Notes

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1 Introduction

This section is devoted to summarize the extent my work at the end of the second year of my PhD.

The aim of this dissertation is to study relationship between several worlds : coarse geometry, groupoids and C^*-algebras. The idea is to transfer ideas that were very profitable to coarse geometry into groupoids, and to try and exploit them to compute K-theory groups of certain groupoids. In this particular section, we will consider several functors between categories which were defined in the litterature, and will recall their construction.

$$\begin{tikzcd}
 & \mathbf{Coarse}\quad (X,\mathcal{E}) \arrow[dl, bend right, "\beta X"'] \arrow[dr, bend left, "G(X)"] \arrow[dd, "{C^*X, C^*_uX}"] & \\
\mathbf{Top}\quad (Z,\mathcal{T}) \arrow[dr, bend right, "C_0(Z)"] & & \mathbf{Grpds}\quad G \arrow[dl, bend left, "C^*_rG"] \\
 & \mathbf{C^*}\quad A \arrow[d] & \\
 & \mathbf{KK} &
\end{tikzcd}$$

Definition 1. Let G be a topological groupoid, and $\mathcal{E}$ a family of subsets of G. $\mathcal{E}$ is said to be a coarse structure on G if

- if E and F are in $\mathcal{E}$, then $EF \in \mathcal{E}$, and $E \cup F \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$
- if $E \in \mathcal{E}$ and $F \subset E$, then $F \in \mathcal{E}$,
- any finite subset is in $\mathcal{E}$ (?) and if $G^{(0)} \in \mathcal{E}$, $\mathcal{E}$ is said to be unital.

1.1 From topological spaces to C^*-algebras

This first section recalls some easy facts, and is mainly directed at enlightening the parallel and the analogy of all the constructions that will be done in the following sections.

Definition 2. Let Y be a set, and $\mathcal{T}$ a subset of $\mathcal{P}(Y)$. $\mathcal{T}$ is called a topology if

- $\emptyset \in \mathcal{T}$,
- if $U \in \mathcal{U}$, then $U^c \in \mathcal{U}$,
- if $\mathcal{U} \subset \mathcal{T}$, then $\cup_{U \in \mathcal{U}} U \in \mathcal{U}$.

Elements of $\mathcal{U}$ are called open sets, and if $x \in X$, we say of any set containing both an open set and x is a neighborhood of x.
A topological space is said to be locally compact iff every point has a relatively compact neighborhood.
A map between two topological spaces $h : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if it respects the topology i.e. if $\forall V \in \mathcal{T}_Y, h^{-1}(V) \in \mathcal{T}_X$.

The natural C^*-algebra associated to a locally compact space Y is $C_0(Y) = \{f : Y \to \mathbb{C}$ continuous s.t. $\lim_\infty f = 0\}$, endowed with the supremum norm $||f|| = \sup_{y\in Y} |f(y)|$.

A continuous map $h : X \to Y$ induces a $*$-homomorphism $C_0(Y) \to C_0(X)$ iff h is proper, that is the inverse image of any compact set is compact. A simple reformulation leads to the following.

Proposition 1. If **Top** is the category of locally compact spaces with morphisms continuous proper maps, and C^* is the category of C^*-algebras with morphisms $*$-homomorphisms, $Y \mapsto C_0(Y)$ is a contravariant functor from **Top** to $\mathbf{C}^*$.

There is a notion of dimension in the **Top**, the covering dimension.

Definition 3. The covering dimension of Y is less than d if, for every covering $\mathcal{V}$ of Y, there exists a covering $\mathcal{U}$ that refines $\mathcal{V}$ and that decomposes into $d+1$ pieces $\mathcal{U} = \mathcal{U}_0 \sqcup ... \sqcup \mathcal{U}_d$ with the property that for $U, V \in \mathcal{U}^{(j)}$, then $U \cap V = \emptyset$.

1.2 From coarse spaces to C^*-algebras

Definition 4. A coarse space is a couple $(X, \mathcal{E})$, where X is a set and $\mathcal{E}$ a coarse structure on the pair groupoid $X \times X$. A coarse map is a map respecting the coarse structure i.e. $h : (X, \mathcal{E}) \to (Y, \mathcal{F})$ such that $\forall F \in \mathcal{F}, (h \times h)^{-1}(F) \in \mathcal{E}$.

A metric space (X, d) naturally inherits a coarse structure from its bounded subsets :

$$\mathcal{E}_X = \{E \subset X \times X : \sup_E d(x, y) < \infty\}.$$

Definition 5. Let $(X, \mathcal{E})$ be a coarse space. A X-module is a pair (H_X, ϕ) where

- H_X is a Hilbert space,
- $\phi : C_0(X) \to \mathcal{L}(H_X)$ is a $*$-homomorphism.
- The module is said to be non-degenerate if $\{\phi(f)\eta : f \in C_0(X), \eta \in H_X\}$ is dense in H_X, and standard if no non-zero function of $C_0X)$ acts as a compact operator of H_X.

Examples If X is a discrete metric space with bounded geometry, $l^2(X) \otimes H$ with action by multiplication on the first factor defines a s.n.d. X-module.
If X is a compact metric space endowed with a measure μ without atoms, then $L^(X, \mu)$ is a s.n.d. X-module.

Definition 6. Let $T \in \mathcal{L}(H_X, H_Y)$ be an operator.

- T is said to be locally compact if $\phi(g)T$ and $T\phi(f)$ are compact operators for all $f \in C_0(X), g \in C_0(Y)$.
- The support of T is the complement of the set of points $(x, y) \in X \times X$ such that there exist $f_x, f_y \in C_0(X), C_0(Y)$ such that $f_x(x) \neq 0, f_y(y) \neq 0$ and $\phi(f_y)T\phi(f_x) = 0$.
- The propagation of T is the smallest $E \in \mathcal{E}$ such that supp $T \subset E$.**ADAPTER**

Let us define the Roe algebra of X form a fixed s.n.d. X-module H_X. It will be shown that, up to unnatural isomorphism, it does not depend on the choice of the s.n.d. X-module.

Definition 7. For any $E \in \mathcal{E}$, define the subspace of locally compact operators with propagation E-controlled :

$$C_E[X, H_X] = \{T \in \mathcal{L}(H_X) : T \text{ is locally compact and supp } T \subset E\}.$$

The Roe algebra is the C^*-algebra

$$C^*(X, H_X) = \overline{\cup_{E \in \mathcal{E}} C_E[X, H_X]}$$

the closure being taken with respect to the operator norm of $\mathcal{L}(H_X)$.

Proposition 2. Let $(X, \mathcal{E}), (Y, \mathcal{F})$ be two coarse spaces, H_X, H_Y two s.n.d. modules over X and Y repsectively, and $h : X \to Y$ a coarse map. Then, for any $F \in \mathcal{F}$, there exists an isometry $V \in \mathcal{L}(H_X, H_Y)$ such that

$$\text{supp } V \subset (h \times id_Y)^{-1}(E).$$

Proof 1. Extend the representation ϕ to $\tilde{\phi} : L^\infty(X) \to \mathcal{L}(H_X)$.

Let $F \in \mathcal{F}$ and $\mathcal{U}$ be a Borel partition of Y such that $\sqcup_{U \in \mathcal{U}} U \times U \subset F$ and every $U \in \mathcal{U}$ is of non-empty interior. If χ_A denotes the characteristic function of A, and because the modules are s.n.d., we can find an isometry

$$V^{(U)} : \chi_{h^{-1}(U)} H_X \to \chi_U H_Y$$

and by standardness $H_X = \bigoplus \chi(h^{-1}(U)) H_X$ and $H_Y = \bigoplus \chi_U H_Y$, so $V = \oplus V^{(U)}$ fits. Indeed,

$$\text{supp } V \subset \sqcup h^{-1}(U) \times U \subset (h \times id_Y)^{-1}(F).$$

□

Now, if H_X and H'_X are two s.n.d. X-modules, apply the preceding lemma to the identity map to have an isometry $V : H_X \to H'_X$ which is supported as close as you want of the diagonal. This induces $Ad_V : C^*(X, H_X) \to C^*(X, H'_X)$ by $Ad_V(T) = VTV^*$, and gives our isomorphism, which is canonical in K-theory.

If we decide to fix a s.n.d. module for every coarse space, we can speak of "the" Roe algebra of X, and we saw that a coarse map between two coarse spaces $h : X \to Y$ induces a $*$-homomorphism $h_* : C^*(X, H_X) \to C^*(Y, H_Y)$.

We will now define a dimension on coarse spaces, after an idea of Gromov **à vérifier**, and that was extensively used in coarse geometry.

Definition 8. A coarse space $(X, \mathcal{E})$ is said to have asymptotic dimension less than d if, for every $E \in \mathcal{E}$, there is a controlled set $F \in \mathcal{F}$ and a family $\mathcal{U}$ of susbsets such that

- $\mathcal{U}$ covers X,
- every $U \in \mathcal{U}$ is F-controlled, i.e. $\sqcup_{U \in \mathcal{U}} U \times U \subset F$,
- we have a decomposition $\mathcal{U} = \mathcal{U}_0 \sqcup ... \sqcup \mathcal{U}_d$ such that every pair of subsets of a $\mathcal{U}^{(j)}$ are E-separated.

1.3 From groupoids to C^*-algebras

We first precise what we consider as morphisms in the category of topological groupoids **Grpds**.

Definition 9. Let G and G' be two topological groupoids.
A generalized morphism between from G to G' is a triple (Z, p, p') where Z is a locally compact space endowed with a right action of G with respect to p and a right action of G' with respect to p' such that :

- the two actions commute : $(g.z).g' = g.(zg'), \forall z \in Z, g \in G_{p(z)}, g' \in G'^{p'(z)}$,
- the action of G' is free and proper,
- $Z \times_{p',s} G' \to Z \times Z$ induces a homeomorphism $Z \rtimes G' \to Z \times_{G^{(0)}} Z$.

If a generalized morphism is invertible, we will say that it is a Morita equivalence.

Examples :

- If $\phi : G \to G'$ is a strict morphism of topological groupoids, i.e. a continuous map which is a natural transformation between the two groupoids seen as categories with inversible arrows, then ϕ naturally defines a generalized morphism. Set $Z = G^{(0)} \times G'$ with actions defined by $g(x, h)g' = (gx, \phi(g)hg')$.
- Let G be a locally compact group. If H is a subgroup of G, then $Z = G/H$ is a Morita equivalence between H and $G/H \rtimes G$.

Let $G \rightrightarrows G^{(0)}$ be a locally compact groupoid, endowed with a Haar system $\lambda = (\lambda^x)_{x \in G^{(0)}}$.

Definition 10. Let X be a locally compact σ-compact space.

- A $C_0(X)$-algebra is a pair (A, θ) where A is a C^*-algebra and $\theta : C_0(X) \to Z(\mathcal{M}(A))$ is a $*$-homomorphism such that $\theta(C_0(X))A = A$.

- A G-algebra is a triple (A, θ, α) where the two first elements form a $C_0(G^{(0)})$-algebra, and $\alpha : s^*A \to r^*A$ is an action of G, i.e. an isomorphism of $C_0(G)$-algebras such that $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$ whenever $(g, g') \in G^{(2)}$.

- if (A, α) is a G-algebra, a G-module E is a pair (E, V) where E is a A-Hilbert-module and $V \in \mathcal{L}(s^*E, r^*E)$ is an unitary such that $V_g \circ V_{g'} = V_{gg'}, \forall (g, g') \in G^{(2)}$.

An important example of G-module is $L^2(G, A)$. Define the $C_0(X, A)$-valued scalar product on $C_c(G, A)$

$$\langle \eta, \xi \rangle(x) = \int_{G^x} \eta(g)^* \xi(g) d\lambda^x(g).$$

The A-Hilbert module $L^2(G, A)$ is the completion of $C_c(G, A)$ under this scalar-product, and the action of G is defined by left-translation :

$$\forall \xi \in L^2(G_{s(g)}, A), (V_g \xi)(h) = \xi(g^{-1}h),$$

which is an isomorphism $s^*L^2(G,A) \to r^*L^2(G,A)$.

As in the group case, the data of a G-algebra A allows one to construct the crossed-product $A \rtimes_r G$, which is a C^*-algebra. Just note that $C_c(G,A)$ is a $*$-algebra for the convolution product

$$\xi * \eta(g) = \int_{G^{r(g)}} \xi(h)\alpha_h(\eta(h^{-1}g))\lambda^{r(g)}(h)$$

and that it acts on $L^2(G,A)$ by the left-regular representation :

$$\forall f \in C_c(G,A), \xi \in L^2(G,A), \lambda(f)\xi = f * \xi.$$

This induces an operator norm, which is often called the reduced norm, on $C_c(G,A)$, $||f||_r := ||\lambda(f)||_{\mathcal{L}(L^2(G,A))}$.

Definition 11. $A \rtimes_r G$ is defined as the $*$-completion of $C_c(G,A)$ for the reduced norm.
When $A = \mathbb{C}$ is the C^*-algebra of complex numbers with the trivial action of G, the reduced cross-product is called the reduced C^*-algebra and is denoted by $C_r^*(G)$.

The reduced cross-product is functorial in A with respect to $*$-homomorphisms but does not respect equivariant exact sequences of G-algebras (unlike the maximal crossed-product, which does). Going to KK-theory, the reduced crossed-product is also functorial $KK^G \to KK$.

Proposition 3 (LeGall [9])**.** There exists a homomorphism $j_G : KK^G(A,B) \to KK(A \rtimes_r G, B \rtimes_r G)$ which is natural with respect to the Kaparov product, i.e. $\forall x \in KK^G(A,B), \forall y \in KK^G(B,C)$, one has :

$$j_G(x \otimes_B y) = j_G(x) \otimes_{B \rtimes_r G} j_G(y).$$

An important remark that we will use later is that $- \rtimes G$ preserves semi-split exact sequences. More precisely, let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an equivariant exact sequence of G-algebras which has a completely positive G-equivariant section, then $0 \longrightarrow A' \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow A'' \rtimes_r G \longrightarrow 0$ is an exact sequence of C^*-algebras.

What about functoriality in the G-variable ? If we reduce our considerations to reduced C^*-algebras, the work of J-L. Tu [16] allows to state the following results.

Definition 12. A generalised morphism (Z,p,p') from G to G' is said to be proper if

- it is locally proper : the action of G is proper,
- for every compact subset $K \subset G'^{(0)}$, $p'^{-1}(K)$ is G-compact.

Proposition 4 (Tu, [16])**.** Let G and G' be two locally compact groupoids with Haar systems, and (Z,p,p') a generalized morphism from G to G'.
If (Z,p,p') is locally proper, the one can construct a C^*-correspondence (E,π) from $C_r^*(G_1)$ to $C_r^*(G_2)$.
Moreover, if the generalized morphism is proper, $\pi : C_r^*(G') \to \mathcal{L}(E)$ maps to the compact operators, so that (Z,p,p') defines an element of $KK(C_r^*(G'),C_r^*(G))$.
These construction are functorial with respect to composition of generalized morphisms and C^*-correspondences.

Some remarks. The result of J-L. Tu applies even if the groupoids are non-Hausdorff. If we actually consider a strict morphism $\phi : G_1 \to G_2$, the generalized morphism is proper iff ϕ restricted to $(G_1)_K^K$ is proper, for every compact subset $K \in G_1^{(0)}$.

1.4 From coarse spaces to groupoids

This section gives details on he construction of the coarse groupoid from [13].

Let $(X,\mathcal{E})$ be a coarse space. The idea is to construct an étale groupoid $G(X)$ extending the pair groupoid $X \times X$. As a topological space,

$$G(X) = \cup_{E \in \mathcal{E}} \overline{E},$$

the closure being taken in $\beta(X\times X)$, so that it is Hausdorff and locally compact. The base space is $G(X)^{(0)} = \beta X$. Now, the firt and second projections can be extended by universal property of the Stone-Cech compactification to give the source and the range map $s,r : G(X) \rightrightarrows \beta X$ respectively, as shown in this commutative diagramm :

$$\begin{tikzcd}
X \times X \arrow[r] \arrow[d, "\iota_{X\times X}"] & X \arrow[r, "\iota_X"] & \beta X \\
\beta(X \times X) \arrow[urr, dotted] & &
\end{tikzcd}.$$

The only non trivial part is to extend the multiplication of the pair groupoid. This is done by extending the inclusion $E \to X \times X$ to $\overline{E} \to \beta X \times \beta X$. The corollary 10.31 of [11] assures that

Lemma 1. The map $r \times s : \overline{E} \to \beta X \times \beta X$ is a topological embedding.

Using this lemma, we can embed $G(X)$ in $\beta X \times \beta X$ and use the pair multiplication in this groupoid, the point being that such a multiplication $G(X) \times_{s,r} G(X) \to G(X)$ is continuous and extend the pair multiplication of $X \times X$.

The following proposition (Proposition 3.5 from [13]) assures that $X \mapsto G(X)$ is a functor from uniformly locally finite coarse spaces to groupoids with generalized morphisms as arrows.

Proposition 5. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces with uniformly locally finite coarse structures, then a coarse map $h : X \to Y$ induces a generalized morphism from $G(X)$ to $G(Y)$:

$$\begin{tikzcd} G(X) \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[dl] \arrow[dr] & G(Y) \arrow[d, shift left] \arrow[d, shift right] \\ \beta X & & \beta Y \end{tikzcd}$$

We now state a result from [13] (lemma 4.4), and give, for the reader's convenience, a more detailed proof than that of the paper.

Proposition 6. Let $(X, \mathcal{E})$ be a uniformly locally finite coarse space, then we have an isomorphism of C^*-algebras

$$C^*(X) \simeq l^\infty(X, \mathfrak{K}) \rtimes_r G(X).$$

Démonstration. Let $D = l^\infty(X, \mathfrak{K})$ and $G = G(X)$. The C^*-algebra $D \rtimes_r G$ is generated by continuous functions $f : \overline{E} \to D$ such that $f(g) \in D_{s(g)} \simeq \mathfrak{K}$ for some $E \in \mathcal{E}$. The crossed product is obtained as the closure in the norm operator defined by the actions of such functions by convolution on $\mathcal{E} = L^2(G, D)$, which defines by definition a faithful map $D \rtimes_r G \to \mathcal{L}(\mathcal{E})$.

Now take the G-invariant ideal $J = C_0(X, \mathfrak{K})$. As $D \to \mathcal{M}(J)$ is faithful, $\mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{E} \otimes_D J)$ is isometric, and we obtain a faithful map by composition $D \rtimes_r G \to \mathcal{L}(\mathcal{E} \otimes_D J)$, and $\mathcal{E} \otimes_D J \simeq L^2(G, J)$.

But $C_0(X)$ acts faithfully by multiplication on $C_0(X, \mathfrak{K})$, hence on $H_X = L^{(}G, J)$, which makes it a n.d.s. X-module and induces a faithful map $C^*(X, H_X) \to \mathcal{L}(H_X)$.

If $T \in C_E[X, H_X]$, define $f(g) = T_{s(g),r(g)}$ when $g \in X \times X$, and extend f by continuity to get $f : \overline{E} \to D$, and $f(g) \in D_{s(g)}$.

If $f : \overline{E} \to D$ is a continuous function such that $f(g) \in D_{s(g)}$, define $T \in C_E[X, H_X]$ by $T\xi = f\xi$. This defines a $*$-homomorphism $D \rtimes_r G \to C^*(X, H_X)$. The previous construction shows it is surjective. But the action of $D \rtimes_r G$ on H_X being faithful, it is also injective, which concludes the proof.

$\square$

2 Controlled K-theory

In this section, we define controlled K-theory in a little more generality than the functor defined in [10], so that the propagation can be indexed by compact subsets of an étale groupoid instead of positive real numbers.

Definition 13. A coarse structure $\mathcal{E}$ is a lattice which is a semi-group. Recall that a lattice is a poset for which every pair (E, E') admits a supremum $E \vee E'$ and an infimum $E \wedge E'$.

Definition 14. A C^*-algebra A is said to be filtered if there exists a coarse structure $\mathcal{E}$ and, for every $E \in \mathcal{E}$, linear subspaces A_E of A such that :

- if $E \leq E'$, then $A_E \subset A_{E'}$, and the inclusion $\phi_E^{E'} : A_E \hookrightarrow A_{E'}$ induces an inductive system of linear spaces,
- A_E is stable by involution,
- for all $E, E' \in \mathcal{E}$, $A_E.A_{E'} \subset A_{EE'}$,
- the union of subspaces is dense in A, i.e. $\overline{\cup_{E\in\mathcal{E}} A_E} = \varinjlim A_E = A$.
- if A is unital, we impose that $1 \in A_E, \forall E \in \mathcal{E}$.

If $(A, \mathcal{E})$ and $(B, \mathcal{E}')$ are two filtered C^*-algebras, a filtered morphism $(\phi, \rho) : (A, \mathcal{E}) \to (B, \mathcal{E}')$ is :

- a non-decreasing map $\rho : \mathcal{E} \to \mathcal{E}'$,
- a $*$-homomorphism $\phi : A \to B$ such that $\phi(A_E) \subset B_{\rho(E)}$ for all $E \in \mathcal{E}$.

2.1 Almost unitaries and almost projections

Definition 15. Let $(A, \mathcal{E})$ be a unital filtered C^*-algebra. Let $\varepsilon \in (0, \frac{1}{4})$ and $E \in \mathcal{E}$ a controlled subset. The set of ε-E-unitaries is the set

$$U^{\varepsilon,E}(A) = \{u \in A_E \text{ s.t. } ||u^*u - 1|| < \varepsilon \text{ and } ||uu^* - 1|| < \varepsilon\}$$

and the set ε-E-projections is the set

$$P^{\varepsilon,E}(A) = \{p \in A_E \text{ s.t. } p = p^* \text{ and } ||p^2 - p|| < \varepsilon\}.$$

We will use the notation $P_n^{\varepsilon,E}(A)$ for $P^{\varepsilon,E}(M_n(A))$, and $U_n^{\varepsilon,E}(A)$ for $U^{\varepsilon,E}(M_n(A))$. Also, $P_\infty^{\varepsilon,E}(A)$ is the algebraic inductive limit of the $P_n^{\varepsilon,E}(A)$ under the natural inclusions

$$\left\{\begin{array}{rcl} P_n^{\varepsilon,E}(A) & \to & P_{n+1}^{\varepsilon,E}(A) \\ p & \mapsto & \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right.$$

and $U_\infty^{\varepsilon,E}(A)$ is the algebraic inductive limit of the $U_n^{\varepsilon,E}(A)$ under the natural inclusions

$$\left\{\begin{array}{rcl} U_n^{\varepsilon,E}(A) & \to & U_{n+1}^{\varepsilon,E}(A) \\ u & \mapsto & \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right. .$$

In order to define controlled K-groups, define the following equivalence relations on $P_\infty^{\varepsilon,E}(A) \times \mathbb{N}$ and $U_n^{\varepsilon,E}(A)$.

- $(p,l) \sim (q,l')$ if there exists a homotopy of almost projections $h \in P_\infty^{\varepsilon,E}(A[0,1])$ and an integer k such that

$$h(0) = \begin{pmatrix} p & 0 \\ 0 & 1_{k+l'} \end{pmatrix} \text{ and } h(1) = \begin{pmatrix} q & 0 \\ 0 & 1_{k+l} \end{pmatrix}$$

- $u \sim v$ if there exists a homotopy of almost unitaries $h \in U_\infty^{3\varepsilon,E\circ E}(A[0,1])$ and an integer k such that $h(0) = u$ and $h(1) = v$.

Denote $[(p,l)]_{\varepsilon,E}$ and $[u]_{\varepsilon,E}$ for the equivalence classes of almost-projections and almost-unitaries. Then, the same proof as [10] shows that $[p,l]_{\varepsilon,E} + [q,l']_{\varepsilon,E} = [diag(p,q), l+l']_{\varepsilon,E}$ and $[u]_{\varepsilon,E} + [v]_{\varepsilon,E} = [diag(u,v)]_{\varepsilon,E}$ induces a group law on the equivalence classes, that we denote $K_0^{\varepsilon,E}(A) = P_\infty^{\varepsilon,E}(A) \times \mathbb{N}/\sim$ and $K_1^{\varepsilon,E}(A) = U_\infty^{\varepsilon,E}(A)/\sim$.

