Questions from Münster

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Contents

Chapter 1

Seminar

These are the notes I took co-corganizing with Erik Guentner and Rufus Willett the seminar of Noncommutative Geometry from Fall 2017 up until now (Fall 2018).

1.1 Cartan subalgebras

The goal of this section is... Historical remarks: aVN and Feldman-Moore,...

The first part will detail J. Renault's work [**?**] on Cartan pairs.

Recall that an element $x \in A$ normalizes a self-adjoint subspace B of A if

$$xBx^* \cup x^*Bx \subset B.$$

The normalizer $N_A(B)$ is the set of all the elements of A that normalize B.

Definition 1.1.1. Let A be a C^*-algebra. A sub-C^*-algebra $B \subseteq A$ is called a Cartan subalgebra of A if:

- B is a maximal abelian self-adjoint subalgebra (MASA) of A;
- B contains an approximate unit for A;
- the normaliser of B in A generates A as a C^*-algebra;
- there is a faithful conditional expectation $E : A \to B$.

The pair (A, B) is referred to as a *Cartan pair.*

Examples:

- $D_n \subset M_n(\mathbb{C})$,
- $C(X) \subset C(X) \rtimes \Gamma$,
- $l^\infty(X) \subset C^*_u(X)$,
- $C_0(G^0) \subset C^*_r(G, \Sigma)$.

Renault obtained the following result in [**?**].

Theorem 1.1.2. Any Cartan pair (A, B) is isomorphic to the Cartan pair

$$(C_r^*(G, \Sigma), C_0(G^0)),$$

where G is an étale topologically prinicipal groupoid with base space G^0 and Σ is a twist over G.

This theorem is very uselful. For instance, it implies that a nuclear C^*-algebra with a Cartan subalgebra satisfies the universal coefficient theorem of Rosenberg and Schochet [**?**]. Indeed, the reduced C^*-algebra of an étale groupoid is nuclear iff it is amenable, in which case it belongs to the bootstrap class [**?**].

The first step in the proof of the theorem is to build, for any inclusion of C^*-algebras $A \subseteq B$ with B unital commutative, an action of $N_A(B)$ by partial homeomorphisms on the spectrum of B. A standard construction then give rise to an étale groupoid $\mathcal{G}_B$ (the groupoid of germs of a *pseudogroup*) of this action. The twist is given by the same kind of construction.

For the second step, one defines a generalized Gelfand transform

$$\{$$

1.1.1 Groupoids of germs

Out of any inclusion of C^*-algebras $A \subseteq B$ with A unital commutative, we construct an action of the normalizer of A in B by partial homeomorphism on X the spectrum of A, i.e. a homomorphism of semigroup

$$\alpha : N_B(A) \to SHomeo(X).$$

If $n \in N_B(A)$ and $x \in Spec(A)$, set

$$\langle \alpha_n(x), a \rangle = \langle x, n^* a n \rangle.$$

This defines a homeomorphism

$$\alpha_n : U_n \to U_{n^*},$$

where $U_n = \{x \in Spec(A), n^*n(x) > 0\}$ such that $\alpha_{nm} = \alpha_n \circ \alpha_m$.

Lemma 1.1.3. If B is abelian and contains an approximate unit, $\alpha : N_A(B) \to PHomeo(X)$ is a homorphism of inverse-semigroups.

In our case, given a Cartan pair (A, B), and $X = Spec(B)$, one defines:

- Σ_B as the quotient of

  $$\{(x, n) \in X \times N_A(B) \text{ s.t. } n^*n(x) > 0\}$$

  by the equivalence relation $(x, n) \sim (x, n')$ when there exist $b, b' \in B$ such that $nb = n'b'$;

- $\mathcal{G}_B$ as the groupoid of germs of the pseudogroup $\alpha(N_A(B))$;

1.1.2 Generalized Gelfand transform

If $(x, n) \in X \times N_A(B)$ such that $n^*n(x) > 0$, and $a \in A$ then

$$\frac{E(n^*a)(x)}{\sqrt{n^*n(x)}}$$

only depends on the class of (x, n) in Σ_B, hence defines a continuous section $\hat{a}$ of the twist Σ_B. The map extends to a $*$-homomorphism

$$\Psi : A \to C_r^*(G_B, \Sigma_B)$$

which is always linear injective and respects the Cartan algebras. Moreover, restricted to B, Ψ coincides with the Gelfand transform $B \to C_0(X)$.

When (A, B) is a Cartan pair, Ψ is an $*$-isomorphism.

1.1.3 Roe algebras

In the case of uniform Roe algebras, White and Willett have obtained in [**?**] rigidity results. The questions are:

- What form can a Cartan subalgebra of $C_u^*(X)$ take?
- Can we describe when it is unique up to unitary equivalence?

The answers they have are the following. If X is an infinite countable metric space with bounded geometry, any Cartan subalgebra of $C_u^*(X)$ is non separable and contains a complete family of orthogonal projections. Interesting examples show that Cartan subalgebras of uniform Roe algebras need not be isomorphic to l^∞. Let us recall:

Definition 1.1.4. A sub-C^*-algebra B of A is a Roe Cartan pair if:

- A is unital;
- A contains the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space as an essential ideal;
- B is a co-separable Cartan subalgebra of A abstractly isomorphic to $l^\infty(\mathbb{N})$. (co-separable means that there is a countable subset of A which generates A together with B)

Theorem 1.1.5. Let (A, B) a Roe Cartan pair. Then there exists a metric space with bounded geometry X such that for any irreducible faithful representation of A on a Hilbert space H, there exists a unitary $u : l^2(X) \to H$ that conjugates A with $C_u^*(X)$, and B with $l^\infty(X)$.
Moreover, if $A = C_u^*(Y)$ for some bounded geometry metric spaxce Y with property A, then X and Y are coarsely isomorphic.

1.2 Dynamical Property (T)

The first thing I will try to do is to justify the use of groupoids. My opinion is that these objects are not loved as much as they deserve. People who very much like short and concise definitions enjoy to say that *groupoids are small categories in which all morphisms are invertible.* This is true, but maybe does not shed light on the reasons people look at such objects.

Groupoids can be thought as a generalisation of both groups and spaces. In that effect, a groupoid G is made of two parts, in our case, two spaces, the *group-like* part G and the *space-like* part G^0. Usually G is called the space of arrows, and G^0 the base space, seen as a subset of G. Any arrow $g \in G$ has a starting point $x \in G^0$ and an ending point $y \in G^0$. This is encoded by two maps $s, r : G \rightrightarrows G^0$ called source and range. Two arrows can be composed as long as the ending point of the first coincides with the starting point of the second. The points of the base space act as units, and every arrow as an inverse with respect to this partial multiplication.

$$\begin{tikzcd} s(g)\bullet \arrow[rr, bend left, "g"] & & \bullet r(g) \arrow[ll, bend left, "g^{-1}"] \end{tikzcd}$$

In our setting, all the spaces will be topological spaces and the maps will be continuous. We will even simplify greatly our life by only looking at second countable, locally compact, étale groupoids with compact base space. From now on, we will only say *étale*, forgetting about all other technical assumptions to gain in clarity.

Being étale means that the range map $r : G \to G^0$ is a local homeomorphism, i.e. for every $g \in G$, there exists a neighborhood U of g such that $r_{|U}$ is a homeomorphism. This implies in particular that every fiber $G^x = r^{-1}(x)$ and $G_x = s^{-1}(x)$ are discrete. When the base space G^0 has the additional property of being totally disconnected, we will say that G is *ample*. Here is a list of examples of étale groupoids.

- A (nice) compact space X defines a trivial groupoid $G = G^0 = X$ and source and target are the identity; in the opposite direction if the base space is a point, the groupoid is a group. One can already see how the notion of groupoid generalises both spaces and groups as promised.

- As an intermediate situation between these two cases, consider a discrete group Γ acting by homeomorphisms on a compact space X. Define the *action groupoid* as follow. Topologically, it is the space $G = X \times \Gamma \rightrightarrows G^0 = X$. The multiplication encodes the action

$$\begin{tikzcd} x \bullet \arrow[r, bend left, "{(x,g)}"] & \bullet\, g.x \arrow[l, bend left, "{(g.x,g^{-1})}"] \arrow[r, bend left, "{(g.x,g')}"] & \bullet g'g.x \end{tikzcd}$$

and this picture gives every element to reconstruct the groupoid.

- If $R \subseteq X \times X$ is an equivalence relation, then R as a canonical structure of groupoid with the base space being the diagonal $R^0 = \{(x,x) \mid x \in X\}$ and the multiplication being the only one possible

$$(x,y)(y,z) = (x,z).$$

- More interesting is the *coarse groupoid* $G(X)$ associated to a discrete countable metric space (X,d) with bounded geometry, that is

$$\sup_{x \in X} |B(x,R)| < \infty \quad \forall R > 0.$$

A nice way of thinking about this condition is to imagine yourself looking at the space with a magnifying glass of prescribed radius, but as great as you wish. Then you should not observe more and more points in your sight as you move around. In other words, the points fitting in the radius of your glass is uniformly bounded.

Now consider the R-diagonals:

$$\Delta_R = \{(x,y) \mid d(x,y) < \infty\} \subseteq X \times X$$

and take their closure $\overline{\Delta_R}$ in $\beta(X \times X)$ (βY being the Stone-Čech compactification of Y). The coarse groupoid is defined topologically as

$$G(X) = \cup_{R>0} \overline{\Delta_R} \rightrightarrows \beta X,$$

and is endowed with the structure of an *ample* groupoid which extend the groupoid $X \times X \rightrightarrows X$ associated with the coarsest equivalence relation on X. The topological property of this groupoid encodes the metric or *coarse* property of the space. For instance, X has property A iff $G(X)$ is amenable, X is coarsely embeddable into a Hilbert space iff $G(X)$ has Haagerup's property, etc.

- The last construction is associated to what is often referred as an *approximated group*, which is the data of $\mathcal{N} = \{\Gamma, \{N_k\}\}$ where Γ is a discrete group, and the N_k's are a tower of finite index normal subgroups with trivial intersection, i.e.

$$N_1 \triangleleft N_2 \triangleleft \ldots \quad \text{s.t.} \quad \cap_k N_k = \{e_\Gamma\} \text{ and } [\Gamma : N_k] < \infty.$$

Then the Γ_k's are finite groups. Set $\Gamma_\infty = \Gamma$ for convenience (which is not usually finite!). For any discrete group Λ, there exists a left-invariant proper metric, which is unique up to coarse equivalence (take any word metric if the group is finitely generated). Let us denote by $|\Lambda|$ the coarse class thus obtained. Then the first object of interest in that case is the coarse space $X_\mathcal{N}$ defined as the *coarse disjoint union*

$$X_\mathcal{N} = \coprod_k |\Gamma_k|.$$

Here the metric is such that $d(|\Gamma_i|, |\Gamma_j|) \to \infty$ as $i + j$ goes to ∞, $i \neq j$.

The second interesting object attached to $\mathcal{N}$ is the HLS (after Higson-Lafforgue-Skandalis [?], where it was first defined to build counter-examples to the Baum-Connes conjecture) groupoid. The base space is the Alexandrov compactification of the integers

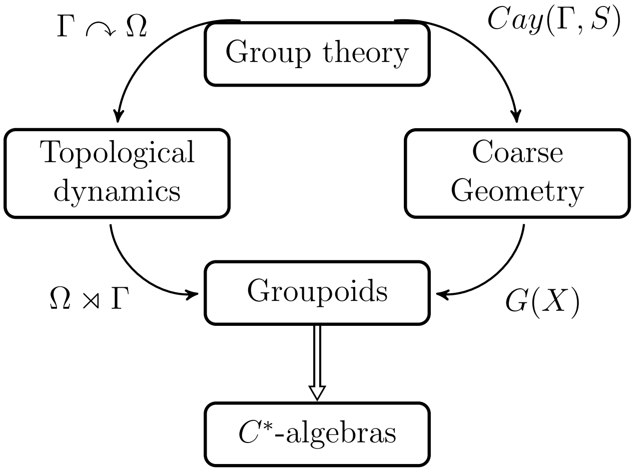
$$G^0_{\mathcal{N}} = \overline{\mathbb{N}},$$

and $G_{\mathcal{N}}$ is a bundle of groups with the fiber of k being Γ_k. The topology is taken to be discrete over the finite base points, and a basis of neighborhood of (∞, γ) is given by

$$\mathcal{V}_{\gamma,N} = \{(k, q_k(\gamma)) \mid k \geq N\} \quad N \in \mathbb{N},$$

where $q_k : \Gamma \to \Gamma_k$ is the quotient map.

