

# Questions from Münster

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# 1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in  $K$ -theory, like determining generators of  $K$ -group of some known  $C^*$ -algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for  $\mathbb{R}^n$ . (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space  $\mathcal{G} = X \rtimes G$ . The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [4], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{ if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}.$$

Indeed, suppose  $Y$  is a  $\mathcal{G}$ -proper cocompact space, then  $Y \rtimes \mathcal{G}$  is a proper groupoid, so there exists a cutt-off function  $c : Y \rightarrow [0, 1]$  such that :

$$\sum_{g \in \mathcal{G}^p(y)} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^p(y)} c(yg) \delta_g$$

which is a  $\mathcal{G}$ -equivariant continuous map. Moreover  $Z_d$  is proper and cocompact, and there exists a  $d$  s.t. the map takes its values in it.

Now if  $\mathcal{G} = X \rtimes G$ ,  $Z_d \simeq X \times Z'_d$  where  $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{ if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$ , so that  $KK^{\mathcal{G}}(\Delta, A) \simeq KK^G(\Delta', A)$ , where  $\Delta$  and  $\Delta'$  are respectively the 0-dimensional part of the equivariant complexes  $Z_d$  and  $Z'_d$ . This is true because the action of  $G$  on  $Z'_d$  is proper and cocompact, see lemma 3.6 of [4]. Now a standard Mayer-Vietoris argument (theorem 3.8 [4]) concludes to show that  $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$ .

As  $C_r^* \mathcal{G} = C_0(X) \rtimes_r G$ , we see that the Baum-Connes assembly map for  $\mathcal{G}$  with coefficients in  $A$  is equivalent to

$$K_*^{top}(G, A) \rightarrow K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

## 1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

## 1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle  $U(2) \rightarrow U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$ . You can foliate the fibers by an irrational rotation  $\theta$ , so that you have an action of  $\mathbb{R}$  on  $C(U(2))$ . Reducing to a complete transversal (take  $SU(2)$ ), the algebra  $C(U(2)) \rtimes \mathbb{R}$  turns out to be Morita equivalent to  $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$  (a general result of foliation groupoids I think).  $\underline{A}$  can be reduced to  $C(\overline{D}) \otimes A_\theta$  and to  $Ind_{\mathbb{T}^2}^{U(2)} A_\theta$ .

Question : Compute the generators of the  $K$ -theory of  $\underline{A}$ .

## 1.3 Foliations

### 1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus  $M = \mathbb{T}^2$ . Its fundamental group  $\Gamma = \mathbb{Z}^2$  acts on its universal cover  $\tilde{M} = \mathbb{R}^2$  by isometries, and the electromagnetic field  $A$  gives a two-form  $w$  (its curvature) on  $\tilde{M}$ , so a 2-cocycle on the fundamental groupoid of  $\tilde{M}$  :

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_{\Delta} \tilde{w}}$$

where  $\Delta$  a geodesic triangle between the 3 points. It turns out that  $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$ , so that  $\tilde{w}$  determines a number  $\theta \in [0, 1)$ , and the twisted reduced algebra of the fundamental groupoid w.r.t.  $\tilde{w}$  is equal to  $A_\theta = C(\mathbb{T}^2) \rtimes_{r, \theta} \mathbb{Z}^2$ . This situation generalizes to general manifold whose fundamental cover are equipped with a line bundle and a connection. We can then associate a 2-cocycle on the fundamental groupoid of  $\tilde{M}$  to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

## 2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let  $G$  be a real reductive group. For all parabolic subgroup  $P$ , there is only one nilpotent normal subgroup  $N$ , and the Levi is defined as  $P = LN$ . The idea of Pierre Julg is to fix first a Levi subgroup  $L$  of  $G$ . Now there is only a finite number of choices for  $N$ , so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group  $W_L = N_G(L)/L$  acts on it by  $w.N = wNw^{-1}$ . Pierre Clare defined a  $C_r^*L$ -module  $C_r^*(G/N)$ , equipped with an action of  $C_r^*G$  by compact operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let  $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$ , where  $L = MA$ ,  $\hat{M}_d$  is the discrete dual of  $M$ , and  $\hat{A} = \mathfrak{a}^*$ . Then  $\sigma \otimes \tau$  is a representation of  $MA = L$ , which we can trivially extend to  $N$  to induce it on  $G$ . Pierre Clare showed the following fact :

$$\text{Ind}_P^G H_{\sigma \otimes \tau \otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma \otimes \tau}.$$

For every  $\tilde{w} \in N_G(L)$ , the operator  $\rho(\tilde{w}) : C_r^*(G/N) \rightarrow C_r^*(G/w.N)$  is well defined and gives a morphism

$$\text{Ad } \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \rightarrow \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because  $C_r^*G$  is acting on  $C^*(G/N)$  by compact operators. This gives a morphism

$$C_r^*G \rightarrow \bigoplus_{[L]} \left( \bigoplus_{N \in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard  $C^*$ -algebraic tools).

The first step would be to prove that

$$\begin{array}{ccc} C_r^*G & \rightarrow & \left( \bigoplus_{N \in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for  $F_N^{-1}(g) = \text{Tr}_{C_r^*L}(T\pi_N(g^{-1}))$ .

### 2.1 In $SL(2, \mathbb{R})$

In this case,  $G$  acts on the Poincaré disc by homographies, and  $P$  can be taken as the stabilizer of a point at infinity, and  $L$  stabilizes a geodesic, that is to say

two points at infinity, so that

$$P_{1,1} \simeq \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad L \simeq \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad N \simeq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing  $P$  amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and  $L$ . Now fix two points at infinity, which gives you  $L$ . You now only have two choices for  $P$ , and the two are exchanges under the action of  $W_L$  on the nilpotent groups.