Definition 16. The controlled K-theory of a filtered C^*-algebra $(A,\mathcal{E})$ is the family of abelian groups $\hat{K}_0(A) = (K_0^{\varepsilon,E}(A))_{\varepsilon\in(0,\frac{1}{4}),E\in\mathcal{E}}$ and $\hat{K}_1(A) = (K_1^{\varepsilon,E}(A))_{\varepsilon\in(0,\frac{1}{4}),E\in\mathcal{E}}$ defined above.

We defined canonical inclusion morphisms : if $\varepsilon, \varepsilon' \in (0,\frac{1}{4})$ and $E, E' \in \mathcal{E}$ such that $\varepsilon < \varepsilon'$ and $E \leq E'$, the natural inclusion $K_*^{\varepsilon,E}(A) \hookrightarrow K_*^{\varepsilon',E'}(A)$ is denoted by $\iota_{\varepsilon,E}^{\varepsilon',E'}$. Notive that $\iota_{\varepsilon',E'}^{\varepsilon'',E''} \circ \iota_{\varepsilon,E}^{\varepsilon',E'} = \iota_{\varepsilon,E}^{\varepsilon'',E''}$ when this expression makes sense.

One has also forgetful inclusions $\iota_{\varepsilon,E} : K_*^{\varepsilon,E} \to K_*(A)$, and $\iota_{\varepsilon',E'} \circ \iota_{\varepsilon,E}^{\varepsilon',E'} = \iota_{\varepsilon,E}$ holds.

Example 1. Some examples that we will use :

- Let $(X,\mathcal{E})$ be a coarse space. The set of controlled subsets $\mathcal{E}$ is our proeminent example.
- Let G be an étale groupoid. Then the set of open relatively compact subsets $\mathcal{E}$ of G is a coarse structure. If G is σ-compact, and A is a G-algebra, $A \rtimes_r G$ is naturally filtered by $\mathcal{E}$: if $E \subset G$ is open relatively compact, define $(A \rtimes -rG)_E = \{f \in C_c(G,A) : \text{ supp}(f) \subset E\}$.
- Let $\mathbb{G}$ be a (compact ?) quantum group (in the sense of ?). Then the set of finite dimensional representations of $\mathbb{G}$ is a coarse structure w.r.t. $\pi \circ \pi' = \pi \otimes \pi'$ as composition, and $\pi \leq \pi'$ if π is equivalent to a subrepresentation of $\mathbb{G}$.
- Gomez-Aparicio :Let G be a locally compact group, and S be the set of representations of G of the form $\lambda \otimes \pi$ for π a finite dimensional representation of G.

2.2 Quantitative objects

In order to study functorial properties of controlled K-theory, we will adapt and study the notion of quantitative object defined in [10].

If $\mathcal{E}$ is a coarse structure, define $\tilde{\mathcal{E}} = (0, \frac{1}{4}) \times \mathcal{E}$. It a poset with respect to the partial order $(\varepsilon, E) < (\varepsilon', E')$ if $\varepsilon < \varepsilon'$ and $E < E'$ for $\varepsilon, \varepsilon' \in (0, \frac{1}{4}), E, E' \in \mathcal{E}$. A poset can always be seen as a category with at most one arrow between objets. Let *Grab* be the category of abelian groups with arrows the homomorphisms of groups.

Definition 17. Let $\mathcal{E}$ be a coarse structure. A quantitative object for $\mathcal{E}$ is a functor $\mathcal{O} : \tilde{\mathcal{E}} \to Grab$.

Example 2. Our example will be controlled K-theory $(\varepsilon, E) \in \tilde{\mathcal{E}} \mapsto K^{\varepsilon,E}(A)$; $\tilde{E} \leq \tilde{E}' \mapsto \iota_{\tilde{E}}^{\tilde{E}'}$ for a filtered C^*-algebra $(A, \mathcal{E})$.

Remark 1. For any non-decreasing map $\rho : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ and any quantitative object $\mathcal{O}$, $\mathcal{O} \circ \rho$ is still a quantitative object.

Definition 18. Let $\mathcal{E}_1, \mathcal{E}_2$ be two coarse structures, $\mathcal{O}, \mathcal{O}'$ two quantitative objects for $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, and $\rho : \tilde{\mathcal{E}}_\rho \to \tilde{\mathcal{E}}_2$ a non-decreasing map with domain a subcategory $\tilde{\mathcal{E}}_\rho$ of $\tilde{\mathcal{E}}_1$. A ρ-controlled morphism is a natural transformation $\mathcal{O}_{|\tilde{\mathcal{E}}_\rho} \to \mathcal{O}' \circ \rho$.

Example 3. The following example is taken from [10]. A control pair (α, h) is a positive real $\alpha > 1$ and $h : (0, \frac{1}{4\alpha}) \to [1, \infty)$ a function bounded above by a non-decreasing function. Define $\rho : (0, \frac{1}{4}) \times \mathbb{R}_*^+ \to \tilde{\mathbb{R}_*^+}$ as $\rho(\varepsilon, R) = (\alpha\epsilon, h_\varepsilon R)$, which defines a non-increasing function. Then a (α, h)-control morphism in the sense of [10] is a ρ-controlled morphism in the sense of the definition above. We will adopt the same langage and call ρ the control pair of the controlled morphism.

We can compose two control pairs $\rho : \tilde{\mathcal{E}}_\rho \subset \mathcal{E}_1 \to \mathcal{E}_2$ and $\rho' : \tilde{\mathcal{E}}'_\rho \subset \mathcal{E}_2 \to \mathcal{E}_3$ in the following way. Let $\tilde{\mathcal{E}}_{\rho' \circ \rho} = \tilde{\mathcal{E}}_\rho \cap \rho^{-1}(\tilde{\mathcal{E}}_{\rho'})$, then $\rho' \circ \rho = \rho' \circ \rho_{|\tilde{\mathcal{E}}_{\rho' \circ \rho}}$.

Remark 2. If $(A, \mathcal{E})$ and $(B, \mathcal{E}')$ are filtered C^*-algebras, ρ a control pair and $\hat{F} : \hat{K}(A) \to K(B)$ a ρ-controlled morphism. Then $\hat{F}$ defines uniquely a morphism $F : K(A) \to K(B)$ by $F = \iota_{\varepsilon,E} \circ F^{\varepsilon,E}$, which is well defined because of the compatibility conditions $\iota_{\varepsilon',E'} \circ \iota_{\varepsilon,E}^{\varepsilon',E'} = \iota_{\varepsilon,E}$.

Remark 3. For étale groupoids, we won't need all control pairs as defined earlier, but something very similar to the coarse case. More precisely, a pair (α, k) where $\alpha \geq 1$ and $h : (0, \frac{1}{4\alpha}) \to \mathbb{N}$ is a map bounded above by a non-decreasing map defines what we call a linear control pair by $\rho(\varepsilon, E) = (\alpha\varepsilon, h_\varepsilon.E$, where $n.E = E.E. \ ... \ .E$ n-times. We will use the notation (α, k) instead of ρ, in order make explicit the constant in computations. Note that linear control pairs are stable by composition.

Remark 4. If $(\phi, \rho) : (A, \mathcal{E}) \to (B, \mathcal{E}')$ is a filtered morphism, it obviously induces a linear controlled morphism

$$\phi_* = \left(\phi_*^{\varepsilon,E} : K_*^{\varepsilon,E}(A) \to K_*^{\varepsilon,\rho(E)}(B)\right)_{\varepsilon \in (0,\frac{1}{4}), E \in \mathcal{E}}.$$

Let α, β, γ be control pairs such that α and $\beta \leq \gamma$. If F is a α-controlled morphism and G a β-controlled morphism $\mathcal{O} \to \mathcal{O}'$, then we write $F \sim_\gamma G$ if, for all $\tilde{E} \in \tilde{\mathcal{E}}_\gamma$, the following diagram commutes :.

$$\begin{tikzcd}
\mathcal{O}_{\tilde{E}} \arrow[r, "F^{\tilde{E}}"] \arrow[d, "G_{\tilde{E}}"] & \mathcal{O}'_{\alpha(\tilde{E})} \arrow[d, "\iota^{\gamma(\tilde{E})}_{\alpha(\tilde{E})}"] \\
\mathcal{O}'_{\beta(\tilde{E})} \arrow[r, "\iota^{\gamma(\tilde{E})}_{\beta(\tilde{E})}"] & \mathcal{O}'_{\gamma(\tilde{E})}
\end{tikzcd}.$$

Definition 19. Let γ be a control pair, and $F : \mathcal{O} \to \mathcal{O}'$ a α-controlled morphism such that $\alpha \leq \gamma$. We say that F is γ-invertible if there exists a controlled morphism $G : \mathcal{O}' \to \mathcal{O}$ such that $G \circ F \sim_\gamma Id_{\mathcal{O}}$ and $F \circ G \sim_\gamma Id_{\mathcal{O}'}$. G is called a γ-inverse for F.

Definition 20. Let, for $j \in \{00, 01, 10, 11\}$, $\mathcal{E}_j$ be coarse structures, ρ_{ij} : $\tilde{\mathcal{E}}_{\rho_{ij}}\tilde{\mathcal{E}}_i \to \tilde{\mathcal{E}}_j$ control pairs, $\mathcal{O}_j$ quantitative object w.r.t. $\tilde{\mathcal{E}}_j$ and ρ_{ij}-controlled morphisms $F_{ij} : \mathcal{O}_i \to \mathcal{O}_j$. The following diagram

$$\begin{tikzcd}
\mathcal{O}_{00} \arrow[r, "F_{00}^{01}"] \arrow[d, "F_{00}^{10}"] & \mathcal{O}_{01} \arrow[d, "F_{10}^{11}"] \\
\mathcal{O}_{10} \arrow[r, "F_{10}^{11}"] & \mathcal{O}_{11}
\end{tikzcd}$$

is said to be ρ-commutative if $F_{10}^{11} \circ F_{00}^{10} \sim_\rho F_{01}^{11} \circ F_{00}^{01}$.

We now detail what these definitions become in the case of an étale groupoid G, with coarse structure given by $\mathcal{E} = \{U \subset G, U \text{ open s.t. } \overline{U} \text{ is compact}\}$.

A quantitative object is a family $(\mathcal{O}_{\varepsilon,E})_{\varepsilon \in (0,\frac{1}{4}), E \in \mathcal{E}}$ of abelian groups equipped with a family of homomorphisms $\iota_{\varepsilon,E}^{\varepsilon',E'} : \mathcal{O}_{\varepsilon,E} \to \mathcal{O}_{\varepsilon',E'}$ such that :

- $\iota_{\varepsilon,E}^{\varepsilon,E} = Id_{\mathcal{O}_{\varepsilon,E}}$,
- $\iota_{\varepsilon',E'}^{\varepsilon'',E''} \circ \iota_{\varepsilon,E}^{\varepsilon',E'} = \iota_{\varepsilon,E}^{\varepsilon'',E''}$ for $0 < \varepsilon \leq \varepsilon' \leq \varepsilon'' < \frac{1}{4}$ and $E \subset E' \subset E''$.

Let (α, k) a linear control pair as in remark 3. A (α, k)-controlled morphism is a family $\hat{F}$ of homomorphisms

$$F^{\varepsilon,E} : \mathcal{O}_{\varepsilon,E} \to O_{\alpha\varepsilon, h_\varepsilon E} \quad , \forall \varepsilon \in (0, \frac{1}{4\alpha}), E \in \mathcal{E}$$

such that $\iota_{\alpha\varepsilon, h_\varepsilon . E}^{\alpha\varepsilon', h_{\varepsilon'}.E'} \circ F^{\varepsilon,E} = F^{\varepsilon',E'} \circ \iota_{\varepsilon,E}^{\varepsilon',E'}$ for $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $E \subset E'$.

If we don't specify any control pair, it is implicit and evident from the context. For a controlled morphism $\hat{F} : \hat{K}(A) \to \hat{K}(B)$, we will denote $F : K(A) \to K(B)$ the unique homomorphism it induces in K-theory.

2.3 Controlled exact sequences

Definition 21. Let $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ be coarse structures, and $\mathcal{O}, \mathcal{O}', \mathcal{O}''$ quantitative objects over them respectively, $\rho : \mathcal{E} \to \mathcal{E}'$ and $\rho' : \mathcal{E}' \to \mathcal{E}''$ control pairs. If

$F : \mathcal{O} \to \mathcal{O}'$ is a ρ-controlled morphism and $F : \mathcal{O}' \to \mathcal{O}''$ is a ρ'-controlled morphism, we say that the sequence

$$\mathcal{O} \xrightarrow{F} \mathcal{O}' \xrightarrow{G} \mathcal{O}''$$

is λ-exact at $\mathcal{O}'$ if $G \circ F = 0$ and for every $\tilde{E} \in \tilde{\mathcal{E}}$, if $x \in \mathcal{O}'_{\rho(\tilde{E})}$ satisfies $G_{\rho(\tilde{E})}(x) = 0$, then there exist $\tilde{E}' \in \mathcal{E}$ such that $\tilde{E}' \geq \tilde{E}$ and $y \in \mathcal{O}_{\tilde{E}'}$ such that $F_{\tilde{E}'}(y) = \iota_{\rho(\tilde{E})}^{\rho(\tilde{E}')}(x)$.

A sequence $\mathcal{O}_0 \longrightarrow O_1 \longrightarrow ... \longrightarrow O_k$ is λ-exact if it is λ-exact everywhere.

2.4 Morita equivalence

Proposition 7. Let $(A, \mathcal{E})$ be a filtered C^*-algebra and H a separable Hilbert space. Then the $*$-morphism

$$A \to A \otimes \mathfrak{K}(H); \quad a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & ... \end{pmatrix}$$

induces a group isomorphism

$$\mathcal{M}_A^{\varepsilon,E} : K^{\varepsilon,E}(A) \to K^{\varepsilon,E}(A \otimes \mathfrak{K}(H))$$

for every $\varepsilon \in (0, \frac{1}{4})$ and $E \in \mathcal{E}$. The family $\mathcal{M}_A = (\mathcal{M}_A^{\varepsilon,E})_{\varepsilon \in (0,\frac{1}{4}), E \in \mathcal{E}}$ is called the controlled Morita equivalence and is a controlled morphism, and it induces the usual Morita equivalence $M_A : K(A) \to K(A \otimes \mathfrak{K}(H))$ in K-theory.

2.5 Controlled exact sequences

We will describe the 6-term controlled exact sequence associated to a completely filtered extension of C^*-algebras. For any extension of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 \ ,$$

we denote $\partial_{J,A}$ the boundary map $K_*(A/J) \to K_{*+1}(J)$. We are bargaining material from [10], where all these properties are defined and proved.

Definition 22. Let $(A, \mathcal{E})$ a filtered C^*-algebra and $J \subset A$ an ideal. If $J_E = A_E \cap J$, the extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is said to be completely filtered if the continuous linear bijection $A_E/J_E \hookrightarrow (A_E + J)/J$ induced by the inclusion $A_E \hookrightarrow A$ is a complete isometry, i.e.

$$\inf_{y \in M_n(J_E)} ||x + y|| = \inf_{y \in M_n(J)} ||x + y|| \quad , \forall n \in \mathbb{N}, x \in M_n(A_E), E \in \mathcal{E}.$$

Proposition 8. There exists a linear control pair (α_D, k_D) such that for any completely filtered extension of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

there exists a (α_D, k_D)-controlled morphism

$$D_{J,A} : \hat{K}(A/J) \to \hat{K}(J)$$

which induces $\partial_{J,A}$ in K-theory.

Theorem 1. There exists a linear control pair (λ, h) such that for any completely filtered extension of C^*-algebras

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{q} A/J \longrightarrow 0$$

the following 6-term exact sequence is (λ, h)-exact

$$\begin{tikzcd} \hat{K}(J) \arrow[r, "\iota_*"] & \hat{K}(A) \arrow[r, "q_*"] & \hat{K}(A/J) \arrow[d, "D_{J,A}"] \\ \hat{K}(A/J) \arrow[u, "D_{J,A}"] & \hat{K}(A) \arrow[l, "q_*"] & \hat{K}(J) \arrow[l, "\iota_*"] \end{tikzcd} .$$

2.6 Tensorisation in KK-theory

If $(B, \mathcal{E})$ is a filtered C^*-algebra and A any C^*-algebra, and if $A \otimes B$ is the spatial tensor product, then $(A \otimes B_E)_{E \in \mathcal{E}}$ defines a filtration of $A \otimes B$ over $\mathcal{E}$. If $\phi : A_1 \to A_2$ is a $*$-homomorphism, we use the notation ϕ_B for the induced $*$-homomorphism $A_1 \otimes B \to A_2 \otimes B$.

In [7], G. Kasparov defined a map

$$\tau_B : KK(A_1, A_2) \to KK(A_1 \otimes B, A_2 \otimes B)$$

for any C^*-algebras A_1 and A_2, which is compatible with the Kasparov product. Any $z \in KK(A_1, A_2)$ defines a morphism

$$K(A_1 \otimes B) \to K(A_2 \otimes B)$$

which is proved in [10] to be induced from a controlled morphism. The following theorem is borrowed from [10].

Theorem 2. Let $\mathcal{E}$ be a coarse strucure. There exists a linear control pair (α_τ, k_τ) such that, for any filtered C^*-algebra $(B, \mathcal{E})$, any C^*-algebras A_1 and A_2 and any K-cycle $z \in KK(A_1, A_2)$, there exists a (α_τ, k_τ)-controlled morphism $\hat{\tau}_B : \hat{K}(A_1 \otimes B) \to \hat{K}(A_2 \otimes B)$ such that :

- $\hat{\tau}_B(z)$ induces right-multiplication by $\tau_B(z)$ in K-theory,
- for any K-cycles $z, z' \in KK(A_1, A_2)$, $\hat{\tau}_B(z + z') = \hat{\tau}_B(z) + \hat{\tau}_B(z')$,
- if $\phi : A_1 \to A_1'$ is a $*$-homomorphism, then $\hat{\tau}_B(\phi^*(z)) = \hat{\tau}_B(z) \circ (\phi_B)_*$ for any $z \in KK(A_1', A_2)$,

- if $\phi : A_2' \to A_2$ is a $*$-homomorphism, then $\hat{\tau}_B(\phi_*(z)) = (\phi_B)_* \circ \hat{\tau}_B(z)$ for any $z \in KK(A_1, A_2')$,
- $\hat{\tau}_B([Id_A]) \sim_{(\alpha_\tau, k_\tau)} Id_{\hat{K}(A \otimes B)}$,
- for any C^*-algebra D, any K-cycle $z \in KK(A_1, A_2)$, $\hat{\tau}_B(\tau_D(z)) = \hat{\tau}_{B\otimes D}(z)$.
- for any semi-split extension $0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$ with boundary element $[\partial_{J,A}] \in KK_1(A/J, J)$, $\hat{\tau}_B([\partial_{J,A}]) = D_{J\otimes B, A\otimes B}$.

This controlled tensorisation map respects Kasparov product. See [10].

Theorem 3. There exists a linear control pair λ such that, for any separable C^*-algebras A_1 and A_2, any filtered C^*-algebra $(B, \mathcal{E})$, the following holds : for any $z \in KK(A_1, A_2)$ and $z' \in KK(A_2, A_3)$,

$$\hat{\tau}_B(z \otimes z') \sim_\lambda \hat{\tau}_B(z') \circ \hat{\tau}_B(z)$$

At least, the controlled tensorisation map is natural with respect to morphism of filtered C^*-algebras. [10]

Proposition 9. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ two coarse structures, $(B_1, \mathcal{E}_1)$ and $(B_2, \mathcal{E}_2)$ filtered C^*-algebras and $(\phi, \rho) : B_1 \to B_2$ a filtered morphism. Then $(\phi_{A_2})_* \circ \hat{\tau}_{B_1} = \hat{\tau}_{B_2}(z) \circ (\phi_{A_1})_*$ for any $z \in KK(A_1, A_2)$.

3 Assembly Maps

In this section, all groupoids $G \rightrightarrows G^{(0)}$ are étale.

3.1 Proper groupoids and proper actions

Definition 23. A topological groupoid $G \rightrightarrows G^{(0)}$ is proper if the map $G \times_s G^{(0)} \to G^{(0)} \times G^{(0)}$ defined by $(g, z) \mapsto (z, gz)$ is a proper map.
A G-space Z is said to be proper if the crossed-product groupoid $Z \rtimes G$ is proper. If the space of orbits Z/G is compact, Z is called G-compact.

Definition 24. A G-space $\underline{E}G$ is called universal if, given any proper G-compact G-space Z, there exists a continuous G-equivariant map $Z \to \underline{E}G$.

We now construct a model for universal spaces of G, based on J.-L.Tu's work [17]. For any compact subset $K \subset G$, define $P_K(G)$ to be the space of probability measures ν with support contained in one and only one fiber G^x for some $x \in G^{(0)}$, and such that if $g, g' \in \text{supp}\ (\nu)$, then $g'g^{-1} \in K$. We endow $P_K(G)$ with the weak-$*$ topology.

The action of G is defined by translation. The momentum map $P_K(G) \to G^{(0)}$ is just the map associating to ν the only x such that supp $(\nu) \subset G^x$. As the fibers are discrete, any $\nu \in P_K(G)$ can be represented as as sum $\nu = \sum_{g \in G^x} \lambda_g(\nu)\delta_g$, where δ_g is the Dirac measure at $g \in G$. The continuous functions λ_g are called coordinate functions and satisfy $\sum_{g \in G^x} \lambda_g(\nu) = 1$ for every $x \in G^{(0)}$. The action of G is given by $g\lambda_h = \lambda_{g^{-1}h}$.

Lemma 2 (Tu,[17])**.** The action of G on $P_K(G)$ is proper and cocompact.

Lemma 3 (Tu,[17])**.** Let Z be a proper G-compact G-space. Then there exists a compact subset $K \subset G$ and a G-equivariant continuous map $Z \to P_K(G)$.

3.2 Equivariant K-homology

We will use the equivariant KK-theory developped by Le Gall in his thesis [9], which is an extension of the usual equivariant KK-theory of Kasparov. Recall that, if A and B are two G-algebras, elements of $KK^G(A, B)$ are homotopy classes of triple (E, π, T) where :

- E is a G-module,
- $\pi : A \to \mathcal{L}_B(E)$ is a $*$-homomorphism,
- $T \in \mathcal{L}_B(E)$ is an adjoinable operator such that the triple satisfies the condition of K-cycle : $\pi(a)(T^2 - T), \pi(a)(T^* - T)[\pi(a), T]$, and $\pi(a)(T - g.T)$ are compact operators if E, for all $a \in A, g \in G$.

Definition 25. Let Y be a proper G-space and B a G-algebra. Then the analytic K-homology of Y with coefficients in B is defined by

$$RK^G(Y, B) = \varinjlim_{Z \subset Y} KK^G(C_0(Z), B),$$

the inductive limit being taken on proper G-compact G-subspaces Z of Y.

The previous lemmas assures that this inductive limit can be somehow restricted when it comes to the analytic $>K$-homology of any universal space $\underline{E}G$:

$$RK^G(\underline{E}G, B) = \varinjlim_{K \subset G \text{ compact}} KK^G(C_0(P_K(G)), B).$$

3.3 Descent functor

There exists a natural transformation $KK^G(A, B) \to KK(A \rtimes_r G, B \rtimes_r G)$ respecting the Kasparov product, which is called the descent functor. The same statement remains true for maximal crossed products. We recall its construction, which was first stated in [9].

Let A and B be two G-algebras and $(E, \pi, T) \in E^G(A, B)$ be a K-cycle. Define :

- E_G to be the completion of $C_c(G, E)$ with respect to the norm $||f|| = \sup_{x \in G^{(0)}} \sum_{G^x} |f(g)|^2$,
- π_G to be the image of $\pi : A \to \mathcal{L}_B(E)$ under the reduced crossed-product functor,
- T_G to be the image of $T \in \mathcal{L}(E)$ under the reduced crossed-product functor.