One of the reasons we use groupoids is that they are convenient to build interesting C^*-algebras. To see their relevance, one may start with the question *What are operator algebraists doing?* A possible answer is that part of Noncommutative Geometry and Operator Algebras are devoted to the construction of interesting classes of C^*-algebras. For instance, *nuclearity* was naturally introduced after Grothendieck's work, followed by a C^*-algebraic formulation. Arises then the question *does there exist nonnuclear C^*-algebras?* A now classical result states that, when Γ is a discrete group, the reduced $C^*_r(\Gamma)$ is nuclear iff Γ is amenable. Calling out a nonamenable group, like any nonabelian free group, produces then a nonnuclear C^*-algebra. This game revealed itself to be very fruitful: study a property in some field and try to apply it to C^*-algebras to see what exotic being can be built out of it. The most common fields that have natural C^*-algbras associated to them are traditionally group theory, coarse geometry and dynamical systems (there are others like foliations etc, but let me just limit myself to these ones). This can be summarized in the following diagram.



Another interesting strategy is to try and translate a property in one of those upper boxes directly in terms of groupoids. Then the property can either be used to build C^*-algebras, either give a new definition in the case of other upper boxes. For instance, that is what we tried to do with Rufus Willett in our work on property T. Property T is originally a group property defined in terms of its unitary representations. In [?], Willett and Yu defined a geometric property T for monogenic discrete metric spaces with bounded geometry. Following their work, our first goal was to try and define a property T for (nice enough) topological groupoids so that in the case of groups and coarse groupoids, it reduces to these notions of property T. It gives then a notion of property T for dynamical systems, by considering property T for the action groupoid $X \rtimes \Gamma$. The second part of the work is dedicated to go down the last arrow, that is studying implications of property T for G to its reduced and maximal C^*-algebras, and even more general completions of $C_c(G)$.

Let us first recall what is property T for discrete groups.

If $\pi : \Gamma \to B(H)$ is a unitary representation of Γ on a separable Hilbert space, say that π almost has invariant vectors if for every pair (F, ε) where F is a finite subset of the group and ε a positive number, there exists a unit vector $\xi \in H$ such that

$$\|s.\xi - \xi\| < \varepsilon \quad \forall s \in F.$$

Definition 1.2.1. A group Γ has property T if every representation that almost has invariant vectors admits a nonzero invariant vector.

This definition is not the original one. Indeed property T was defined by Kazhdan in order to prove that *some* lattices in *some* Lie groups were finitely generated. It seemed a very specific property and application, but it turned out that property T gave very nice applications. Here are some of the most spectacular the author is aware of.

- Margulis supperrigidity theorem (about this, see Monod's [**?**] beautiful generalization, which Erik called the most beautiful paper he ever read);

- existence of expander: for any infinite approximated group (in the sense of the examples above) Γ, the space $X_{\mathcal{N}}$ is an expander;

- existence of Kazdhan projections which are very wild objects one should only approach with care;

- more generally, property T was for a long time an obstruction to the Baum-Connes conjecture, up until the work of Lafforgue ([**?**], [**?**]). It still gives interesting properties for diverse crossed-product constructions as we will see.

One can prove easily that finite groups have T. Indeed, in that case, take the finite subset to be the whole group and look intensely at the identity

$$\|s.\xi - \xi\|^2 = 2(1 - Re\langle s.\xi, \xi\rangle).$$

If ξ is (Γ, ε)-invariant for ε sufficiently small, then the above identity implies that $\frac{1}{|\Gamma|}\sum_{s\in\Gamma} s.\xi$ is nonzero because its inner-product with ξ will have real part close to 1. But ξ is invariant.

Now take $\Gamma = \mathbb{Z}$ and look at the left-regular representation, i.e. $H = l^2\Gamma$ and

$$(s.\xi)(x) = \xi(s^{-1}x).$$

Then if $\xi_n = \frac{1}{|F_n|}\chi_{F_n} \in H$ is the characteristic function of F_n normalized to be a unit vector, one can check that

$$\sup_{s\in F} \|s.\xi_n - \xi\| \to 0 \text{ as } n \to \infty$$

so that the regular representation always almost has invariant vectors. But it never has nonzero invariant ones, so that $\mathbb{Z}$ does not have T. This proof actually works for every infinite amenable group.

The moral of this story is that if one wants to find infinite groups with property T, one has to look at nonamenable groups. Maybe $\mathbb{F}_2$ or $SL(2,\mathbb{Z})$? Actually not: they both surject to $\mathbb{Z}$ which does not have T, and this is an obstruction to having T as is obvious from the definition.

Finding infinite groups with property T is actually a hard problem. Here are some examples, without any proofs since these would go out of scope for these notes.

- $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$ if $n \geq 3$;

- $Sp(n,1)$ and its lattices, which gives examples of infinite hyperbolic (in the sense of Gromov) groups having property T;

- $Aut(\mathbb{F}_5)$ and $Out(\mathbb{F}_5)$ by a recent result of Nowak and Ozawa [**?**]. Their proof is interesting in that they use numerical computations to reach their result using a previous result of Ozawa [**?**];

- $SO(p,q)$ with $p > q \geq 2$ and $SO(p,p)$ with $p \geq 3$. More generally, any real Lie group with real rank at least two, and all their lattices. Also, any simple algebraic group over a local field of rank at least two have T.

To define property T for groupoids, we need to choose what kind of representations we are looking at, and to decide what are the invariant vectors.

A representation will be a $*$-homomorphism $\pi : C_c(G) \to B(H)$. A vector $\xi \in H$ is called invariant if

$$f.\xi = \Psi(f).\xi \quad \forall f \in C_c(G).$$

The subspace of invariant vectors is denoted by H^π and its orthogonal complement, the space of coinvariants, is denoted by H_π.

Here Psi... Groups

Let $\mathcal{F}$ be a family of representations.

Definition 1.2.2. *G has property T if there exists a pair (K,ε) where $K \subseteq G$ is compact and $\varepsilon > 0$ such that, for every $\pi \in \mathcal{F}$, there exists $f \in C_K(G)$ such that $\|f\|_I \leq 1$ and*

$$\|f.\xi - \Psi(f).\xi\| < \varepsilon\|\xi\| \quad \forall \xi \in H_\pi.$$

The first thing we did was to study what were the relationships between groupoid property T and other property T.

- if $G = \Gamma$ is a discrete group, Γ has property T iff G has property T (in the groupoid sense);

- if X is a coarsely geodesic metric space, then X has geometric property T iff $G(X)$ has property T;

- in the case of a topological action, $X \rtimes \Gamma$ has property T iff Γ has T w.r.t. the family $\mathcal{F}_X$ of representations $\pi : \mathbb{C}[\Gamma] \to B(H)$ s.t. there exists a representation $\rho : C(X) \to B(H)$ such that (ρ, π) is covariant. This hypothesis simplifies in the case where there exists a invariant ergodic probability measure on X; in that case property T for $X \rtimes \Gamma$ and for Γ are equivalent;

- in the case of an approximated group Γ, then $G_{\mathcal{N}}$ has property T iff Γ has T. This may sound disappointing, but if one refines the result, one gets the nice following property: Γ has property τ w.r.t. $\mathcal{N}$ iff $G_{\mathcal{N}}$ has T w.r.t. the family of representations that extend to the reduced C^*-algebra of G.

The last part of the work is devoted to the existence of Kazdhan projections. Recall, if $\mathcal{F}$ is a family of representations, $C^*_{\mathcal{F}}(G)$ is the C^*-algebra obtained as the completion of $C_c(G)$ w.r.t. the norm

$$\|a\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \{\|\pi(a)\|\}.$$

A Kazdhan projection $p \in C^*_{\mathcal{F}}(G)$ is a projection such that its image in any of the representations in $\mathcal{F}$ is the orthogonal projection on the invariant vectors.

Theorem 1.2.3. Let G be compactly generated. Then if G has property T w.r.t. $\mathcal{F}$, there exists a Kazdhan projection $p \in C^*_{\mathcal{F}}(G)$.

This gives an obstruction to inner-exactness. Denote by F the closed G-invariant subset

$$\{x \in G^0 \mid G^x \text{ is infinite }\}$$

and U its complement.

Theorem 1.2.4. Let G be compactly generated and with property T. If one can find a sequence of points $(x_i)_i \subset U$ such that, for every compact subset $K \subset G$, K only intersects a finite number of orbits $G.x_i = r(s^{-1}(x_i))$, then G is not inner-exact. In fact it is not K-inner-exact. in particular, at least one of the groupoids G, $G_{|U}$ or $G_{|U}$ does not satisfy the Baum-Connes conjecture.

1.2.1 Kazdhan projections and failure of K-exactness

For $K \subset G$, $C_K(G)$ denotes the continuous functions supported in K.

Theorem 1.2.5. Let G be an étale groupoid whose reduced C^*-algebra contains a non trivial Kazdhan projection p. Suppose there exists an invariant probability measure on G^0 and that there exists an open subset $U \subset G^0$ not equal to G^0 containing a sequence of points (x_i) such that:

- x_i has finite orbit ($x_i \in G^0_{fin}$);
- for every compact $K \subset U$, the orbits $Gx_i = r(G_{x_i})$ ultimately don't intersect K;

then $C^*_r(G)$ is not K-exact.

Proof. Denote by M_i the finite dimensional C^*-algebra $B(l^2 G_{x_i})$ and $\lambda_i : C^*_r(G) \to M_i$ the corresponding left regular representation. We will show that the sequence

$$0 \longrightarrow C^*_r(G) \otimes \oplus M_i \longrightarrow C^*_r(G) \otimes \prod M_i \xrightarrow{q} C^*_r(G) \otimes \prod M_i / \oplus M_i \longrightarrow 0$$

is not exact in K-theory. We shall call q the last map in this diagram.

Define the following $*$-morphism

$$\phi \begin{cases} C^*_r(G) & \to & C^*_r(G) \otimes (\prod M_i) \\ x & \mapsto & x \otimes (\lambda_i(x))_i \end{cases}$$

Claim: the image of ϕ is contained in the kernel of q.

Let $x \in C^*_r(G)$ and $\epsilon > 0$. Let $K \subset G$ be a compact subset and $a \in C_K(G)$ such that $\|x - a\|_r < \varepsilon$. Let ϕ_i be the $*$-homomorphism defined in the same fashion as ϕ only with the first i components of $\phi(x)$ equated to zero. Denote by $\overline{x}$ the class of x in $C^*_r(G) \otimes \prod M_i / \oplus M_i$. Then $\overline{\phi(x)} = \overline{\phi_i(x)}$. Also, as the orbits G_{x_i} are ultimately disjoint, there is a i_0 such that $\lambda_i(a) = 0$ and thus $\phi_i(a) = 0$ for all $i > i_0$. This ensures

$$\|\overline{\phi(x)}\| = \|\overline{\phi_i(x)}\| = \|\overline{\phi_i(x)} - \overline{\phi_i(a)}\| < \varepsilon$$

hence $\overline{\phi(x)} = 0$.