FIGURE 1 – Choices for the Levi subgroup

### 3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let  $\mathcal{N}$  be the smallest class of  $C^*$ -algebras containing  $\mathbb{C}$ , closed under countable inductive limits, stable by  $KK$ -equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in  $\mathcal{N}$ , so is the third). Here is the classical theorem :

**Théorème 1** (Universal Coefficient Theorem). Let  $A$  and  $B$  be two separable  $C^*$ -algebras, where  $A$  is in  $\mathcal{N}$ . Then there is a short exact sequence

$$0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(K_*(A), K_*(B)) \longrightarrow KK_*(A, B) \longrightarrow \text{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ??
- The second map is given by the boundary element associated to any impair  $K$ -cycle. Namely, if  $z \in KK^1(A, B)$ , let  $(H_B, \pi, T)$  be a  $K$ -cycle representing  $z$ , and  $P$  the associated projector  $P = \frac{1+T}{2}$ . Define the pull-back

$$E^{(\pi, T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$

Then the boundary of the following extension

$$0 \longrightarrow \mathfrak{K}_B \longrightarrow E^{(\pi, T)} \longrightarrow A \longrightarrow 0$$

is given by  $\partial = - \otimes z : K_*(A) \longrightarrow K_*(B)$  which depends only on  $z$ . The map is just  $z \mapsto \partial$

- If  $\partial = 0$ , then the sequence associated to  $z$  splits and we have exact sequences

$$0 \longrightarrow K_*(B) \longrightarrow K_*(E^{(\pi, T)}) \longrightarrow K_*(A) \longrightarrow 0$$

which gives an element of  $\text{Ext}_{\mathbb{Z}}^1(K_*(A), K_*(B))$ .

#### 3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Mischenko and Kasparov.

**Proposition 1.** Let  $G$  be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold  $M$ . Then

$$K^{top}(G) \xrightarrow{\mu} K(C_r^*G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for  $G$ .



The original point being that G. Yu can prove this (how?) without using the heavy machinery of the Dirac Dual-Dirac method, nor anything related to  $KK^G$ -theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

**Proposition 2.** Let  $G$  be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture holds for  $G$ .

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^*P_d(X_{2k})) \xrightarrow{\beta} K(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k})))$$

is an isomorphism.

Here are some details : first decompose  $G = G_0 \cup G_1$  into two subspaces, which are not necessarily subgroups, such that each is a  $R$ -disjoint union of bounded subsets (in fact finite since  $G$  is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define  $\prod^R C^*(P_d(X_{2k})) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), \text{prop}(T_{2k}) \leq R)\}$ , so that  $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$ , and each  $X_{2k}$  coarsely embeds into some  $\mathbb{R}^{m_k}$ . The isomorphism of  $\beta \circ \mu$  implies the injectivity of  $\mu$ , and by cutting and pasting,  $\mu$  can be shown to be injective for  $G$  so that Novikov is satisfied.

## 4 Funky questions, ideas of talks

### 4.1 Expanders

Here are some interesting questions I had after a talk on expanders.

#### 4.1.1 Plan of the talk

I first gave a motivation for considering expanders. Namely, we are interested in the following network theory problem : can we construct a network as big as we want, such that the cost is controlled and which is not subject to easy failure ?

Building a network as big as we want means we want to consider a family of graphs  $X_j = (V_j, E_j)$  such that  $|V_j| \rightarrow +\infty$ , and controlling the cost means that  $\deg(X_j) < k$  for all  $j$ . But what does "not easily subject to failure" means ? For this, I want to explain why we should ask our family to stay well connected and why the second value of the discrete Laplacian is a good way to measure that.

The idea is to relate the Laplacian to the uniform random walk on the graph, and to show that  $\lambda_1(X)$  controls the speed of convergence of the uniform random walk to the stationary measure which is the uniform probability on the graph, given by  $\nu(x) = C \cdot \deg(x)$ .

A family of graphs satisfying the previous conditions and such that  $\lambda_1(X_j) > c > 0$  is called an expander. If time allows, one can then elaborate on metric properties of this type of graphs. The impossibility to embed them coarsely into any separable Hilbert space, and the relations to the Baum-Connes conjecture are close to my work.

#### 4.1.2 Questions

- Paolo Pigato : What is the dynamic at the limit ?
- Anne Briquet : Is  $\lambda_1(X)$  such a good way to measure the connectedness of a graph, if you consider the phenomenon of cutoff for finite Markov Chains.

### 4.2 Ideas of funky talks

- What is the relation between the Fourier transform and quantum groups ?
- What is the relation between the Runge Kutta methode and renormalization in QFT ?
- What is the relation between Brownian motion and second quantization ?

## 5 A list of books

A list of books I like about general knowledge in science :

- L'aventure des nombres, Godefroy
- L'autobiographie de Paul Levy, Laurent Schwartz, et Yuri Manin.
- Recoltes et semailles, Grothendieck.
- Lee Smolin, The trouble with physics, the rise of String theory, the fall of a Science, and what comes next,
- Julian Barbour, The End of Time, The next revolution in Physics,
- Carlo Rovelli, Et si le temps n'existait pas, un peu de science subversive,
- Mandlebrot, The (Mis)Behaviour of markets, Fractals and Chaos, the Mandelbrot set and beyond, The fractal geometry of nature.
- Manjit Kumar :
- Amir Alexander, Infinitesimal : How a Dangerous Mathematical Theory Shaped the Modern World
- Ian Stewart, Does God play dice ?
- History of Statistics, Stielger
- Logicomix

Overview and more specialized books :

- Moonshine beyond the Monster, Terry Gannon
- Le theoreme d'uniformisation, Saint-Gervais
- Invitation aux mathematiques de Fermat, Hellgouarch
- Rached Mneime, tous ses livres !
- Hubbard West pour les equa diff
- Noether's theorem, Yvette K
- Nother's wonderful theorem
- The annus mirabellus of Einstein
- The Road to Reality, Sir Roger Penrose

## 6 Seminar

### 6.1 Cartan subalgebras

Out of any inclusion of  $C^*$ -algebras  $A \subseteq B$  with  $A$  unital commutative, we construct an action of the normalizer of  $A$  in  $B$  by partial homeomorphism on  $X$  the spectrum of  $A$ , i.e. a homomorphism of semigroup

$$\alpha : N_B(A) \rightarrow S\text{Homeo}(X).$$

If  $n \in N_B(A)$  and  $x \in \text{Spec}(A)$ , set

$$\langle \alpha_n(x), a \rangle = \langle x, n^* a n \rangle.$$

This defines a homeomorphism

$$\alpha_n : U_n \rightarrow U_{n^*},$$

where  $U_n = \{x \in \text{Spec}(A), n^* n(x) > 0\}$  such that  $\alpha_{nm} = \alpha_n \circ \alpha_m$ .

