

Questions from Münster

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Table des matières

1	Simple examples for Baum-Connes for groupoids	3
1.1	Non commutative tori	3
1.2	Principal bundle over $U(2)$	4
1.3	Foliations	4
1.4	An example from physics	4
2	Parabolic induction and Hilbert modules	5
2.1	In $SL(2, \mathbb{R})$	5
3	Universal Coefficient Theorem	7
3.1	Other questions	7
4	Funky questions, ideas of talks	9
4.1	Expanders	9
4.1.1	Plan of the talk	9
4.1.2	Questions	9
4.2	Ideas of funky talks	9
5	A list of books	10
6	Seminar	10
6.1	Cartan subalgebras	10
6.2	Classification and the UCT	10
7	Groups	11
8	C^*-algebras	11
9	Noncommutative geometry	11
9.1	Why $SU_q(2)$?	11
9.2	TQFT	13
9.3	Reminder	13

1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in K -theory, like determining generators of K -group of some known C^* -algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for \mathbb{R}^n . (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space $\mathcal{G} = X \rtimes G$. The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [?], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{ if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}.$$

Indeed, suppose Y is a \mathcal{G} -proper cocompact space, then $Y \rtimes \mathcal{G}$ is a proper groupoid, so there exists a cutt-off function $c : Y \rightarrow [0, 1]$ such that :

$$\sum_{g \in \mathcal{G}^p(y)} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^p(y)} c(yg) \delta_g$$

which is a \mathcal{G} -equivariant continuous map. Moreover Z_d is proper and cocompact, and there exists a d s.t. the map takes its values in it.

Now if $\mathcal{G} = X \rtimes G$, $Z_d \simeq X \times Z'_d$ where $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{ if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$, so that $KK^{\mathcal{G}}(\Delta, A) \simeq KK^G(\Delta', A)$, where Δ and Δ' are respectively the 0-dimensional part of the equivariant complexes Z_d and Z'_d . This is true because the action of G on Z'_d is proper and cocompact, see lemma 3.6 of [?]. Now a standard Mayer-Vietoris argument (theorem 3.8 [?]) concludes to show that $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$.

As $C_r^* \mathcal{G} = C_0(X) \rtimes_r G$, we see that the Baum-Connes assembly map for \mathcal{G} with coefficients in A is equivalent to

$$K_*^{top}(G, A) \rightarrow K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle $U(2) \rightarrow U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$. You can foliate the fibers by an irrational rotation θ , so that you have an action of \mathbb{R} on $C(U(2))$. Reducing to a complete transversal (take $SU(2)$), the algebra $C(U(2)) \rtimes \mathbb{R}$ turns out to be Morita equivalent to $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$ (a general result of foliation groupoids I think). \underline{A} can be reduced to $C(\overline{D}) \otimes A_\theta$ and to $Ind_{\mathbb{T}^2}^{U(2)} A_\theta$.

Question : Compute the generators of the K -theory of \underline{A} .

1.3 Foliations

1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus $M = \mathbb{T}^2$. Its fundamental group $\Gamma = \mathbb{Z}^2$ acts on its universal cover $\tilde{M} = \mathbb{R}^2$ by isometries, and the electromagnetic field A gives a two-form w (its curvature) on \tilde{M} , so a 2-cocycle on the fundamental groupoid of \tilde{M} :