Let $p \in C^*_r(G)$ the Kazdhan projection. Then $P = \phi(p)$ goes to zero in the right side of the sequence above. Let us show that its class in K-theory does not come form an element in $K_0(C^*_r(G) \otimes \oplus M_i)$.

The invariant probability measure on G^0 induces a trace τ on $C^*_r(G)$. Define τ_i to be the trace $\tau \otimes tr$ on $C^*(G) \otimes M_i$, where tr is the normalized trace on M_i. It is easy to see that $\tau_n(P) = \tau(p) > 0$. But if $z \in K_0(C^*_r(G) \otimes \oplus M_i)$, $\tau_n(z)$ is ultimatley zero. This implies that the non triviality of P ensures the non K-exactness of the sequence above in K-theory.

□

This result gives interesting examples of non K-exact C^*-algebras:

- if X is an expander, the coarse groupoid of X satisfies the hypothesis above, so that the uniform Roe algebra $C_u^*(X) \cong C_r^*(G)$ is not K-exact; in particular, if Γ contains an expander almost isometrically, its reduced errrrr no?
- if Γ is a residually finite group with property (τ), then any HLS groupoid associated to an approximating sequence of Γ satisfies the hypothesis above so that $C_r^*(G)$ is not K-exact.

1.3 Classification and the UCT

For A a simple unital C^*-algebra, the Elliot invariant is:

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \to S(K_0(A)))\,,$$

here $T(A)$ is the trace space and r_A the paring $r_A(\tau)([p]) = [\tau(p)]$.

Elliot's conjecture: Separable, simple, nuclear are classifiable by Elliot's invariants.

Theorem 1.3.1. Separable, simple, unital, nuclear, $\mathcal{Z}$-stable, UCT algebras are classifiable by Elliot's invariants.

An example of a classification theorem: Elliot's theorem,

Theorem 1.3.2. Let A and B unital AF-algebras and

$$\alpha : K_0(A) \to K_1(A)$$

a unital order isomorphism, i.e.

$$\alpha(K_0(A)_+) \subseteq K_0(B)_+ \quad \text{and} \quad \alpha([1_A]) = [1_B].$$

Then there exists a unital $*$-isomorphism $\phi; A \to B$ such that $\phi_* = \alpha$.

1.4 C^*-simplicity

1.4.1 General introduction

Let Γ be a discrete group. We will recall two equivalence relations on the set(?) of unitary representations of Γ, which are group homomorphisms

$$\pi : \Gamma \to U(H_\pi)$$

where $U(H_\pi)$ stands for the unitary group of a complex Hilbert space H_π. We will refer to such a representation as (π, H_π) or even just π or H_π if no confusion is possible.

Let π and σ be two representations of Γ.

- $\pi \simeq \sigma$ iff there exists a unitary $u : H_\pi \to H_\sigma$ such that
$$u\pi_\gamma u^* = \sigma_\gamma \quad \forall \gamma \in \Gamma.$$
- $\pi \approx \sigma$ iff there exists a sequence of unitaries $u_n : H_\pi \to H_\sigma$ such that
$$\|u_n \pi_\gamma u_n^* - \sigma_\gamma\| \to 0 \quad \forall \gamma \in \Gamma.$$

Fact: It turns out that for a lot of groups (e.g. finite, abelian, compact, simple Lie groups,...), these two notions coincide

$$\pi \approx \sigma \quad \text{iff} \quad \pi \simeq \sigma \quad \text{for } \pi, \sigma \text{ irreducible.}$$

Let $\hat{\Gamma}$ be the collection of all representations of Γ. A very hard problem is to describe

$$\hat{\Gamma}/\approx .$$

It can be done sometimes, e.g. for $\mathbb{Z}$ the irreducible representations are given by the circle, and any representation decomposes more or less uniquely into these.

Let us recall that the (left) regular representation

$$\lambda : \Gamma \to U(l^2\Gamma)$$

is defined by $\lambda_g(\delta_h) = \delta_{gh}$. The reduced C^*-algebra $C_r^*(\Gamma)$ is the closure under the operator norm of the image of the regular representation, i.e.

$$C_r^*(\Gamma) = \overline{span\{\lambda_\gamma\}_{\gamma \in \Gamma}}.$$

A representation π is tempered if it extends to a $*$-representation of $C_r^*(\Gamma)$. This happens iff the linear extension

$$\pi : \mathbb{C}[\Gamma] \to B(H_\pi)$$

satisfies $\|\pi(a)\| \leq \|\lambda(a)\|, \forall a \in \mathbb{C}[\Gamma]$.

Fact: All representations are tempered iff the group is amenable.

Another (very hard) problem is to describe

$$\hat{\Gamma}_r/\approx .$$

Definition 1.4.1. Γ is C^*-simple if $C^*_r(\Gamma)$ is simple, i.e. admits no proper two sided closed ideal.

Theorem 1.4.2 (Voiculescu)**.** Γ is C^*-simple iff $\hat{\Gamma}_r/\approx$ is a point.

Examples of C^*-simple groups:

- Non abelian free groups;
- Torsion free hyperbolic groups;
- $PSL(n,\mathbb{Z})$;
- Thompson's group V.

Non C^*-simple examples

Recall that a group Γ if the trivial representation

$$1_\Gamma : \Gamma \to U(\mathbb{C}) = \mathbb{S}^1; \gamma \mapsto id = 1;$$

is tempered. As a consequence, non trivial amenable groups are not C^*-simple as $1 \approx \lambda$ ($dim(l^2\Gamma) \neq 1$).

More generally if there exists an amenable normal subgroup $K \lhd \Gamma$, then the quasi regular representation

$$\lambda_{\Gamma/K} : \Gamma \to U(l^2(\Gamma/K));\ \lambda_{\Gamma/K}(\gamma)(\delta_{xK}) = \delta_{\gamma xK};$$

is tempered, hence if K is not trivial, Γ is not C^*-simple. In particular any semi-direct product $K \rtimes H$ with K amenable and non trivial is not C^*-simple.

Amenability being stable by extensions and increasing unions, any group has a largest normal amenable subgroup $R \lhd \Gamma$ called the amenable radical. The previous discussion shows that if Γ is C^*-simple, then $R = \{e\}$. The converse does not hold and was completely answered by Kennedy et al.

How to prove C^*-simplicity?

á la Powers [**?**].

Definition 1.4.3. A group Γ is a *Powers group* if for every finite subset $F \subset \Gamma$ there exists a partition

$$\Gamma = C \coprod D$$

and a finite number of elements $\gamma_1, ..., \gamma_n \in \Gamma$ with

- $\gamma C \cap C = \emptyset$ for every $\gamma in F$:
- $\gamma_i D \cap \gamma_j D = \emptyset$ for every $i \neq j$.

Examples:

- The free group on two generators $\mathbb{F}_2$ (Powers [**?**]);

- Many other examples using "North-South" type dynamics (De la Harpe, Bridson, Osin).

Let us write a few words about the technique Powers used. For $\mathbb{F}_2 = \langle a, b\rangle$, let

$$\tau : C_r^*(\mathbb{F}_2) \to \mathbb{C}; a \mapsto \langle \delta_e, a\delta_e\rangle$$

be the canonical tracial state.

Theorem 1.4.4 (Powers [?]). For every $a \in C_r^*(\Gamma)$,

$$\tau(x) = \lim \frac{1}{mn} \sum_{i=1,nj=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}$$

Corollary 1.4.5. $\mathbb{F}_2$ is C^*-simple.

Proof. Let $J \lhd C_r^*(\mathbb{F}_2)$ be an ideal. For $x \in C_r^*(\mathbb{F}_2)$ let $x_{mn} = \sum_{i=1,nj=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}$. If $x \in J$ then $(x^*x)_{mn} \in J$ so $\tau(x^*x)1_{C_r^*(\mathbb{F}_2)} \in \overline{J}^{\|\ \|}$. If J is not trivial, it contains a non zero element x, which forces $1_{C_r^*(\mathbb{F}_2)} \in J$ as $\tau(x^*x) > 0$. This ensures that $J = C_r^*(\mathbb{F}_2)$ and we are done. □

Corollary 1.4.6. $C_r^*(\mathbb{F}_2)$ has a unique tracial state.

Proof. Let τ' be a tracial state on $C_r^*(\mathbb{F}_2)$. Then for $x \in C_r^*(\mathbb{F}_2)$,

$$\tau'(x) = \tau'(x_{mn}) \to \tau'(\tau(x)1) = \tau(x)\tau'(1) = \tau(x).$$

□

1.4.2 Definitions

We only consider discrete countable groups, usually denoted by Γ.

Definition 1.4.7. A group is said to be C^*-simple if its reduced C^*-algebra is simple, i.e. has no proper closed two sided ideals.

A motivation for the interest toward such a notion can be the following result of Murray and Von Neumman: the Von Neumman algebra $L(\Gamma)$ is simple (no proper weakly closed two sided ideals) iff it is a factor iff Γ is ICC (infinite conjugacy classes, i.e. all non trivial conjugacy classes are infinite). Another one is that simplicity is one out of the 5 criteria (unital simple separable UCT with finite nuclear dimension) needed in the classification theorem obtained by Winter et. al.

Recall that, given two unitary representations of Γ, we say that π is weakly contained in σ and write

$$\pi < \sigma$$

if every positive type function associated to π can be approximated uniformly on compact sets by finite sums of such things associated to σ. In other words, if for every $\xi \in H_\pi$, $F \subseteq \Gamma$ finite and every $\varepsilon > 0$, there exists $\eta_1, \eta_2, ..., \eta_k$ such that

$$|\langle \pi(s)\xi, \xi\rangle - \sum_i \langle \sigma(s)\eta_i, \eta_i\rangle| < \varepsilon \quad \forall s \in F.$$

Remark: one can restricts to convex combinations of normalized positive type functions.

If $\pi < \sigma$, then the identity $\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ extends to a surjective $*$-morphisms

$$C^*_\sigma(\Gamma) \to C^*_\pi(\Gamma).$$

Indeed, it suffices to show that for every $a \in \mathbb{C}[\Gamma]$,

$$\|\pi(a)\| \leq \|\sigma(a)\|.$$

As $\|\pi(a)\|^2 = \|\pi(a^*a)\|$, we can suppose a positive. Then

$$\begin{aligned}\langle \pi(s)\xi, \xi\rangle &\leq \sum_i t_i \langle \sigma(s)\eta_i, \eta_i\rangle + \varepsilon \\ &\leq \|\sigma(a)\| + \varepsilon\end{aligned}$$

hence $\|\pi(a)\| \leq \|\sigma(a)\| + \varepsilon$, and let just ε go to zero.

Definition 1.4.8. A group Γ is C^*-simple if its reduced C^*-algebra is simple (i.e. has no proper closed two sided ideal).

Theorem 1.4.9. If Γ has a non trivial amenable normal subgroup, then it is not C^*-simple.

Proof. Let N be a normal amenable subgroup of Γ. Let (F_k) be a sequence of Folner sets for N, and

$$\xi_k = \frac{1}{|F_k|}\chi_{F_k} \in l^2(\Gamma)$$

Then

$$\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle = 2 - \frac{|F_k \Delta s F_k|}{|F_k|}$$

wich is 0 if $s \notin N$, and goes to 1 as n goes to infinity if $s \in N$. In other words

$$\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle \to \langle \lambda_{\Gamma/N}(s)\delta_{eN}, \delta_{eN}\rangle,$$

which shows that $\lambda_{\Gamma/N} < \lambda_\Gamma$. This gives us a surjective $*$-morphism

$$\phi : C^*_r(\Gamma) \to C^*_{\Gamma/N}(\Gamma).$$

A faster way which still works out when the ambient group is only locally compact is to point out that, N being amenable,

$$1_N < \lambda_N,$$

ensures by induction

$$Ind_N^\Gamma 1_N = \lambda_{\Gamma/N} < Ind_N^\Gamma \lambda_N = \lambda_\Gamma.$$

But if $n \in N$ is non trivial, $\lambda_\Gamma(n)$ is non trivial and sent to $\lambda_{\Gamma/N}(n) = 1$ via ϕ, so that $Ker\ \phi$ is a proper ideal in $C^*_r(\Gamma)$.