If  $A$  is maximal abelian in  $B$ , and other conditions, then  $B$  is shown to be isomorphic to the twisted reduced  $C^*$ -algebra of the groupoid of stalks of  $N_B(A)$ . This can be seen as an extension of the Gelfand transform

$$\begin{cases} B & \rightarrow C_r^*(G) \\ b & \mapsto \end{cases}$$

## 6.2 Classification and the UCT

For  $A$  a simple unital  $C^*$ -algebra, the Elliot invariant is :

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \rightarrow S(K_0(A))),$$

here  $T(A)$  is the trace space and  $r_A$  the paring  $r_A(\tau)([p]) = [\tau(p)]$ .

**Elliot's conjecture :** Separable, simple, nuclear are classifiable by Elliot's invariants.

**Théorème 2.** Separable, simple, unital, nuclear,  $\mathcal{Z}$ -stable, UCT algebras are classifiable by Elliot's invariants.

An example of a classification theorem : Elliot's theorem,

**Théorème 3.** Let  $A$  and  $B$  unital AF-algebras and

$$\alpha : K_0(A) \rightarrow K_1(A)$$

a unital order isomorphism, i.e.

$$\alpha(K_0(A)_+) \subseteq K_0(B)_+ \quad \text{and} \quad \alpha([1_A]) = [1_B].$$

Then there exists a unital  $*$ -isomorphism  $\phi : A \rightarrow B$  such that  $\phi_* = \alpha$ .

## 7 Groups

- Amenable, a-T-menable, property T, with a diagram
- Mapping class groups
- Profinite groups, locally profinite groups,  $Aut(\overline{\mathbb{Q}}/\mathbb{Q})$
- Automorphism of a regular tree, the Grigorchuk group,
- Lamplighter groups  $H^\Gamma \rtimes \Gamma$ , usually

$$\oplus \mathbb{Z}_2 \rtimes \mathbb{Z}.$$

And groupoids :

- the coarse groupoid  $G(X)$  : étale (even ample) with totally disconnected basis  $\beta X$ . Dynamical asymptotic dimension of asymptotic dimension of  $X$ . A-T-menable iff  $X$  has property A
- HLS groupoid associated to a sequence of finite metric spaces  $X_n$  equipped with maps  $X_n \rightarrow \Gamma$  to a finitely generated group  $\Gamma = \langle S \rangle$ .
- groupoids of germs of semigroup of partial homeomorphisms acting on a topological space
- holonomy groupoids of a foliation
- action groupoids  $X \rtimes \Gamma$ , principal bundles groupoids  $P \times_G P$ , where  $P \rightarrow X$  is a  $G$ -bundle
- equivalence relation groupoids

## 8 $C^*$ -algebras

- CAR algebra  $C^*\langle a_i, a_i a_j + a_j a_i = \delta_{ij} \rangle$  or  $\bigotimes M_2(\mathbb{C})$  or

$$\varinjlim \begin{cases} M_{2^n}(\mathbb{C}) & \rightarrow M_{2^{n+1}}(\mathbb{C}) \\ a & \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{cases}$$

Class of  $C^*$ -algebras.

- The bootstrap class  $\mathcal{B}$
- The class  $\mathcal{N}$  of  $C^*$ -algebras  $A$  such that the map

$$\alpha_{A,B} : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B)$$

is an isomorphism for every  $C^*$ -algebra  $B$  such that  $K_*(B)$  is a free abelian group. In [?], it is shown that  $\mathcal{N}$  contains all of the bootstrap class.

- Non exact  $C^*$ -algebra : to my knowledge only one example is known : the reduced  $C^*$ -algebra  $C_r^*(\Gamma)$  of a finitely generated group whose Cayley graph contains expander. Using Ozawa's result [?], one can construct finite dimensional  $C^*$ -algebras  $M_{X_n}$  such that

$$0 \rightarrow C_r^*(\Gamma) \otimes \bigoplus M_{X_n} \rightarrow C_r^*(\Gamma) \otimes \prod M_{X_n} \rightarrow C_r^*(\Gamma) \otimes \left( \prod M_{X_n} / \bigoplus M_{X_n} \right) \rightarrow 0$$

is not exact in the middle.

The problem of the existence of such a group is an interesting question, which was stated by Gromov and proved rigorously by several people in the wake of this.

One can define analog of approximation properties in the setting of  $K$ -theory.

- $A$  is  $K$ -nuclear if the class of the natural map

$$p_{A,B} : A \otimes_{\max} B \rightarrow A \otimes_{\min} B$$

is invertible as an element of  $KK(A \otimes_{\max} B, A \otimes_{\min} B)$ .

- $G$  is  $K$ -amenable if the class of the regular representation

$$\lambda_G : C_{\max}^*(G) \rightarrow C_r^*(G)$$

is invertible as an element of  $KK(C_{\max}^*(G), C_r^*(G))$ .

For instance, Skandalis proves in [?] that, if  $\Lambda$  is an infinite hyperbolic property T group, then  $C_r^*(\Lambda)$  is not  $K$ -nuclear. In particular, it is not  $KK$ -equivalent to a nuclear  $C^*$ -algebra, and cannot be Bootstrap. This completely renders proving the Baum-Connes conjecture by mean of Dirac-Dual-Dirac method hopeless. An example of such a group is given by any lattice in  $Sp(n, 1)$  for instance. (higher rank algebraic semisimple groups?)

After developing a restriction principle for groupoids, a natural question was to find a  $C^*$ -algebra coming from a groupoid crossed-product that we were able to prove that it satisfied the Künneth formula, while still not being a consequence of previous results. One could have started with the so called HLS groupoid  $G_{\mathcal{N}}(\Gamma)$  associated to a residually finite finitely generated group  $\Gamma$  and a nested sequence of decreasing normal subgroups of finite index  $\mathcal{N}$ .