Then $(E_G, \pi_G, T_G) \in E(A \rtimes_r G, B \rtimes_r G)$ and the map $(E, \pi, T) \mapsto (E_G, \pi_G, T_G)$ induces a homorphism of abelian groups

$$j_G : KK^G(A, B) \to KK(A \rtimes_r G, B \rtimes_r G)$$

satisfying $j_G(z \otimes_D z') = j_G(z) \otimes_{D \rtimes_r G} j_G(z')$ for any $z \in KK^G(A, D), z' \in KK^G(D, B)$.

3.4 The assembly map

If Z is a proper G-compact G-space, $Z \rtimes G$ is a proper groupoid by definition, hence there exists a cutoff function $c : Z \to [0, 1]$ such that $\int_G^{p(z)} c(zg) d\lambda^x(g) = 1$. The function $g \mapsto c(r(g))^{\frac{1}{2}} c(s(g))^{\frac{1}{2}}$ defines a projection in $C_0(Z) \rtimes_r G$ which we denote by $\mathcal{L}_Z$. If $Z = P_K(G)$, then $\mathcal{L}_Z = \mathcal{L}_K$.

Definition 26. The assembly map for G with coefficients in B is defined as the inductive limit of the maps $\mu^{(Z)}_{G,B} : KK^G(C_0(Z), B) \to K(B \rtimes_r B)$ given by

$$\mu^{(Z)}_{G,B}(z) = [\mathcal{L}_Z] \otimes_{C_0(Z) \rtimes G} j_G(z),$$

that is $\mu_{G,B} = \varinjlim \mu^{(Z)}_{G,B}$ (one has to check that theses maps respects the inductive systems, which they do).

Another definition : with Ad...

3.5 The Baum-Connes conjecture

Conjecture 1 (Baum-Connes conjecture)**.** Let G be an étale groupoid, and A a G-algebra.
The Baum-Connes conjecture for G with coefficients in A is the following claim : $\mu_{G,A}$ is an isomorphism of abelian groups.
The Baum-Connes conjecture with coefficients is : for all G-algebras A, $\mu_{G,A}$ is an isomorphism of abelian groups.

The conjecture was first stated in [1] and in [2]. The statement is a descendent of the Connes-Kasparov conjecture, which is simply, when one knows the Baum-Connes conjecture, the latter for almost connected groups (i.e. locally compact groups G such that G/G_0 is compact, where G_0 is the connected component of the identity).

Here is the status of the conjecture :

- The Connes-Kasparov conjecture was established by J. Chabert, S. Echterhoff and R. Nest, who proved that μ_G is an isomorphism for every secound countable almost connected group G, and every group of k-rational points of a linear algebraic group over a local field of characteristic 0.

- N. Higson and G. Kasparov proved that the conjecture with coefficients holds for groups having Haagerup property.
- V. Lafforgue proved the conjecture with coefficients for hyperbolic groups.
- V. Lafforgue showed the conjecture for every semi-simple Lie group, real or p-adic reductive, and discrete cocompact subgroup of real rank 1 Lie groups or of $SL(3,k)$ for any local field k.

For nice topological groupoids, the conjecture is known to be false in full generality [6], but the following result was shown by J-L. Tu in [14]

Theorem 4. Let G be a locally compact σ-compact Hausdorff groupoid. Then if G is a-T-menable, the Baum-Connes conjecture with coefficients holds for G.

3.6 The Coarse Assembly map

The construction of the Coarse Assembly map is very similar to the Assembly map for groupoids, but relies on different functors.

In [13] is defined the Roe algebra $C^*(X,B)$ of a discrete metric space with bounded geometry X with coefficients in an arbitrary C^*-algebra B. For $R>0$, define $C_R[X,B]$ to be the involutive subspace of operators $T \in \mathcal{L}_B(l^2(X) \otimes H \otimes B)$ such that $T_{xy} \in \mathfrak{K}(H \otimes B)$ and $T_{xy}=0$ as soon as $d(x,y)>R$, and $\mathbb{C}[X,B] = \cup_{R>0} C_R[X,B]$.

Definition 27. The Roe algebra $C^*(X,B)$ is defined as the Hibert B-module obtained after the completion of $\mathbb{C}[X,B]$ with respect to the operator norm in $\mathcal{L}_B(l^2(X) \otimes H \otimes B)$.

This construction is functorial : any $*$-homomorphism $\phi : A \to B$ gives rise to a $*$-homomorphism $\phi_X : C^*(X,A) \to C^*(X,B)$. Moreover this functoriality extends to KK-theory : there exists a natural transformation

$$\sigma_X : KK(A, B) \to KK(C^*(X, A), C^*(X, B))$$

which respects the Kasparov product, i.e. $\sigma(z \otimes_D z') = \sigma_X(z) \otimes_{C^*(X,D)} \sigma_X(z') \quad, \forall z \in KK(A, D), z' \in KK(D, B)$.

Let X be a coarse space with bounded geometry. The Rips complex is the inductive system of finite dimensional simplicial complexes

$$P_E(X) = \{\eta \in \mathrm{Prob}(X) \text{ s.t. supp } \eta \subset E\} \quad, \forall E \in \mathcal{E}_X,$$

endowed with the $*$-weak topology. It is an inductive system with respect to inclusion of entourages.

The coarse homology group of X with coefficients in an arbitrary C^*-algebra B is defined as

$$KX_*(X, B) = \varinjlim_{E \in \mathcal{E}_X} RK(P_E(X), B)$$

where the limit is taken along the inductive system $\{P_E(X)\}_{E \in \mathcal{E}_X}$.

Let $E \in \mathcal{E}_X$. Any simplex $\eta \in P_E(X)$ can be written as a finite sum $\eta = \sum_x \lambda_x(\eta)\delta_x$ by boundedness of the geometry. Here, δ_x is the Dirac measure at $x \in X$. The functions $\lambda_x : \eta \mapsto \lambda_x(\eta)$ are continuous and satisfisfy $\sum \lambda_x(\eta) = 1, \forall \eta \in P_E(X)$, so that

$$V_0 \begin{cases} C_0(P_E(X)) & \to & l^2(X) \otimes C_0(P_E(X)) \\ f & \mapsto & (\lambda_x^{\frac{1}{2}} f)_{x \in X} \end{cases}$$

is an isometry of $C_0(P_E(X))$-modules, and $V_0 V_0^* \otimes id_H$ defines a projection in $l^2(X) \otimes H \otimes C_0(P_E(X))$ with finite propagation, so class $[\mathcal{L}_E] \in K_0(C^*(X, C_0(P_E(X)))$.

Definition 28. The Coarse Assembly map $\mu : KX_*(X, B) \to K_*(C^*(X, B))$ for X with coefficients in B is defined as

$$\forall z \in RK(P_E(X), B), \mu_{X,B}(z) = [\mathcal{L}_E] \otimes_{C^*(X, C_0(P_E(X)))} \sigma_X(z).$$

The Coarse Baum-Connes conjecture is the following claim.

Conjecture 2 (Coarse Baum-Connes conjecture)**.** For any coarse space X with bounded geometry, the Coarse Assembly map $\mu_{X,\mathbb{C}}$ is an isomorphism.

Here are some remarks about the status of the conjecture :

- the conjecture is known to hold for any coarse space that admits a coarse embedding into a separable Hilbert space[20].
- couterexamples have been constructed [6],
- Let Γ be a finitely generated group, and $|\Gamma|$ the coarse space associated to the word-length metric (any two left invariant metric on Γ are coarsely equivalent). Then the Coarse Baum-Connes conjecture for $|\Gamma|$ implies the Novikov conjecture for higher signatures for Γ.

Here is another description of the Coarse Assembly map which will be of some interest for us.

Let $E \in \mathcal{E}_X$ be an entourage. Take a cycle $(H, \pi, T) \in E(C_0(P_E(X)), \mathbb{C})$, and equip the finite dimensional simplicial complex $P_E(X)$ with a metric which restricts to the spherical metric on simplices, so that H is a $P_E(X)$-module that we can suppose standard non-degenrate. Then

$$T' = \sum \lambda_x^{\frac{1}{2}} T \lambda_x^{\frac{1}{2}}$$

is a compact perturbation of T in $\mathcal{L}(H)$ with bounded propagation, and is invertible modulo $C^*(P_E(X), H)$, so defines a class $[T'] \in K_0(C^*(P_E(X), H))$. But $P_E(X)$ and X are coarsely equivalent, the barycentric map $P_E(X) \to X$ is a coarse equivalence which is covered by V_0, so that

$$(Ad_{V_0})_* : K_*(C^*(P_E(X), H)) \to K_*(C^*(X, l^2(X) \otimes H))$$

is an isomorphism, which does only depend on the coarse class of the barycentric map.

With that in mind, one can show that $\mu_X([H, \pi, T]) = (Ad_{V_0})_*[T']$.

The point of the article [13] is to show that the Coarse Assembly map is equivalent to the Assembly map for the coarse groupoid $G = G(X)$ with coefficients in the G-algebra $l^\infty = l^\infty(X, \mathfrak{K})$.

More precisely, $Z = P_{\overline{E}}(G)$ is a proper cocompact G-space with ismomorphic fibers $Z_x \simeq P_E(X)$, and $l^\infty_x \simeq \mathfrak{K}$, so that the inclusion of groupoid $\{x\} \hookrightarrow G$ induces a morphism $\iota_Z : RK^G(Z, l^\infty) \to KK(Z_x, \mathfrak{K})$ which is actually an isomorphism. Recall that one can construct an $*$-isomorphism $\Psi_X : l^\infty \rtimes_r G \to C^*(X)$, and $\mu_X^{(Z_x)} \circ \iota_Z = (\Psi_X)_* \circ \mu_{G,l^\infty}^{(Z)}$ holds and respects inductive limits over Z to give the following commutative diagram :

$$\begin{tikzcd}
K_*^{top}(\underline{E}G, l^\infty) \arrow[r, "\mu_{G,l^\infty}"] \arrow[d, "\varinjlim \iota_Z"] & K_*(l^\infty \rtimes_r G) \arrow[d, "(\Psi_X)_*"] \\
KX_*(X) \arrow[r, "\mu_X"] & K_*(C^*(X))
\end{tikzcd}$$

with vertical arrows being isomorphism.

A key point is to use how analytical properties of G translate coarse properties of X. In particular,
- in [13] is proved that X admits a coarse embedding into Hilbert space iff $G(X)$ is a-T-menable,
- in [13] is proved that X has property A iff $G(X)$ is amenable,
- in [3], M. Finn-Sell shows that if X admits a coarse fibered embedding, then $G_\partial = G(X)_{|\partial\beta X}$ is a-T-menable.

All of these results, combined with theorem 4, give examples of coarse spaces satisfying the Coarse Baum-Connes conjecture and the Boundary Coarse Baum-Connes conjecture.

4 Controlled assembly maps for coarse spaces

We will first construct a controlled assembly map for coarse space $(X, \mathcal{E})$. In this section, X will be a discrete metric space with bounded geometry, and $\mathcal{E}$ is the coarse structure generated by its controlled subsets. We also fix a separable Hilbert space H. For $R > 0$, Δ_R is $\{(x, y) \in X \times X \text{ s.t. } d(x, y) < R\}$.

Recall first the construction of the Roe algebra of X with coefficients in a C^*-algebra B, which can be found in [13]. H_B denotes the standard B-Hilbert module $H \otimes B$.

For any positive number $R > 0$, define the family of linear subspaces

$C_R[X, B] = \{T \in \mathcal{L}(l^2(X) \otimes H_B) \text{ s.t. } T_{xy} \in \mathfrak{K}(H_B) \text{ and } T_{xy} = 0 \text{ for } (x, y) \notin \Delta_R\}$

and $C^*(X, B)$ is the completion of $\cup_{R>0} C_R[X, B]$ for the operator norm in $\mathcal{L}(l^2(X) \otimes H_B)$. Remark that the C^*-algebra $C^*(X, B)$ is filtered by $\mathcal{E}$, and also by $\mathbb{R}_+^*$, seen as a coarse structure.

For $\phi : A \to B$, we use the notation ϕ_X for the induced $*$-homomorphism $C^*(X, A) \to C^*(X, B)$.

4.1 Controlled descent functor

Every K-cycle $z \in KK(A, B)$ can be represented as a triplet (H_B, π, T) where :

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B.
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator.
- T and π satisfy the K-cycle condition, i.e. $[T, \pi(a)]$, and $\pi(a)(T^2 - id_{H_B})$ are compact operators over H_B for all $a \in A$.

4.1.1 Odd case

For $z \in KK_1(A, B)$, represented by $(H_B, \pi, T) \in E(A, B)$, define $P = (\frac{1+T}{2}) \in \mathcal{L}(H_B)$ and $P_X \in \mathcal{L}(H_{C^*(X,B)})$, and

$$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in C^*(X, A), y \in C^*(X, B) \otimes \mathfrak{K}\}$$

which is a C^*-algebra which make the following sequence exact :

$$0 \longrightarrow C^*(X, B) \otimes \mathfrak{K} \longrightarrow E^{(\pi,T)} \longrightarrow C^*(X, A) \longrightarrow 0 \ .$$

Let us show that the controlled boundary map $D^{(\pi,T)} = D_{C^*(X,B)\otimes\mathfrak{K}, E^{(\pi,T)}}$ only depends on the class z.
Let $(H_B, \pi_j, T_j), j = 0, 1$ two K-cycles which are homotopic via $(H_{B[0,1]}, \pi, T)$. We denote e_t the evaluation at $t \in [0, 1]$ for an element of $B[0, 1]$, and set $y_t = e_t(y)$ for such a y. The $*$-morphism

$$\phi : \begin{cases} E^{(\pi,T)} & \to \quad E^{(\pi_t,T_t)} \\ (x, y) & \mapsto \quad (x, y_t) \end{cases}$$

satisfies $\phi(C^*(X,K_{B[0,1]}))\subset K_{C^*(X,B)}$ and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(X,K_{B[0,1]}) & \longrightarrow & E^{(\pi,T)} & \longrightarrow & C^*(X,A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_{|C^*(X,K_{B[0,1]})}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & C^*(X,B)\otimes\mathfrak{K} & \longrightarrow & E^{(\pi_t,T_t)} & \longrightarrow & C^*(X,A) & \longrightarrow & 0 \end{array}.$$

According to [10], remark 3.7., the following holds

$$D_{C^*(X,B)\otimes\mathfrak{K},E^{(\pi_t,T_t)}}=\phi_*\circ D_{C^*(X,B[0,1])\otimes\mathfrak{K},E^{(\pi,T)}}.$$

As $id\otimes e_t$ gives a homotopy between $id\otimes e_0$ and $id\otimes e_1$, and as if two $*$-morphisms are homotopic, then they are equal in controlled K-theory,

$$D_{C^*(X,B)\otimes\mathfrak{K},E^{(\pi_0,T_0)}}=D_{C^*(X,B)\otimes\mathfrak{K},E^{(\pi_1,T_1)}}$$

holds, and the boundary of the extension $E^{(\pi,T)}$ depends only on z.

Definition 29. We define the Roe transformation $\hat{\sigma}_X$ as

$$\hat{\sigma}_X(z)=\mathcal{M}^{-1}_{C^*(X,B)\otimes\mathfrak{K}}\circ D_{C^*(X,B)\otimes\mathfrak{K},E^{(\pi,T)}}\quad,\forall z\in KK_1(A,B)$$

which is a (α_D,k_D)-controlled morphism $\hat{K}(C^*(X,A))\to\hat{K}(C^*(X,B))$, because the Morita equivalence preserves the filtration.

Proposition 10. Let A and B two C^*-algebras. There exists a control pair (α_X,k_X) such that for every $z\in KK_1(A,B)$, there exists a (α_X,k_X)-controlled morphism

$$\hat{\sigma}_X(z):\hat{K}_*(C^*(X,A))\to\hat{K}_{*+1}(C^*(X,B))$$

such that

(i) $\hat{\sigma}_X(z)$ induces right multiplication by $\sigma_X(z)$ in K-theory;

(ii) $\hat{\sigma}_X$ is additive, i.e.

$$\hat{\sigma}_X(z+z')=\hat{\sigma}_X(z)+\hat{\sigma}_X(z').$$

(iii) For every $*$-homomorphism $f:A_1\to A_2$,

$$\hat{\sigma}_X(f^*(z))=\hat{\sigma}_X(z)\circ f_{X,*}$$

for all $z\in KK_1(A_2,B)$.

(iv) For every $*$-homomorphism $g:B_1\to B_2$,

$$\hat{\sigma}_X(g_*(z))=g_{X,*}\circ\hat{\sigma}_X(z)$$

for all $z\in KK_1(A,B_1)$.

(v) Let $0\to J\to A\to A/J\to 0$ be a semi-split extension of C^*-algebras and $[\partial_J]\in KK_1(A/J,J)$ be its boundary element. Then

$$\hat{\sigma}_X([\partial_J])=D_{C^*(X,J),C^*(X,A)}.$$

Proof 2. (i) The K-cycle $[\partial_{K_{C^*(X,B)},E^{(\pi,T)}}] \in KK_1(C^*(X,A),C^*(X,B))$ implementing the boundary of the extension $E^{(\pi,T)}$ induces the map σ_X by definition, and modulo Morita equivalence, which immediately gives the first point.

(ii) If z, z' are elements of $KK_1(A,B)$, represented by two K-cycles (H_B,π_j,T_j), and if (H_B,π,T) is a K-cycle representing the sum $z+z'$, then $E^{(\pi,T)}$ is naturally isomorphic to the extension sum of the $E_j := E^{(\pi_j,T_j)}$, namely

$$0 \longrightarrow K_{C^*(X,B)} \longrightarrow D \longrightarrow C^*(X,A) \longrightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j,E_i) \right\}.$$

Naturality of the controlled boundary maps [10] ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result.

(iii) Let $z \in KK_1(A_2,B)$, represented by a cycle (H_B,π,T). Representing $f^*(z)$ is $(H_B,f^*\pi,T)$ with off course $f^*\pi = \pi\circ f$. The map

$$\phi : \begin{cases} E^{f^*(\pi,T)} & \to \quad E^{(\pi,T)} \\ (x, P_X(f^*\pi)(x)P_X + y) & \to \quad (f_X(x), P_X(f^*\pi)(x)P_X + y) \end{cases}$$

satisfies

- $\phi(K_{C^*(X,B)}) \subset K_{C^*(X,B)}$, and makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & K_{C^*(X,B)} \arrow[r] \arrow[d, "="] & E^{f^*(\pi,T)} \arrow[r] \arrow[d, "\phi"] & C^*(X,A_1) \arrow[r] \arrow[d, "f_X"] & 0 \\ 0 \arrow[r] & K_{C^*(X,B)} \arrow[r] & E^{(\pi,T)} \arrow[r] & C^*(X,A_2) \arrow[r] & 0 \end{tikzcd}.$$

- It intertwines the sections of the two extensions.

Remark 3.7 of [10] assures that

$$D_{K_{C^*(X,B)},E^{f^*(\pi,T)}} = D_{K_{C^*(X,B)},E^{(\pi,T)}} \circ f_{X,*}$$

, and the claim is clear from composition by $\mathcal{M}^{-1}_{C^*(X,B)}$.

(iv) Let $\mathcal{E} = H_{B_1} \otimes_g B_2$, which is a countably generated Hilbert B_2-module. The homomorphism $g : B_1 \to B_2$ gives rise to $g_* : \mathcal{L}_{B_1}(H_{B_1}) \to \mathcal{L}_{B_2}(\mathcal{E})$, which preserves compact operators : $g_*(K_{B_1}) \subset K(\mathcal{E})$. We have a similar statement for $g_X : C^*(X,B_1) \to C^*(X,B_2)$. We denote $\mathcal{E}_X$ the Hilbert $C^*(X,B_2)$-module $\mathcal{C}^*(X,E) \simeq H_{C^*(X,B_1)} \otimes_g (C^*(X,B_2))$.

Let $z \in KK(A,B_1)$ be represented by the K-cycle (H_{B_1},π,T). Then $(H_{B_1} \otimes_g B_2, g_*\circ\pi, g_*(T)) = (\mathcal{E},\tilde{\pi},\tilde{T})$ represents $g_*(z)$.

The map $(x,y) \mapsto (x,(g_X)_*(y))$ induces $\Psi : E^{(\pi,T)} \to E^{g_*(\pi,T)}$ such that

$$\Psi(x, P_X\pi_X(x)P_X + y) \mapsto (x, \tilde{P}_X\tilde{\pi}_X(x)\tilde{P}_X + (g_X)_*(y)).$$

Indeed, the functor $A \mapsto C^*(X, A)$ commutes with pull-back by $*$-homomorphisms, and $(g_X) * \circ \pi_X = (g_* \circ \pi)_X = \tilde{\pi}_X$ and $(g_X)_*(P_X) = g_*(P)_X = \tilde{P}_X$ so that

$$(g_X)_*(P_X \pi_X(x) P_X) = \tilde{P}_X \tilde{\pi}_X(x) \tilde{P}_X.$$

Now, by the stabilisation lemma, we know that the countably generated Hilbert module $\mathcal{C}^(X, E)$ sits as a complemented module of $H_{C^*(X,B_2)}$, and there exists a projection $p \in L(H_{C^*(X,B_2)})$ such that $pH_{C^*(X,B_2)} \simeq \mathcal{E}_X$ and $pK_{C^*(X,B_2)}p \simeq K(\mathcal{E}_X)$. Let ψ be the composition $K_{C^*(X,B_1)} \to_{(g_X)_*} K(\mathcal{E}_X) \to K_{C^*(X,B_2)}$. In this particular case, we can give an explicit description of ψ. The map defined on basic tensor products $(x_j)_j \otimes b \mapsto (g(x_j)b)_j$ extends to an isometric embedding $\mathcal{E}_X \to H_{C^*(X,B_2)}$, under which $b\theta_{e_i,e_j}$ is mapped to $g(b)\theta_{u_i,u_j}$, where $\{e_j\}$ and $\{u_j\}$ are respectively the canonical orthogonal basis of $H_{C^*(X,B_1)}$ and $H_{C^*(X,B_2)}$. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{C^*(X,B_1)} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & C^*(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_{C^*(X,B_2)} & \longrightarrow & E^{g_*(\pi,T)} & \longrightarrow & C^*(X,A) & \longrightarrow & 0 \end{array}.$$

and Ψ intertwines the two filtered sections by the previous relation. Moreover, $\Psi_{|K_{C^*(X,B_1)}} \subset K_{C^*(X,B_2)}$, so that we can again apply the remark 3.7 of [10] to state

$$D_{K_{C^*(X,B_2)},E^{g_*(\pi,T)}} = \psi_* \circ D_{K_{C^*(X,B_1)},E^{(\pi,T)}},$$

which we compose by the Morita equivalence on the left $M^{-1}_{C^*(X,B_2)}$

$$\hat{\sigma}_X(g_*(z)) = M^{-1}_{C^*(X,B_2)} \circ g_{X,*} \circ D_{K_{C^*(X,B_1)},E^{(\pi,T)}}.$$

The homomorphisms inducing the Morita equivalence make the following diagram commutes,

$$\begin{array}{ccc} C^*(X,B_1) & \xrightarrow{g_X} & C^*(X,B_2) \\ \downarrow & & \downarrow \\ K_{C^*(X,B_1)} & \xrightarrow{\psi} & K_{C^*(X,B_2)} \end{array},$$

and $\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ M^{-1}_{C^*(X,B_1)} \circ D_{K_{C^*(X,B_1)},E^{(\pi,T)}} = g_{X,*} \circ \hat{\sigma}(z)$.

(v) Let $q : A \to A/J$ be the quotient map and (H_J, π, T) be a cycle representing $[\partial_J]$. Then we apply remark 3.7 of [10] to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(X,J) & \longrightarrow & C^*(X,A) & \longrightarrow & C^*(X,A/J) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle s\circ q_X} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_{C^*(X,J)} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & C^*(X,A/J) & \longrightarrow & 0 \end{array},$$

where the first vertical arrow is the canonical mapping that induces the Morita equivalence.

