.

12.4 TQFT

We recalled the definitions of a monoidal category, a braided category, and a symmetric monoidal category. The two main examples are the category of bordisms $Bord^d$ in dimension d, and the category of vector spaces over a field k. The first talk focused on topological quantum fields theories in dimension 1 and 2.

Définition 2. A TQFT in dimension d is a monoidal symmetric functor

$$Z : Bord_d \to Vect_k.$$

The two main results we showed are :

- there is an equivalence of categories

$$TQFT_1 \cong Vect_k$$

  obtained as $Z \mapsto Z(pt)$.
- there is an equivalence of categories

$$TQFT_2 \cong Frob_k$$

  obtained as $Z \mapsto Z(\mathbb{S}^1)$.

A nice example in dimension 2 : $Z(\mathbb{S}^1) = \mathbb{C}[t]/(t^2 - 1)$ is the Frobenius algebra given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad \epsilon(1) = 0 \quad \epsilon(t) = 1.$$

Then the handle element is $h = 2t$ and

$$Z(\Sigma_g) = \begin{cases} 2^g & \text{if } g \text{ is odd} \\ 0 & \text{if } g \text{ is even.} \end{cases}$$

The second talk was directed towards extended field theories. First recall some higher category theory : n-categories, etc... And an extented TFT is a symmetric monoidal functor between symmetric monoidal n-categories

$$Z : Cob_n \to \mathcal{C}.$$

Then the following theorem was proved in [2].

Théorème 4. The evaluation functor

$$Z \mapsto Z(*)$$

establishes a bijective correspondance between extended n-dimensional TFT and fully dualizable objects of $\mathcal{C}$.

We now give an application of this result to the Jones polynomial. In [4], Witten gives an interpretation of the Jones polynomial, an isotopy invariant of links, as induced from a 3-dimensional TFT. The drawback of this article (for us) is that Witten uses Physical TFT's, i.e. gauge theories. The Jones polynmial is then shown to be the value of the partition function of a gauge field theory on $\mathbb{S}^3$ with gauge group $SU(2)$. I propose to rewrite this result in our setting as an exercise.

A link is a disjoint union of embedding of the circle into $\mathbb{S}^3$

$$\mathcal{L} = \{\text{embeddings } \coprod_{i=1}^{k} \mathbb{S}^1 \hookrightarrow \mathbb{S}^3\}.$$

we will often make no distinction between the embedding and its image in the 3-sphere, which we will denote by L. The Jones polynomial of a link L is defined as an isotopy invariant polynomial $V : \mathcal{L} \to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ satisfying the Skein relations

$$-t^{\frac{1}{2}}V_+ + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_0 + t^{-\frac{1}{2}}V_- = 0.$$

To a link L one can associated the 3-manifold $M_L = \mathbb{S}^3 - L$. Consider the extended TFT

$$Z^{(n)} : Cob_3 \to \mathcal{C}$$

given by $Z() = V_n$ where is the fundamental representation of $\mathfrak{su}(n)$. By the cobordism theorem, it is enough to define the TFT on all of Cob_3. Then

$$\phi(V_L) = Z^{(2)}(M_L),$$

where $\phi : \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \to \mathbb{C}$ is the evaluation at a root of unity $q \in \mathbb{C}^\times$. This can be proved by showing that $Z^{(n)}(M_L)$ satisfies the skein relation

$$-q^{\frac{n}{2}}V_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_0 + q^{-\frac{n}{2}}V_- = 0$$

12.5 Reminder

A locally ringed space is a topological space X together with a sheaf or ring $\mathcal{O}_X$ over X such that all stalks are local rings, ie have a unique maximal ideal.

For R a ring, $X = Spec(R)$ denotes the topological space obtained as the set of prime ideals of R endowed with the Zariski topology, i.e. the topology generated by the closed subsets

$$V_I = \{J \text{ ideals in } R \text{ s.t. } I \subset J\}.$$

Equivalently, a basis of open subsets is given by

$$D_f = \{J \text{ ideals in } R \text{ s.t. } f \notin J\}$$

for every $f \in R$. Let S_f be the multiplicative domain given by the powers of f. Then define a sheaf of ring over X by

$$\mathcal{O}_X(D_f) = S_f^{-1}R.$$

It is called the structural sheaf of $Spec(R)$. Any locally ringed space isomorphic to

$$(Spec(R), \mathcal{O}_{Spec(R)})$$

with R commutative is called an affine variety.

Note : the functor *Spec* gives an antiequivalence of categories between the categories of commutative rings and the category of affine varieties.

Définition 3. A scheme is a locally ringed space locally isomorphic to an affine variety.