One always has the following exact sequence of  $*$ -algebras

$$0 \rightarrow \oplus \mathbb{C}[\Gamma_n] \rightarrow C_c(G) \rightarrow \mathbb{C}[\Gamma] \rightarrow 0$$

which induces the following exact sequence of  $C^*$ -algebras

$$0 \rightarrow \oplus \mathbb{C}[\Gamma_n] \rightarrow C_r^*(G) \rightarrow C_{\mathcal{N}}^*(\Gamma) \rightarrow 0$$

where  $C_{\mathcal{N}}^*(\Gamma)$  is the completion of  $\mathbb{C}[\Gamma]$  w.r.t. to the norm

$$\|x\|_{\mathcal{N}} = \sup_{N \in \mathcal{N}} \|\lambda_N(x)\| \quad x \in \mathbb{C}[\Gamma]$$

induced by the quasi-regular representations  $\lambda_N : C_{max}^*(\Gamma) \rightarrow \mathcal{L}(l^2(\Gamma/N))$ .

Now this exact sequence intertwines the Baum-Connes assembly maps, and the Baum-Connes conjecture for  $G_{\mathcal{N}}(\Gamma)$  is equivalent to  $\mu_{\Gamma, \mathcal{N}}$  being an isomorphism.

- If  $\Gamma = \mathbb{F}_2$  and

$$N_n = \cap \ker \phi$$

for  $\phi$  running across all group homomorphisms from  $\Gamma$  to a finite group of cardinality less than  $n$ , then  $C_{\mathcal{N}}^*(\Gamma) \cong C_{max}^*(\Gamma)$  and  $G$  satisfies the Baum-Connes conjecture, is ample and satisfies the restriction condition. So we get that  $C_r^*(G)$  satisfies the Künneth formula. It is still a result that one can get using the fact that  $\Gamma$  being a-T-menable, it is  $K$ -amenable. Hence  $C_{max}^*(\Gamma)$  and  $C_r^*(\Gamma)$  are  $KK$ -equivalent and bootstrap, so that  $C_r^*(G)$  also is by extension stability of bootstrapness. A remark of R. Willett is worth mentioning :  $\mathbb{F}_2$  being the fundamental group of the wedge of two circles, it is  $KK$ -equivalent to  $C(\mathbb{S}^1 \wedge \mathbb{S}^1)$ .

- One can artificially try to get rid of bootstrapness by spatially tensoring this exact sequence by  $C_r^*(\Lambda)$  for a infinite hyperbolic property T group. One then get the extension

$$0 \rightarrow \oplus \mathbb{C}[\Gamma_n] \otimes_{min} C_r^*(\Lambda) \rightarrow C_r^*(G \times \Lambda) \rightarrow C_{\mathcal{N}}^*(\Gamma) \otimes_{min} C_r^*(\Lambda) \rightarrow 0.$$

The restriction principle applies for the groupoid  $G_{\mathcal{N}}(\Gamma) \times \Lambda$ , and induces that its reduced  $C^*$ -algebra satisfies the Künneth formula. But then again, one can deduce this from a previous result, namely the restriction principle for groups. Indeed, apply it to  $\Lambda$  with coefficient on the trivial bootstrap  $\Lambda$ -algebra  $C_r^*(G)$ .

- Bekka shows that ??

## 9 Baum-Connes

Compact groups, or better : proper groupoids : Green-Julg.

Connes-Kasparov : proof by representation theory (Wasserman, etc)

Kasparov's Conjecture : towards Higson-Kasparov paper and the proof for Haagerup (J-L. Tu's general version in  $KK$ -theory, plus the beautiful result that amenability implies bootstrap)

Ideas from Coarse geometry, and Yu and Roe's work, Skandalis-Tu-Yu etc.

## 10 GPOTS & NCGOA 2018

**10.1 Arnaud Brothier : some representations of the Thompson group**

**10.2 Piotr Nowak : Property T for  $Out(\mathbb{F}_n)$**

**10.3 Wilhem Winter : Relative nuclear dimension**

**10.4 Rufus Willett : Exactness and exotic crossed-product**

## 11 Coarse geometry & dynamics

## 12 Noncommutative geometry

### 12.1 Basic objects and constructions

Mainly, I'm interested in  $*$ -algebras  $A$  (and their completions) which are  $k$ -algebras equipped with an involution  $*$ . Usually,  $k = \mathbb{C}$  is the field of complex numbers. A very famous example of  $*$ -algebra is the algebra of the quantum harmonic oscillator,

$$\mathcal{H} = k\langle x, y \rangle / (xy - yx = 1).$$

When  $k = \mathbb{C}$ , one often represent  $A$  as a sub- $*$ -algebra of the bounded operators on a Hilbert space  $\mathcal{L}(H)$ , and complete w.r.t. to the norm. Note that not all complex  $*$ -algebras admit such a representation.

For instance, for  $\mathcal{H}$ , one easily get that

$$[x, P(y)] = P'(y) \quad \forall P \in \mathbb{C}[t]$$

Then if  $\|\cdot\|$  is a multiplicative norm on  $\mathcal{H}$ , it satisfies

$$2\|x\| \|y\| \geq \|x^ny\| \quad \forall n > 0.$$

Basic construction :

- separation-completion : in our sense, a norm can be degenerate. Being multiplicative, the annihilator of any norm is a closed ideal in  $A$ , so that there is an induced (classical/ nondegenerate) norm on the quotient algebra. The separation-completion is defined to be the completion of the quotient w.r.t.