□

4.1.2 Even case

We can now define $\hat{\sigma}_X$ for even K-cycles. Let A and B be two C^*-algebras. Let $[\partial_{SB}] \in KK_1(B, SB)$ be the K-cycle implementing the boundary of the extension $0 \to SB \to CB \to B \to 0$, and $[\partial] \in KK_1(\mathbb{C}, S)$ be the Bott generator. As $z \otimes_B [\partial_{SB}]$ is an odd K-cycle, we can define

$$\hat{\sigma}_X(z) := \tau_{C^*(X,B)}([\partial]^{-1}) \circ \hat{\sigma}_X(z \otimes [\partial_{SB}]).$$

Here τ_D refers to the (α_τ, k_τ)-controlled map $\hat{K}(A_1 \otimes D) \to \hat{K}(A_2 \otimes D)$, that H. Oyono-Oyono and G. Yu constructed in [10] for any C^*-algebras D, A_1, A_2 and $z \in KK_*(A_1, A_2)$. It enjoys many natural properties, and induces right multiplication by $\tau_D(z) \in KK(A_1 \otimes D, A_2 \otimes D)$ in K-theory. We can see that, if we set $\alpha_J = \alpha_\tau \alpha_D$ and $k_J = k_\tau * k_D$, $\hat{\sigma}(z)$ is (α_X, k_X)-controlled.

Proposition 11. Let A and B two C^*-algebras. For every $z \in KK_*(A, B)$, there exists a control pair (α_X, k_X) and a (α_X, k_X)-controlled morphism

$$\hat{\sigma}_X(z) : \hat{K}(C^*(X, A)) \to \hat{K}(C^*(X, B))$$

of the same degree as z, such that

(i) $\hat{\sigma}_X(z)$ induces right multiplication by $\sigma_X(z)$ in K-theory;

(ii) $\hat{\sigma}_X$ is additive, i.e.

$$\hat{\sigma}_X(z + z') = \hat{\sigma}_X(z) + \hat{\sigma}_X(z').$$

(iii) For every $*$-homomorphism $f : A_1 \to A_2$,

$$\hat{\sigma}_X(f^*(z)) = \hat{\sigma}_X(z) \circ f_{X,*}$$

for all $z \in KK_*(A_2, B)$.

(iv) For every $*$-homomorphism $g : B_1 \to B_2$,

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ \hat{\sigma}_X(z)$$

for all $z \in KK_*(A, B_1)$.

(v) $\hat{\sigma}_X([id_A]) \sim_{(\alpha_X, k_X)} id_{\hat{K}(C^*(X,A))}$

Proof 3. The point (iii) is a consequence of the previous proposition 19, and of the equality $f^*(x) \otimes y = f^*(x \otimes y)$.

□

We now show that the Roe transform respects in a quantitative way the Kasparov product.

Proposition 12. There exists a control pair (α_R, k_R) such that for every C^*-algebras A, B and C, and every $z \in KK(A, B), z' \in KK(B, C)$, the controlled equality

$$\hat{\sigma}_X(z \otimes_B z') \sim_{\alpha_R, k_R} \hat{\sigma}_X(z') \circ \hat{\sigma}_X(z)$$

holds.

5 Controlled assembly maps for étale groupoids

5.1 Kasparov transform

Let A and B be two $\mathcal{G}$-C^*-algebras, and H a separable Hilbert space, $l^2(\mathbb{Z})$ for instance, and $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. The standard Hilbert module over B is denoted by $H_B = H_{\mathcal{G}} \otimes B$, and K_B is the algebra of compact operators for H_B, i.e. $K(H) \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Every K-cycle $z \in KK^G(A, B)$ can be represented as a triplet (H_B, π, T) where :

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B.
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator.
- T and π satisfy the K-cycle condition, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ and $\pi(a)(g.T - T)$ are compact operators over H_B for all $a \in A, g \in \mathcal{G}$.

Set $T_{\mathcal{G}} = T \otimes id_{B \rtimes \mathcal{G}} \in \mathcal{L}_{B \rtimes \mathcal{G}}(H_B \otimes (B \rtimes \mathcal{G})) \simeq \mathcal{L}_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$, and $\pi_G : A \rtimes \mathcal{G}v \to L_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$. Then, according to Le Gall [9], $(H_{B \rtimes \mathcal{G}, \pi_{\mathcal{G}}, T_{\mathcal{G}}})$ represents the K-cycle $j_{\mathcal{G}}(z) \in KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$. Let us construct a controlled morphism associated to z,

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G}),$$

which induces right multiplication by $j_{\mathcal{G}}(z)$ in K-theory.

5.1.1 Odd case

Let us first do the work for $z \in KK_1^{\mathcal{G}}(A, B)$. Let (H_B, π, T) be a K-cycle representing z. Set $P = \frac{1+T}{2}$ and $P_{\mathcal{G}} = P \otimes id_{B \rtimes \mathcal{G}}$. We define

$$E^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in A \rtimes \mathcal{G}, y \in K_{B \rtimes \mathcal{G}}\}$$

a C^*-algebra which is filtered by

$$E_U^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in (A \rtimes \mathcal{G})_U, y \in K \otimes (B \rtimes \mathcal{G})_U\}$$

which gives us a filtered extension

$$0 \longrightarrow K_{B \rtimes_r \mathcal{G}} \longrightarrow E^{(\pi,T)} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

and semi split by $s : \left\{ \begin{array}{lll} A \rtimes_r \mathcal{G} & \to & E^{(\pi,T)} \\ x & \mapsto & (x, P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}}) \end{array} \right.$.

Let us show that the controlled boundary map of this extension does not depend on the representant chosen, but only on the class z.
Let $(H_B, \pi_j, T_j), j = 0, 1$ two K-cycles which are homotopic via $(H_{B[0,1]}, \pi, T)$. We denote e_t the evaluation at $t \in [0, 1]$ for an element of $B[0, 1]$, and set $y_t = e_t(y)$ for such a y. The $*$-morphism

$$\phi : \left\{ \begin{array}{lll} E^{(\pi,T)} & \to & E^{(\pi_t,T_t)} \\ (x, y) & \mapsto & (x, y_t) \end{array} \right.$$

satisfies $\phi(K_{B[0,1]\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$ and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B[0,1]\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A\rtimes_r\mathcal{G} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_{|K_{B[0,1]\rtimes_r\mathcal{G}}}} & & \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi_t,T_t)} & \longrightarrow & A\rtimes_r\mathcal{G} & \longrightarrow & 0 \end{array}.$$

According to [10], remark 3.7., the following holds

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_t,T_t)}} = \phi_* \circ D_{K_{B[0,1]\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As $id\otimes e_t$ gives a homotopy between $id\otimes e_0$ and $id\otimes e_1$, and as if two $*$-morphisms are homotopic, then they are equal in controlled K-theory,

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_0,T_0)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_1,T_1)}}$$

holds, and the boundary of the extension $E^{(\pi,T)}$ depends only on z.

Definition 30. The controlled Kasparov transform of an element $z \in KK_1^{\mathcal{G}}(A,B)$ is defined as the compostion

$$J_{red,\mathcal{G}}(z) = \mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}} \circ D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As the boundary map is a (α_D, k_D)-controlled morphism and the Morita equivalence preserves the filtration, $J_{red,\mathcal{G}(z)}$ is (α_D, k_D)-controlled.

Proposition 13. Let A and B two $\mathcal{G}$-C^*-algebras. There exists a control pair (α_J, k_J) such that for every $z \in KK_1^{\mathcal{G}}(A,B)$, there exists a (α_J, k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}_*(A\rtimes_r\mathcal{G}) \to \hat{K}_{*+1}(B\rtimes_r\mathcal{G})$$

such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z+z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_1^G(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_1^G(A, B_1)$.

(v) Let $0 \to J \to A \to A/J \to 0$ be a semi-split equivariant extension of $\mathcal{G}$-algebras and $[\partial_J] \in KK_1^{\mathcal{G}}(A/J, J)$ be its boundary element. Then

$$J_{\mathcal{G}}([\partial_J]) = D_{J\rtimes_r G, A\rtimes_r\mathcal{G}}.$$

Proof 4. (i) The K-cycle $[\partial_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}] \in KK_1(A\rtimes_r\mathcal{G}, B\rtimes_r\mathcal{G})$ implementing the boundary of the extension $E^{(\pi,T)}$ induces the map $j_{red,\mathcal{G}}$ by definition, and modulo Morita equivalence, which immediately gives the first point.

(ii) If z, z' are elements of $KK_1^G(A,B)$, represented by two K-cycles (H_B, π_j, T_j), and if (H_B, π, T) is a K-cycle representing the sum $z+z'$, then $E^{(\pi,T)}$ is naturally isomorphic to the extension sum of the $E_j := E^{(\pi_j,T_j)}$, namely

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow D \longrightarrow A\rtimes_r\mathcal{G} \longrightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j, E_i) \right\}.$$

Naturality of the controlled boundary maps [10] ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result.

(iii) Let $z \in KK_1^{\mathcal{G}}(A_2, B)$, represented by a cycle (H_B, π, T). Representing $f^*(z)$ is $(H_B, f^*\pi, T)$ with off course $f^*\pi = \pi\circ f$. The map

$$\phi : \begin{cases} E^{f^*(\pi,T)} & \to \quad E^{(\pi,T)} \\ (x, P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) & \to \quad (f_{\mathcal{G}}(x), P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) \end{cases}$$

satisfies

- $\phi(K_{B\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$, and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{f^*(\pi,T)} & \longrightarrow & A_1\rtimes_r\mathcal{G} & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\phi} & & \big\downarrow{f_{\mathcal{G}}} & & \\ 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A_2\rtimes_r\mathcal{G} & \longrightarrow & 0 \end{array}.$$

- It intertwines the sections of the two extensions.

Remark 3.7 of [10] assures that

$$D_{K_{B\rtimes_r\mathcal{G}},E^{f^*(\pi,T)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}} \circ f_{\mathcal{G},*}$$

, and the claim is clear from composition by $\mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}}$.

(iv) Let $\mathcal{E} = H_{B_1}\otimes_g B_2$, which is a countably generated Hilbert B_2-module. The homomorphism $g : B_1 \to B_2$ gives rise to $g_* : \mathcal{L}_{B_1}(H_{B_1}) \to \mathcal{L}_{B_2}(\mathcal{E})$, which preserves compact operators : $g_*(K_{B_1}) \subset K(\mathcal{E})$. We have a similar statement for $g_G : B_1\rtimes\mathcal{G} \to B_2\rtimes\mathcal{G}$. We denote $\mathcal{E}_G$ the Hilbert $B_2\rtimes\mathcal{G}$-module $\mathcal{E}\rtimes\mathcal{G} \simeq H_{B_1\rtimes\mathcal{G}}\otimes_g (B_2\rtimes\mathcal{G})$.

Let $z \in KK^{\mathcal{G}}(A, B_1)$ be represented by the K-cycle (H_{B_1}, π, T). Then $(H_{B_1}\otimes_g B_2, g_*\circ\pi, g_*(T)) = (\mathcal{E}, \tilde{\pi}, \tilde{T})$ represents $g_*(z)$.

The map $(x,y) \mapsto (x, (g_G)_*(y))$ induces $\Psi : E^{(\pi,T)} \to E^{g_*(\pi,T)}$ such that

$$\Psi(x, P_G\pi_G(x)P_G + y) \mapsto (x, \tilde{P}_G\tilde{\pi}_G(x)\tilde{P}_G + (g_G)_*(y)).$$

Indeed, the crossed-product functor commutes with pull-back by $\mathcal{G}$-morphisms, and $(g_G) * \circ \pi_G = (g_* \circ \pi)_G = \tilde{\pi}_G$ and $(g_G)_*(P_G) = g_*(P)_G = \tilde{P}_G$ so that

$$(g_G)_*(P_G \pi_G(x) P_G) = \tilde{P}_G \tilde{\pi}_G(x) \tilde{P}_G.$$

Now, by the stabilisation lemma of Le Gall [9], we know that the countably generated Hilbert module $\mathcal{E}_G$ sits as a complemented module of $H_{B_2 \rtimes \mathcal{G}}$, and there exists a projection $p \in L(H_{B_2 \rtimes \mathcal{G}})$ such that $pH_{B_2 \rtimes \mathcal{G}} \simeq \mathcal{E}_{\mathcal{G}}$ and $pK_{B_2 \rtimes \mathcal{G}} p \simeq K(\mathcal{E}_G)$. Let ψ be the composition $K_{B_1 \rtimes \mathcal{G}} \to_{(g_G)_*} K(\mathcal{E}_{\mathcal{G}}) \to K_{B_2 \rtimes \mathcal{G}}$. In this particular case, we can give an explicit description of ψ. The map defined on basic tensor products $(x_j)_j \otimes b \mapsto (g(x_j)b)_j$ extends to an isometric embedding $\mathcal{E}_{\mathcal{G}} \to H_{B_2 \rtimes \mathcal{G}}$, under which $b\theta_{e_i,e_j}$ is mapped to $g(b)\theta_{u_i,u_j}$, where $\{e_j\}$ and $\{u_j\}$ are respectively the canonical orthogonal basis of $H_{B_1 \rtimes \mathcal{G}}$ and $H_{B_2 \rtimes \mathcal{G}}$. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B_1 \rtimes \mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_{B_2 \rtimes \mathcal{G}} & \longrightarrow & E^{g_*(\pi,T)} & \longrightarrow & A \rtimes \mathcal{G} & \longrightarrow & 0 \end{array} .$$

and Ψ intertwines the two filtered sections by the previous relation. Moreover, $\Psi_{|K_{B_1 \rtimes \mathcal{G}}} \subset K_{B_2 \rtimes \mathcal{G}}$, so that we can again apply the remark 3.7 of [10] to state

$$D_{K_{B_2 \rtimes \mathcal{G}}, E^{g_*(\pi,T)}} = \psi_* \circ D_{K_{B_1 \rtimes \mathcal{G}}, E^{(\pi,T)}},$$

which we compose by the Morita equivalence on the left $M^{-1}_{B_2 \rtimes \mathcal{G}}$

$$J_{\mathcal{G}}(g_*(z)) = M^{-1}_{B_2 \rtimes \mathcal{G}} \circ g_{G,*} \circ D_{K_{B_1 \rtimes \mathcal{G}}, E^{(\pi,T)}}.$$

The homomorphisms inducing the Morita equivalence make the following diagram commutes,

$$\begin{array}{ccc} B_1 \rtimes \mathcal{G} & \xrightarrow{g_{\mathcal{G}}} & B_2 \rtimes \mathcal{G} \\ \downarrow & & \downarrow \\ K_{B_1 \rtimes \mathcal{G}} & \xrightarrow{\psi} & K_{B_2 \rtimes \mathcal{G}} \end{array} ,$$

and $J_{\mathcal{G}}(g_*(z)) = g_{G,*} \circ M^{-1}_{B_1 \rtimes \mathcal{G}} \circ D_{K_{B_1 \rtimes \mathcal{G}}, E^{(\pi,T)}} = g_{G,*} \circ J_{\mathcal{G}}(z)$.

(v) Let $q : A \to A/J$ be the quotient map and (H_J, π, T) be a cycle representing $[\partial_J]$. Then we apply remark 3.7 of [10] to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \rtimes \mathcal{G} & \longrightarrow & A \rtimes \mathcal{G} & \longrightarrow & A/J \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle s \circ q_{\mathcal{G}}} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_{J \rtimes \mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A/J \rtimes \mathcal{G} & \longrightarrow & 0 \end{array} ,$$

where the first vertical arrow is the canonical mapping that induces the Morita equivalence.

□

5.1.2 Even case

We can now define $J_{\mathcal{G}}$ for even K-cycles. Let A and B be two $\mathcal{G}$-algebras. Let $[\partial_{SB}] \in KK_1(B, SB)$ be the K-cycle implementing the boundary of the extension $0 \to SB \to CB \to B \to 0$, and $[\partial] \in KK_1(\mathbb{C}, S)$ be the Bott generator. As $z \otimes_B [\partial_{SB}]$ is an odd K-cycle, we can define

$$J_{\mathcal{G}}(z) := \tau_{B \rtimes \mathcal{G}}([\partial]^{-1}) \circ J_{\mathcal{G}}(z \otimes [\partial_{SB}]).$$

Here τ_D refers to the (α_τ, k_τ)-controlled map $\hat{K}(A_1 \otimes D) \to \hat{K}(A_2 \otimes D)$, that H. Oyono-Oyono and G. Yu constructed in [10] for any C^*-algebras D, A_1, A_2 and $z \in KK_*(A_1, A_2)$. It enjoys many natural properties, and induces right multiplication by $\tau_D(z) \in KK(A_1 \otimes D, A_2 \otimes D)$ in K-theory. We can see that, if we set $\alpha_J = \alpha_\tau \alpha_D$ and $k_J = k_\tau * k_D$, $J_{\mathcal{G}}(z)$ is (α_J, k_J)-controlled.

Proposition 14. Let A and B two $\mathcal{G}$-C^*-algebras. For every $z \in KK_*^{\mathcal{G}}(A, B)$, there exists a control pair (α_J, k_J) and a (α_J, k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}(A \rtimes_r \mathcal{G}) \to \hat{K}(B \rtimes_r \mathcal{G})$$

of the same degree as z, such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z + z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_*^G(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_*^G(A, B_1)$.

(v) $J_G([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A \rtimes \mathcal{G})}$

Proof 5. The point (iii) is a consequence of the previous proposition 19, and of the equality $f^*(x) \otimes y = f^*(x \otimes y)$.

□

We now show that the controlled Kasparov transform respects in a quantitative way the Kasparov product.

Proposition 15. There exists a control pair (α_K, k_K) such that for every $\mathcal{G}$-C^*-algebras A, B and C, and every $z \in KK^{\mathcal{G}}(A, B), z' \in KK^{\mathcal{G}}(B, C)$, the controlled equality

$$J_{\mathcal{G}}(z \otimes_B z') \sim_{\alpha_K, k_K} J_{\mathcal{G}}(z') \circ J_{\mathcal{G}}(z)$$

holds.

Proof 6. We will use the following fact : there exists a positive integer d such that every cycle $z \in KK^{\mathcal{G}}(A,B)$ has decomposition property (d). For more details, we send to the appendice of the article of V. Lafforgue [8] where H. Oyono-Oyono shows that claim. We just need to know that z satisfies the decomposition property (d) if there exist $d+1$ $\mathcal{G}$-C^*-algebras A_j and d cycles $\alpha_j \in KK^{\mathcal{G}}(A_{j-1}, A_j), j = 1, d$ such that $A_0 = A$, $A_d = B$ and each α_j is either coming from a $*$-morphism $A_{j-1} \to A_j$, or there is a $*$-morphism $\theta_j : A_j \to A_{j-1}$ such that $\alpha_j \otimes_{A_j} [\theta_j] = 1$ in $KK^G(A_{j-1}, A_{j-1})$.

This property reduces the proof to the special case of α being the inverse of a morphism in $KK^{\mathcal{G}}$-theory : $\alpha \otimes [\theta] = 1$, then :

$$\begin{array}{rcl} J_{\mathcal{G}}(\alpha \otimes z) & \sim_{\alpha_J^2, k_J * k_J} & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\alpha \otimes [\theta]) \\ & \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\theta_*(\alpha)) \\ & \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ \theta_{\mathcal{G},*} \circ J_{\mathcal{G}}(\alpha) \\ & \sim & J_{\mathcal{G}}(\theta^*(\alpha \otimes z)) \circ J_{\mathcal{G}}(\alpha) \\ & \sim & J_{\mathcal{G}}(z) \circ J_{\mathcal{G}}(\alpha) \end{array}$$

because $\theta^*(\alpha \otimes z) = \theta^*(\alpha) \otimes z = 1 \otimes z = z$. The control on the propagation of the first line follows from remark 2.5 of [10] and point (v), the other lines are equal by points (iii) and (iv). As d is uniform for all locally compact groupoids with Haar systems, a simple induction concludes, and (α_K, k_K) can be taken to be $(d\alpha_J^{2d}, (k_J * k_J)^{*d})$. □

5.2 Quantitative assembly maps

6 A quantitative Künneth formula

In this section, we present an application of the previous results to a Künneth formula for crossed products by an étale groupoid.

Recall that for A and B two C^*-algebras, one can define a homomorphism

$$\alpha_{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \quad ; \quad (x, y) \mapsto x \otimes \tau_A(y),$$

where τ_A is the external Kapsarov product.

Definition 31. A C^*-algebra A is said to satisfy the Künneth formula if, for every C^*-algebra B such that $K_*(B)$ is a free abelian group, $\alpha_{A,B}$ is an isomorphism.

If A satifies the Künneth formula, then, for any C^*-algebra B, one has the following exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow Tor(K_*(A), K_*(B)) \longrightarrow 0$$

If $(A, \mathcal{E})$ is now filtered, $(A \otimes B, \mathcal{E})$ is also filtered, and one can define a controlled morphism

$$\hat{\alpha}_{A,B} : \hat{K}_*(A) \otimes K_*(B) \to \hat{K}_*(A \otimes B) \quad ; \quad (x, y) \mapsto \hat{\tau}_A(y)(x),$$

which induces $\alpha_{A,B}$ in K-theory.

Definition 32. A filtered C^*-algebra $(A, \mathcal{E})$ is said to satisfy the quantitative Künneth formula if, for every C^*-algebra B such that $K_*(B)$ is a free abelian group, $\hat{\alpha}_{A,B}$ is a controlled isomorphism.

The remainder of the section is devoted to prove the following theorem.

Theorem 5. Let G be an σ-compact étale groupoid and A a G-algebra. Suppose that

- G satisfies the Baum-Connes conjecture with coefficients,
- for every compact subgroupoids H of G, $A \rtimes_r H$ satisfies the Künneth formula.

Then $A \rtimes_r G$ satisfies the quantitative Künneth formula.

The following corollary is obvious from the theorem.[3]

Corollary 1. Let X be a proper discrete metric space with bounded geometry which admits a fibered coarse embedding into a Hilbert space and A a C^*-algebra. If A satisfies the Künneth formula, then $A \otimes K(l^2(X))$ satifies the quantitative Künneth formula.

6.1 Baum-Connes and the Künneth formula

If A is a G-algebra, and B a C^*-algebra, $A \otimes B$ naturally inherits a G-algebra structure with trivial action of G on the B factor.

Let Z be a G-proper space. Define the homomorphism :

$$\alpha_{A,B}^{G,Z} : RK_*^G(Z,A) \otimes K_*(B) \to RK_*^G(Z, A \otimes B) \quad ; \quad (x,y) \mapsto x \otimes \tau_A(y),$$

which respects inductive limits w.r.t. inclusions of G-proper spaces, so that it induces

$$\alpha_{A,B}^{G} : K_*^{top}(G,A) \otimes K_*(B) \to K_*^{top}(G, A \otimes B).$$

To prove theorem 5, we will need the following result.

Theorem 6. Let E be an open relatively compact subset of G and P_E. If for all compact subgroupoids H of G, $A \rtimes_r H$ satisfies the Künneth formula, then $\alpha_{A,B}^{G,P_E(G)}$ is an isomorphism for all C^*-algebras B such that $K_*(B)$ is a free abelian group.

Démonstration. Let $Z_0 \subset Z_1 \subset ... \subset Z_n$ be the skeleton decomposition of $P_E(G)$.

Let us prove by induction that $\alpha_j = \alpha_{A,B}^{G,Z_j}$ is an isomorphism. By a standard argument similar to the proof of theorem 5, it is sufficient to prove the statement for $j = 0$.

Let $U \subset Z_0$ be a G-compact G-proper space. There exists a compact subgroupoid H of G and a proper H-space such that $U = \mathrm{Ind}_H^G V$. The following diagram is commutative :

$$\begin{tikzcd}
RK_*^G(U,A) \otimes K_*(B) \arrow[r, "\alpha_{A,B}^{G,U}"] \arrow[d, "\mathrm{Res}_H^G"] & RK^G(U, A \otimes B) \arrow[d, "\mathrm{Res}_H^G"] \\
RK_*^H(V, \mathrm{Res}_H^G A) \otimes K_*(B) \arrow[r, "\alpha_{\mathrm{Res}_H^G A,B}^{H,V}"] \arrow[d, "\mu_{H,\mathrm{Res}_H^G A \otimes id}"] & RK_*^H(V, \mathrm{Res}_H^G A \otimes B) \arrow[d, "\mu_{H,\mathrm{Res}_H^G A}"] \\
K_*(\mathrm{Res}_H^G A \rtimes_r H) \otimes K_*(B) \arrow[r, "\alpha_{\mathrm{Res}_H^G A,B}"] & K_*\left((\mathrm{Res}_H^G A \otimes B) \rtimes_r H\right)
\end{tikzcd}$$

The first vertical arrows are isomorphisms by property of the restriction functor, the last horizontal one and the last vertical ones are by hypothesis, so that $\alpha_{A,B}^{G,U}$ is an isomorphism. By taking the inductive limit on G-proper G-compact spaces $U \subset Z_0$, we get that $\alpha_{A,B}^{G,Z_0}$ is an isomorphism. $\square$

During the past years, there has been a growing interest on the links between several conjectures involving assembly maps. This report will focus on the link between the coarse Baum-Connes conjecture and the Novikov conjecture. If Γ is a finitely generated group, the descent principle assures that if the coarse Baum-Connes assembly map for Γ as a metric space with the word length is an isomorphism, then the Baum-Connes assembly map for Γ is rationnaly injective, thus the Novikov conjecture holds for Γ.