13 Langlands

A modular form of weight k is a section of

$$\Lambda^{k+2}T^*M.$$

The projective space of the $\mathbb{N}$-graded algebra

$$A = \bigoplus \Lambda^{k+2}T^*M$$

is the compactification of the modular curve

$$\mathbb{P}(A) \cong \tilde{\mathcal{C}}.$$

If $F = \mathbb{Q}$ and $G = GL_2$, the finite part of the adele

$$\mathbb{A}_f = \prod_{\text{finite places}} F_\nu = \prod_{p \in \mathcal{P}} \mathbb{Q}_p$$

is ? ? and $G(\hat{\mathbb{Z}})$ is the maximal compact of $G(\mathbb{A}_f)$ with $G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$ being two copies of the upper half plane $\mathbb{H}$, and $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ is the modular curve.

Is the right part $G(\mathbb{A}_f)/G(K)$ is isomorphic to the inductive limit $G(\mathbb{Z}/p^k\mathbb{Z})$?

Yes if $G = GL_1$:

$$\varinjlim \mathbb{Z}/p^k\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p.$$

14 Haagerup property, cocycles and the mapping class group

If Σ is a closed oriented connected surface (with marked points), we denote by $Mod(\Sigma)$ its so-called mapping class group.

In [1] are used bounded representations of the mapping class group parametrized by a complex number $z \in \mathbb{D}$:

$$\pi_z : \Gamma \to \mathcal{L}(H).$$

Here, H is the Hilbert space obtained as the free Hilbert space generated by multicurves having a finite number of intersections with a fixed triangulation τ of Σ.

15 Hawaii

15.1 HLS groupoids

Let $(\Gamma, \mathcal{N})$ be an *approximated group* and $G_{\mathcal{N}}$ its associated HLS groupoid. Then :

- $G_{\mathcal{N}}$ is amenable iff Γ is amenable,

- if $G_{\mathcal{N}}$ is a-T-menable, then Γ is a-T-menable. The converse doesn't hold : in [**?**], the authors construct an approximated pair $(\mathbb{F}_2, \mathbb{N})$ such that the assembly map $\mu_{G_{\mathcal{N}},r}$ is not surjective, even if $\mathbb{F}_2$ is a-t-menable.

- $G_{\mathcal{N}}$ has T iff Γ has T,

- the algebraic exact sequence

$$0 \longrightarrow \bigoplus_n \mathbb{C}[\Gamma_n] \longrightarrow C_c(G_{\mathcal{N}}) \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$

  extends to

$$0 \longrightarrow \bigoplus_n C_r^*(\Gamma_n) \longrightarrow C_r^*(G_{\mathcal{N}}) \longrightarrow C_{r,\infty}^*(\Gamma) \longrightarrow 0 \quad ,$$

  where the right side algebra is the completion of $\mathbb{C}[\Gamma]$ w.r.t. the norm

$$||x||_{r,\infty} = \sup\{||y||_r : q(y) = x\} \quad \forall x \in \mathbb{C}[\Gamma].$$

  This is not an exotic crossed product functor, but one can still define an assembly map $\mu_{\Gamma,r,\infty}$ as the composition of $\mu_{\Gamma,max}$ with the induced at the level of K-theory of the quotient map $C_{max}^*(\Gamma) \to C_{r,\infty}^*(\Gamma)$. This exact sequence and the one induced by the decomposition of $G^0 = \overline{\mathbb{N}}$ is $\mathbb{N}$ and ∞ intertwines the assembly maps so that the next point follows :

- $G_{\mathcal{N}}$ satisfies BC iff Γ satifies BC for $\mu_{\Gamma,r,\infty}$.

- If Γ has T, then if μ_Γ is injective (which is the case for all closed subgroups of connected Lie groups), then $\mu_{\mathcal{G}_\mathcal{N}}$ fails to be surjective.

- **Congruence subgroup property.** If Γ has c.s.p. , then the assembly map fails to be surjective for any HLS groupoid $G_\mathcal{N}(\Gamma)$. If one can find such a groupoid which is a-T-menable for $SO(n,1)$, then this would imply Serre's c.s.p. conjecture : Any lattice in $SO(n,1)$ does not have c.s.p.

A useful fact from [**?**] :

$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$

is exact implies that the cone C_γ of the natural inclusion $\gamma : J \to C_\beta$ has vanishin K-groups :

$$K_*(C_\gamma) = 0.$$

16 Mayer-Vietoris

17 Quantum groups

18 Property T

19 Number theory

20 Fock spaces, CuntzKrieger algebras, and second quantization

21 Grothendieck and tensor products, the origin of nuclearity

This section is based on a talk given by Gilles Pisier, and his (exceptionally good) survey article.

Grothendieck started his work in functional analysis. While this is well known, I wanted to write a little post about how his work is important in my field.

Grothendieck did his Licence (his "undergrad") in the south of France, in the city of Montpellier.
If $x = \sum_j \alpha_j \otimes \beta_j$,

$$||x||_\wedge = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_\wedge = \sup\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_H = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

14 fundamental norms.

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