the induced norm. Let us say that if  $\alpha$  is such a norm, we denote by  $A_\alpha$  the associated separation-completion. Any inequality

$$\alpha(x) \leq \beta(x) \quad \forall x \in A$$

induces an inclusion of annihilator  $N_\beta \subset N_\alpha$ , and gives a canonical quotient map

$$A_\beta \rightarrow A_\alpha.$$

The basic class of examples comes from completion of the complex group ring  $\mathbb{C}[\Gamma]$ . For any family of unitary representations  $\mathcal{F}$ , one can define the  $*$ -norm

$$\|x\|_{\mathcal{F}} = \sup\{\|\pi(x)\| : \pi \in \mathcal{F}\}$$

on  $\mathbb{C}[\Gamma]$ . The separation-completion is a  $C^*$ -algebra denoted  $C_{\mathcal{F}}^*(\Gamma)$ . For instance, if  $\mathcal{F}$  consists of all unitary representations of  $\Gamma$ , then one gets the maximal  $C^*$ -algebra  $C_{max}^*(\Gamma)$ , while if the family is reduced to the left regular representation  $\lambda_\Gamma$ , one gets the reduced  $C^*$ -algebra  $C_r^*(\Gamma)$ . By inclusion, one gets the canonical quotient map

$$\lambda_\Gamma : C_{max}^*(\Gamma) \rightarrow C_r^*(\Gamma).$$

Crossed-product : the basic ingredients are a  $*$ -algebra  $H$  endowed with a coassociative coproduct

$$\Delta : H \rightarrow H \otimes H,$$

and a  $C^*$ -algebra  $A$  on which  $H$  acts via a  $*$ -homomorphism

$$\alpha : A \rightarrow A \otimes H$$

such that  $(1 \otimes \Delta)\alpha = (\alpha \otimes 1)\alpha$ . The crossed-product is a twisted version of the tensor product.

$$(a \otimes x)(a' \otimes y) := (a \otimes 1_{M(H)})\alpha(a')(1_{M(A)} \otimes xy)$$

## 12.2 Quantum groups

A  $C^*$ -bialgebra is a pair  $(H, \Delta)$  where  $H$  is a  $C^*$ -algebra and

$$\Delta : H \rightarrow M(\tilde{H} \otimes_{min} H + H \otimes_{min} \tilde{H}, H \otimes_{min} H)$$

is a non-degenerate  $*$ -homomorphism such that  $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ .

A  $H$ -algebra is a pair  $(A, \alpha)$  where  $A$  is a  $C^*$ -algebra and

$$\alpha : A \rightarrow M(\tilde{A} \otimes_{min} H, A \otimes_{min} H)$$

such that  $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha$ . Its principal map is

$$\Psi : \begin{cases} A \otimes_{alg} A & \rightarrow & M(A \otimes_{min} H) \\ x \otimes y & \mapsto & (x \otimes 1_{M(H)})\alpha(y) \end{cases}$$

Let  $(H, \Delta)$  be a  $C^*$ -bialgebra and  $(A, \alpha)$  a  $H$ -algebra, with principal map

$$\Psi : A \otimes A \rightarrow M(A \otimes_{min} H).$$

- free if the range of  $\Psi$  is strictly dense in  $M(A \otimes_{min} H)$
  - proper if the range of  $\Psi$  is contained in  $A \otimes_{min} H$
  - principal if  $\Psi(A \otimes_{alg} A)$  is a norm dense subset of  $A \otimes_{min} H$
- principal = free and proper



### 12.3 Why $SU_q(2)$ ?

Apparently, some people are interested in deformation of classical Lie groups such as  $SU_q(2)$ , which is the Hopf algebra generated by 3 generators  $E, F, K$  satisfying the relations

$$R.$$

I wanted to understand where these relations are coming from, which led me to interesting ideas developed by several people, including Yuri Manin. The idea is to define  $SU_q(2)$  as a special group like object of the automorphism group of some noncommutative space, the quantum plane.

Let  $k$  be a field. The free (noncommutative)  $k$ -algebra on  $n$  generators is denoted by  $k\langle x_1, \dots, x_n \rangle$ .

**Définition 1.** A quadratic algebra

$$A = \bigoplus_{i \geq 0} A_i$$

is a  $\mathbb{N}$ -graded finitely generated algebra such that :

- $A_0 = k$ , and  $A_1$  generates  $A$ ,
- the relations on generators are in  $A_1 \otimes A_1$ .

The quadratic algebra  $A$  is said to be a Frobenius algebra of dimension  $d$  if moreover

- $A_d = k$  and  $A_i = 0$  for all  $i > d$ ,
- the multiplication map

$$m : A_i \otimes A_{d-i} \rightarrow A_d$$

is a perfect duality.

The main example is the quantum plane

$$\mathbb{A}_q^2 = k\langle x, y \rangle / (xy - qyx)$$

where  $q \in k^\times$ . More generally, the quantum space of dimension  $n|m$  is

$$\mathbb{A}_q^{n|m} = k\langle x_1, \dots, x_n, \eta_1, \dots, \eta_m \rangle / (x_i x_j - q x_j x_i, q \eta_i \eta_j + \eta_j \eta_i).$$

This example is suppose to come from physics. In quantum field theories, physicists deal with two kind of particles, bosons and fermions, and use commuting variables for one type, and anticommuting for the other. One object they appeal to are called supermanifolds, which are manifolds enriched with anticommuting variables. Formally, it means they look at ringed spaces  $(X, \mathcal{O})$  locally isomorphic to  $(\mathbb{R}^n, C^\infty[\eta_1, \dots, \eta_m])$ , where  $C^\infty[\eta_1, \dots, \eta_m]$  is the free sheaf of rings generated by anticommuting variables  $\eta_i$  over the smooth complex valued functions  $C^\infty(\mathbb{R}^n)$ .

Remark that a quadratic algebra  $A$  is a quotient of  $k\langle x_1, \dots, x_n \rangle$  by elements  $r_\alpha \in A_1 \otimes A_1$ , which we will denote as

$$A = k\langle x_1, \dots, x_n \rangle / (r_\alpha)$$

or

$$A = \langle A_1, R_A \rangle$$

with  $R_A \subseteq A_1 \otimes A_1$ .

Manin defines the quantum dual of a quadratic algebra as

$$A^! = k\langle x^i \rangle / (r^\beta)$$

where  $r_{ij}^\beta r_\alpha^{ij} = 0$ , i.e.  $R_{A^!} = R_A^\perp$ . Then, the quantum endomorphisms between two quadratic algebra is

$$\text{Hom}(A, B) = k\langle z_i^j \rangle / (r_\alpha^\beta)$$

where  $r_\alpha^\beta = r_\alpha^{ij} r_{kl}^\beta z_i^k z_j^l$ . If  $\text{End}(A) = \text{Hom}(A, A)$ , then  $\text{End}(A)$  satisfies the universal property to be initial in the category of  $k$ -algebras  $(B, \beta)$  endowed with an algebra homomorphism  $\beta : A \rightarrow A \otimes B$ .