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_{\Delta} \tilde{w}}$$

where Δ a geodesic triangle between the 3 points. It turns out that $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$, so that \tilde{w} determines a number $\theta \in [0, 1)$, and the twisted reduced algebra of the fundamental groupoid w.r.t. \tilde{w} is equal to $A_\theta = C(\mathbb{T}^2) \rtimes_{r, \theta} \mathbb{Z}^2$. This situation generalizes to general manifold whose fundamental cover are equipped with a line bundle and a connection. We can then associate a 2-cocycle on the fundamental groupoid of \tilde{M} to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let G be a real reductive group. For all parabolic subgroup P , there is only one nilpotent normal subgroup N , and the Levi is defined as $P = LN$. The idea of Pierre Julg is to fix first a Levi subgroup L of G . Now there is only a finite number of choices for N , so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group $W_L = N_G(L)/L$ acts on it by $w.N = wNw^{-1}$. Pierre Clare defined a C_r^*L -module $C_r^*(G/N)$, equipped with an action of C_r^*G by compact operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$, where $L = MA$, \hat{M}_d is the discrete dual of M , and $\hat{A} = \mathfrak{a}^*$. Then $\sigma \otimes \tau$ is a representation of $MA = L$, which we can trivially extend to N to induce it on G . Pierre Clare showed the following fact :

$$\text{Ind}_P^G H_{\sigma \otimes \tau \otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma \otimes \tau}.$$

For every $\tilde{w} \in N_G(L)$, the operator $\rho(\tilde{w}) : C_r^*(G/N) \rightarrow C_r^*(G/w.N)$ is well defined and gives a morphism

$$\text{Ad } \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \rightarrow \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because C_r^*G is acting on $C^*(G/N)$ by compact operators. This gives a morphism

$$C_r^*G \rightarrow \bigoplus_{[L]} \left(\bigoplus_{N \in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard C^* -algebraic tools).

The first step would be to prove that

$$\begin{array}{ccc} C_r^*G & \rightarrow & \left(\bigoplus_{N \in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for $F_N^{-1}(g) = \text{Tr}_{C_r^*L}(T\pi_N(g^{-1}))$.

2.1 In $SL(2, \mathbb{R})$

In this case, G acts on the Poincaré disc by homographies, and P can be taken as the stabilizer of a point at infinity, and L stabilizes a geodesic, that is to say

two points at infinity, so that

$$P_{1,1} \simeq \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad L \simeq \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad N \simeq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing P amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and L . Now fix two points at infinity, which gives you L . You now only have two choices for P , and the two are exchanges under the action of W_L on the nilpotent groups.

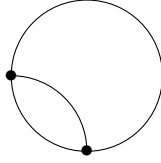


FIGURE 1 – Choices for the Levi subgroup

3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let \mathcal{N} be the smallest class of C^* -algebras containing \mathbb{C} , closed under countable inductive limits, stable by KK -equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in \mathcal{N} , so is the third). Here is the classical theorem :

Théorème 1 (Universal Coefficient Theorem). Let A and B be two separable C^* -algebras, where A is in \mathcal{N} . Then there is a short exact sequence

$$0 \rightarrow Ext_{\mathbb{Z}}^1(K_*(A), K_*(B)) \rightarrow KK_*(A, B) \rightarrow Hom(K_*(A), K_*(B)) \rightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ??
- The second map is given by the boundary element associated to any impair K -cycle. Namely, if $z \in KK^1(A, B)$, let (H_B, π, T) be a K -cycle representing z , and P the associated projector $P = \frac{1+T}{2}$. Define the pull-back

$$E^{(\pi, T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$

Then the boundary of the following extension

$$0 \rightarrow \mathfrak{K}_B \rightarrow E^{(\pi, T)} \rightarrow A \rightarrow 0$$

is given by $\partial = - \otimes z : K_*(A) \rightarrow K_*(B)$ which depends only on z . The map is just $z \mapsto \partial$

- If $\partial = 0$, then the sequence associated to z splits and we have exact sequences

$$0 \rightarrow K_*(B) \rightarrow K_*(E^{(\pi, T)}) \rightarrow K_*(A) \rightarrow 0$$

which gives an element of $Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))$.

3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Mischenko and Kasparov.

Proposition 1. Let G be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold M . Then

$$K^{top}(G) \xrightarrow{\mu} K(C_r^*G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for G .

The original point being that G. Yu can prove this (how?) without using the heavy machinery of the Dirac Dual-Dirac method, nor anything related to KK^G -theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

Proposition 2. Let G be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture holds for G .

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^*P_d(X_{2k})) \xrightarrow{\beta} K(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k})))$$

is an isomorphism.