Following ideas of M. Gromov, G. Yu introduced new coarse concepts in the study of these assembly maps. He was able to prove the coarse Baum-Connes conjecture for proper metric spaces with finite asymptotic dimension [19], which is a coarse analogue of the topological covering dimension. Later on, in a paper with Guenter and R. Tessera [4], they defined decomposition complexity for metric spaces, which is a broad generalization of asymptotic dimension. In particular, proper metric spaces with finite asymptotic dimension are of finite decomposition complexity. At the end of [4], as concluding remarks, the authors point out that one should be able to derive a new proof of the coarse Baum-Connes conjecture for spaces with finite decomposition complexity. We should emphasize that this is already known : a space which is finitely decomposable has property (A), hence verifies the coarse Baum-Connes conjecture by the work of G. Yu. [20] But the techniques of this proof is highly analytical, it uses a Dirac-Dual Dirac type construction, which involves infinite dimensional analysis. The suggestion of [4] is to give a geometrical proof, using a coarse Mayer-Vietoris argument in the spirit of the proof of the Baum-Connes conjecture for spaces with finite asymptotic dimension.

Such a proof was given in the setting of algebraic K-theory in a paper of D. A. Ramras, R. Tessera and G. Yu where they established the integral Novikov conjecture for algebraic K-theory of group rings $R[\Gamma]$ when the group Γ has FDC (finite decomposition complexity). Their proof uses the continuously controlled algebraic K-theory groups very intensively : their key lemma is a vanishing theorem of these groups. In a series of papers [10][5], H. Oyono-Oyono and G. Yu developed an analogue of this controlled K-theory for operator algebras, which they named quantitative K-theory. It consists of a family of groups $\hat{K}(A) = (K^{\epsilon,r}(A))$ for $r \geq 0, \epsilon \in (0, \frac{1}{4})$ and A a filtered C^*-algebra, which we shall describe later. They were able to define quantitative assembly maps that factorize the usual ones, and to give equivalence between isomorphisms of the assembly map and quantitative statements.

Following the route of these articles [10][5], we will define quantitative assembly maps for étale groupoids with a proper length. These assembly maps are equivalent to the coarse quantitative assembly maps for proper metric spaces X defined in [5] if one takes $G = G(X)$, the coarse groupoid of X. We give also quantitative statements equivalent to a certain isomorphism. **(rerédiger ce paragraphe de façon plus précise une fois les résultats écrits)**

7 Review of quantitative K-theory

This section presents basic constructions of quantitative K-theory for operator algebras that we shall use. For more details, see the original article of H. Oyono-Oyono and G. Yu.[10] We will refer either to quantitative or controlled K-theory for the same object, namely a family of abelian groups $\hat{K}(A) = (K^{\epsilon,R}(A))$ where $R > 0, 0 < \epsilon < \frac{1}{4}$, defined for a filtered C^*-algebra A. The motivating idea is to keep track of propagation of an operator while taking his (possibly higher) index. The main example is that of Roe algebras.

7.1 Roe algebras and filtration

Let (X, d) be a discrete proper metric space, i.e. its closed ball are compact, that is uniformly bounded, so that for every $R > 0$, there exists an integer $N \geq 0$ such that every ball of radius R contains less than N elements. A X-module is a hilbert space H equiped with a C^*-morphism $\rho : C_0(X) \to \mathcal{L}(H)$. To lighten notations, we write fx instead of $\rho(f)x$ if $f \in C_0(X)$ and $x \in X$. All these definitions can be found in [12]

Definition 33. Let H be a X-module.

- An operator $T \in \mathcal{L}(H)$ is locally compact if for every $f \in C_0(X)$, fT and Tf are compact operators, where f is understood as a multiplication operator.
- An operator $T \in \mathcal{L}(H)$ is of finite propagation bounded by $R > 0$ if for every pair of functions $f, g \in C_0(X)$ such that $d(\text{supp } f, \text{supp } g) > R$, $fTg = 0$.
- We denote by $C_R[X]$ the set of locally compact operators with finite propagation bounded by R. The Roe algebra of X is $C^*(X)$, the closure of $\cup_{R>0} C_R[X]$ in the operator topology of $\mathcal{L}(H)$.

A simple example is given by $l^2(X) \otimes H$ with H a separable Hilbert space, in which case $C_R[X]$ is the algebra of operators $(T_{xy})_{x,y\in X}$ such that $T_{x,y} \in K(H)$ for every $x, y \in X$, and $T_{xy} = 0$ as soon as $d(x, y) > R$.
Remark : one coulde replace Hilbert spaces by Hilbert modules E over a C^*-algebra B in this definition, $\mathcal{L}(H)$ by adjoinable operators $\mathcal{L}_B(E)$ and $K(H)$ by compact operators $K_B(E)$, to obtain $C^*(X, B)$, the Roe algebra with coefficients in B. The Roe algebra $C^*(X, B)$ enjoys functorial properties in B.

This example motivates the following definition.

Definition 34. A C^*-algebra A is said to be filtered if there are closed $*$-stable linear subspaces A_R for every $R > 0$ such that

- $A_s \subset A_r$ when $s \leq r$,
- $\cup_{R>0} A_R$ is dense in A,
- $A_s.A_r \subset A_{s+r}$ for every $r, s \geq 0$,
- $\forall r > 0, 1 \in A_r$ when A is unital.

A C^*-morphism between filtered C^*-algebras $\phi : A \to B$ is filtered if $\phi(A_R) \subset B_R$ for every $R > 0$.

If A is a non-unital C^*-algebra, let A^+ be the unital C^*-algebra containing A as a two-sided ideal, defined as :

$$A^+ = \{(a, \lambda) \in A \times \mathbb{C}\}$$
$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a, \lambda)^* = (a^*, \overline{\lambda})$$

with the norm operator

$$||(a, \lambda)|| = \sup\{||ax + \lambda x|| : x \in A, ||x|| = 1\}.$$

When A is not unital and filtered by $(A_R)_{R>0}$, A^+ is filtered by $A_R^+ = \{(x, \lambda) : x \in A_R, \lambda \in \mathbb{C}\}$.

7.2 Definition of quantitative K-theory

Let A be unital and filtered, and $\epsilon \in (0, \frac{1}{4}), R > 0$.

- p is a ϵ-R-projection if $p \in A_R$ and $||p^2 - p|| < \epsilon$.
- u is a ϵ-R-unitary if $u \in A_R$ and $||u^*u - 1|| < \epsilon$ and $||uu^* - 1|| < \epsilon$.

A ϵ-R-projection has its spectrum localized around 0 and 1, with a spectral gap in between, which allows to define a true projection $\kappa(p)$ by functional calculus.

Let $P_n^{\epsilon,R}(A)$ be the set of ϵ-R-projections and $U_n^{\epsilon,R}(A)$ the set of ϵ-R-unitaries of $M_n(A)$. We can also define the inductive limits $P_\infty^{\epsilon,R}(A) = \varinjlim P_n^{\epsilon,R}(A)$ and $U_\infty^{\epsilon,R}(A) = \varinjlim U_n^{\epsilon,R}(A)$ under the inductive system of morphisms

$$p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \text{ and } u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The following defines equivalence relations on $P_\infty^{\epsilon,R}(A) \times \mathbb{N}$ and $U_\infty^{\epsilon,R}(A)$:

- $(p, m) \sim (q, n)$ if there exists $h \in P_\infty^{\epsilon,R}(A[0,1])$ and $k > 0$ such that $h(0) = \begin{pmatrix} p & 0 \\ 0 & I_{m+k} \end{pmatrix}$ and $h(0) = \begin{pmatrix} q & 0 \\ 0 & I_{n+k} \end{pmatrix}$.
- $u \sim v$ if there exists $h \in U_\infty^{3\epsilon,2R}(A[0,1])$ such that $h(0) = u$ and $h(1) = v$.

Definition 35. Let $\epsilon \in (0, \frac{1}{4}), R > 0$.

- $K_0^{\epsilon,R}(A) = P_\infty^{\epsilon,R}(A) \times \mathbb{N}/\sim$ if A is unital, or $\{[p, n] \in P_\infty^{\epsilon,R}(A^+) \times \mathbb{N}$ such that $\mathrm{tr}\rho_A(\kappa(p)) = n\}$ if A is not unital.
- $K_1^{\epsilon,R}(A) = U_\infty^{\epsilon,R}(A^+)/\sim$ where $A = A^+$ if A is unital.

These are abelian groups (see [10],[5]) with laws described by $[p, m]_{\epsilon,R} + [q, n]_{\epsilon,R} = [diag(p, q), m + n]_{\epsilon,R}$ and $[u]_{\epsilon,R} + [v]_{\epsilon,R} = [diag(u, v)]_{\epsilon,R}$.

Definition 36. To be more flexible, we allow our morphisms between quantitative objets to increase propagation, but in a controlled way. That control is implemented by what Oyono-Oyono and Yu call a control pair.

- A control pair (α, h) is a positive number $\alpha > 1$ and a map $h : (0, \frac{1}{4\alpha}) \to \mathbb{R}_*^+$ such that $h \leq F$ for F a non-increasing function. We define a partial order on the control pair by $(\alpha, h) \leq (\beta, k)$ if $\alpha \leq \beta$ and $h \leq k$ on $(0, \frac{1}{4\beta})$.

- A quantitative object is a family of abelian groups $\hat{H} = (H^{\epsilon,R})_{\epsilon\in(0,\frac{1}{4}),R>0}$ and group morphisms $\phi_{\epsilon,R}^{\epsilon',R'} : G^{\epsilon,R} \to G^{\epsilon',R'}$ any time $R \leq R'$ and $\epsilon \leq \epsilon'$, such that $\phi_{\epsilon,R}^{\epsilon,R} = id_{H^{\epsilon,R}}$ and $\phi_{\epsilon_1,R_1}^{\epsilon_2,R_2} \circ \phi_{\epsilon_0,R_0}^{\epsilon_1,R_1} = \phi_{\epsilon,R}^{\epsilon_2,R_2}$.
- For a control pair (α, k) and two quantitative objects H and H', a (α, k)-controlled (quantitative) morphism $F : H \to H'$ is a family of group morphisms

$$F^{\epsilon,R} : H^{\epsilon,R} \to H'^{\alpha\epsilon,k_\epsilon R} \quad \forall \epsilon \in (0, \frac{1}{4\alpha}), R > 0.$$

We have natural morphisms of abelian groups $\iota_{\epsilon,R}^{\epsilon',R'} : K_*^{\epsilon,R}(A) \to K_*^{\epsilon',R'}(A)$ if $\epsilon \leq \epsilon', R \leq R'$ and $\iota_{\epsilon,R} : K_*^{\epsilon,R}(A) \to K_*(A)$, and $\iota_{\epsilon',R'} \circ \iota_{\epsilon,R}^{\epsilon',R'} = \iota_{\epsilon,R}$ holds. This fact gives sense to saying that a controlled morphism induced a morphism in K-theory.

Example 4. The basic example of quantitative object is of course quantitative K-theory.
If $\phi : A \to B$ is a filtered C^*-morphism, it naturally induces a $(1, 1)$-controlled morphism $\phi_* : \hat{K}(A) \to \hat{K}(B)$.
Another (important) examples will be that of the controlled Morita equivalence and quantitative boundary maps.

7.3 Morita equivalence

As in classical K-theory, we have an isomorphism which we call the (controlled) Morita equivalence.

Proposition 16. Let A be a filtered C^*-algebra and H a separable Hilbert space. We denote by K_A the C^*-algebra of compact operators of the standard Hilbert module H_A, which is C^*-isomorphic to $A\otimes K(H)$. Let e be any rank-one projection in $K(H)$. Then the C^*-morphism

$$\begin{cases} A & \to & K_A \\ a & \mapsto & a \otimes e \end{cases}$$

induces an $\mathbb{Z}_2$-graded isomorphism

$$M_A^{\epsilon,R} : K^{\epsilon,R}(A) \to K^{\epsilon,R}(K_A)$$

for every $R > 0$ and $0 < \epsilon < \frac{1}{4}$.

7.4 Quantitative boundary maps

Let $0 \to J \to A \to A/J \to 0$ be an extension of C^*-algebras. We denote $\partial_{J,A} : K_*(A/J) \to K_{*+1}(J)$ the associated boundary map.

Proposition 17. There exist a control pair (α_D, k_D) such that for any semi-split extension of filtered C^*-algebras $0 \to J \to A \to A/J \to 0$, there exists a (α_D, k_D)-controlled morphism of odd degree

$$D_{J,A} : \hat{K}(A/J) \to \hat{K}(J) \quad \epsilon \in (1, \frac{1}{4\alpha_D}), R > 0$$

which induces $\partial_{J,A}$ in K-theory.

We have to recall what we will refer as the remark 3.7 of [10], a property of quantitative assembly maps that we shall intensively.
Let $\phi : A \to B$ be a filtered morphism between two filtered C^*-algebras. We suppose we have extensions $0 \to I \to A \to A/I \to 0$ and $0 \to J \to B \to B/J \to 0$. If

- $\phi(I) \subset J$
- ϕ intertwines two completely positive sections of the extensions : $\phi \circ s_A = s_B \circ \phi_I^J$, where ϕ_I^J is the induced map $A/I_rightarrowB/J$.

then $D_{J,B} \circ (\phi_I^J)_* = \phi_* \circ D_{I,A}$.
METTRE LA RQ 3.7 de OY2 et celle 1.8 **sur le** λ

8 Quantitative statements

8.1 A reminder on groupoids action

For the rest of the article, $\mathcal{G}$ will denote a topological groupoid, $X = \mathcal{G}^{(0)}$ its space of units, $s, r : \mathcal{G} \rightrightarrows X$ respectively the source and target maps. The fibers over $x \in X$ are denoted by $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$. In this report, all the groupoids are supposed to be locally compact, σ-compact and étale, that is r is a local homeomorphism, which entails that the fibers are discrete, hence $\mathcal{G}$ is naturally endowed with a Haar system $(\lambda^x)_{x\in X}$.
For any $C(X)$-algebra B, $B_x = B \otimes_{ev_x} \mathbb{C}$ is the fiber over $x \in X$, where ev_x is the evaluation at x. The pull-back r^*B and s^*B are respectively defined to be $B \otimes_r C_0(\mathcal{G})$ and $B \otimes_s C_0(\mathcal{G})$. They are $C(\mathcal{G})$-algebras with fiber over $g \in \mathcal{G}$ equal to $B_{r(g)}$ (resp. $B_{s(g)}$).
An action of $\mathcal{G}$ on B is an isomorphism of $C(\mathcal{G})$-algebras $\alpha : s^*B \to r^*B$ which respects the product : $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$ for all composable pairs (g, g').
For a right B-Hilbert module $\mathcal{E}$, we can also define the pull-backs $r^*\mathcal{E} = \mathcal{E} \otimes_r C_0(\mathcal{G})$ and $s^*\mathcal{E} = \mathcal{E} \otimes_s C_0(\mathcal{G})$: they are respectively right and left s^*B-Hilbert modules (the s^*B-module structure on $r^*\mathcal{E}$ being obtained by the inverse of the action $\alpha^{-1} : r^*B \to s^*B$) with fiber over g given by $\mathcal{E}_{r(g)}$ and $\mathcal{E}_{s(g)}$.
An action of $\mathcal{G}$ on $\mathcal{E}$ is a unitary $U \in \mathcal{L}(s^*\mathcal{E}, r^*\mathcal{E})$ such that $U_g U_{g'} = U_{gg'}$ for all composable pairs (g, g').
We will call any $C(X)$-algebra B endowed with an action of $\mathcal{G}$ a **$\mathcal{G}$-algebra**.
If we have at our disposition a family of representations $\mathcal{F}$ of $\mathcal{G}$, we can then form the crossed-product $B \rtimes_{\mathcal{F}} \mathcal{G}$: it is the completion-reduction of $C_c(\mathcal{G}, B)$ for the norm operator $||f||_{\mathcal{F}} = \sup_{\pi\in\mathcal{F}} ||\pi(f)||$. Classical examples are that of the reduced crossed product $B \rtimes_r \mathcal{G}$ for which $\mathcal{F}$ is the left-regular representation, and the maximal crossed-product $B \rtimes_{max} \mathcal{G}$ for which $\mathcal{F}$ is the family of all unitary representations. If $\phi : A \to B$ is a $\mathcal{G}$-morphism, $f \mapsto \phi \circ f$ gives rise to a C^*-morphism $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$. The crossed-product is easily seen to be a functor from $\mathcal{G}$-algebras to C^*-algebras.
The formula

$$\langle \psi, \eta \rangle_x = \int_{\mathcal{G}^x} \langle \psi(g), \eta(g) \rangle d\lambda^x(g)$$

defines a $C(X)$-hermitian product on $C_c(\mathcal{G}, B)$, which we can complete to get the right-$B \rtimes \mathcal{G}$-Hilbert module $\mathcal{E} \rtimes \mathcal{G}$. So the functor $B \mapsto B \rtimes \mathcal{G}$ carries to $\mathcal{E} \mapsto \mathcal{E} \rtimes \mathcal{G}$.

We will be mainly interested in the properties of the functor $B \mapsto B \rtimes \mathcal{G}$, and the majority of the constructions performed in this section are carriable with "good" crossed product functors. We will discuss the property needed for a crossed-product functor to be "good".
The first property is **semi-split–exactness**. We recall that an extension of $\mathcal{G}$-algebra

$$0 \longrightarrow J \longrightarrow A \xrightarrow{p} A/J \longrightarrow 0$$

is said to be semi-split if there exists an equivariant completely positive map $s : A/J \to A$ such that $p \circ s = 1$. The functor $- \rtimes \mathcal{G}$ is said to be semi-split exact if for any such extension,

$$0 \longrightarrow J \rtimes \mathcal{G} \longrightarrow A \rtimes \mathcal{G} \xrightarrow{p_{\mathcal{G}}} A/J \rtimes \mathcal{G} \longrightarrow 0$$

is a semi-split extension of C^*-algebras. The second property implies the first. We say that $-\rtimes\mathcal{G}$ has the CP-property if it preserves completely positive maps, so that from any $\mathcal{G}$-completely positive map $\phi : A \to B$, we can induce a CP map $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$. This assures that the action of $\mathcal{G}$ on any $\mathcal{G}$-module $\mathcal{E}$ descends to an isomorphism $K(\mathcal{E}) \rtimes \mathcal{G} \simeq K(\mathcal{E} \rtimes \mathcal{G})$. **A FINIR, faire une preuve entre autre ...**

8.2 The Baum-Connes conjecture

The more general setting of the Baum-Connes conjecture [15] is that of a locally compact σ-compact Hausdorff groupoid $\mathcal{G}$ endowed with a Haar system, together with a coefficient C^*-algebra B acted upon by $\mathcal{G}$, which give rise to an assembly map

$$\mu_r : K_*^{top}(\mathcal{G}, B) \to K_*(B \rtimes_r \mathcal{G}).$$

The left hand side $K_*^{top}(\mathcal{G}, B)$ is the K-homology of the classifying space $\mathcal{E}\mathcal{G}$ for proper actions of $\mathcal{G}$ with coefficients in B. We give a sketch of the construction when $\mathcal{G}$ is étale. Let $d \geq 0$ and $P_d(\mathcal{G})$ be the Rips complex of $\mathcal{G}$, i.e. the space of probabilities supported on a fiber $\mathcal{G}^x$ for a $x \in \mathcal{G}^{(0)}$

$$P_d(\mathcal{G}) = \{p \in \mathcal{P}(\mathcal{G}) : \exists x \in \mathcal{G}^{(0)}, r^*p = \delta_x, \operatorname{supp} p \subset B(e_x, d)\},$$

endowed with the weak-$*$ topology. Then $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B)$ is defined to be the inductive limite of $KK^{\mathcal{G}}(C_0(X), B)$ for X $\mathcal{G}$-proper cocompact $\mathcal{G}$-spaces. If $d \leq d'$, we have a morphism $KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B) \to KK^{\mathcal{G}}(C_0(P_{d'}(\mathcal{G})), B)$ naturally induced by the inclusion $P_d(\mathcal{G}) \subset P_{d'}(\mathcal{G})$, and the K-homology of $\mathcal{G}$ is defined as

$$K_*^{top}(\mathcal{G}, B) = \lim_{d \to \infty} KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B).$$

In his thesis [9], P.-Y. Le Gall constructed the Kasparov transform for the action of a groupoid

$$j_{\mathcal{G}} : KK^{\mathcal{G}}(A, B) \to KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$$

for any $\mathcal{G}$-C^*-algebras A and B. It is also in this paper that equivariant KK-theory for groupoids and the corresponding Kasparov product are defined. One can then give an formula for the assembly map, namely if $z \in KK^{G}(C_0(X), B)$ for a $\mathcal{G}$-proper $\mathcal{G}$-space X of $P_d(\mathcal{G})$, then

$$\mu_r(z) = [\mathcal{L}_X] \otimes_{C_0(X) \rtimes_r \mathcal{G}} j_{\mathcal{G}}(z) \in K_*(B \rtimes_r \mathcal{G})$$

holds, where $[\mathcal{L}_X]$ is the class of a canonical element associated to X which is to be thought of as a Miscenko bundle over $C_0(X) \rtimes_r \mathcal{G}$.

The remaining of this section will be devoted to the construction of a controlled Kasparov transformation for every $z \in KK^{\mathcal{G}}(A, B)$:

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G})$$

which is of course a controlled morphism which induces right multiplication by $j_{\mathcal{G}}(z)$ in K-theory. This will allow us to define a bunch of quantitative assembly maps

$$\mu_{\mathcal{G}}^{\epsilon, R} : K^{top}(\mathcal{G}, B) \to K^{\epsilon, R}(B \rtimes \mathcal{G})$$

inducing the assembly map in K-theory, and to study the relation between the quantitative Baum-Connes conjecture and the classical one for $\mathcal{G}$.

8.3 Length, propagation and controlled six-terms exact sequence

Let $\mathcal{G}$ be a locally compact groupoid with base $\mathcal{G}^{(0)} = X$, a compact space, endowed with a Haar system $\lambda = (\lambda^x)_{x\in X}$. We suppose that $\mathcal{G}$ comes with a proper length l, that is a family of application $(l^x)_{x\in X}$ defined on the fibers $\mathcal{G}^x$ with values in $\mathbb{R}_+$, such that

$$\begin{aligned}
&l^x(e_x) = 0\\
&l^{r(\gamma)}(\gamma) = l^{s(\gamma)}(\gamma^{-1})\\
&l^x(\gamma_1^{-1}\gamma_2) \leq l^x(\gamma_1) + l^x(\gamma_2).
\end{aligned}$$

That length allows us to define a filtration on crossed-product algebras of $\mathcal{G}$ by

$$(A \rtimes \mathcal{G})_r = \{f \in C_c(\mathcal{G}, A) : \text{ supp } f \subset \cup_{x\in X} B_x(r)\}$$

for any $\mathcal{G}$-algebra A. Here, $B_x(r)$ is the ball $\{\gamma \in \mathcal{G}^x : l^x(\gamma) \leq r\}$, and $\rtimes$ can be either the reduced cross-product $\rtimes_r$ or the maximal one $\rtimes_{max}$. Recall that $A \rtimes \mathcal{G}$ is functorial in A, from the category of $\mathcal{G}$-C^*-algebras with $\mathcal{G}$-equivariant C^*-morphisms to the category of C^*-algebras with C^*-morphisms. For $\phi : A \to B$ a $\mathcal{G}$-equivariant C^*-morphism, we denote by $\phi_{\mathcal{G}} : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$ the induced C^*-morphism.