If one does that to the quantum plane  $\mathbb{A}_q^2$ , one still doesn't find quite  $M_q(2)$  : half of the relations are missing. Also

$$(\mathbb{A}_q^{2|0})^! = \mathbb{A}_q^{0|2}?$$

Exercise.

## 12.4 TQFT

We recalled the definitions of a monoidal category, a braided category, and a symmetric monoidal category. The two main examples are the category of bordisms  $\text{Bord}^d$  in dimension  $d$ , and the category of vector spaces over a field  $k$ . The first talk focused on topological quantum fields theories in dimension 1 and 2.

**Définition 2.** A TQFT in dimension  $d$  is a monoidal symmetric functor

$$Z : \text{Bord}_d \rightarrow \text{Vect}_k.$$

The two main results we showed are :

- there is an equivalence of categories

$$\text{TQFT}_1 \cong \text{Vect}_k$$

obtained as  $Z \mapsto Z(pt)$ .

- there is an equivalence of categories

$$\text{TQFT}_2 \cong \text{Frob}_k$$

obtained as  $Z \mapsto Z(\mathbb{S}^1)$ .

A nice example in dimension 2 :  $Z(\mathbb{S}^1) = \mathbb{C}[t]/(t^2 - 1)$  is the Frobenius algebra given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad \epsilon(1) = 0 \quad \epsilon(t) = 1.$$

Then the handle element is  $h = 2t$  and

$$Z(\Sigma_g) = \begin{cases} 2^g & \text{if } g \text{ is odd} \\ 0 & \text{if } g \text{ is even.} \end{cases}$$

The second talk was directed towards extended field theories. First recall some higher category theory :  $n$ -categories, etc... And an extended TFT is a symmetric monoidal functor between symmetric monoidal  $n$ -categories

$$Z : Cob_n \rightarrow \mathcal{C}.$$

Then the following theorem was proved in [3].

**Théorème 4.** The evaluation functor

$$Z \mapsto Z(*)$$

establishes a bijective correspondance between extended  $n$ -dimensional TFT and fully dualizable objects of  $\mathcal{C}$ .

We now give an application of this result to the Jones polynomial. In [5], Witten gives an interpretation of the Jones polynomial, an isotopy invariant of links, as induced from a 3-dimensional TFT. The drawback of this article (for us) is that Witten uses Physical TFT's, i.e. gauge theories. The Jones polynomial is then shown to be the value of the partition function of a gauge field theory on  $\mathbb{S}^3$  with gauge group  $SU(2)$ . I propose to rewrite this result in our setting as an exercise.

A link is a disjoint union of embedding of the circle into  $\mathbb{S}^3$

$$\mathcal{L} = \{\text{embeddings } \prod_{i=1}^k \mathbb{S}^1 \hookrightarrow \mathbb{S}^3\}.$$

we will often make no distinction between the embedding and its image in the 3-sphere, which we will denote by  $L$ . The Jones polynomial of a link  $L$  is defined as an isotopy invariant polynomial  $V : \mathcal{L} \rightarrow \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$  satisfying the Skein relations

$$-t^{\frac{1}{2}}V_+ + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_0 + t^{-\frac{1}{2}}V_- = 0.$$

To a link  $L$  one can associated the 3-manifold  $M_L = \mathbb{S}^3 - L$ . Consider the extended TFT

$$Z^{(n)} : Cob_3 \rightarrow \mathcal{C}$$

given by  $Z() = V_n$  where is the fundamental representation of  $\mathfrak{su}(n)$ . By the cobordism theorem, it is enough to define the TFT on all of  $Cob_3$ . Then

$$\phi(V_L) = Z^{(2)}(M_L),$$

where  $\phi : \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \rightarrow \mathbb{C}$  is the evaluation at a root of unity  $q \in \mathbb{C}^\times$ . This can be proved by showing that  $Z^{(n)}(M_L)$  satisfies the skein relation

$$-q^{\frac{n}{2}}V_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_0 + q^{-\frac{n}{2}}V_- = 0$$

## 12.5 Reminder

A locally ringed space is a topological space  $X$  together with a sheaf or ring  $\mathcal{O}_X$  over  $X$  such that all stalks are local rings, ie have a unique maximal ideal.

For  $R$  a ring,  $X = \text{Spec}(R)$  denotes the topological space obtained as the set of prime ideals of  $R$  endowed with the Zariski topology, i.e. the topology generated by the closed subsets

$$V_I = \{J \text{ ideals in } R \text{ s.t. } I \subset J\}.$$

Equivalently, a basis of open subsets is given by

$$D_f = \{J \text{ ideals in } R \text{ s.t. } f \notin J\}$$

for every  $f \in R$ . Let  $S_f$  be the multiplicative domain given by the powers of  $f$ . Then define a sheaf of ring over  $X$  by

$$\mathcal{O}_X(D_f) = S_f^{-1}R.$$

It is called the structural sheaf of  $\text{Spec}(R)$ . Any locally ringed space isomorphic to

$$(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$$

with  $R$  commutative is called an affine variety.

Note : the functor  $\text{Spec}$  gives an antiequivalence of categories between the categories of commutative rings and the category of affine varieties.

**Définition 3.** A scheme is a locally ringed space locally isomorphic to an affine variety.

## 13 Langlands

A modular form of weight  $k$  is a section of

$$\Lambda^{k+2}T^*M.$$

The projective space of the  $\mathbb{N}$ -graded algebra

$$A = \bigoplus \Lambda^{k+2}T^*M$$

is the compactification of the modular curve

$$\mathbb{P}(A) \cong \tilde{\mathcal{C}}.$$

If  $F = \mathbb{Q}$  and  $G = GL_2$ , the finite part of the adele

$$\mathbb{A}_f = \prod_{\text{finite places}} F_\nu = \prod_{p \in \mathcal{P}} \mathbb{Q}_p$$

is ?? and  $G(\hat{\mathbb{Z}})$  is the maximal compact of  $G(\mathbb{A}_f)$  with  $G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$  being two copies of the upper half plane  $\mathbb{H}$ , and  $G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)$  is the modular curve.

Is the right part  $G(\mathbb{A}_f)/G(K)$  is isomorphic to the inductive limit  $G(\mathbb{Z}/p^k\mathbb{Z})$  ?