Here are some details : first decompose $G = G_0 \cup G_1$ into two subspaces, which are not necessarily subgroups, such that each is a R -disjoint union of bounded subsets (in fact finite since G is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define $\prod^R C^*(P_d(X_{2k})) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), \text{prop}(T_{2k}) \leq R)\}$, so that $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$, and each X_{2k} coarsely embeds into some \mathbb{R}^{m_k} . The isomorphism of $\beta \circ \mu$ implies the injectivity of μ , and by cutting and pasting, μ can be shown to be injective for G so that Novikov is satisfied.

4 Funky questions, ideas of talks

4.1 Expanders

Here are some interesting questions I had after a talk on expanders.

4.1.1 Plan of the talk

I first gave a motivation for considering expanders. Namely, we are interested in the following network theory problem : can we construct a network as big as we want, such that the cost is controlled and which is not subject to easy failure ?

Building a network as big as we want means we want to consider a family of graphs $X_j = (V_j, E_j)$ such that $|V_j| \rightarrow +\infty$, and controlling the cost means that $\deg(X_j) < k$ for all j . But what does "not easily subject to failure" means ? For this, I want to explain why we should ask our family to stay well connected and why the second value of the discrete Laplacian is a good way to measure that.

The idea is to relate the Laplacian to the uniform random walk on the graph, and to show that $\lambda_1(X)$ controls the speed of convergence of the uniform random walk to the stationary measure which is the uniform probability on the graph, given by $\nu(x) = C \cdot \deg(x)$.

A family of graphs satisfying the previous conditions and such that $\lambda_1(X_j) > c > 0$ is called an expander. If time allows, one can then elaborate on metric properties of this type of graphs. The impossibility to embed them coarsely into any separable Hilbert space, and the relations to the Baum-Connes conjecture are close to my work.

4.1.2 Questions

- Paolo Pigato : What is the dynamic at the limit ?
- Anne Briquet : Is $\lambda_1(X)$ such a good way to measure the connectedness of a graph, if you consider the phenomenon of cutoff for finite Markov Chains.

4.2 Ideas of funky talks

- What is the relation between the Fourier transform and quantum groups ?
- What is the relation between the Runge Kutta methode and renormalization in QFT ?
- What is the relation between Brownian motion and second quantization ?

5 A list of books

A list of books I like about general knowledge in science :

- L'aventure des nombres, Godefroy
- L'autobiographie de Paul Levy, Laurent Schwartz, et Yuri Manin.
- Recoltes et semailles, Grothendieck.
- Lee Smolin, The trouble with physics, the rise of String theory, the fall of a Science, and what comes next,
- Julian Barbour, The End of Time, The next revolution in Physics,
- Carlo Rovelli, Et si le temps n'existait pas, un peu de science subversive,
- Mandelbrot, The (Mis)Behaviour of markets, Fractals and Chaos, the Mandelbrot set and beyond, The fractal geometry of nature.
- Manjit Kumar :
- Amir Alexander, Infinitesimal : How a Dangerous Mathematical Theory Shaped the Modern World
- Ian Stewart, Does God play dice ?

6 Seminar

6.1 Cartan subalgebras

Out of any inclusion of C^* -algebras $A \subseteq B$ with A unital commutative, we construct an action of the normalizer of A in B by partial homeomorphism on X the spectrum of A , i.e. a homomorphism of semigroup

$$\alpha : N_B(A) \rightarrow SHomeo(X).$$

If $n \in N_B(A)$ and $x \in Spec(A)$, set

$$\langle \alpha_n(x), a \rangle = \langle x, n^* a n \rangle.$$

This defines a homeomorphism

$$\alpha_n : U_n \rightarrow U_{n^*},$$

where $U_n = \{x \in Spec(A), n^* n(x) > 0\}$ such that $\alpha_{nm} = \alpha_n \circ \alpha_m$.