If $0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0$ is a semi-split exact sequence of $\mathcal{G}$-C^*-algebras, then $0 \longrightarrow J \rtimes \mathcal{G} \xrightarrow{\phi_{\mathcal{G}}} A \rtimes \mathcal{G} \xrightarrow{\psi_{\mathcal{G}}} A/J \rtimes \mathcal{G} \longrightarrow 0$ is a flitered semi-split exact sequence. From this, we can state the following proposition.

Proposition 18. There exists a control pair (λ, h) such that for every semi-split extension of $\mathcal{G}$-C^*-algebras

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0\ ,$$

the following diagrams (λ, h)-commutes and are (λ, h)-exact

$$\begin{array}{ccccc}
\hat{K}_0(J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_r \mathcal{G})\\
\uparrow & & & & \downarrow\\
\hat{K}_1(A/J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_r \mathcal{G})
\end{array} ,$$

$$\begin{array}{ccccc}
\hat{K}_0(J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_{max} \mathcal{G})\\
\uparrow & & & & \downarrow\\
\hat{K}_1(A/J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_{max} \mathcal{G})
\end{array} .$$

8.4 The Kasparov transform

Let A and B be two $\mathcal{G}$-C^*-algebras, and H a separable Hilbert space, $l^2(\mathbb{Z})$ for instance, and $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. The standard Hilbert module over B is denoted by $H_B = H_{\mathcal{G}} \otimes B$, and K_B is the algebra of compact operators for H_B, i.e. $K(H) \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Every K-cycle $z \in KK^G(A, B)$ can be represented as a triplet (H_B, π, T) where :

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B.
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator.
- T and π verify the K-cycle condition, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ and $\pi(a)(g.T - T)$ are compact operator over H_B for all $a \in A, g \in \mathcal{G}$.

Set $T_{\mathcal{G}} = T \otimes id_{B \rtimes \mathcal{G}} \in \mathcal{L}_{B \rtimes \mathcal{G}}(H_B \otimes (B \rtimes \mathcal{G})) \simeq \mathcal{L}_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$, and $\pi_G : A \rtimes \mathcal{G}v \to L_{B \rtimes \mathcal{G}}(H_{B \rtimes \mathcal{G}})$. Then, according to Le Gall [9], $(H_{B \rtimes \mathcal{G}, \pi_{\mathcal{G}}, T_{\mathcal{G}}})$ represents the K-cycle $j_{\mathcal{G}}(z) \in KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$. Let us construct a controlled morphism associated to z,

$$J_{\mathcal{G}}(z) : \hat{K}(A \rtimes \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G}),$$

which induces right multiplication by $j_{\mathcal{G}}(z)$ in K-theory.

8.4.1 Odd case

Let us first do the work for $z \in KK_1^{\mathcal{G}}(A, B)$. Let (H_B, π, T) be a K-cycle representing z. Set $P = \frac{1+T}{2}$ and $P_{\mathcal{G}} = P \otimes id_{B \rtimes \mathcal{G}}$. We define

$$E^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in A \rtimes \mathcal{G}, y \in K_{B \rtimes \mathcal{G}}\}$$

a C^*-algebra which is filtered by

$$E_R^{(\pi,T)} = \{(x, P_G \pi_G(x) P_{\mathcal{G}} + y) : x \in (A \rtimes \mathcal{G})_R, y \in K \otimes (B \rtimes \mathcal{G})_R\}$$

which gives us a filtered extension

$$0 \longrightarrow K_{B \rtimes_r \mathcal{G}} \longrightarrow E^{(\pi,T)} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

and semi split by $s : \left\{ \begin{array}{lll} A \rtimes_r \mathcal{G} & \to & E^{(\pi,T)} \\ x & \mapsto & (x, P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}}) \end{array} \right.$.

Let us show that the controlled boundary map of this extension does not depend on the representant chosen, but only on the class z.
Let $(H_B, \pi_j, T_j), j = 0, 1$ two K-cycles which are homotopic via $(H_{B[0,1]}, \pi, T)$. We denote e_t the evaluation at $t \in [0, 1]$ for an element of $B[0, 1]$, and set $y_t = e_t(y)$ for such a y. The $*$-morphism

$$\phi : \left\{ \begin{array}{lll} E^{(\pi,T)} & \to & E^{(\pi_t,T_t)} \\ (x, y) & \mapsto & (x, y_t) \end{array} \right.$$

satisfies $\phi(K_{B[0,1] \rtimes_r \mathcal{G}}) \subset K_{B \rtimes_r \mathcal{G}}$ and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B[0,1] \rtimes_r \mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \Big\downarrow \phi_{|K_{B[0,1] \rtimes_r \mathcal{G}}} & & \Big\downarrow \phi & & \Big\downarrow = & & \\ 0 & \longrightarrow & K_{B \rtimes_r \mathcal{G}} & \longrightarrow & E^{(\pi_t,T_t)} & \longrightarrow & A \rtimes_r \mathcal{G} & \longrightarrow & 0 \end{array} .$$

According to [10], remark 3.7., the following holds

$$D_{K_{B\rtimes_r\mathcal{G},E^{(\pi_t,T_t)}}} = \phi_* \circ D_{K_{B[0,1]\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As $id\otimes e_t$ gives a homotopy between $id\otimes e_0$ and $id\otimes e_1$, and as if two $*$-morphisms are homotopic, then they are equal in controlled K-theory,

$$D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_0,T_0)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi_1,T_1)}}$$

holds, and the boundary of the extension $E^{(\pi,T)}$ depends only on z.

Definition 37. The controlled Kasparov transform of an element $z \in KK_1^{\mathcal{G}}(A,B)$ is defined as the compostion

$$J_{red,\mathcal{G}}(z) = \mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}} \circ D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}.$$

As the boundary map is a (α_D, k_D)-controlled morphism and the Morita equivalence preserves the filtration, $J_{red,\mathcal{G}(z)}$ is (α_D, k_D)-controlled.

Proposition 19. Let A and B two $\mathcal{G}$-C^*-algebras. There exists a control pair (α_J, k_J) such that for every $z \in KK_1^{\mathcal{G}}(A,B)$, there exists a (α_J, k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}_*(A \rtimes_r \mathcal{G}) \to \hat{K}_{*+1}(B \rtimes_r \mathcal{G})$$

such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z+z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_1^G(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_1^G(A, B_1)$.

(v) Let $0 \to J \to A \to A/J \to 0$ be a semi-split equivariant extension of $\mathcal{G}$-algebras and $[\partial_J] \in KK_1^{\mathcal{G}}(A/J, J)$ be its boundary element. Then

$$J_{\mathcal{G}}([\partial_J]) = D_{J\rtimes_r G, A\rtimes_r\mathcal{G}}.$$

Proof 7. (i) The K-cycle $[\partial_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}] \in KK_1(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ implementing the boundary of the extension $E^{(\pi,T)}$ induces the map $j_{red,\mathcal{G}}$ by definition, and modulo Morita equivalence, which immediately gives the first point.

(ii) If z, z' are elements of $KK_1^G(A,B)$, represented by two K-cycles (H_B, π_j, T_j), and if (H_B, π, T) is a K-cycle representing the sum $z+z'$, then $E^{(\pi,T)}$ is naturally isomorphic to the extension sum of the $E_j := E^{(\pi_j,T_j)}$, namely

$$0 \longrightarrow K_{B\rtimes_r \mathcal{G}} \longrightarrow D \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j, E_i) \right\}.$$

Naturality of the controlled boundary maps [10] ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result.

(iii) Let $z \in KK_1^{\mathcal{G}}(A_2, B)$, represented by a cycle (H_B, π, T). Representing $f^*(z)$ is $(H_B, f^*\pi, T)$ with off course $f^*\pi = \pi \circ f$. The map

$$\phi : \left\{ \begin{array}{lll} E^{f^*(\pi,T)} & \to & E^{(\pi,T)} \\ (x, P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) & \to & (f_{\mathcal{G}}(x), P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) \end{array} \right.$$

satisfies

- $\phi(K_{B\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$, and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{f^*(\pi,T)} & \longrightarrow & A_1 \rtimes_r \mathcal{G} & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{\phi} & & \Big\downarrow{f_{\mathcal{G}}} & & \\ 0 & \longrightarrow & K_{B\rtimes_r\mathcal{G}} & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A_2 \rtimes_r \mathcal{G} & \longrightarrow & 0 \end{array}.$$

- It intertwines the sections of the two extensions.

Remark 3.7 of [10] assures that

$$D_{K_{B\rtimes_r\mathcal{G}}, E^{f^*(\pi,T)}} = D_{K_{B\rtimes_r\mathcal{G}}, E^{(\pi,T)}} \circ f_{\mathcal{G},*}$$

, and the claim is clear from composition by $\mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}}$.

(iv) Let $\mathcal{E} = H_{B_1} \otimes_g B_2$, which is a countably generated Hilbert B_2-module. The homomorphism $g : B_1 \to B_2$ gives rise to $g_* : \mathcal{L}_{B_1}(H_{B_1}) \to \mathcal{L}_{B_2}(\mathcal{E})$, which preserves compact operators : $g_*(K_{B_1}) \subset K(\mathcal{E})$. We have a similar statement for $g_G : B_1 \rtimes \mathcal{G} \to B_2 \rtimes \mathcal{G}$. We denote $\mathcal{E}_G$ the Hilbert $B_2 \rtimes \mathcal{G}$-module $\mathcal{E} \rtimes \mathcal{G} \simeq H_{B_1 \rtimes \mathcal{G}} \otimes_g (B_2 \rtimes \mathcal{G})$.

Let $z \in KK^{\mathcal{G}}(A, B_1)$ be represented by the K-cycle (H_{B_1}, π, T). Then $(H_{B_1} \otimes_g B_2, g_* \circ \pi, g_*(T)) = (\mathcal{E}, \tilde{\pi}, \tilde{T})$ represents $g_*(z)$.

The map $(x, y) \mapsto (x, (g_G)_*(y))$ induces $\Psi : E^{(\pi,T)} \to E^{g_*(\pi,T)}$ such that

$$\Psi(x, P_G \pi_G(x) P_G + y) \mapsto (x, \tilde{P}_G \tilde{\pi}_G(x) \tilde{P}_G + (g_G)_*(y)).$$

Indeed, the crossed-product functor commutes with pull-back by $\mathcal{G}$-morphisms, and $(g_G) * \circ \pi_G = (g_* \circ \pi)_G = \tilde{\pi}_G$ and $(g_G)_*(P_G) = g_*(P)_G = \tilde{P}_G$ so that

$$(g_G)_*(P_G \pi_G(x) P_G) = \tilde{P}_G \tilde{\pi}_G(x) \tilde{P}_G.$$

Now, by the stabilisation lemma of Le Gall [9], we know that the countably generated Hilbert module $\mathcal{E}_G$ sits as a complemented module of $H_{B_2\rtimes\mathcal{G}}$, and there exists a projection $p \in L(H_{B_2\rtimes\mathcal{G}})$ such that $pH_{B_2\rtimes\mathcal{G}} \simeq \mathcal{E}_\mathcal{G}$ and $pK_{B_2\rtimes\mathcal{G}}p \simeq K(\mathcal{E}_G)$. Let ψ be the composition $K_{B_1\rtimes\mathcal{G}} \to_{(g_G)_*} K(\mathcal{E}_\mathcal{G}) \to K_{B_2\rtimes\mathcal{G}}$. In this particular case, we can give an explicit description of ψ. The map defined on basic tensor products $(x_j)_j \otimes b \mapsto (g(x_j)b)_j$ extends to an isometric embedding $\mathcal{E}_\mathcal{G} \to H_{B_2\rtimes\mathcal{G}}$, under which $b\theta_{e_i,e_j}$ is mapped to $g(b)\theta_{u_i,u_j}$, where $\{e_j\}$ and $\{u_j\}$ are respectively the canonical orthogonal basis of $H_{B_1\rtimes\mathcal{G}}$ and $H_{B_2\rtimes\mathcal{G}}$. This gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_{B_1\rtimes\mathcal{G}} \arrow[r] \arrow[d, "\psi"] & E^{(\pi,T)} \arrow[r] \arrow[d, "\Psi"] & A \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & K_{B_2\rtimes\mathcal{G}} \arrow[r] & E^{g_*(\pi,T)} \arrow[r] & A \rtimes \mathcal{G} \arrow[r] & 0 \end{tikzcd} .$$

and Ψ intertwines the two filtered sections by the previous relation. Moreover, $\Psi_{|K_{B_1\rtimes\mathcal{G}}} \subset K_{B_2\rtimes\mathcal{G}}$, so that we can again apply the remark 3.7 of [10] to state

$$D_{K_{B_2\rtimes\mathcal{G}},E^{g_*(\pi,T)}} = \psi_* \circ D_{K_{B_1\rtimes\mathcal{G}},E^{(\pi,T)}},$$

which we compose by the Morita equivalence on the left $M^{-1}_{B_2\rtimes\mathcal{G}}$

$$J_\mathcal{G}(g_*(z)) = M^{-1}_{B_2\rtimes\mathcal{G}} \circ g_{G,*} \circ D_{K_{B_1\rtimes\mathcal{G}},E^{(\pi,T)}}.$$

The homomorphisms inducing the Morita equivalence make the following diagram commutes,

$$\begin{tikzcd} B_1 \rtimes \mathcal{G} \arrow[r, "g_\mathcal{G}"] \arrow[d] & B_2 \rtimes \mathcal{G} \arrow[d] \\ K_{B_1\rtimes\mathcal{G}} \arrow[r, "\psi"] & K_{B_2\rtimes\mathcal{G}} \end{tikzcd} ,$$

and $J_\mathcal{G}(g_*(z)) = g_{G,*} \circ M^{-1}_{B_1\rtimes\mathcal{G}} \circ D_{K_{B_1\rtimes\mathcal{G}},E^{(\pi,T)}} = g_{G,*} \circ J_\mathcal{G}(z)$.

(v) Let $q : A \to A/J$ be the quotient map and (H_J, π, T) be a cycle representing $[\partial_J]$. Then we apply remark 3.7 of [10] to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \rtimes \mathcal{G} \arrow[r] \arrow[d] & A \rtimes \mathcal{G} \arrow[r] \arrow[d, "s\circ q_\mathcal{G}"] & A/J \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & K_{J\rtimes\mathcal{G}} \arrow[r] & E^{(\pi,T)} \arrow[r] & A/J \rtimes \mathcal{G} \arrow[r] & 0 \end{tikzcd} ,$$

where the first vertical arrow is the canonical mapping that induces the Morita equivalence.

□

8.4.2 Even case

We can now define $J_\mathcal{G}$ for even K-cycles. Let A and B be two $\mathcal{G}$-algebras. Let $[\partial_{SB}] \in KK_1(B, SB)$ be the K-cycle implementing the boundary of the extension $0 \to SB \to CB \to B \to 0$, and $[\partial] \in KK_1(\mathbb{C}, S)$ be the Bott generator. As

$z \otimes_B [\partial_{SB}]$ is an odd K-cycle, we can define

$$J_{\mathcal{G}}(z) := \tau_{B\rtimes\mathcal{G}}([\partial]^{-1}) \circ J_{\mathcal{G}}(z \otimes [\partial_{SB}]).$$

Here τ_D refers to the (α_τ, k_τ)-controlled map $\hat{K}(A_1 \otimes D) \to \hat{K}(A_2 \otimes D)$, that H. Oyono-Oyono and G. Yu constructed in [10] for any C^*-algebras D, A_1, A_2 and $z \in KK_*(A_1, A_2)$. It enjoys many natural properties, and induces right multiplication by $\tau_D(z) \in KK(A_1 \otimes D, A_2 \otimes D)$ in K-theory. We can see that, if we set $\alpha_J = \alpha_\tau \alpha_D$ and $k_J = k_\tau * k_D$, $J_{\mathcal{G}}(z)$ is (α_J, k_J)-controlled.

Proposition 20. Let A and B two $\mathcal{G}$-C^*-algebras. For every $z \in KK_*^{\mathcal{G}}(A, B)$, there exists a control pair (α_J, k_J) and a (α_J, k_J)-controlled morphism

$$J_{red,\mathcal{G}}(z) : \hat{K}(A \rtimes_r \mathcal{G}) \to \hat{K}(B \rtimes_r \mathcal{G})$$

of the same degree as z, such that

(i) $J_{red,\mathcal{G}}(z)$ induces right multiplication by $j_{red,\mathcal{G}}(z)$ in K-theory ;

(ii) $J_{red,\mathcal{G}}$ is additive, i.e.

$$J_{red,\mathcal{G}}(z + z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) For every $\mathcal{G}$-morphism $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

for all $z \in KK_*^{G}(A_2, B)$.

(iv) For every $\mathcal{G}$-morphism $g : B_1 \to B_2$,

$$J_{red,\mathcal{G}}(g_*(z)) = g_{\mathcal{G},red,*} \circ J_{red,\mathcal{G}}(z)$$

for all $z \in KK_*^{G}(A, B_1)$.

(v) $J_G([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A\rtimes\mathcal{G})}$

Proof 8. The point (iii) is a consequence of the previous proposition 19, and of the equality $f^*(x) \otimes y = f^*(x \otimes y)$.

□

We now show that the controlled Kasparov transform respects in a quantitative way the Kasparov product.

Proposition 21. There exists a control pair (α_K, k_K) such that for every $\mathcal{G}$-C^*-algebra A, B and C, and every $z \in KK^{\mathcal{G}}(A, B)$, $z' \in KK^{\mathcal{G}}(B, C)$, the controlled equality

$$J_{\mathcal{G}}(z \otimes_B z') \sim_{\alpha_K, k_K} J_{\mathcal{G}}(z') \circ J_{\mathcal{G}}(z)$$

holds.

Proof 9. We will use the following fact : there exists a positive integer d such that every cycle $z \in KK^{\mathcal{G}}(A, B)$ has decomposition property (d). For more details, we send to the appendice of the article of V. Lafforgue [8] where H. Oyono-Oyono shows that claim. We just need to know that z satisfies the decomposition property (d) if there exist $d + 1$ $\mathcal{G}$-C^*-algebras A_j and d cycles

$\alpha_j \in KK^{\mathcal{G}}(A_{j-1}, A_j), j = 1, d$ such that $A_0 = A$, $A_d = B$ and each α_j is either coming from a $*$-morphism $A_{j-1} \to A_j$, or there is a $*$-morphism $\theta_j : A_j \to A_{j-1}$ such that $\alpha_j \otimes_{A_j} [\theta_j] = 1$ in $KK^G(A_{j-1}, A_{j-1})$.

This property reduces the proof to the special case of α being the inverse of a morphism in $KK^{\mathcal{G}}$-theory : $\alpha \otimes [\theta] = 1$, then :

$$\begin{array}{rcl} J_{\mathcal{G}}(\alpha \otimes z) & \sim_{\alpha_J^2, k_J * k_J} & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\alpha \otimes [\theta]) \\ & \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ J_{\mathcal{G}}(\theta_*(\alpha)) \\ & \sim & J_{\mathcal{G}}(\alpha \otimes z) \circ \theta_{\mathcal{G},*} \circ J_{\mathcal{G}}(\alpha) \\ & \sim & J_{\mathcal{G}}(\theta^*(\alpha \otimes z)) \circ J_{\mathcal{G}}(\alpha) \\ & \sim & J_{\mathcal{G}}(z) \circ J_{\mathcal{G}}(\alpha) \end{array}$$

because $\theta^*(\alpha \otimes z) = \theta^*(\alpha) \otimes z = 1 \otimes z = z$. The control on the propagation of the first line follows from remark 2.5 of [10] and point (v), the other lines are equal by points (iii) and (iv). As d is uniform for all locally compact groupoids with Haar systems, a simple induction concludes, and (α_K, k_K) can be taken to be $(d\alpha_J^{2d}, (k_J * k_J)^{*d})$. □

8.5 Quantitative assembly maps

Following the article of J.-L. Tu [15], we recall that a locally compact, σ-compact and Hausdorff groupoid G, endowed with a Haar system λ, is said to be proper if there exists a cut-off function $c : G^{(0)} \to \mathbb{R}_+$ continuous such that

- for all compact subset K of $G^{(0)}$, supp $c \cap s(G^K)$ is compact,
- $\int_{G^x} c(s(g))d\lambda^x(g) = 1, \forall x \in G^{(0)}$.

If moreover $G^{(0)}/G$ is also compact, reducing the first condition to "supp c compact", then $g \mapsto \sqrt{c(r(g))c(s(g))}$ defines a projection in $C_c(G)$ for convolution, which gives an element $[\mathcal{L}_G] \in K_0(C^*G)$.

Now when X is a locally compact space which is $\mathcal{G}$-proper and $\mathcal{G}$-compact, the groupoid $X \rtimes \mathcal{G}$ is proper with a compact orbit base space ($X/\mathcal{G}$ is compact). We can then define $[\mathcal{L}_X] \in K_0(C_0(X) \rtimes_r \mathcal{G})$ as the class of the projector for $G = X \rtimes \mathcal{G}$.

8.5.1 Classifying space for proper actions

We remind the construction of a classifying space for proper actions for a σ-compact étale groupoid, which can be found in [17] and [5].
If $d \geq 0$, we set

$$P_d(\mathcal{G}) = \{p \in Prob(\mathcal{G}) : \exists x \in \mathcal{G}^{(0)}, r^*p = \delta_x \text{ and } l^x(g) \leq d,\ \forall g \in \text{supp } p\}$$

endowed with the $*$-weak topology, and with the natural action of $\mathcal{G}$ by translation.
If $p \in P_d(\mathcal{G})$ such that $r^*p = \delta_x$ for a certain $x \in \mathcal{G}^{(0)}$, we can write

$$p = \sum_{g \in \mathcal{G}^x} \lambda_g(p)\delta_g.$$

If we set $\phi^2(p) = \lambda_{e_x}(p) \geq 0$, we have $\phi \in C_0(P_d(\mathcal{G}))$ and $(g.\phi^2)(p) = \lambda_g(p)$. Now define :

$$\mathcal{L}_d = \sum_{g \in \mathcal{G}^x} \phi.(g.\phi) \in C(X, C_0(P_d(\mathcal{G}))) \subset C(\mathcal{G}, C_0(P_d(\mathcal{G})))$$

because X is compact. **? ?** $\mathcal{L}_d$ is a projection of $C_0(P_d(\mathcal{G})) \rtimes_r \mathcal{G}$ with finite propagation, and defines a class $[\mathcal{L}_d]_{\epsilon,R} \in K_0^{\epsilon,R}(C_0(P_d(\mathcal{G})) \rtimes_r \mathcal{G}))$ for all $R > 0, 0 < \epsilon < 1/4$.

Definition 38. Let B be a $\mathcal{G}$-algebra, and $R > 0, 0 < \epsilon < 1/4, d > 0$. The local quantitative assembly map for $\mathcal{G}$ is defined as the composition of $J_{\mathcal{G}}$ with the evaluation at $[\mathcal{L}_d]$:

$$\mu_B^{\epsilon,R,d} \left\{ \begin{array}{rcl} KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B) & \to & K_*^{\epsilon,R}(B \rtimes \mathcal{G}) \\ z & \mapsto & J_{\mathcal{G}}^{\epsilon,R}(z)([\mathcal{L}_d]_{\epsilon,R}) \end{array} \right.$$

Remarks

(1) The assembly map is defined for all reasonnable crossed-products by $\mathcal{G}$. In particular for the reduced one and the maximal one, so that we have two different assembly, which we would distinguish writing $J_{\mathcal{G},r}$ and $J_{\mathcal{G},max}$ if necessary.