□

After the talk, Erik Guentner suggested the following proof. It is even shorter and doesn't assume any knowledge about weak containment or induction of representations. It is a weakening of the following fact: when Γ is amenable, the trivial representation $1_\Gamma : C^*_{max}(\Gamma) \to \mathbb{C}$ extends to the reduced C^*-algebra.

Indeed let $a \in \mathbb{C}[\Gamma]$ and (F_n) be a sequence of Folner sets for the support of a. Define $\xi_n = \frac{1}{|F_n|^{\frac{1}{2}}} \chi_{F_n} \in l^2(\Gamma)$. Then, suppose a is positive, and compute

$$\begin{aligned} \langle a\xi_n, \xi_n \rangle &= \sum_{s \in \text{ supp } a} a_\gamma \frac{|F_n \cap sF_n|}{|F_n|} \\ &\to \|a\|_{1_\Gamma} \end{aligned}$$

so that $\|a\|_r \leq \|a\|_{1_\Gamma}$.

Now if N is a normal amenable subgroup of Γ...

We saw that $\mathbb{F}_2$ is C^*-simple, yet it has a copy of $\mathbb{Z}$ as an amenable subgroup (non normal), and a normal (non amenable) subgroup: the commutator subgroup, which is an infinite rank free group, $\langle [x, y] : x, y \in \mathbb{F}_2 \rangle = \mathbb{F}([a^n, b^m]; n, m)$. Both conditions are necessary.

This result led to following (false) conjecture: a group is C^*-simple iff it has no non trivial amenable normal subgroups.

1.4.3 Completely positive maps

If A and B are C^*-algebra, then a linear map $\phi : A \to B$ is called completely positive if

$$\phi^{(n)}(a) = (\phi(a_{ij}))_{ij} \geq 0 \quad \forall a \in M_n(A)_+.$$

Denote $CP(A, B)$ the normed vector space of completely positive maps form A to B. $S(A)$ denotes the state space of A, endowed with the weak-$*$ topology (it's then a convex subspace of A^*, compact when A is unital).

Then:

- $CP(C(X), C(Y)) \cong C(Y, P(X))$ via $\mu_y(f) = \Phi(f)(y)$;

- $CP(A, C(Y)) \cong C(Y, S(A))$ via $\omega_y(f) = \Phi(a)(y)$;

- What about $CP(C(X), B)$? Continuous sections on the continuous field of C^*-algebras $\bigoplus_{\omega \in S(B)} B(H_\omega)$.

1.4.4 Injective C^*-algebras

Recall that an abelian group M is injective if, given any injective homomorphism of abelian group $A \hookrightarrow B$, any homomorphism $A \to M$ extends to a homomorphism $B \to M$. In words: any homomorphism into M extends to super-objects. We will often use the following commutative diagram

$$\begin{tikzcd} B \arrow[dr, dotted, "\exists"] & \\ A \arrow[u, hook] \arrow[r] & M \end{tikzcd}$$

to represent this situation. We will now turn to a analog notion in the C^*-algebraic world.

Definition 1.4.10. A C^*-algebra M is *injective* if, given an inclusion of C^*-algebra $A \subset B$, any injective $*$-homomorphism $A \to M$ extends to B by a contractive completely positive (CCP) map.

$$\begin{tikzcd} B \arrow[dr, dotted, "\exists\ ccp"] & \\ A \arrow[u, hook] \arrow[r] & M \end{tikzcd}$$

Even if the straight arrow are here supposed to be $*$-homomorphism, Stinespring's dilation theorem ensures that we can suppose all the arrows to be only CCP maps. We will say that M is Γ-injective if Γ acts by automorphisms on all the C^*-algebras in the diagram, and all the arrows are Γ-equivariant.

We will define a particular class of compact spaces acted upon by Γ, called Γ-boundaries, and show that there exists a maximal Γ-boundary $\partial_F\Gamma$, called the *Thurston boundary.*

The first **major goal** of this presentation is to show that $C(\partial_F\Gamma)$ is Γ-injective.

Description of commutative injective algebras

Lemma 1.4.11. If M is injective and $S \subset M$, define

$$Ann_M(S) = \{m \in M \mid sm = 0 \forall s \in S.$$

Then there exists a projection $p \in M$ satisfying $Ann_M(S) = pM$.

Proof. This is true if $M = B(H)$ for some Hilbert space. In the general case, embed M unitally in some $B(H)$. By injectivity of M, there exists a CCP map $E : B(H) \to M$ such that $E(m) = m, \forall m \in M$ so $M \subset dom(E)$ (multiplicative domain). There exists a projection $p \in B(H)$ with $Ann_{B(H)}(S) = pB(H)$ (take the projection on $\cap_{s\in S} Ker(s)$). If $s \in S$,

$$sE(p) = E(sp) = 0 \text{ hence } E(p) \in Ann_M(S).$$

Moreover if $m \in Ann_M(S) \subset pB(H)$, $pm = m$ and

$$E(p)m = E(pm) = E(m) = m$$

so that for $m = E(p)$, we get $E(p)$ is a projection. This also proves that $E(p)Ann_M(S) = Ann_M(S)$. A slight fiddling ensures then that $Ann_M(S) = E(p)M$. $\square$

Corollary 1.4.12. Let X be a compact Hausdorff space. If $C(X)$ is injective then X is Stonean, i.e. $\overline{U}$ is open for every open subset $U \subset X$.

Proof. Let $U \subset X$ be open, and $S = C_0(U)$. By the previous lemma, there exists a projection $p \in C(X)$ such that $Ann_{C(X)}(S) = pC(X)$. But p cannot be anyone else than the characteristic function of $\overline{U}^c$ so that $1 - p = \chi_{\overline{U}}$ is continuous and $\overline{U}$ is open. $\square$

Note: Infinite compact Stonean spaces are not metrizable (not even second countable). Suppose the contrary and get a sequence $x_i \to x$ in X and open sets $U_n = B(x_n, \varepsilon_n)$, with ε_n such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ for every $n \neq m$. Set $U = \cup_n U_{2n}$, then $x \in \overline{U}$ ($\overline{U}$ is open) so $x_n \in \overline{U}$ for large n but $x_n \notin \overline{U}$ for n odd.

1.4.5 Furstenburg boundary

If Γ is a discrete group acting on a compact Hausdorff space X (we will just say that X is a Γ-space), the space of probability measures $Prob(X)$ endowed with the weak-$*$ topology is homeomorphic to the state space $S(C(X))$ with the topology of simple convergence. We identify X with a closed subspace of $Prob(X)$ with the help of the Dirac masses ($X \hookrightarrow Prob(X); x \mapsto \delta_x$ is an embedding). Recall that the action can be extended to $Prob(X)$, which is then a Γ-space by Banach-Alaoglu's theorem.

Definition 1.4.13. A Γ-space X is:

- *minimal* if the only Γ-invariant closed subset of X are itself and $\emptyset$;
- *strongly proximal* if $\overline{\Gamma.\mu}^{weak-*}$ contains δ_x for some $x \in X$;
- a Γ-boundary if it is minimal and strongly proximal

$$X \subset \overline{\Gamma.\mu}^{weak-*} \quad \forall \mu \in Prob(X).$$

Example: Let $SL(2,\mathbb{Z})$ act on the projective line $\mathbb{R}P^1$ (the quotient of $\mathbb{R}^2 \backslash \{0\}$ by the group of dilations) given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Most $g \in Sl(2,\mathbb{Z})$ are acting hyperbolically (two distinct eigenspaces, one expansive one contractive). Take $\mu \in Prob(\mathbb{R}P^1)$, and a generic element $g \in SL(2,\mathbb{Z})$. As n goes to ∞,

$$g^n.\mu \to_{weak-*} \delta_{\text{Expanding eigenspace}}$$

unless $\mu(\{\text{contractive eigenspace}\}) > 0$, hence

$$\{\delta_{\text{Expanding eigenspace}}\}_{g \in SL(2,\mathbb{Z})} \subset \overline{\Gamma.\mu}^{wk-*}.$$

Exercise: the set of these is dense in $\mathbb{R}P^1 \subset Prob(\mathbb{R}P^1)$.

Theorem 1.4.14 (Furstenburg)**.** There exists a Γ-boundary $\partial_F \Gamma$ (now called the Furstenburg boundary) such that for any Γ-boundary X there exists a continuous Γ-equivariant surjection $\partial_F \Gamma \twoheadrightarrow X$.

Proof. Let $\mathcal{B}$ be the class of all Γ-boundaries. It is non empty as it contains the point space. Take

$$Z = \prod_{Y \in \mathcal{B}} Y$$

which is compact by Tychonoff's theorem. Equip Z by the diagonal Γ-action.

- It is strongly proximal: for any $\mu \in Prob(Z)$, a diagonal argument gives a weak-$*$ convergent net $g_i.\mu \to \delta_z$ for some $z \in Z$.
- It is not minimal, but Zorn's lemma ensures the existence of a minimal closed Γ-invariant subset $\partial_F \Gamma$ of Z.

We obtain the desired map as the composition of the inclusion $\partial_F\Gamma \hookrightarrow Z$ with the projection on the X-factor $Z \twoheadrightarrow X$. □

Theorem 1.4.15 (Kalantar-Kennedy)**.** $C(\partial_F\Gamma)$ is Γ-injective.

Lemma 1.4.16. There exists a bijective correspondence between the completely positive maps from $C(X)$ to $C(Y)$ and the continous maps from Y to $Prob(X)$. The statement remains true if one asks for equivariance. **send to a previous section on CP maps**

Lemma 1.4.17 (Furstenburg)**.** Let X and Y be two Γ-boundaries. Then any Γ-equivariant map $X \to Prob(Y)$ has image in Y, i.e. any UCP map $C(X) \to C(Y)$ is a $*$-homomorphism! Moreover there is at most one such map.

Proof. Take $\mu : X \to Prob(Y)$. The image $\mu(X) \subset Prob(Y)$ is a closed Γ-invariant subspace: by strong proximality of Y, there exists $y \in Y$ such that

$$\delta_y \in \overline{\Gamma.\mu_x}^{wk-*} \subset \mu(X).$$

By minimality of Y, $\overline{\Gamma.\mu_x}^{wk-*} \cap Y = Y$, By minimality of X, $\mu^{-1}(Y) = X$ i.e. $\mu(X) \subset Y$.

Let $\mu, \eta : X \to Prob(Y)$ be two such maps. Then $\frac{1}{2}\mu + \frac{1}{2}\eta$, μ and η all take values in Y so that they are all equal. □

Corollary 1.4.18. Any equivariant UCP map $C(\partial_F\Gamma) \to C(\partial_F\Gamma)$ is the identity.

Recall that if A is a unital Γ-algebra, its state space $S(A)$ is convex compact Γ-space.

Proposition 1.4.19 (Gleason)**.** Let $Z \subset S(A)$ be a Γ-invariant closed convex subspace, which is minimal w.r.t. these properties. (Such a thing exists by Zorn's lemma.) Then

$$\partial_{ex} Z = \{\phi \in Z \mid \phi \text{ is not a non trivial convex combination of anything in } Z\}$$

is a Γ-boundary.