If A is maximal abelian in B , and other conditions, then B is shown to be isomorphic to the twisted reduced C^* -algebra of the groupoid of stalks of $N_B(A)$. This can be seen as an extension of the Gelfand transform

$$\begin{cases} B & \rightarrow C_r^*(G) \\ b & \mapsto \end{cases}$$

6.2 Classification and the UCT

For A a simple unital C^* -algebra, the Elliot invariant is :

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \rightarrow S(K_0(A))),$$

here $T(A)$ is the trace space and r_A the paring $r_A(\tau)([p]) = [\tau(p)]$.

Elliot's conjecture : Separable, simple, nuclear are classifiable by Elliot's invariants.

Théorème 2. Separable, simple, unital, nuclear, \mathcal{Z} -stable, UCT algebras are classifiable by Elliot's invariants.

An example of a classification theorem : Elliot's theorem,

Théorème 3. Let A and B unital AF-algebras and

$$\alpha : K_0(A) \rightarrow K_1(A)$$

a unital order isomorphism, i.e.

$$\alpha(K_0(A)_+) \subseteq K_0(B)_+ \quad \text{and} \quad \alpha([1_A]) = [1_B].$$

Then there exists a unital $*$ -isomorphism $\phi: A \rightarrow B$ such that $\phi_* = \alpha$.

7 Groups

- Amenable, a-T-menable, property T, with a diagram
- Mapping class groups
- Profinite groups, locally profinite groups, $\text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$
- Automorphism of a regular tree, the Grigorchuk group,
- Lamplighter groups $H^\Gamma \rtimes \Gamma$, usually

$$\oplus \mathbb{Z}_2 \rtimes \mathbb{Z}.$$

8 C^* -algebras

- CAR algebra $C^*\langle a_i, a_i a_j + a_j a_i = \delta_{ij} \rangle$ or $\bigotimes M_2(\mathbb{C})$ or

$$\varinjlim \left\{ \begin{array}{ccc} M_{2^n}(\mathbb{C}) & \rightarrow & M_{2^{n+1}}(\mathbb{C}) \\ a & \mapsto & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{array} \right.$$

9 Noncommutative geometry

9.1 Why $SU_q(2)$?

Apparently, some people are interested in deformation of classical Lie groups such as $SU_q(2)$, which is the Hopf algebra generated by 3 generators E, F, K satisfying the relations

$$R.$$

I wanted to understand where these relations are coming from, which led me to interesting ideas developed by several people, including Yuri Manin. The idea is to define $SU_q(2)$ as a special group like object of the automorphism group of some noncommutative space, the quantum plane.

Let k be a field. The free (noncommutative) k -algebra on n generators is denoted by $k\langle x_1, \dots, x_n \rangle$.

Définition 1. A quadratic algebra

$$A = \oplus_{i \geq 0} A_i$$

is a \mathbb{N} -graded finitely generated algebra such that :

- $A_0 = k$, and A_1 generates A ,
- the relations on generators are in $A_1 \otimes A_1$.

The quadratic algebra A is said to be a Frobenius algebra of dimension d if moreover

- $A_d = k$ and $A_i = 0$ for all $i > d$,
- the multiplication map

$$m : A_i \otimes A_{d-i} \rightarrow A_d$$

is a perfect duality.

The main example is the quantum plane

$$\mathbb{A}_q^2 = k\langle x, y \rangle / (xy - qyx)$$

where $q \in k^\times$. More generally, the quantum space of dimension $n|m$ is

$$\mathbb{A}_q^{n|m} = k\langle x_1, \dots, x_n, \eta_1, \dots, \eta_m \rangle / (x_i x_j - q x_j x_i, q \eta_i \eta_j + \eta_j \eta_i).$$

This example is suppose to come from physics. In quantum field theories, physicists deal with two kind of particles, bosons and fermions, and use commuting variables for one type, and anticommuting for the other. One object they appeal to are called supermanifolds, which are manifolds enriched with anticommuting variables. Formally, it means they look at ringed spaces (X, \mathcal{O}) locally isomorphic to $(\mathbb{R}^n, C^\infty[\eta_1, \dots, \eta_m])$, where $C^\infty[\eta_1, \dots, \eta_m]$ is the free sheaf of rings generated by anticommuting variables η_i over the smooth complex valued functions $C^\infty(\mathbb{R}^n)$.