(2) The bunch of assembly maps $\mu_B^{\epsilon,R,d}$ induces the Baum-Connes assembly map for $\mathcal{G}$ in K-theory : the following diagram commutes

$$\begin{tikzcd} KK^G(C_0(P_d(\mathcal{G})), B) \arrow[r, "\mu_B^{\epsilon,R,d}"] \arrow[dr, "\mu_{\mathcal{G}}^d"'] & K_*^{\epsilon,R}(B \rtimes \mathcal{G}) \arrow[d, "\iota_{\epsilon,R}"] \\ & K_*(B \rtimes \mathcal{G}) \end{tikzcd}$$

because $J_{\mathcal{G}}(z)$ induces the right multiplication by $j_{\mathcal{G}}(z)$ and also $\mu_{\mathcal{G}}^d(z) = [\mathcal{L}_d] \otimes j_{\mathcal{G}}(z)$. But, as $\mathcal{L}_{d'|P_d(\mathcal{G})} = \mathcal{L}_d$ as soon as $d \leq d'$, this diagram commutes with inductive limit over d.

(3) In [5], H. Oyono-Oyono and G. Yu defined a bunch of local quantitative coarse assembly maps for a metric space X. For the sake of simplicity, we take X to be discrete and uniformly bounded. Let $\mathcal{C}$ be its coarse structure, that is the set of all its controlled subsets. Then, for any C^*-algebras A and B and a K-cycle $z \in KK(A, B)$, they construct a controlled morphism

$$\sigma_X(z) : \hat{K}(C^*(X, A)) \to \hat{K}(C^*(X, B)).$$

There exists a projection P_X with finite propagation, and the local quantitative assembly map is defined as

$$A_{X,B}^{\epsilon,r,d}(z) = \sigma_X^{\epsilon,r}(z)([P_X]_{\epsilon,r})$$

for $z \in KK(C_0(P_d(X)), B)$, where $P_d(X)$ is the classical Rips complex of X. This bunch of assembly maps induce the usual coarse assembly map of X

$$A_{X,B} : KX_*(X, B) \to K_*(C^*(X, B)$$

in K-theory. Now let $\mathcal{G}$ be the coarse groupoid of X. It is an étale groupoid with compact base space $\mathcal{G}^{(0)} = \beta X$, the Stone-Cech compactification of X defined as

$$\mathcal{G} := \cup_{E\in\mathcal{C}}\overline{E},$$

where $\overline{E}$ is the closure of E in $\beta(X\times X)$. A classical result of G. Skandalis, J.-L. Tu and G. Yu [13] claims that the coarse Baum-Connes conjecture for X with coefficients in B is equivalent to the Baum-Connes conjecture for the groupoid G with coefficient in $l^\infty(X, K_B)$. More precisely, there is an isomorphism of C^*-algebras $\Psi_B : l^\infty(X, K_B) \rtimes_r \mathcal{G} \simeq C^*(X, B)$ and the following diagram commutes :

$$\begin{tikzcd}
KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G}), l^\infty(X, K_B)) \arrow[r, "\mu^d_{\mathcal{G}, l^\infty(X,K_B)}"] \arrow[d, "\iota^*"] & K_*(l^\infty(X, K_B) \rtimes_r G) \arrow[d, "(\Psi_B)_*"] \\
KK_*(C_0(P_d(X), B) \arrow[r, "A^d_{X,B}"] & K_*(C^*(X, B))
\end{tikzcd}$$

where the left vertical arrow comes from the inclusion of groupoid $\iota : \{x\} \to \mathcal{G}$ for any $x \in X$. We claim that we can prove a controlled analogue of this result which induces it in K-theory.
To prove this, we shall describe Ψ more precisely. For any C^*-algebra B, let $\tilde{B} = l^\infty(X, K_B)$. It is naturaly a $\mathcal{G}$-algebra, and the fiber over any $x \in \beta X$ is easily seen to be $\tilde{B}_x = B$. Now, if $f \in C_c(\mathcal{G}, \tilde{B})$, as $\overline{E}$ are the compact-open of $\mathcal{G}$, f is continuous over a $\overline{E}$, so it is just a bounded function over E.
Define for $g = (x, y) \in X \times X \subset \mathcal{G}$, $\Psi_B(f)_{xy} = f(g)(x) \in \tilde{B}$, so that $\Psi_B(f) = (\Psi_B(f)_{xy})_{x,y\in X}$ is a locally compact operator of finite propagation (its support is in E). This is a $*$-morphism which extends to the annouced isomorphism. Moreover, $\tilde{B}$ is naturally a C^*-subalgebra of both $\tilde{B} \rtimes_r \mathcal{G}$ and $C^*(X, B)$, and the two inclusion commute modulo Ψ_B. We have a commutative diagram :

$$\begin{tikzcd}
 & B \arrow[ddr, bend left, "\iota_3^B"] & \\
 & \tilde{B} \arrow[u, "ev_x"] \arrow[dl, hook, "\iota_1^B"'] \arrow[dr, hook, "\iota_2^B"] & \\
\tilde{B} \rtimes_r \mathcal{G} \arrow[rr, "\Psi_B"] & & C^*(X, B)
\end{tikzcd}$$

Off course, Ψ_B induces $\Psi_{B*} : \mathcal{L}_{\tilde{B}\rtimes_r\mathcal{G}}(H_{B\rtimes_r\mathcal{G}}) \to \mathcal{L}_{C^*(X,B)}(\mathcal{E})$ where $\mathcal{E} = H_{B\rtimes_r\mathcal{G}} \otimes_{\Psi_B} C^*(X, B)$.
Let A and B be two C^*-algebra and $z \in KK_1^{\mathcal{G}}(\tilde{A}, \tilde{B})$, represented by $(H_{\tilde{B}}, \psi, T)$. As $T_{\mathcal{G}} = (\iota_1)_*(T)$, we have $(\Psi_B)_*(T_{\mathcal{G}}) = (\iota_2)_*(T) = (T_x)_X$. Also, the relations $(\iota_1^A)_* \circ \psi = \psi_{\mathcal{G}} \circ \iota_1^A$ and $(\iota_2^A)_* \circ \psi_x = (\psi_x)_X \circ \iota_2^A$ are easy to derive, which lead to $(\Psi_B)_* \circ \psi_G \circ \iota_1^A = (\iota_2^B)_* \circ \psi_x = (\psi_x)_X \circ \Psi_A \circ \iota_1^A$. By extending $\mathcal{G}$-equivariantly to $\tilde{A} \rtimes_r G$, we have $(\Psi_B)_*(\psi_{\mathcal{G}}(a)) = (\psi_x)_X(\Psi_A(a))$. The map $(x, y) \mapsto (\Psi_A(x), (\Psi_B)_*(y))$ induces a morphism $\Psi_E : E^{(\psi,T)} \to E^{((\psi_x,T_x))}$ which sends $(x, P_{\mathcal{G}}\psi_{\mathcal{G}}(x)P_{\mathcal{G}} + y)$ to

$(\Psi_A(x),(P_x)_X(\psi_x)_X(\Psi_A(x))(P_x)_X+(\Psi_B)_*(y))$ by the previous computations. This map makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\tilde{B}\rtimes\mathcal{G}} & \longrightarrow & E^{(\psi,T)} & \longrightarrow & \tilde{A}\rtimes_r\mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle(\Psi_B)_*} & & \downarrow{\scriptstyle\Psi_E} & & \downarrow{\scriptstyle\Psi_A} & & \\ 0 & \longrightarrow & K_{C^*(X,B)} & \longrightarrow & E^{(\psi_x,T_x)} & \longrightarrow & C^*(X,A) & \longrightarrow & 0 \end{array}.$$

Now the remark 3.7 of [10] gives $((\Psi_B)_*)_*\circ D^{K_{\tilde{B}\rtimes\mathcal{G}}}_{\tilde{A}\rtimes_r\mathcal{G}}=D^{K_{C^*(X,B)}}_{C^*(X,A)}\circ(\Psi_A)_*$, and if we compose by the Morita equivalence, we get

$$\sigma(\iota^*(z))\circ(\Psi_A)_*=(\Psi_B)_*\circ J_{\mathcal{G}}(z),$$

where $\iota^*(z)$ is indeed the class of (H_B,ψ_x,T_x).

As $C_0(P_d(\mathcal{G}))$ is a $\mathcal{G}$-algebra whose fiber over any $w\in\beta X$ is isomorphic to $C_0(P_d(X))$, if $A=C_0(P_d(\mathcal{G}))$, then $(\Psi_A)_*[\mathcal{L}_d]\in K_0^{\epsilon,R}(C^*(X,C_0(P_d(X)))$ which is equal to $[P_X]$, and gives the result.

$$(\Psi_B)_*\circ\mu_{\mathcal{G}}^{\epsilon,R}(z)=A^{\epsilon,R}(\iota^*(z)).$$

This, passing to K-theory, implies the result of [13].

8.6 Quantitative statements

Proposition 22. Let A be a $\mathcal{G}$-algebra.
If the following statement is true :

•(Quantitative Injectivity) $\forall d\geq 0$, there exists $\epsilon\in(0,\frac{1}{4})$ such that, for all $r\geq r_{d,\epsilon}$, there exists $d'\geq d$ such that if $x\in KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G})),A)$ satisfies $\mu_{\mathcal{G}}^{\epsilon,R,d}(x)=0\in K^{\epsilon,R}$, then $x=0$ in $KK^{\mathcal{G}}(C_0(P_{d'}(\mathcal{G})),A)$;

then $\mu_{\mathcal{G},A}$ is injective.

On the other hand, if this statement is true :

•(Quantitative Surjectivity) there exists $\epsilon'\in(0,\frac{1}{4})$ such that $\forall r'\geq r_{d,\epsilon},\exists\epsilon,r$ such that $\epsilon'\leq\epsilon<\frac{1}{4}$ and $r_{d,\epsilon}\leq r\leq r'$, such that for all $y\in K_*^{\epsilon',r'}(A\rtimes\mathcal{G}),\exists x\in KK_*^{\mathcal{G}}(C_0(P_d(\mathcal{G})),A)$ such that $\mu_{\mathcal{G},A}^{\epsilon',r',d}=\iota_{\epsilon,r}^{\epsilon',r'}(y)$;

then $\mu_{\mathcal{G},A}$ is surjective.

Proof 10. Let $x\in KK(C_0(P_d(\mathcal{G})),A)$ which satisfies $\mu_{\mathcal{G},A}(x)=0$, then $\iota_{\epsilon,r}\circ\mu_{\mathcal{G},A}^{\epsilon,r,d}(x)=0$. By remark 1.18 of [10], there exists a universal $\lambda>0$ and a certain $r'>0$ such that

$$\begin{aligned} 0 &= \iota_{\epsilon,r}^{\lambda\epsilon,r'}\circ\mu_{\mathcal{G},A}^{\epsilon,r,d}(x)\\ &= \iota_{\epsilon,r}^{\lambda\epsilon,r'}(J_{\mathcal{G}}^{\epsilon,r}(x)([\mathcal{L}_d]_{\epsilon,r}))\\ &= J_{\mathcal{G}}^{\lambda\epsilon,r'}(x)([\mathcal{L}_d]_{\lambda\epsilon,r'})\\ &= \mu_{\mathcal{G},A}^{\lambda\epsilon,r',d}(x). \end{aligned}$$

But then the quantitative injectivity condition assures that $x = 0$ in $KK^{\mathcal{G}}(C_0(P_{d'}), A)$ and $x = 0$ in the inductive limite over d $K^{top}(G, A)$.
The second point is immediate.

□

This kind of statement leads us to define the following proprieties, following [5].

- $QI_{\mathcal{G},B}(d, d', R, \epsilon)$: for any $x \in KK^{\mathcal{G}}(C_0(P_d(\mathcal{G})), B)$, $\mu_{\mathcal{G}}^{\epsilon,R}(x) = 0$ implies $\iota_d^{d'}(x) = 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), B)$.
- $QS_{\mathcal{G},B}(d, R, R', \epsilon, \epsilon')$: for any $y \in K^{\epsilon,R}(B \rtimes \mathcal{G})$, there exists $x \in KK^{\mathcal{G}}(P_d(\mathcal{G}), B)$ such that $\mu_{\mathcal{G}}^{\epsilon',R'}(x) = \iota_{\epsilon,R}^{\epsilon',R'}(y)$.

Theorem 7. Let B a $\mathcal{G}$-algebra, and $\tilde{B} = l^\infty(X, K_B)$. Then $\mu_{\mathcal{G},\tilde{B}}$ is injective if and only if for all $d, \epsilon, r \geq r_{\mathcal{G},d,\epsilon}$, there exists $d' \geq d$ such that $QI_{\mathcal{G},B}(d, d', \epsilon, R)$.

To prove the theorem, we will need a serie of lemmas.

Lemma 4. Let Z be a $\mathcal{G}$-compact proper $\mathcal{G}$-space such that the anchor map $p : Z \to \mathcal{G}^{(0)}$ is locally injective, and let (B_j) be a countable family of $\mathcal{G}$-algebras. Then the projection $\prod_j B_j \otimes K \to B_j \otimes K$ induces an isomorphism

$$KK^{\mathcal{G}}(C_0(Z), \prod_j B_j \otimes K) \to \prod_j KK^{\mathcal{G}}(C_0(Z), B_j \otimes K).$$

Proof 11. Let $B_\infty = \prod B_j \otimes K$ and $p_k : B\infty \to B_k$ the projection.
Let $(\mathcal{E}, \varphi, F) \in E^{\mathcal{G}}(C_0(Z), B_\infty)$ be a cycle such that every

$$(\mathcal{E}_k, \varphi_k, F_k) = (p_k)_*(\mathcal{E}, \varphi, F)$$

is homotopic to 0. According to [5], we can choose a homotopy which is C-Lipschitz on the Calkin algebra for a universal constant $C > 0$, hence ([18], Lemma 17.3.3) we can find a family of compact operators $T_{s,t} \in K(\mathcal{E})$ such that $||F_s - F_t + T_{s,t}|| \leq C|s - t|$. But $t \mapsto F'_t = F_t + T_{0,t}$ is a compact perturbation of $s \mapsto F_s$ in $\mathcal{L}(\mathcal{E})$ which is C-Lipschitzian. Up to replace $(F_s)_s$ with (F'_t), we can suppose the homotopies are uniformly Lipschitzian, and $\tilde{F} = \prod F_j$ defines a bounded operator.
We now use an idea of [17], lemma 3.6. Namely, using the local injectivity of p, we show that F can be supposed to commute with φ and $\mathcal{G}$. For the sake of completness, we recall the proof. First, choose a finite open cover $(U_j)_j$ of a compact fundamental domain K for the action of $\mathcal{G}$ such that $p_{|U_j}$ is injective, and take compactly supported continuous functions $\phi_j : Z \to \mathbb{R}_+$ such that supp $\phi_j \subset U_j$ and $K \subset \cup \phi_j^{-1}(0, +\infty)$. We can suppose $\sum_{j,g \in \mathcal{G}^{p(z)}} \phi_j(zg) = 1, \forall z \in Z$. Now define $F'_x = \sum_{j,g \in \mathcal{G}^x} \alpha_g(\phi^{\frac{1}{2}} F_{s(g)} \phi^{\frac{1}{2}})$. It is an $\mathcal{G}$-invariant operator which commutes with the action of $C_0(Z)$.
Now we can see that $(\prod_j \mathcal{E}, \prod \varphi_j, \prod_j F_j)$ defines a cycle as $[\varphi(a), \tilde{F}] = 0$ and $\varphi(a)(\alpha_g(F_{s(g)} - \tilde{F}_{r(g)}) = 0$. Moreover it is unitarly equivalent to $(\mathcal{E}, \varphi, \tilde{F})$, and homotopic to 0.
For the surjectivity, just take $[(\prod \mathcal{E}_j, \prod \varphi_j, \prod F_j)]$ as a preimage of $\prod_j [(\mathcal{E}_j, \varphi_j, F_j)]$, using the previous construction.

□

Lemma 5. Let $\mathcal{G}$ be a locally compact, σ-compact étale groupoid, $\{B_j\}_{j\geq 0}$ a family of $\mathcal{G}$-algebras and K the algebra of compact operators over a separable Hilbert space. Set $\Delta = P_d(\mathcal{G})$, then we have an $\mathbb{Z}_2$-graded ismorphism

$$KK^{\mathcal{G}}(C_0(\Delta), \prod_j B_j \otimes K) \simeq \prod_j KK^{\mathcal{G}}(C_0(\Delta), B_j)$$

Proof 12. For all j and any locally compact $\mathcal{G}$-space X, the projection $\prod_j B_j \otimes K \to B_j \otimes K$ induces a morphism

$$\Theta^X : KK^G(C_0(X), \prod_j B_j \otimes K) \to \prod_j KK^G(C_0(X), B_j \otimes K).$$

Let $X_0 \subset X_1 \subset ... \subset X_n$ be the n-skeleton decomposition associated to the simplicial structure of the Rips complex Δ and let $Z_j = C_0(X_j)$, $Z^j_{j-1} = C_0(X_j - X_{j-1})$ and $\Theta_j = \Theta^{X_j}$. We will show the claim by induction on the dimension of Δ.

The extension of $\mathcal{G}$-algebras $0 \to Z^j_{j-1} \to Z_j \to Z_{j-1} \to 0$ gives a commutative diagram with exact lines :

$$\begin{tikzcd}
KK_*(Z^j_{j-1}, \prod_j B_j \otimes K) \arrow[r, "\delta"] \arrow[d, "\Theta^j_{j-1}"] & KK_*(Z_{j-1}, \prod_j B_j \otimes K) \arrow[r] \arrow[d, "\Theta_{j-1}"] & KK_*(Z_j, \prod_j B_j \otimes K) \arrow[r] \arrow[d, "\Theta_j"] & KK_*(Z^j_{j-1}, \prod_j \arrow[d, "\Theta^j_{j-}"] \\
\prod_j KK_*(\tilde{Z}^j_{j-1}, B_j \otimes K) \arrow[r, "\delta"] & \prod_j KK_*(\tilde{Z}_{j-1}, B_j \otimes K) \arrow[r] & \prod_j KK_*(\tilde{Z}_j, B_j \otimes K) \arrow[r] & \prod_j KK_*(\tilde{Z}^j_{j-1},
\end{tikzcd}$$

The five lemma assures that if Θ_{j-1} and Θ^j_{j-1} are isomorphisms, then so is Θ_j. Moreover, because Δ is a typed simplicial simplex (see [17]), $X_j - X_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where $\mathring{\sigma}_j$ denotes the interior of the standard simplex, and Σ_j is the set of centers of j-simplices of X_j. Bott periodicty assures then that, if Θ_{j-1} is an isomorphism, then so is Θ^j_{j-1}. By induction, proving that Θ_0 is an isomorphism concludes the proof, which is essentially the content of lemma 4 : X_0 is a $\mathcal{G}$-compact proper $\mathcal{G}$-space, and its anchor map is just the target map $r : \mathcal{G} \to \mathcal{G}^{(0)}$, which is supposed to be étale, so locally injective.

□

We can now prove the theorem 7.

Proof 13. Let $x \in KK^{\mathcal{G}}(P_d(\mathcal{G}), \tilde{B})$ such that $\mu_{G,\tilde{B}}(x) = 0$. Then, as the quantitative assembly maps factorize $\mu_{G,\tilde{B}}$, there exist $\epsilon > 0$ and $R \geq r_{\mathcal{G},\tilde{B},d}$, such that $\mu^{\epsilon,R}_{\mathcal{G},\tilde{B}}(x) = 0$. Using the isomorphism of lemma 5 and the Morita equivalence, we can identify x with $(x_j)_j$ under $KK^{\mathcal{G}}(P_d(\mathcal{G}), \tilde{A}) \simeq \prod_j KK^{\mathcal{G}}(P_d(\mathcal{G}), A)$. Now let $d' \geq d$ such that $QI_A(d, d', \epsilon, R)$ holds. That assures that $x_j = 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), B)$, and $x = 0$.

For the converse, suppose one can find d, ϵ, R such that $QI_{\mathcal{G},A}(d, d', \epsilon, R)$ is NOT true for all $d' \geq d$. Then one can extract a increasing sequence d_j diverging to $+\infty$ and $x_j \in KK^{\mathcal{G}}(P_d(\mathcal{G}), A)$ such that $\mu^{\epsilon,R}_{\mathcal{G},\tilde{B}}(x_j) = 0$ and $x_j \neq 0$ in

$KK^{\mathcal{G}}(P_{d_j}, A)$. Let $x \in KK^{\mathcal{G}}(P_d, \tilde{A})$ be the image of $(x_j) \in \prod KK^{\mathcal{G}}(P_d, A)$. We have $\mu_{\mathcal{G},\tilde{A}}(x) = 0$, and $x \neq 0$ in $KK^{\mathcal{G}}(P_{d'}(\mathcal{G}), \tilde{A})$ for all $d' \geq d$, so $\mu_{\mathcal{G},\tilde{A}}$ is not injective.

□

We also have a theorem relating quantitative surjectivity for $\mu_{\mathcal{G},B}$ and surjectivity of $\mu_{\mathcal{G},\tilde{B}}$.

Theorem 8. Let B a $\mathcal{G}$-algebra, and $\tilde{B} = l^\infty(X, K_B)$. Then there exists $\lambda > 1$ such that $\mu_{\mathcal{G},\tilde{B}}$ is onto if and only if for any $0 < \epsilon < \frac{1}{4\lambda}$ and $R > 0$, there exist $R' \geq \max(R, r_{\mathcal{G},d,\epsilon})$ and $d > 0$ such that $QS_{B,\mathcal{G}}(d, R, R', \epsilon, \lambda\epsilon)$ holds.

Proof 14. Let $\lambda > 0$ the universal constant of remark 1.18 of [10] : for any C^*-algebra and $x, y \in K^{\epsilon,R}(A)$ such that $\iota_{\epsilon,R}x = \iota_{\epsilon,R}y$, there exists $R' \geq R$ such that $\iota_{\epsilon,R}^{\lambda\epsilon,R'}x = \iota_{\epsilon,R}^{\lambda\epsilon,R'}y$.

Let $y \in K_*(\tilde{B} \rtimes \mathcal{G})$, and take $z \in K^{\epsilon,R}$, where $R > 0, \epsilon < \frac{1}{4\lambda}$, such that $\iota_{\epsilon,R}z = y$. The projection on the j^{th} component $\tilde{B} \to K_B$ used in K-theory then composed with Morita equivalence gives a map $K^{\epsilon,R}(\tilde{B} \rtimes \mathcal{G}) \to K^{\epsilon,R}(B \rtimes \mathcal{G})$, and z_j denotes the image of z under this map. We can pick d and $R' \geq \max(r_{\mathcal{G},d,\epsilon})$ such that $QS(d, R, R', \epsilon, \lambda\epsilon)$: for every j, there exists $x_j \in KK^{\mathcal{G}}(P_d, B)$ such that $\mu_{\mathcal{G},B}^{d,\lambda\epsilon,R'}(x_j) = \iota_{\epsilon,R}^{\lambda\epsilon,R'}z_j$. As $KK^{\mathcal{G}}(P_d, \tilde{B}) \simeq \prod_j KK^{\mathcal{G}}(P_d, B)$, (x_j) can be taken as an element x of $KK^{\mathcal{G}}(P_d, \tilde{B})$. Naturality of the assembly maps, and compatibility of quantitative assembly maps with the usual one assures that $\mu_{\mathcal{G},\tilde{B}}(x) = z$, whereby $\mu_{\mathcal{G},\tilde{B}}$ is onto.

Suppose that there exist $0 < \epsilon < \frac{1}{4\lambda}$ and $R > 0$ such that for every positive numbers $d > 0$ and $R' \geq \max(r_{\mathcal{G},B}, R)$, $QS(d, R, R', \epsilon, \lambda\epsilon)$ does not hold. Let (d_j) and (R_j) be unbounded increasing sequences of positive numbers and $y_j \in K^{\epsilon,R}(B \rtimes \mathcal{G})$ such that $\iota_{\epsilon,R}^{\lambda\epsilon,R_j}(y_j)$ is not in the range of $\mu_{\mathcal{G},B}^{d_j,\lambda\epsilon,R_j}$. Let $y \in K^{\epsilon,R}(\tilde{B} \rtimes \mathcal{G})$ be an element such that its image with the previous map coincides with y_j. If there exists $x \in KK^{\mathcal{G}}(P_s, \tilde{B})$ for a $s \geq d$ such that $\iota_{\epsilon,R}(y) = \mu_{\mathcal{G},\tilde{B}}^s(x)$ then there would exists a $R' \geq R$ such that

$$\iota_{\epsilon,R}^{\lambda\epsilon,R'}(y) = \mu_{\mathcal{G},\tilde{B}}^{s,\lambda\epsilon,R'}(x) = \iota_{\epsilon,R}^{\lambda\epsilon,R'} \circ \mu_{\mathcal{G},\tilde{B}}^{s,\epsilon,R}(x).$$

Now choose j such that $d_j \geq s$ and $R_j > R'$, and compose the previous equality with $\iota_{\lambda\epsilon,R'}^{\lambda\epsilon,R_j}$ and $q_s^{d_j}$ to obtain $\iota_{\epsilon,R}^{\lambda\epsilon,R_j}(y_j) = \mu_{\mathcal{G},B}^{d_j,\lambda\epsilon,R_j}$ which contradicts our assumption.