Proof. There is a barycenter map $\beta : Prob(Z) \to Z$ such that

$$\int_Z f d\mu = f(\beta(\mu)) \quad \forall f \in C(Z) \text{ affine.}$$

Indeed, if $\mu = \delta_z$, $\beta(\mu) = z$ and if $\mu = \sum \alpha_i \delta_{z_i}$ with $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1$, then $\beta(\mu) = \sum \alpha_i z_i$. Finite convex combinations are weak-$*$ dense in $Prob(Z)$ by the Hahn-Banach separation theorem. As β is weak-$*$ continuous, and affine so uniformly weak-$*$

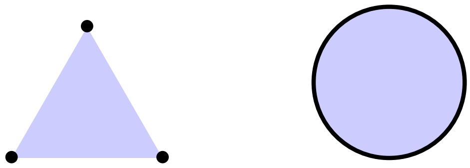


Figure 1.1: Two examples with Z in blue and $\partial_{ex}Z$ in black.

continuous, it extends to the whole space $Prob(Z)$.

Note: β is Γ-equivariant continuous and satisfies $\beta(\mu) = z \in \partial_{ex}Z$ iff $\mu = \delta_z$.

Then, for any $\mu \in Prob(Z)$,

$$\beta(\overline{conv(\Gamma\mu)}) = \overline{conv(\Gamma\beta(\mu))} = Z,$$

the first equality coming from continuity, Γ-equivariance and affinity. Now, $\partial_{ex}Z$ is minimal, and if $\mu \in \partial_{ex}Z$, then

$$AFINIR$$

□

We are now ready for the main result of this section.

Theorem 1.4.20 (Kalantar-Kennedy)**.** $C(\partial_F\Gamma)$ is Γ-injective.

Proof. First, observe that $l^\infty(\Gamma)$ is Γ-injective. Let indeed $A \subset B$ be an inclusion of C^*-algebras and $\phi : A \to l^\infty(\Gamma)$ a $*$-homomorphism. Then $ev_e \circ \phi$ is a state on A, so it extends to a state Ψ on B. Define $\tilde{\phi} : B \to l^\infty(\Gamma)$ by

$$\tilde{\phi}(b)(\gamma) = \Psi(\gamma^{-1}.b).$$

Then Ψ is a UCP Γ-equivariant map that extends ϕ.

Now, producing ucp equivariant maps

$$C(\partial_F\Gamma) \xrightarrow{\alpha} l^\infty(\Gamma) \xrightarrow{\beta} C(\partial_F\Gamma)$$

is sufficient to conclude, as their composition must be the identity by corollary 1.4.18.

Define $\alpha : C(\partial_F\Gamma) \to l^\infty(\Gamma)$ by fixing $\mu \in Prob(\partial_F\Gamma)$ and set

$$\alpha(f)(\gamma) = \mu(\gamma^{-1}.f).$$

By Gleason's theorem 1.4.19, there is a Γ-boundary $X \subset S(l^\infty(\Gamma))$. By universal property of $\partial_F\Gamma$, we have an equivariant surjection $\partial_F\Gamma \twoheadrightarrow X \subset S(l^\infty(\Gamma))$. By duality, we get a Γ-equivariant ucp map

$$\Psi : l^\infty(\Gamma) \to C(\partial_F\Gamma)$$

and we are done. □

As a final remark, one can point out that this last proof used the following useful fact: if B is injective and $\phi : A \to B$ is a split injective Γ-ucp map, then A is injective. We use this with $A = C(\partial_F\Gamma)$ and $B = l^\infty(\Gamma)$.

1.4.6 Dynamical characterization of C^*-simplicity

(Facts we are using:
$C(\partial_F\Gamma)$ is Γ-injective, in particular any Γ-equivariant u.c.p. $C(\partial_f\Gamma) \to A$ is split, so is an isometric embedding,
$\partial_F\Gamma$ is totally disconnected,)

The goal of this section is to prove the following theorem.

Theorem 1.4.21. Γ is C^*-simple iff the action of Γ on $\partial_F\Gamma$ is free.

Let's do first the forward direction.

Suppose the action is free. First, to show $C_r^*(\Gamma)$ is simple, it is enough to show that any representation

$$\pi : C_r^*(\Gamma) \to B(H)$$

is injective.

By Arveson's extension theorem, π extends to a u.c.p. map

$$\phi : C(\partial_F\Gamma) \rtimes_r \Gamma \to B(H).$$

Its restriction ϕ_0 to $C(\partial_F\Gamma)$ is Γ-equivariant, because $C(\partial_F\Gamma)$ is in the multiplicative domain of ϕ_0, and thus must be an isometric embedding, by Γ-injectivity of $C(\partial_F\Gamma)$ (it is split because $\mathbb{C} \subseteq B(H)$). The equivariant u.c.p. map ϕ_0 is an isomorphism onto its image: extend its inverse form $im\ \phi_0$ to $im\ \phi$ and denote the resulting u.c.p map by τ.

Claim: $\Psi = \tau \circ \phi$ is the canonical expectation $E : C(\partial_F\Gamma) \rtimes_r \Gamma \to C(\partial_F\Gamma)$ which is faithful. This implies π is injective.

Let's end up with the claim.

- $\Psi_{|C(\partial_F\Gamma)} = id_{C(\partial_F\Gamma)}$. Indeed, τ is the inverse of $\phi_0 = \phi_{C(\partial_F\Gamma)}$.
- If $\gamma \neq e_\Gamma$, the action being free, for every x there exists a function $f \in C(\partial_F\Gamma)$ such that
$$f(x) \neq 0 \quad \text{and} \quad f(s^{-1}x) = 0.$$
Now $C(\partial_F\Gamma)$ is in the multiplicative domain of Ψ, so
$$\Psi(\lambda_s)f = \Psi(\lambda_s f) = \Psi((sf)\lambda_s) = (sf)\Psi(\lambda_s)$$
which evaluated at x gives $\Psi(\lambda_s)(x) = 0$, for all x, so $\Psi(\lambda_s) = 0$.

The other direction is more intricated. It consists in two steps:

1. if $x \in \partial_F\Gamma$, then the stabilizer Γ_x is amenable, which implies that $\lambda_{\Gamma/\Gamma_x} < \lambda_\Gamma$,

2. if X is a X is a Γ-boundary, and $\gamma \neq 0$ such that $int(X_s) \neq \emptyset$, then $\lambda_\Gamma \not< \lambda_{\Gamma/\Gamma_x}$, so that the kernel of $C_r^*(\Gamma) \to C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma)$ is a non trivial two sided closed ideal.

This, together with the fact that $\partial_F\Gamma$ is topologically free iff it is free, concludes the proof.

First bullet:

- there exists a Γ_x-equivariant injective $*$-homomorphism

$$\rho : l^\infty(\Gamma_x) \to l^\infty(\Gamma)$$

  defined by $\rho(f)(ts_i) = f(t)$ for every $t \in \Gamma_x$, $\{s_i\}_i$ being a system of representatives of the right cosets $\Gamma_x \backslash \Gamma$.

- there exists a Γ_x-equivariant u.c.p. map

$$\psi : l^\infty \to C(\partial_F\Gamma),$$

  by universal property of $\partial_F\Gamma$, and the fact that the spectrum of $l^\infty(\Gamma)$ is $\beta\Gamma$. (for any compact Γ-space, there exists a Γ-map $\partial_F\Gamma \to P(X)$. take the dual of this map for $X = \beta\Gamma$).

- The composition $\phi = ev_x \circ \psi \circ \rho$ defines a Γ_x-invariant state on $l^\infty(\Gamma_x)$, which concludes the proof.

Second bullet:

This needs a lemma:

Lemma 1.4.22. Let X be a Γ-boundary. For every non empty subset of X, every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma \backslash \{e_\Gamma\}$ such that

$$\min_{t \in F} \mu(tU^c) < \varepsilon \quad \forall \mu \in P(X).$$

Proof. Let $x \in U$. By strong proximality, there exists $t_\mu \neq e_\Gamma$ such that

$$\delta_x(U) - \mu(t_\mu U) = \mu(t_\mu U^c) < \varepsilon,$$

and by continuity of the action

$$V_\mu = \{\nu \in P(X) \mid \nu(t_\mu U^c) < \varepsilon\}$$

is a neighboorhood of μ. By compactness of $P(X)$ in the weak-$*$ topology, we can extract a finite cover such that

$$P(X) = \cup_{i=1,m} V_{\mu_i}.$$

Then $F = \{t_{\mu_1}, ..., t_{\mu_m}\}$ fills the requirements of the lemma.

∎

Suppose the action is not topologically free and let $s \neq e_\Gamma$ such that the interior U of X_s is not empty. Let F the finite subset given by the lemma for U and $\varepsilon = \frac{1}{3}$. Suppose

$$\lambda_\Gamma < \lambda_{\Gamma/\Gamma_x}.$$

We will show this is absurd by looking at the coefficient $c_\gamma = \langle \lambda_\Gamma(\gamma)\delta_e, \delta_e\rangle$, which is 0 unless $\gamma = e_\Gamma$.

On the finite subset $K = \{tst^{-1}\}_{t\in F}$, approximate c_γ up to ε by a convex combination

$$\sum_{j=1,n} \alpha_j \langle \lambda_{\Gamma/\Gamma_x}(\gamma)\xi_j, \xi_j\rangle$$

of coefficients of the quasi regular representation. Set

$$\mu_j = \sum_{y\in\Gamma.x} |\xi_j(y)|^2 \delta_y \in P(X) \quad \text{and} \quad \mu = \sum \alpha_j \mu_j,$$

where we identify $\Gamma.x$ with Γ/Γ_x. **A FINIR**

Questions:

- Can we get a more direct proof for the last implication? (without representation theory)

- It is not known in general wether the action of Γ on $\partial_F\Gamma$ is amenable. If X is a Γ-space such that one of the stabilizer is not amenble, the action cannot be amenable. Is it true that, if Γ is exact, this is the only obstruction for the amenability of the action?

1.4.7 Another proof

The last subsection uses representation theory (induction) which makes one wonder if this could be avoided. While the implication

$$\partial_F\Gamma \text{ is free } \Rightarrow \Gamma \text{ is } C^*\text{-simple}$$

is still good enough if one wants to stay clear of representation theoretic lingo, the other direction can be proven in another way.

This proof is taken from a set of notes that Ozawa wrote after giving lectures for the "Annual Meeting of Operator Theory and Operator Algebras" at Tokyo university, 24–26 December 2014.

For X a compact Γ-space and H a subgroup of Γ, we denote by:

- $E_x : C(X) \rtimes_r \Gamma \to C^*_r(\Gamma)$ the canonical conditional expectation onto $C^*_r(\Gamma)$ given by extending the evaluation at x,

- $E_H : C_r^*(\Gamma) \to C_r^*(H)$ the canonical conditional expectation given by $E(\lambda_s) = \delta_{s\in H}$,

- τ_H the canoncical trace $C_r^*(H) \to \mathbb{C}$.

The first thing one can show is the following.

Proposition 1.4.23. Let X be a Γ-boundary, then

$$C(X) \rtimes_r \Gamma$$

is simple.

Proof. It is enough to show that any quotient map

$$\pi : C(X) \rtimes_r \Gamma \to B$$

is injective. By C^*-simplicity, π restricts to an isomorphism on $C_r^*(\Gamma)$ so that the canoncial trace τ is well defined on $\pi(C_r^*(\Gamma)$. Seeing $\mathbb{C}$ as the sub-C^*-algebra of constant functions in $C(\partial_F\Gamma)$, we can extend τ to B.

$$\begin{tikzcd}
C(X) \rtimes_r \Gamma \arrow[r, "\pi"] & B \arrow[dr, dotted, "\phi"] & \\
C_r^*(\Gamma) \arrow[u, hook] \arrow[r, "\cong"] & \pi(C_r^*(\Gamma)) \arrow[u, hook] \arrow[r, "\tau"] & \mathbb{C} \subseteq C(\partial_F\Gamma)
\end{tikzcd}$$

Now $\phi \circ \pi$ restricts to a Γ-u.c.p. map $C(X) \to C(\partial_F\Gamma)$ which can only be the inclusion. This ensures that

$$C(X) \subseteq Dom(\phi \circ \pi).$$

As ϕ extends τ, $\phi \circ \pi$ is the canonical conditional expectation $C(X) \rtimes_r \Gamma \to C(X)$ which is faithful. In particular, π is faithful, and is injective. □

Applying this to $X = \partial_F\Gamma$, we get that $C(\partial_F\Gamma) \rtimes_r \Gamma$ is simple. In that case, every stabilizer

$$\Gamma_x = \{s \in \Gamma \mid sx = x\} \quad \forall x \in \partial_F\Gamma$$

is amenable. Moreover, the strong stabilizer

$$\Gamma_x^0 = \{s \in \Gamma \mid \exists U \text{ neighborhood of } x \text{ s.t. } s_U = id_U\}$$

is a normal subgroup of Γ_x.(In particular, is is amenable.) In that case, we will apply the following proposition.