Yes if  $G = GL_1$  :

$$\varinjlim \mathbb{Z}/p^k\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p.$$

## 14 Haagerup property, cocycles and the mapping class group

If  $\Sigma$  is a closed oriented connected surface (with marked points), we denote by  $Mod(\Sigma)$  its so-called mapping class group.

In [1] are used bounded representations of the mapping class group parametrized by a complex number  $z \in \mathbb{D}$  :

$$\pi_z : \Gamma \rightarrow \mathcal{L}(H).$$

Here,  $H$  is the Hilbert space obtained as the free Hilbert space generated by multicurves having a finite number of intersections with a fixed triangulation  $\tau$  of  $\Sigma$ .

## 15 Hawaii

### 15.1 HLS groupoids

Let  $(\Gamma, \mathcal{N})$  be an *approximated group* and  $G_{\mathcal{N}}$  its associated HLS groupoid. Then :

- $G_{\mathcal{N}}$  is amenable iff  $\Gamma$  is amenable,
- if  $G_{\mathcal{N}}$  is a-T-menable, then  $\Gamma$  is a-T-menable. The converse doesn't hold : in [?], the authors construct an approximated pair  $(\mathbb{F}_2, \mathbb{N})$  such that the assembly map  $\mu_{G_{\mathcal{N}}, r}$  is not surjective, even if  $\mathbb{F}_2$  is a-t-menable.
- $G_{\mathcal{N}}$  has T iff  $\Gamma$  has T,
- the algebraic exact sequence

$$0 \longrightarrow \bigoplus_n \mathbb{C}[\Gamma_n] \longrightarrow C_c(G_{\mathcal{N}}) \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$

extends to

$$0 \longrightarrow \bigoplus_n C_r^*(\Gamma_n) \longrightarrow C_r^*(G_{\mathcal{N}}) \longrightarrow C_{r, \infty}^*(\Gamma) \longrightarrow 0 ,$$

where the right side algebra is the completion of  $\mathbb{C}[\Gamma]$  w.r.t. the norm

$$\|x\|_{r, \infty} = \sup\{\|y\|_r : q(y) = x\} \quad \forall x \in \mathbb{C}[\Gamma].$$

This is not an exotic crossed product functor, but one can still define an assembly map  $\mu_{\Gamma, r, \infty}$  as the composition of  $\mu_{\Gamma, max}$  with the induced at the level of  $K$ -theory of the quotient map  $C_{max}^*(\Gamma) \rightarrow C_{r, \infty}^*(\Gamma)$ . This exact sequence and the one induced by the decomposition of  $G^0 = \overline{\mathbb{N}}$  is  $\mathbb{N}$  and  $\infty$  intertwines the assembly maps so that the next point follows :

- $G_{\mathcal{N}}$  satisfies BC iff  $\Gamma$  satisfies BC for  $\mu_{\Gamma, r, \infty}$ .

- If  $\Gamma$  has T, then if  $\mu_\Gamma$  is injective (which is the case for all closed subgroups of connected Lie groups), then  $\mu_{G_N}$  fails to be surjective.
- **Congruence subgroup property.** If  $\Gamma$  has c.s.p. , then the assembly map fails to be surjective for any HLS groupoid  $G_N(\Gamma)$ . If one can find such a groupoid which is a-T-menable for  $SO(n, 1)$ , then this would imply Serre's c.s.p. conjecture : Any lattice in  $SO(n, 1)$  does not have c.s.p.

A useful fact from [?] :

$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$

is exact implies that the cone  $C_\gamma$  of the natural inclusion  $\gamma : J \rightarrow C_\beta$  has vanishin K-groups :

$$K_*(C_\gamma) = 0.$$

**16 Mayer-Vietoris**

**17 Quantum groups**

**18 Property T**

**19 Number theory**

**20 Fock spaces, CuntzKrieger algebras, and second quantization**

## 21 Representations of groupoids

**Mettre les references et des rappels sur les champs continus et mesurables d'espaces de Hilbert.**

This section is a reminder on the different notions of representations for groupoids that exists. Let us first begin by a reminder on continuous fields of  $C^*$ -algebras and Hilbert spaces. All this material can be found in Dixmier's book[2].

A continuous field of Banach spaces over a topological space  $X$  is a pair

$$E = (\{E_x\}_{x \in X}, \Gamma_E)$$

where :

- $E_x$  is a Banach space, with norm denoted  $\| \cdot \|_x$ ,
- $\Gamma_E$  is a linear subspace of  $\prod_{x \in X} E_x$  such that  $x \mapsto \|\gamma(x)\|_x$  is continuous (or upper semi-continuous according to Lafforgue) for every  $\gamma \in \Gamma_E$ , and if every  $\sigma \in \prod_{x \in X} E_x$  is locally uniformly approximable by sections, i.e. if for every  $x \in X$  and  $\varepsilon > 0$ , there exists  $\gamma \in \Gamma_E$  and a neighborhood  $U$  of  $x$  such that  $\sup_{y \in U} \|\sigma - \gamma\|_y < \varepsilon$ , then  $\sigma \in \Gamma_E$ ,
- $\{\gamma(x)\}_{x \in X}$  is dense in  $E_x$ .

The elements of  $\Gamma_E$  are called the continuous sections of  $E$ . A continuous section is said to be bounded if

$$\sup_x \|\gamma(x)\|_x < \infty.$$

The space of continuous bounded sections with the sup-norm is a Banach space.

A continuous field of  $C^*$ -algebras over a locally compact space  $X$  is a pair

$$(\{A_x\}_{x \in X}, \Gamma_A)$$

where :

- $A_x$  is a  $C^*$ -algebra,
- $\Gamma_A \subset \prod_{x \in X} A_x$  is a  $*$ -algebra such that  $x \mapsto \|\gamma(x)\|$  is continuous for every  $\gamma \in \Gamma_A$ , and if every  $\sigma \in \prod_{x \in X} A_x$  is locally uniformly approximable by sections, i.e. if for every  $x \in X$  and  $\varepsilon > 0$ , there exists  $\gamma \in \Gamma_A$  such that  $\|\sigma - \gamma\| < \varepsilon$ , then  $\sigma \in \Gamma_A$ ,
- $\{\gamma(x)\}_{x \in X}$  is dense in  $A_x$ .