□

In particular, if the groupoid G satisfies the Baum-Connes conjecture with coefficients, it satisfies the quantitative Baum-Connes conjecture. Interesting examples follow from the result of J-L. Tu [14] that a-T-menable groupoids satisfy the Baum-Connes conjecture with coefficients. In particular,

- amenable groupoids are a-T-menable.

- Let X be a uniformly discrete metric space with bounded geometry. Then, if X is coarsely embeddable into a separable Hilbert space, $G(X)$ is a-T-menable.[13]

- If X admits a fibred coarse embedding into Hilbert space, then $G(X)_{|\partial\beta X}$ is a-t-menable.[3] For interesting examples of this type, recall the definition of a box space. Let Γ be a finitely generated group, and $\mathcal{N}$ a family of nested normal subgroups with trivial intersection, which have finite index in Γ. Take the coarse union of the quotients to construct a coarse space $X_{\mathcal{N}}(\Gamma) = \cup_{H\in\mathcal{N}}\Gamma/H$. Then, $X_{\mathcal{N}}(\Gamma)$ admits a fibred coarse embedding if and only if Γ is a-T-menable. But if $X_{\mathcal{N}}$ is an expander, it cannot be coarsely embedded into a Hilbert space, so just take an a-T-menable group which has a box space X which is an expander to get a coarse space that is not coarsely embeddable into Hilbert space ($SL(2,\mathbb{Z})$ works), but admits a fibred coarse embedding.

We recall the following definition from [5].

Definition 39. Let B be a filtered C^*-algebra.

- $PA_B(\epsilon,\epsilon',R,R')$: for every $x\in K_*^{\epsilon,R}(B)$ such that $\iota_{\epsilon,R}(x)=0$ in $K_*(B)$, then $\iota_{\epsilon,R}^{\epsilon',R'}(x)=0$ in $K_*^{\epsilon',R'}(B)$.
- B is said to satisfy the Persistance Approximation Property (PAP) if for every $R>0$ and $\epsilon\in(0,\frac{1}{4})$, there exists $R'\geq 0$ and $\epsilon'\in[\epsilon,\frac{1}{4})$ such that $PA(\epsilon,\epsilon',R,R')$ holds.

Theorem 9. Up to some hypothesis, if $\mu_{G,l^\infty(\mathbb{N},K_A)}$ is onto and $\mu_{G,A}$ is one-to-one, then, for a universal constant λ_{PA}, for all $\epsilon\in(0,\frac{1}{4\lambda_{PA}})$ and $R>0$, there exists $R'\geq R$ such that $PA_{A\rtimes G}(\epsilon,\lambda_{PA}\epsilon,R,R')$ holds.

Proof 15. We denote $l^\infty(\mathbb{N},K_A)$ by $\tilde{A}$.
Assume the statement does not holds : there exists ϵ and R such that $PA(\epsilon,\epsilon',R,R')$ is not true for every $R'\geq R$. Then we can extract a increasing unbounded sequence of positive numbers R_j and elements $x_j\in K_*^{\epsilon,R}(A\rtimes G)$ such that $\iota_{\epsilon,R}(x_j)=0$ and $\iota_{\epsilon,R}^{\lambda_{PA},R_j}(x)\neq 0$.
According to the LEMMA, there exists $x\in K_*^{\alpha\epsilon,h_\epsilon R}(\tilde{A}\rtimes G)$ such that $p_j(x)=x_j$ where p_j is the composition of the projection on the j^{th} factor $\tilde{A}\rtimes G\to K_A\rtimes G$ and the Morita equivalence in K-theory. By naturality of the quantitative assembly maps, the following diagram commutes

$$\begin{tikzcd} KK^G(P_d,\tilde{A}) \arrow[r, "\mu^{d,\alpha\epsilon,h_\epsilon R}_{G,\tilde{A}}"] \arrow[dr, "\mu^d_{G,\tilde{A}}"'] & K^{\alpha\epsilon,h_\epsilon R}(\tilde{A}\rtimes G) \arrow[d, "\iota_{\alpha\epsilon,h_\epsilon R}"] \\ & K(\tilde{A}\rtimes G) \end{tikzcd}$$

If $\iota_{\alpha\epsilon,h_\epsilon R}(x)$ is in the range of $\mu_{G,\tilde{A}}$, there exists a $d>0$ and $z\in KK^G(P_d,\tilde{A})$ such that $\mu_{G,\tilde{A}}(z)=\iota_{\alpha\epsilon,h_\epsilon R}(x)$. Denote the image of z under the isomorphism $KK^G(P_d,\tilde{A})\simeq\prod KK^G(P_d,A)$ by $(z_j)_j$. By naturality, $\mu^d_{G,\tilde{A}}(z_j)=\iota^{\lambda\epsilon,R_j}_{\alpha\epsilon,h_\epsilon R}(x_j)\neq 0$ and $\mu_{G,A}(z)=0$ so that $\mu_{G,A}$ is not injective.

$\square$

9 Decomposition complexity

Let G be alocally compact étale groupoid with a proper length l. Fix $s > 0$, and $G_s = \{g \in G : l^{r(g)}(g) \leq s\}$. For each subset U of X, we can define the s-neiborghood of U :

$$U_s = \{x \in X : \exists g \in G_s \text{ s.t. } r(g) \in U \text{ and } s(g) = x\}.$$

Definition 40. Let G be an étale groupoid with compact base space X, and $\mathcal{F}$ a family of groupoids . We say that G decomposes over $\mathcal{F}$ if, for every open relatively compact subset K of G, every $R > 0$, there exists a decomposition $X = X^{(1)} \cup X^{(2)}$ of $K^{(0)} := r(K) \cup s(K)$ into (open ?) subspaces such that each subspace $X^{(i)}$ is a disjoint union $\sqcup_j X_j^{(i)}$ satisfying the following conditions :

- if $j \neq j'$, $X_{j,R}^{(i)} \cap X_{j'}^{(i)} = \varnothing$
- there exists $s > 0$ such that for every i, j, the groupoid $G_{j,s}^{(i)}$ generated by $G_{|X_j^{(i)}} \cap G_s$ is relatively compact and is in $\mathcal{F}$.

Definition 41. Let $R > 0$. An étale groupoid is said to be R-decomposable over $\mathcal{F}$ if for every open relatively compact subset K of G, every $R > 0$, there exists an open cover of $K^{(0)} := r(K) \cup s(K) \subset V^{(0)} \cup V^{(1)}$ such that each subspace $X^{(i)}$ is a disjoint union $\sqcup_j X_j^{(i)}$ satisfying the following conditions :

- if $j \neq j'$, $X_{j,R}^{(i)} \cap X_{j'}^{(i)} = \varnothing$
- there exists $s > 0$ such that for every i, j, the groupoid $G_{j,s}^{(i)}$ generated by $G_{|X_j^{(i)}} \cap G_s$ is in $\mathcal{F}$.

We write $\mathcal{G} \xrightarrow{R} \mathcal{F}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of C^*-algebras.

Definition 42. We say that $\mathcal{A}$ is decomposable over $\mathcal{B}$ if for every A in $\mathcal{A}$ and every $\epsilon > 0$ and every finite subset F of A, there exist two C^*-algebras $B^{(0)}$ and $B^{(1)}$ in $\mathcal{B}$ and ccp maps Ψ and Φ

$$\begin{tikzcd} & B^{(0)} \oplus B^{(1)} \arrow[dr, "\Phi"] & \\ A \arrow[ur, "\Psi"] & & A \end{tikzcd}$$

such that $\Phi_{|B^{(i)}}$ is of order zero and

$$||\Phi \circ \Psi(a) - a|| < \epsilon, \forall a \in F.$$

9.1 Examples and applications

- **Motivating example.** Let $G = G(X)$ the coarse groupoid associated to a uniformly discrete metric space X with bounded geometry. Then G has FDC iff X has FDC.

- **Dynamic asymptotic dimension.** Recall that an étale groupoid is said to have dynamic asymptotic dimension d if it is the smallest integer such that for every relatively compact subset K of G, there exist open subsets U_0, ...,U_d of $G^{(0)}$ such that they cover $r(K) \cup s(K)$ and for each j, $G_{|U_j} \cap K$ generates a relatively compact subgroupoids of G. Then if G has finite dynamic asymptotic dimension, it has FDC.

- **Nuclear dimension :** Obtaining a bound on $\dim_{nuc}(A \rtimes G)$ w.r.t. $\dim_{nuc}(A \rtimes G_1)$ and $\dim_{nuc}(A \rtimes G_2)$, maybe $\dim_{cov}(G^{(0)})$.
- **Mayer Vietoris :** $A \rtimes G_1$ and $A \rtimes G_1$ are a coercive pair, and we have quantitative Mayer-Vietoris.

10 Mayer-Vietoris exact sequences and controlled cutting and pasting

10.1 Mayer-Vietoris sequence in K-homology

We first recall how to construct a Mayer-Vietoris element out of any pull back diagram of C^*-algebras.

Let us decompose X into two open sets U_0 and U_1 and consider the C^*-algebras $A = C_0(X)$, $A_j = C_0(U_j)$ where $U_{01} = U_0 \cap U_1$, and the cone C of $A_0 \oplus A_1$, with the canonical morphisms $\alpha : C \to A$ and $\beta : C \to SA_{01}$. The first is an homotopy equivalence, and the Mayer-Vietoris boundary is defined as the dotted arrow in the following commutative diagram

$$\begin{tikzcd} KK^G_*(C,B) \arrow[r, "\beta^*"] \arrow[d, "\alpha^*"] & KK^G_*(SA_{01},B) \arrow[d, "{[\partial_{A_{01}}]\otimes -}"] \\ KK^G_*(A,B) \arrow[r, dotted] & KK^G_{*+1}(A_{01},B) \end{tikzcd}$$

where the right vertical arrow is the Bott element $[\partial_{A_{01}}] \in KK^G_1(A_{01}, SA_{01})$.

At the level of controlled K-theory, Oyono-Oyono and Yu introduced a notion of Mayer-Vietoris pair in a filtered C^*-algebra A weaker than that of a pull-back diagram. Recall that a R-controlled weak Mayer-Vietoris pair for A is a quadruple $(\Delta_0, \Delta_1, A_0, A_1)$ such that for some constant c :

- if $x \in M_n(A_s)$ for $s \leq R$ and any $n > 0$, there exists $x_j \in M_n(\Delta_j \cap A_s)$ such that $x = x_0 + x_1$ and $||x_j|| \leq c||x||$,
- A_j is filtered by $(A_j \cap A_s)_{s\geq 0}$ and $C^*N^{(R,5R)}_{\Delta_j} \subset A_j$
- for any $\epsilon > 0$, if $x \in M_n(A_{0,s})$, $y \in M_n(A_{1,s})$ such that $||x - y|| < \epsilon$, there exists $z \in M_n(A_{0,s} \cap A_{1,s})$ with $||z - x|| < \epsilon$ and $||z - y|| < \epsilon$.

For example, take G to be an étale groupoïd with proper length l, with compact base space X. The convolution algebra C^*_rG is filtered by $(C_c(G_R))_{R>0}$, $G_R = l^{-1}[0, R)$. Fix $R > 5r$. If V is an open subset of X, set

- $V^R = \{r(g) : g \in G_V \cap G_R\}$,
- $\Delta_V = C_0(G_V \cap G_R)$
- $G^V_{V,R} = \langle G^V_V \cap G_R \rangle$

then $(\Delta_{V_0}, \Delta_{V_1}, C^*_r(G_0), C^*_r(G_1))$ is a r-controlled Mayer-Vietoris pair for C^*_rG when $X = V_0 \cup V_1$, and $G_j = G^{V_j^R,(R)}_{V_j^R}$.

The existence of a controlled Mayer-Vietoris pair is nice, because even if the C^*-algebra is simple, it can possess such a decomposition, and the following result gives a way to compute the K-theory analogous to the situation of a classical Mayer-Vietoris decomposition :

Theorem 10. For every positive c, there exists a control pair (λ, h) such that for any filtered C^*-algebra A which has a R-controlled Mayer-Vietoris pair $(\Delta_0, \Delta_1, A_0, A_1)$ the following sequence is (λ, h)-exact at order R

$$\begin{tikzcd}
\hat{K}_0(A_0 \cap A_1) \arrow[r] & \hat{K}_0(A_0) \oplus \hat{K}_0(A_1) \arrow[r] & \hat{K}_0(A) \arrow[d, "D"] \\
\hat{K}_1(A) \arrow[u, "D"] & \hat{K}_1(A_0) \oplus \hat{K}_1(A_1) \arrow[l] & \hat{K}_1(A_0 \cap A_1) \arrow[l]
\end{tikzcd}$$

where D is a controlled Mayer-Vietoris boundary.

To go back to our example, if $X = V_0 \cup V_1$, we simulteanously have two decompositions : that arising from the decomposition of the Rips simplex into two open sets $P_d(G_{V^R}^{V^R,(R)})$, and that of the controlled Mayer Vietoris pair. The aim of this section is to show that the quantitative assembly maps respects these two exact sequences in a precise way.

Actually, in this particular case, the quantitative Mayer-Vietoris exact sequence should hold at all orders, which would allows us to state a much stronger result, with more interesting applications : a Künneth formula for crossed product algebras of étale groupoids.

A remark : There is a seemingly harmful parallel between the controlled Mayer-Vietoris decomposition and the nuclear dimension of the reduced C^*-algebra. Namely, to show that the pair satisfies the Mayer-Vietoris conditions, we make use of the completely positive map induced by

$$\begin{cases} C_c(G) & \to & C_c(G) \\ f & \mapsto & \phi_0 \circ r \ .f.\ \phi_0 \circ s \end{cases}$$

where ϕ_0 is any continuous function $X \to [0,1]$ with support in V_0 which is 1 on some compact $K \subset V_0$.
Could we push the analogy further ?

10.2 Standard modules

The aim of this section is to develop a notion of non-degenerate standard modules over a groupoid analogous to non-degenerate standard modules over coarse spaces.

Let us first recall the coarse case. Let X be a discrete metric space with bounded geometry.

Definition 43. A X-module is a Hilbert space H_X equipped with a $*$-representation $\phi : C_0(X) \to \mathcal{L}(H_X)$. The X-module (H_X, ϕ) is said to be :

- standard if $\phi(C_0(X))H_X$ is dense in H_X,
- non-degenerate if $\forall f \in C_0(X), \phi(f) \in \mathfrak{K}(H_X) \implies f = 0$.

The usefulness of n.d.s. X-modules comes from the following lemma :

Lemma 6. Let X and Y be two discrete metric spaces with bounded geometry, and $h : X \to Y$ a coarse map. Then, for any two standard modules H_X and H_Y over X and Y respectively, there exists an isometry which covers h, i.e. for any $\epsilon > 0$, there exists $V \in \mathcal{L}(H_X, H_Y)$ such that

$$\text{supp } V \subset \{(x, y) \in X \times Y, d(h(x), y) < \epsilon\}.$$

Remark : We can induce V on the Roe algebras by $\forall T \in C^*(X, H_X), Ad_V(T) := VTV^* \in C^*(Y, H_Y)$, and the preceding lemma entails that

$$(Ad_V)_* : K(C^*(X, H_X)) \to K(C^*(Y, H_Y))$$

only depends on the coarse class of h. We directly see that the K-theory of the Roe-algebras of X do not depend on the standard modules if they are n.d.s. : one just need to take an isometry covering the identity.

We can actually show that taking the Roe algebra is a functor. Choose, for any coarse space X a n.d.s. X-module H_X, and consider the category $Coarse$ of coarse spaces with morphisms coarse maps, and the category KK with objects C^*-algebras, and morphisms defined by KK-theory, $Hom_{KK}(A, B) = KK(A, B)$. Then $X \mapsto C^*(X, H_X)$ and $(h : X \to Y) \mapsto (Ad_V)_* \in KK(C^*(X, H_X), C^*(Y, H_Y))$ defines a functor $Coarse \to KK$, which does not depend on the choices $X \mapsto H_X$ being made.

We will focus ont the following result :

Theorem 11. Let X be a finite simplicial complex, and B a C^*-algebra. There exists $\varepsilon_0 \in (0, \frac{1}{4})$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists $R_\varepsilon > 0$ s.t.

$$\forall R \in (0, R_\varepsilon),\ Ind^{\varepsilon,R}_{X,B} : KK(C(X), B) \to K^{\varepsilon,R}(B \otimes \mathfrak{K}(H_X))$$

is an isomorphism.

We wish to prove a similar result for "nice" groupoids. Let us recall the definition of G-simplicial complex from [17].

Definition 44. Let G be a locally compact groupoid. A G-simplicial complex of dimension less than n is a triple (Z, Δ, p) where

- Z is a locally compact space of vertices and $p : Z \to G^{(0)}$ a locally injective map, which is an anchor map for an action of G,
- Δ is a closed G-invariant subset of the probability measures $P(Z)$ on Z, equipped with the weak-$*$ topology, with the property that every element of Δ has a support contained in a fiber of p and has at most $n+1$ elements. Such a support is call a simplex of Δ. Moreover, if $\nu \in \Delta, \mu \in P(Z)$ and $supp(\mu) \subset supp(\nu)$ then $\mu \in \Delta$.

The first step is to defined what is a s.n.d. G-module.

Definition 45. A s.n.d. G-module is a triple (E, ϕ, V) where E is a $C_0(G^{(0)})$-algebra, $\phi : C_0(G^{(0)}) \to \mathcal{L}(E)$ is a $*$-representation and V is an action of G on E such that

- E decomposes as a external tensor product $E \simeq E^{(0)} \otimes E^{(1)}$ of G-modules such that there exists $\phi : C_0(G^{(0)}) \to \mathcal{L}(E^{(0)})$ and $\phi = \phi_0 \otimes id_{E^{(1)}}$

- No non-zero function of $C_0(G^{(0)})$ acts as a compact operator via ϕ_0, and $\overline{\phi_0(C_0(G^{(0)}))E^{(0)}} = E^{(0)}$
- For all compact subgroupoids K of G, there exists an isomorphism of K-modules $\mathrm{Res}_K^G E^{(1)} \simeq L^2(K) \otimes H_K$ where H_K is a separable Hilbert space.

Now we want to show the following :

Lemma 7. Let G be a locally compactly induced groupoid, and E, E' any two s.n.d. G-modules, then for any compact subset K of G, there exists an isometry $V \in \mathcal{L}(E, E')$ such that

$$supp\ V \subset (s \times r)(K).$$

Proof 16. Let E a n.d.s. G-module, and decompose the base space $G^{(0)}$ into G-invariant subset which are locally induced by compact subgroupoids of G :

$$G^{(0)} = \cup_{j=1}^{J} G \times_{K_j} U_j.$$

By the third hypothesis of being s.n.d., there exist a Hilbert space H_j and an isomorphism of Hilbert modules $F_j : \mathrm{Res}_{K_j}^G E^{(1)} \to L^2(K_j) \otimes H_j$, but the induction $\mathrm{Ind}_{K_j}^G L^2(G/K_j) \otimes L^2(K_j) \otimes H_j$ is non canonically isomorphic to $L^2(G) \otimes H_j$, so that E is isomorphic to $E^{(0)} \otimes L^2(G) \otimes H_j$.

□

Corollary 2. Let G and G' two locally compactly induced groupoids, E, E' two s.n.d. modules over G and G' respectively and (Z, p, p') a generalized morphism from G to G' which respects condition ??. For any compact subset K, there exists an isometry $V \in \mathcal{L}(E, E')$ such that

$$supp\ V \subset (p \circ (p')^{-1}(s(K)) \times r)(K).$$

11 Nuclear dimension

In their paper **reference !**, E. Guentner, R. Willett and G. Yu defined asymptotic dimension for étale groupoids, and the theorem which we are intersted in is the following.

Theorem 12. Let G be an étale groupoid with finite asymptotic dimension, and with base space $G^{(0)}$ of finite covering dimension. If G is free, then the following inequality holds :

$$\dim_{nuc}^{+1}(C_r^*G) \leq \dim_{cov}^{+1}(G^{(0)}).\mathrm{asdim}^{+1}(G).$$

The article points out that freeness is mainly technical, and that one could somehow could get rid of it. That is what we entail to do.

The point of the proof is to construct, out of any compact subset $K \subset G$, an almost invariant partition of unity subordinate to a covering fulfilling the definition of asymptotic dimension. This PDU is use to construct a completely positive factorisation of the identity of C_r^*G through the reduced C^*-algebra of the open relatively compact subgroupoids generated by restiction to the cover and intersection with K. Here freeness does not intervene. **VERIFY**

Freeness is only used for this paticular result.

Proposition 23. Let G a free étale groupoid, and H an open relatively compact subgroupoid of G. Then

$$\dim_{nuc} C_r^*H = \dim_{cov} H^{(0)}.$$

Of course, the result should not hold without any assumption on G, more precisely, I think we should ask something about the isotropy bundle of G.

Definition 46. The isotropy bundle of G is the closed subgroupoid $\mathcal{J} = (r \times s)^{-1}(\Delta)$, i.e. the group bundle of the stabilizers $\mathcal{J} = \cup G_x^x$. Here $\Delta \subset G^{(0)} \times G^{(0)}$ is the diagonal of the base space.

Remark : If G is a (locally compact) transitive groupoid, then P. Muhly, J. Renault and D. Williams **ref !** have shown that C_r^*G is isomorphic to $C_r^*H \otimes \mathfrak{K}(L^2(\mu))$ for H any of the group statbilizers, which are all isomorphic, and μ a measure on $G^{(0)}$, so that

$$\dim_{nuc}^{+1}(C_r^*G) \leq \dim_{nuc}^{+1}(C_r^*H).\dim_{nuc}^{+1}(\mathfrak{K}(L^2(\mu))).$$

by Prop. 2.3 of **WZ**.
Let $H^{(0)}/H$ be the base space quotiented by the equivalence relation induced by H, and let $[x] = r(s^{-1}(x))$ denotes the equivalence class of $x \in H^{(0)}$, $\pi : H^{(0)} \to H^{(0)}/H$ the canonical projection map.

Proposition 24. Let H be a compact étale groupoid. Then $X = H^{(0)}/H$ is compact and Hausdorff, moreover there exists a structure of $C(X)$-algebra on C_r^*H with fibers isomorphic to $C_r^*(H_x^x) \otimes \mathfrak{K}(l^2([x]))$ for any $x \in H^{(0)}$.

Démonstration. Define

$$\left\{ \begin{array}{rcl} C(X) & \to & Z(\mathcal{M}(C_r^*H)) \\ f & \mapsto & f \circ \pi \end{array} \right. ,$$

which defines a $C(X)$-structure on C_r^*H.
Its fiber over $t \in X$ is given by the quotient by the ideal $C(H^{(0)} - t)C_r^*H$ i.e. $C^*H(t)$, as $H^{(0)} - t$ is an open H-invariant subset. But $H(t)$ is a principal groupoid so we can apply the remark to get $C_r^*H(t) \simeq C_r^*H_x^x \otimes \mathfrak{K}(l^2(t))$ where x is any point of t.

□

Proposition 25. Let H be a compact étale groupoid.
If, for all $x \in H^{(0)}$, $C_r^*H_x^x$ is a nuclear C^*-algebra, then C_r^*H is nuclear.
If $\sup_{x \in H^{(0)}} \dim_{nuc}(C_r^*H_x^x) \leq d$ then

$$\dim_{nuc}(C_r^*H) \leq (\dim_{cov}(H^{(0)}) + 1)(d + 1) - 1.$$

Démonstration. **LATER** □

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