Proposition 1.4.24. Let X be a minimal compact Γ-space. If

$$C(X) \rtimes_r \Gamma$$

is simple and there exists $x \in X$ such that Γ_x^0 is amenable, then X is topologically free.

Proof. By minimality, topological freeness is equivalent to $\Gamma_x^0 = 1$ for some x.

Indeed, if $\Gamma_x^0 = 1$ for some x, every non trivial group element cannot fix any neighborhood of x hence for every $s \neq e_\Gamma$, we get a sequence of points that converge to x which are not fixed by s. By minimality,

$$X_s = \{y \in X \mid sy \neq y\}$$

is a non empty dense open set of X for every $s \neq e_\Gamma$. By Baire category's theorem,

$$\cap_{s\in\Gamma\setminus\{e\}} X_s$$

is dense in X so that X is topologically free.

Let us show that $\Gamma_x^0 = 1$. Define a representation

$$\rho : C(X) \rtimes_r \Gamma \to B(l^2(\Gamma/\Gamma_x^0))$$

by $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(s\gamma.x)\delta_{s\gamma\Gamma_x^0}$. It is clearly covariant on the algebraic crossed-product.

To prove ρ extends to the whole crossed-product, i.e. $\|\rho(a)\| \leq \|a\|_{C(X)\rtimes_r\Gamma}$, it is enough to show that

$$\langle\ \rho(a)\delta_{\Gamma_x^0}, \delta_{\Gamma_x^0}\ \rangle \leq \|a\|_{C(X)\rtimes_r\Gamma}$$

because $\delta_{\Gamma_x^0}$ is cyclic. This follows from the fact that the latter is the composition $\tau \circ E_{\Gamma_x^0} \circ E_x$ of 3 u.c.p maps (so contractive).

Pick up x such that Γ_x^0 is amenable and $s \in \Gamma$ arbitrary that fixes some neighborhood of x: there exists a neighborhood U of x such that $s_{|U} = id_U$. Let $f \in C(X)$ be nonzero and supported in U. Let us compute

$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0}.$$

- If $\gamma.x \in U$, then $s\gamma.x = \gamma.x$ and

$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(\gamma.x)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}.$$

- If $\gamma.x \notin U$, $f(\gamma.x) = 0 = f(s\gamma.x)$, so that $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}$.

This shows that $\rho(f(\lambda_s - 1)) = 0$. By injectivity, $\lambda_s = 1$ and $s = e_\Gamma$ hence $\Gamma_x^0 = 1$ and we are done. □

1.4.8 Thompson's group V is C^*-simple

In this section, we prove that Thompson's group V is C^*-simple. Recall that V is defined as the group of piecewise linear bijections of $[0, 1)$ with finitely many points of non differentiability, all of which are dyadic rational numbers. Such a function f is entirely determined by two partitions

$$[0,1) = \coprod_{i=1}^{n} I_i = \coprod_{i=1}^{n} J_i$$

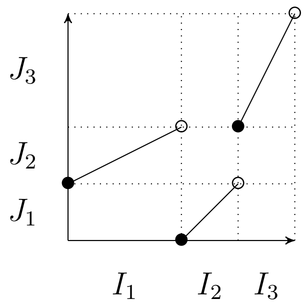


Figure 1.2: The graph of $\begin{pmatrix} I_1 & I_2 & I_3 \\ J_2 & J_1 & J_3 \end{pmatrix}$

and a bijection $\begin{pmatrix} I_1 & \dots & I_n \\ J_{\sigma(1)} & \dots & J_{\sigma(n)} \end{pmatrix}$. The intervals I_i and J_i are of the type $[a, a+2^{-n})$, with a dyadic rational in $[0,1)$. Then f is defined on I_i as the only linear increasing function applying I_i to $J_{\sigma(i)}$.

In order to prove that V is C^*-simple, we will:

- realize V as a countable group of homeomorphisms of the Cantor set;
- use the following result of Le Boudec and Matte-Bon ([**?**], thm 3.7):

  Theorem 1.4.25. Let X be a Hausdorff locally compact space and Γ be a countable subgroup of $Homeo(X)$. Suppose that for every non empty open subset $U \subset X$, the rigid stabilizer
  $$\Gamma_U = \{\gamma \in \Gamma \mid \gamma x = x \ \forall x \notin U\}$$
  is non amenable. Then Γ is C^*-simple.

Let G be an ample groupoid with compact base space. We also always suppsose that groupoids are second countable, Hausdorff and locally compact. Recall that a bisection $U \subset G$ is a set such that s and r are homoeomorphisms when restricted to U. In particular, any open bisection U induces a partial homeomorphism

$$\alpha_U \left\{ \begin{array}{rcl} s(U) & \to & r(U) \\ x & \mapsto & r \circ s_{|U}^{-1}(x) \end{array} \right.$$

The topological full group $[\![G]\!]$ is defined as the set of bisections U of G such that $s(U) = r(U) = G^0$. The operations are defined by

$$e = G^0, \quad UV = \{gg' \mid g \in U, g' \in V \text{ s.t. } s(g) = r(g')\}, \quad U^{-1} = \{g^{-1} \mid g \in U\}.$$

Recall that a Cantor space is any compact metrizable totally disconnected space without any isolated points. It is a standard fact that they are all homeomorphic. A model for Ω is the countable product A^X, where

- A is a finite set, often reffered to as the *alphabet*;
- X is a countable set.

Then elements of Ω are infinite words indexed by X. Denote by Ω_f the set of finite words

$$\Omega_f = \coprod_{\text{finite } F \subset X} A^F,$$

then the topology on Ω is the one generated by the *cylinders*

$$C_a = \{w \in \Omega \mid w(x) = a(x)\ \forall x \in F = supp(a)\}.$$

For finite words $a \in \Omega_f$, $l(a)$ denotes their length, and if $F = \mathbb{N}$, $x \in \Omega$, ax denotes the concatenation of a and x, i.e. the word obtained by first saying a and then x.

Examples:

1. Let Γ a countable discrete group acting on a Hausdorff compact space X by homeomorphisms. Then $[\![X \rtimes \Gamma]\!]$ consists of the bisections of the type

$$S = \coprod U_i \times \{\gamma_i\}$$

where $X = \coprod_{i=1}^n U_i = \coprod_{i=1}^n \gamma_i U_i$.

2. Let $\mathbb{Z}$ act on the Cantor space $\Omega = \{0,1\}^{\mathbb{Z}}$ by Bernoulli shift

$$n(a_i)_i = (a_{i+n})_i \quad \forall n \in \mathbb{Z}, a \in \Omega.$$

Then $[\![\Omega \rtimes \mathbb{Z}]\!]$ consists of homeomorphisms $\phi : \Omega \to \Omega$ such that there exists a continuous function $n : \Omega \to \mathbb{Z}$ such that

$$\phi(x) = n(x).x \quad \forall x \in \Omega.$$

3. Let $\Omega = \{0,1\}^{\mathbb{N}}$ be another model for the Cantor space. Define $T : \Omega \to \Omega$ continuous to be the shift

$$T(a_0, a_1, ...) = (a_1, a_2, ...).$$

Let G_2 be the so-called *Cuntz* or *Renault-Deaconu* groupoid defined by

$$\{(x, m-n, y) \mid x, y \in \Omega, m, n \in \mathbb{N} \text{ s.t. } T^m x = T^n y\}.$$

Exercise: The reduced C^*-algebra of G_2 is isomorphic to the Cuntz algebra

$$O_2 = C^*\langle s_1, s_2 \mid s_1 s_1^* + s_2 s_2^* = 1, s_1^* s_1 = s_2^* s_2 = 1\rangle.$$

The open sets

$$U_{a,b} = \{(ax, l(a) - l(b), bx) \mid x \in \Omega\}$$

define compact open bisections which cover G_2 when a, b run across Ω_f.

Then $[\![G_2]\!]$ consists of the bisections of the type

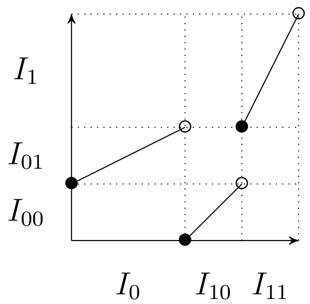
$$S = \coprod_{i=1}^n U_{a_i, b_i}$$

where $\Omega = \coprod_{i=1,n} C_{a_i} = \coprod_{i=1,n} C_{b_i}$.

If for $a \in \Omega_f$, $I_a = [\overline{a}, \overline{a} + 2^{-l(a)}) \subset [0,1)$, then

$$\begin{cases} [\![G_2]\!] & \to & V \\ \coprod_{i=1}^{n} U_{a_i,b_i} & \mapsto & \begin{pmatrix} I_{a_1} & \dots & I_{a_n} \\ I_{b_1} & \dots & I_{b_n} \end{pmatrix} \end{cases}$$

is an isomorphism of groups.



$$S = U_{0,01} \coprod U_{10,00} \coprod U_{11,1} \text{ corresponds to } \begin{pmatrix} I_0 & I_{10} & I_{11} \\ I_{00} & I_{01} & I_1 \end{pmatrix}$$

Figure 1.3: The isomorphism $[\![G_2]\!] \cong V$

The last example realizes V as a countable subgroup of homeomorphsims of Ω. If $U = C_a$ is a cylinder for $a \in \Omega_f$, then the rigid stabilizer V_U is isomorphic to V. But V contains a nonabelian free groups, hence is nonamenable. The above theorem ensures that V is thus C^*-simple.

1.5 Weakly and non-weakly band dominated operators

1.5.1 Approximation of band dominated operators

In the following, X denotes a discrete metric space (e.g. $\mathbb{Z}$) with bounded geometry. This last requirement means that, for each positive number r, the cardinality of the r-balls is uniformly bounded, i.e. the number $N_r = \sup_{x\in X}|B(x,r)|$ is finite. For $T \in B(l^2X)$, define the *matrix coefficients* of T by

$$T_{xy} = \langle \delta_x, T\delta_y\rangle \quad \forall x, y \in X.$$

Think of T as a matrix (T_{xy}) indexed by X. The propagation of such a T will then be the (possibly infinite) number

$$prop(T) = \inf\{d(x,y) \mid T_{xy} \neq 0\}.$$

If the propagation of T is finite, we will say that T is *bounded* or has *finite propagation. Band dominated operators* are the norm limits of bounded operators. They form a C^*-algebra $C^*_u(X)$, called the *uniform Roe algebra* of X.

Questions:

1. If T is band dominated, how can we approximate it by bounded operators?
2. How can we recognize when T is band dominated?

The two next numbers will give partial answers to these two questions. Or at least try to explain why they are not trivial.

1.5.2 Approximation by bounded operators

For $T \in B(l^2X)$ band dominated, define $T^{(n)}$ to be the operator with matrix coefficients

$$T^{(n)}_{xy} = \begin{cases} T_{xy} & \text{if } d(x,y) \leq n \\ 0 & \text{otherwise.}\end{cases}$$

We hope that T^n converges to T in norm as n goes to ∞.