On the other hand a  $C_0(X)$ -algebra is a  $C^*$ -algebra  $A$  endowed with a nondegenerate  $*$ -homomorphism

$$\phi : C_0(X) \rightarrow M(Z(A)).$$

Any field  $(\{A_x\}_{x \in X}, \Gamma_A)$  over  $X$  defines a  $C_0(X)$ -algebra

$$C^*(\Gamma_A) := \{\gamma \in \Gamma_A \text{ s.t. } x \mapsto \|\gamma(x)\| \in C_0(X)\}.$$

A  $C_0(X)$ -algebra  $A$  is continuous if  $x \mapsto \|a_x\|$  is continuous for each  $a \in A$ . Here,  $a_x$  denotes the image of  $a$  under the map  $A \rightarrow A/\phi(I_x)A$ , with  $I_x$  the ideal of functions vanishing at  $x$ .

There is a correspondence between these two notions.

Let  $G$  be a locally compact groupoid. Renault defines a representation of  $G$  as the following data :

- a measure  $\mu$  on  $G^0$ ,
- a measurable field of Hilbert spaces  $(\mathcal{H}, \mu)$  over  $G^0$ ,
- a family of bounded operators  $L_g : \mathcal{H}_{s(g)} \rightarrow \mathcal{H}_{r(g)}$  for each  $g \in G$  satisfying  $L_{g_1} L_{g_2} = L_{g_1 g_2}$  for every  $(g_1, g_2) \in G^2$ ,  $L_{e_x} = id_{\mathcal{H}_x}$ , and

$$g \mapsto \langle L_g(\xi_{s(g)}), \eta_{r(g)} \rangle$$

is measurable for every pair of measurable sections.

The main examples are the left and right regular representations, and the trivial one. The left regular representation  $\lambda$  is defined as the family of operators

$$\lambda_g : \begin{cases} L^2(G^x, \lambda^x) & \rightarrow & L^2(G^x, \lambda^x) \\ \xi & \mapsto & [\gamma \mapsto \xi_{g^{-1}\gamma}] \end{cases}$$

whereas the trivial representation is defined as

$$\tau_g : \begin{cases} H_{s(g)} = \mathbb{C} & \rightarrow & H_{r(g)} = \mathbb{C} \\ \xi & \mapsto & \xi \end{cases}$$

Recall the following : given a  $C_0(G^0)$ -Hilbert module  $E$ , a unitary representation  $G$  is a unitary

$$V \in \mathcal{L}_{s^* C_0(G^0)}(s^* E, r^* E)$$

which satisfies  $V_1 V_2 = \Delta V$ , where :

- $V_i$  is the  $C_0(G^2)$ -operator induced from  $V$  by the projection  $p_i : G^2 \rightarrow G$ ,
- $\Delta$  is the (comultiplication) map  $C_0(G) \rightarrow M(C_0(G^2))$ -operator induced by the multiplication  $\Delta : G^2 \rightarrow G$ .

Fiberwise this gives you a more restrictive class than the representations in the sense of Renault. Indeed, in the case of a trivial groupoid over a locally compact space  $X$ , the spectral theorem ensures that any  $*$ -representation on a Hilbert space  $H$

$$\pi : C_0(X) \rightarrow \mathcal{L}(H)$$



disintegrates into a representation in the sense of Renault on a field of Hilbert space. However, our  $\{V_g\}$  gives a continuous field of representation over a continuous field of Hilbert space, which is a priori stronger.

As in the case of groups, one can try to define a integrated representation, by

$$(V(f)\xi)_x = \int_{g \in G^x} f(g)V_g(\xi_{s(g)})d\lambda^x(g) \quad f \in C_c(G), \xi \in E.$$

This defines a map  $C_c(G) \rightarrow \mathcal{B}(E)$ , where  $\mathcal{B}(E)$  denotes the bounded operator of  $E$  seen as a Banach space. But this map is not even multiplicative!

Instead, consider  $G^0$  to be discrete, and

$$V(f)_{xy} = \sum_{g \in G_y^x} f(g)V_g,$$

so that

$$(V(f)\xi)_x = \sum_{y \in X} V(f)_{xy}\xi_y = \sum_{g \in G^x} f(g)V_g(\xi_{s(g)}),$$

but this time

$$V(f * g)_{xz} = \sum_{y \in X} V(f)_{xy}V(g)_{yz}.$$

If  $G^0$  is not discrete, suppose there is a measure  $\mu$  on  $G^0$ . Then

$$V(f * g)_{xz} = \int_X V(f)_{xy}V(g)_{yz}d\mu(y).$$

Some facts.

Let  $G$  be proper and  $V \in \mathcal{L}_{s^*C_0(G^0)}(s^*E, r^*E)$  be a unitary representations. If  $c : G^0 \rightarrow [0, 1]$  is a cutoff function, any vector  $\xi \in E$  can be averaged as to get an invariant one :

$$(\bar{\xi})_x = \int_{G^x} c(x\gamma)V_\gamma(\xi_{s(\gamma)})d\lambda^x(\gamma).$$

A simple computation and  $(L_g)^*\lambda^{s(g)} = \lambda^{r(g)}$  gives indeed the  $\bar{\xi}$  is  $G$ -invariant. This ensures that any proper groupoid has property T.

## 22 Grothendieck and tensor products, the origin of nuclearity

This section is based on a talk given by Gilles Pisier, and his (exceptionally good) survey article.

Grothendieck started his work in functional analysis. While this is well known, I wanted to write a little post about how his work is important in my field.

Grothendieck did his Licence (his "undergrad") in the south of France, in the city of Montpellier.

If  $x = \sum_j \alpha_j \otimes \beta_j$ ,

$$\|x\|_{\wedge} = \inf\{\|\alpha_j\| \|\beta_j\| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$\|x\|_{\wedge} = \sup\{\|\alpha_j\| \|\beta_j\| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$\|x\|_H = \inf\{\|\alpha_j\| \|\beta_j\| : x = \sum_j \alpha_j \otimes \beta_j\}$$

14 fundamental norms.

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