Remark that a quadratic algebra A is a quotient of $k\langle x_1, \dots, x_n \rangle$ by elements $r_\alpha \in A_1 \otimes A_1$, which we will denote as

$$A = k\langle x_1, \dots, x_n \rangle / (r_\alpha)$$

or

$$A = \langle A_1, R_A \rangle$$

with $R_A \subseteq A_1 \otimes A_1$.

Manin defines the quantum dual of a quadratic algebra as

$$A^! = k\langle x^i \rangle / (r^\beta)$$

where $r_{ij}^\beta r_\alpha^{ij} = 0$, i.e. $R_{A^!} = R_A^\perp$. Then, the quantum endomorphisms between two quadratic algebra is

$$Hom(A, B) = k\langle z_i^j \rangle / (r_\alpha^\beta)$$

where $r_\alpha^\beta = r_\alpha^{ij} r_{kl}^\beta z_i^k z_j^l$. If $End(A) = Hom(A, A)$, then $End(A)$ satisfies the universal property to be initial in the category of k -algebras (B, β) endowed with

an algebra homomorphism $\beta : A \rightarrow A \otimes B$.

If one does that to the quantum plane \mathbb{A}_q^2 , one still doesn't find quite $M_q(2)$: half of the relations are missing. Also

$$(\mathbb{A}_q^{2|0})^! = \mathbb{A}_q^{0|2}?$$

Exercise.

9.2 TQFT

We recalled the definitions of a monoidal category, a braided category, and a symmetric monoidal category. The two main examples are the category of bordisms $Bord^d$ in dimension d , and the category of vector spaces over a field k . The first talk focused on topological quantum fields theories in dimension 1 and 2.

Définition 2. A TQFT in dimension d is a monoidal symmetric functor

$$Z : Bord_d \rightarrow Vect_k.$$

The two main results we showed are :

- there is an equivalence of categories

$$TQFT_1 \cong Vect_k$$

obtained as $Z \mapsto Z(pt)$.

- there is an equivalence of categories

$$TQFT_2 \cong Frob_k$$

obtained as $Z \mapsto Z(\mathbb{S}^1)$.

A nice example in dimension 2 : $Z(\mathbb{S}^1) = \mathbb{C}[t]/(t^2 - 1)$ is the Frobenius algebra given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad \epsilon(1) = 0 \quad \epsilon(t) = 1.$$

Then the handle element is $h = 2t$ and

$$Z(\Sigma_g) = \begin{cases} 2^g & \text{if } g \text{ is odd} \\ 0 & \text{if } g \text{ is even.} \end{cases}$$

9.3 Reminder

A locally ringed space is a topological space X together with a sheaf or ring \mathcal{O}_X over X such that all stalks are local rings, ie have a unique maximal ideal.

For R a ring, $X = Spec(R)$ denotes the topological space obtained as the set of prime ideals of R endowed with the Zariski topology, i.e. the topology generated by the closed subsets

$$V_I = \{J \text{ ideals in } R \text{ s.t. } I \subset J\}.$$

Equivalently, a basis of open subsets is given by

$$D_f = \{J \text{ ideals in } R \text{ s.t. } f \notin J\}$$

for every $f \in R$. Let S_f be the multiplicative domain given by the powers of f . Then define a sheaf of ring over X by

$$\mathcal{O}_X(D_f) = S_f^{-1}R.$$

It is called the structural sheaf of $\text{Spec}(R)$. Any locally ringed space isomorphic to

$$(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$$

with R commutative is called an affine variety.

Note : the functor Spec gives an antiequivalence of categories between the categories of commutative rings and the category of affine varieties.

Définition 3. A scheme is a locally ringed space locally isomorphic to an affine variety.