As an example, take $X = \mathbb{Z}$ with its canonical metric (given by the absolute value). Each $f \in C(\mathbb{S}^1)$ gives rise to a multiplication operator $M_f \in B(L^2(\mathbb{S}^1))$, and by Fourier transform to a convolution operator $T_f \in B(l^2\mathbb{Z})$. It is the operator of norm $\|f\|_\infty$ with matrix coefficients $(T_f)_{xy}$ proportional to $\hat{f}(x-y)$.
In particular, if $f = \sum_{n=-N}^{N} \lambda_n z^n$ is a trigonometric polynomial, then T_f is bounded as $\hat{f}(n) = 0$ for $|n| > N$. This ensures that every T_f is band dominated, as every continuous functions is a uniform limit of trigonometric polynomials. For such operators, our naive guess

$$\text{“ } T^{(n)}_f \to_{\|\|} T_f \text{ ”}$$

is equivalent to

$$\text{“ } \sum_{n=-N}^{N} \hat{f}(n) z^n \to_{\|\|_\infty} f \text{ ”}$$

which is false. It is even worse: one can have $\|T_f\| = 1$ while $\|T_f^{(n)}\|$ goes to ∞ (Baire category argument, see [?] p. 167) and this implies (by uniform boundedness theorem) that $(T_f^{(n)})_n$ does not even converges to T_f in the strong operator topology.

1.5.3 Weakly band dominated operators

Definition 1.5.1. An operator $T \in B(l^2X)$ has (r, ε)-propagation if for every subsets $A, B \subset X$ such that $d(A, B) > r$,

$$\|\chi_A T \chi_B\| < \varepsilon.$$

T is weakly band dominated if, for every $\varepsilon > 0$, there is $r > 0$ such that T has (r, ε)-propagation.

Note: Bounded implies weakly band dominated, therefore, weakly band dominated being a closed condition, band dominated implies weakly band dominated, as the intuition suggests.

Question: Does weakly bounded implies bounded?

This was claimed without proof for spaces with finite asymptotic dimension by J. Roe ca '97, and actually proved

- by Rabinovich-Roch-Silbermann in '00 for $X = \mathbb{Z}^n$ [?];
- by Špakula-Tikuisis in '16 for finite asymptotic dimension (and a bit more, finite decomposition complexity spaces for the curious reader) [?];
- by Špakula-Zhang in '18 for spaces with property A [?].

We have no counterexamples to this date (25 jan. 2019).

Theorem 1.5.2 (Folklore)**.** The following are equivalent:

1. T is weakly band dominated;
2. for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $f \in l^\infty(X)_1$ and $Lip(f) \leq \delta$ then $\|[T, f]\| < \varepsilon$.

Proof. Let us start with the reverse implication. say $d(A, B) > r$, then there exists $f \in l^\infty(X)_1$ satisfying $0 \leq f \leq 1$, $f_{|A} = 1$, $f_{|B} = 0$ and $Lip(f) < \frac{1}{r}$. Then $f\chi_A = \chi_A$ and $f\chi_B = 0$ so that

$$\chi_A T \chi_B = \chi_A [f, T] \chi_B$$

and $\|\chi_A T \chi_B\| \leq \frac{1}{r}$. □

1.5.4 Characterizing membership in the Roe algebra

The main goal of this section is to prove the following result, after the work of Spakula and Tikuisis.

Theorem 1.5.3. Consider the following properties of an operator $b \in B(l^2X)$.

1. $\lim \|[b, f_n]\| = 0$ for every very lipschitz sequence $(f_n) \subset C_b(X)$;
2. b is quasi local;
3. $[b, g] \in \mathfrak{K}(l^2X)$ for every Higson function $g \in C_h(X)$;
4. $b \in C^*_u(X)$.

Then (4) $\implies$ (1) $\iff$ (2) $\iff$ (3). Moreover if X has finite asymptotic dimension, then (4) is equivalent to all of these.

Some remarks are in order.

- These results grew out of a question of John Roe, who asked about the implication (2) $\implies$ (4) when X has *finite asymptotic dimension* (FAD).
- The theorem in [**?**] is better: (2) $\implies$ (4) when X has straight *finite decomposition complexity* (FDC), which is much weaker than FAD. For instance, $\mathbb{Z} \wr \mathbb{Z}$ has FDC but not FAD, while FAD always implies FDC.
- There is a follow up paper which shows (2) $\implies$ (4) when X has property A, an even weaker condition. This last result will be treated in a following number.

Let us understand the conditions better.

Very Lipschitz condition

Recall that a function f is Lipschitz if its Lipschitz modulus

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

is finite. More precisely, a function f is L-Lipschitz if $Lip(f) \leq L \iff |f(x) - f(y)| \leq Ld(x, y), \quad \forall x \neq y$.

A sequence $(f_n) \subset l^\infty(X)$ is *very Lipschitz* if

- the sequence is uniformly bounded: $\exists C > 0$ such that $\|f_n\| \leq C$;
- $\lim Lip(f_n) = 0$.

With this notation, the condition (1) is equivalent to

$$\forall \varepsilon > 0, \exists L > 0 \text{ s.t. if } \|f\| \leq 1 \text{ and } Lip(f) \leq L \text{ then } \|[b, f]\| < \varepsilon.$$

Indeed, one direction is obvious, and suppose there exists $\varepsilon > 0$ such that for every $L > 0$ there is a $f \in l^\infty(X)$ with $\|f\| \leq 1$, $Lip(f) \leq L$ and $\|[b, f]\| \geq \varepsilon$. Take $L = \frac{1}{n}$ to get a very Lipschitz sequence (f_n) with $\|[b, f]\| \geq \varepsilon > 0$, which contradicts (1).

Quasi-locality

$b \in B(l^2X)$ is quasi-local
iff $\forall \varepsilon > 0$, b has finite ε-propagation
iff $\forall \varepsilon > 0, \exists r > 0$ such that $\forall f, g \in l^\infty(X)$, if $\|f\|, \|g\| \leq 1$ and $d(supp(f), supp(g)) \geq r$ then $\|fbg\| < \varepsilon$.

Higson functions

$g \in C_h(X)$
iff $\forall \varepsilon > 0, \forall r > 0$, there exists a finite subset $F \subset X$ such that if $x, y \notin F$ and $d(x, y) \leq r$, then $|g(x) - g(y)| \leq \varepsilon$.

Roe's question on conditions (2) and (4)

(4) $\implies$ (2) is not difficult. In short, quasi-locality is a closed condition, which is obviously satisfied by finite propagation bounded operators.

Closed condition If $\forall \delta > 0$, there is a quasi-local operator b' such that $\|b - b'\| < \delta$, then b is quasi-local.

Finite propagation operators are quasi-local If $\xi \in l^2(X)$ is finitely supported and $prop(b) \leq r$, then $supp(b\xi) \subset N_r(supp(\xi))$, and so if $d(supp(f), supp(g)) > r$, then $gbf = 0$.

(4) $\implies$ (1) is again not too hard.
Condition (1) is closed and is satisfied by finite propagation operators. This follows from elementary estimates and a calculation of the kernel of the commutator.

Lemma 1.5.4. If $b \in B(l^2X)$ such that $|b(x, y)| \leq C$ and $prop(b) \leq r$, then $\|b\| \leq CN_r$

Lemma 1.5.5. Let b as above and $f \in l^\infty(X)$. The kernel of $[b, f]$ is

$$[b, f](x, y) = b(x, y)(f(x) - f(y))).$$

Now (4) $\implies$ (1) follows: if $prop(b) \leq r$, then $prop([b, f]) \leq r$ and $|[b, f](x, y)| \leq CLip(f)r$ so that also $\|[b, f]\| \leq CrN_rLip(f)$. As for the lemmas:

1.6 Noncommutative geometry

1.6.1 Basic objects and constructions

Mainly, I'm interested in $*$-algebras A (and their completions) which are k-algebras equipped with an involution $*$. Usually, $k = \mathbb{C}$ is the field of complex numbers. A very famous example of $*$-algebra is the algebra of the quantum harmonic oscillator,

$$\mathcal{H} = k\langle x, y\rangle/(xy - yx = 1).$$

When $k = \mathbb{C}$, one often represent A as a sub-$*$-algebra of the bounded operators on aHilbert space $\mathcal{L}(H)$,and complete w.r.t. to the norm. Note that not all complex $*$-algebras admit such a representation.

For instance, for$\mathcal{H}$, one easily get that

$$[x, P(y)] = P'(y) \quad \forall P \in \mathbb{C}[t]$$

Then if $|| \ ||$ is a multiplicative norm on$\mathcal{H}$, it satisfies

$$2||x|| \ ||y|| \geq n \quad \forall n > 0.$$

Basic construction:

- separation-completion: in our sense, a norm can be degenerate. Being multiplicative, the annhiliator of any norm is a closed ideal in A, so that there is an induced (classical/ nondegenerate) norm on the quotient algebra. The separation-completion is defined to be the completion of the quotient w.r.t. the induced norm. Let us say that if α is such a norm, wedenote by A_α the associated separation-completion. Any inequality

  $$\alpha(x) \leq \beta(x) \quad \forall x \in A$$

  induces an inclusion of annilihator $N_\beta \subset N_\alpha$, and gives a canonical quotient map

  $$A_\beta \to A_\alpha.$$

  The basic class of examples comes from completion of the complex group ring $\mathbb{C}[\Gamma]$. For any family of unitary representations $\mathcal{F}$, one can define the $*$-norm

  $$||x||_{\mathcal{F}} = \sup\{||\pi(x)|| : \pi \in \mathcal{F}\}$$

  on $\mathbb{C}[\Gamma]$. The separation-completion is a C^*-algebra denoted $C^*_{\mathcal{F}}(\Gamma)$. For instance, if $\mathcal{F}$ consists of all unitary representations of Γ, then one gets the maximal C^*-algebra $C^*_{max}(\Gamma)$, while if the family is reduced to the left regular representation λ_Γ, one gets the reduced C^*-algebra $C^*_r(\Gamma)$. By inclusion, one gets the canonical quotient map

  $$\lambda_\Gamma : C^*_{max}(\Gamma) \to C^*_r(\Gamma).$$

Crossed-product: the basic ingredients are a $*$-algebra H endowed with a coassociative coproduct

$$\Delta : H \to H \otimes H,$$

and a C^*-algebra A on which H acts via a $*$-homomorphism

$$\alpha : A \to A \otimes H$$

such that $(1 \otimes \Delta)\alpha = (\alpha \otimes 1)\alpha$. The crossed-product is a twisted version of the tensor product.

$$(a \otimes x)(a' \otimes y) := (a \otimes 1_{M(H)})\alpha(a')(1_{M(A)} \otimes xy)$$

1.6.2 Quantum groups

A C^*-bialgebra is a pair (H,Δ) where H is a C^*-algebra and

$$\Delta : H \to M(\tilde{H} \otimes_{min} H + H \otimes_{min} \tilde{H}, H \otimes_{min} H)$$

is a non-degenerate $*$-homomorphism such that $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$.

A H-algebra is a pair (A,α) where A is a C^*-algebra and

$$\alpha : A \to M(\tilde{A} \otimes_{min} H, A \otimes_{min} H)$$

such that $(\alpha\otimes 1)\alpha = (1\otimes\Delta)\alpha)$. Its principal map is

$$\Psi : \begin{cases} A \otimes_{alg} A & \to \quad M(A \otimes_{min} H) \\ \quad x \otimes y & \mapsto \quad (x \otimes 1_{M(H)})\alpha(y) \end{cases}$$

Let (H,Δ) be a C^*-bialgebra and (A,α) a H-algebra, with principal map

$$\Psi : A\otimes A \to M(A\otimes_{min} H).$$

- free if the range of Ψ is strictly dense in $M(A\otimes_{min} H)$
- proper if the range of Ψ is contained in $A\otimes_{min} H$
- principal if $\Psi(A\otimes_{alg} A)$ is a norm dense subset of $A\otimes_{min} H$

principal = free and proper

1.6.3 Why $SU_q(2)$?

Apparently, some people are interested in deformation of classical Lie groups such as $SU_q(2)$, which is the Hopf algebra generated by 3 generators E, F, K satisfying the relations

$$R.$$

I wanted to understand where these relations are coming from, which led me to interesting ideas developed by several people, including Yuri Manin. The idea is to define $SU_q(2)$ as a special group like object of the automorphism group of some noncommutative space, the quantum plane.

Let k be a field. The free (noncommutative) k-algebra on n generators is denoted by $k\langle x_1, ..., x_n\rangle$.

Definition 1.6.1. A quadratic algebra

$$A = \oplus_{i\geq 0} A_i$$

is a $\mathbb{N}$-graded finitely generated algebra such that:

- $A_0 = k$, and A_1 generates A,
- the relations on generators are in $A_1 \otimes A_1$.

The quadratic algebra A is said to be a Frobenius algebra of dimension d if moreover

- $A_d = k$ and $A_i = 0$ for all $i > d$,
- the multiplication map
$$m : A_i \otimes A_{d-i} \to A_d$$
is a perfect duality.

The main example is the quantum plane

$$\mathbb{A}_q^2 = k\langle x, y\rangle/(xy - qyx)$$

where $q \in k^\times$. More generally, the quantum space of dimension $n|m$ is

$$\mathbb{A}_q^{n|m} = k\langle x_1, .., x_n, \eta_1, ..., \eta_m\rangle/(x_i x_j - q x_j x_i, q\eta_i\eta_j + \eta_j\eta_i).$$

This example is suppose to come from physics. In quantum field theories, physicists deal with two kind of particles, bosons and fermions, and use commuting variables for one type, and anticommuting for the other. One object they appeal to are called supermanifolds, which are manifolds enriched with anticommuting variables. Formally, it means they look at ringed spaces $(X, \mathcal{O})$ locally isomorphic to $(\mathbb{R}^n, C^\infty[\eta_1, ..., \eta_m])$, where $C^\infty[\eta_1, ..., \eta_m]$ is the free sheaf of rings generated by anticommuting variables η_i over the smooth complex valued functions $C^\infty(\mathbb{R}^n)$.

Remark that a quadratic algebra A is a quotient of $k\langle x_1, ..., x_n\rangle$ by elements $r_\alpha \in A_1 \otimes A_1$, which we will denote as

$$A = k\langle x_1, ..., x_n\rangle/(r_\alpha)$$

or

$$A = \langle A_1, R_A\rangle$$

with $R_A \subseteq A_1 \otimes A_1$.

Manin defines the quantum dual of a quadratic algebra as

$$A^! = k\langle x^i\rangle/(r^\beta)$$

where $r^\beta_{ij} r^{ij}_\alpha = 0$, i.e. $R_{A^!} = R_A^\perp$. Then, the quantum endormorphisms between two quadratic algebra is

$$Hom(A, B) = k\langle z_i^j\rangle/(r_\alpha^\beta)$$

where $r_\alpha^\beta = r_\alpha^{ij} r_{kl}^\beta z_i^k z_j^l$. If $End(A) = Hom(A, A)$, then $End(A)$ satisfies the universal property to be intial in the category of k-algebras (B, β) endowed with an algebra homomorphism $\beta : A \to A \otimes B$.

If one does that to the quantum plane $\mathbb{A}_q^2$, one stil doesn't find quite $M_q(2)$: half of the relations are missing. Also

$$(\mathbb{A}_q^{2|0})^! = \mathbb{A}_q^{0|2}?$$

Exercise.

1.6.4 TQFT

Motivations

This section is aimed at being an introduction to *Topological Field Theories*. One of the difficulties of this particular topic is that it comes from different areas and can be attacked in different ways. The following is my attempt to make sense out of the large amount of information available on the subject. In particular, I do not claim exhaustivity or expertise.

The starting point are probably *path integral* formulations in Physics. In Statistical Mechanics and in Quantum Physics, the values predicted by the theory can often be written as expectation of the type

$$\mathbb{E}[\exp(-\int V(q(t))) \quad \text{or} \quad \mathbb{E}[\exp \frac{i}{h} S(q)].]$$

Physically, one tries to define a positive function on the phase space M (such as an energy (Hamiltonian) H or an action S, the integral of a Lagrangian). The probability distribution of the system should then be

$$\frac{1}{Z_M} e^{-\beta H(\omega)} D\omega \quad \text{or} \quad \frac{1}{Z_M} e^{\frac{i}{h} S(\omega)} D\omega.$$

The first case is the one known as Gibbs measures, and describes the behaviour of a system in contact with a thermostat at inverse temperature β. The second case is the so called Feynman integral of quantum mechanics. The reason these formulas are used is that systems should satisfy some *minimization principle*. In the classical case, the observed trajectories are the minima of the energy function, whereas in the quantum case the observe deviations from the classical trajectories up to the amplitude $e^{iS/h}$.

That point is exactly where it starts to be complicated. Physicists want to define something propoptional to these exponential, and the measure $D\omega$ is supposed to be a reference measure with nice invariance properties. In the finite dimensional case, the natural measure would be the Lebesgue measure. But no such thing exists on a general functional space, which makes the definition above useless.

It turns out that these integral have very interesting invariance properties, notably in topology. More precisely, the *partition function* Z_M gives topological invariant when M is a closed manifolds. This fact gave motivation to mathematicians to study more attentively these functional integrals. While you can try to define integrals analytically (see **REFERENCES**), there exists an algebraic approach which proposes intuitively to define the partition function on simple manifolds (possibly with border) and in coherent manner so that the partition function of a closed manifold can be computed by cutting it in simple pieces, computing the corresponding values, and reassembling these to get the final result. Mathematically, this requires:

- describing an algebraic structure on the family of *n-dimensional manifolds*, and giving generators;
- setting up the value of the partition function;

- showing that all of this makes sense.

This is accomplished by considering the n-category of bordisms and defining the (algebraic) partition function Z to be a nice functor between the latter and the n-category of finite dimensional vector spaces over some fields. In dimension 1 and 2, it will be understood that only one value needs to be fixed (respectively on the point space and on the circle), while this still holds in higher dimension under more hypothesis.

Summary of the talks

We recalled the definitions of a monoidal category, a braided category, and a symmetric monoidal category. The two main examples are the category of bordisms $Bord^d$ in dimension d, and the category of vector spaces over a field k. The first talk focused on topological quantum fields theories in dimension 1 and 2.

Definition 1.6.2. A TQFT in dimension d is a monoidal symmetric functor

$$Z : Bord_d \to Vect_k.$$

The two main results we showed are:

- there is an equivalence of categories

  $$TQFT_1 \cong Vect_k$$

  obtained as $Z \mapsto Z(pt)$.

- there is an equivalence of categories

  $$TQFT_2 \cong Frob_k$$

  obtained as $Z \mapsto Z(\mathbb{S}^1)$.

A nice example in dimension 2: $Z(\mathbb{S}^1) = \mathbb{C}[t]/(t^2 - 1)$ is the Frobenius algebra given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad \epsilon(1) = 0 \quad \epsilon(t) = 1.$$

Then the handle element is $h = 2t$ and

$$Z(\Sigma_g) = \begin{cases} 2^g & \text{if } g \text{ is odd} \\ 0 & \text{if } g \text{ is even.} \end{cases}$$

The second talk was directed towards extended field theories. First recall some higher category theory: n-categories, etc... And an extented TFT is a symmetric monoidal functor between symmetric monoidal n-categories

$$Z : Cob_n \to \mathcal{C}.$$

Then the following theorem was proved in [**?**].

Theorem 1.6.3. The evaluation functor

$$Z \mapsto Z(*)$$

establishes a bijective correspondance between extended n-dimensional TFT and fully dualizable objects of $\mathcal{C}$.

We now give an application of this result to the Jones polynomial. In [**?**], Witten gives an interpretation of the Jones polynomial, an isotopy invariant of links, as induced from a 3-dimensional TFT. The drawback of this article (for us) is that Witten uses Physical TFT's, i.e. gauge theories. The Jones polynmial is then shown to be the value of the partition function of a gauge field theory on $\mathbb{S}^3$ with gauge group $SU(2)$. I propose to rewrite this result in our setting as an exercise.

A link is a disjoint union of embedding of the circle into $\mathbb{S}^3$

$$\mathcal{L} = \{\text{embeddings } \coprod_{i=1}^{k} \mathbb{S}^1 \hookrightarrow \mathbb{S}^3\}.$$

we will often make no distinction between the embedding and its image in the 3-sphere, which we will denote by L. The Jones polynomial of a link L is defined as an isotopy invariant polynomial $V : \mathcal{L} \to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ satisfying the Skein relations

$$-t^{\frac{1}{2}} V_+ + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_0 + t^{-\frac{1}{2}} V_- = 0.$$

To a link L one can associated the 3-manifold $M_L = \mathbb{S}^3 - L$. Consider the extended TFT

$$Z^{(n)} : Cob_3 \to \mathcal{C}$$

given by $Z() = V_n$ where is the fundamental representation of $\mathfrak{su}(n)$. By the cobordism theorem, it is enough to define the TFT on all of Cob_3. Then

$$\phi(V_L) = Z^{(2)}(M_L),$$

where $\phi : \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \to \mathbb{C}$ is the evaluation at a root of unity $q \in \mathbb{C}^\times$. This can be proved by showing that $Z^{(n)}(M_L)$ satisfies the skein relation

$$-q^{\frac{n}{2}} V_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) V_0 + q^{-\frac{n}{2}} V_- = 0$$

1.6.5 Reminder

A locally ringed space is a topological space X together with a sheaf or ring $\mathcal{O}_X$ over X such that all stalks are local rings, ie have a unique maximal ideal.

For R a ring, $X = Spec(R)$ denotes the topological space obtained as the set of prime ideals of R endowed with the Zariski topology, i.e. the topology generated by the closed subsets

$$V_I = \{J \text{ ideals in } R \text{ s.t. } I \subset J\}.$$

Equivalently, a basis of open subsets is given by

$$D_f = \{J \text{ ideals in } R \text{ s.t. } f \notin J\}$$

for every $f \in R$. Let S_f be the multiplicative domain given by the powers of f. Then define a sheaf of ring over X by

$$\mathcal{O}_X(D_f) = S_f^{-1}R.$$

It is called the structural sheaf of $Spec(R)$. Any locally ringed space isomorphic to

$$(Spec(R), \mathcal{O}_{Spec(R)})$$

with R commutative is called an affine variety.

Note: the functor *Spec* gives an antiequivalence of categories between the categories of commutative rings and the category of affine varieties.

Definition 1.6.4. A scheme is a locally ringed space locally isomorphic to an affine variety.

Chapter 2

Zoology of groups and C^*-algebras, and other wild creatures

2.1 A list of books

A list of books I like about general knowledge in science:

- L'aventure des nombres, Godefroy
- L'autobigraphie de Paul Levy, Laurent Schwartz, et Yuri Manin.
- Recoltes et semailles, Grothendieck.
- Lee Smolin, The trouble with physics, the rise of String theory, the fall of a Science, and what comes next,
- Julian Barbour, The End of Time, The next revolution in Physics,
- Carlo Rovelli, Et si le temps n'existait pas, un peu de science subversive,
- Mandlebrot, The (Mis)Behaviour of markets, Fractals and Chaos, the Mandelbrot set and beyond, The fractal geometry of nature.
- Manjit Kumar:
- Amir Alexander, Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World
- Ian Stewart, Does God play dice?
- History of Statistics, Stielger
- Logicomix

Overview and more specialized books:

- Moonshine beyond the Monster, Terry Gannon
- Le theoreme d'uniformisation, Saint-Gervais
- Invitation aux mathematiques de Fermat, Hellgouarch
- Rached Mneime, tous ses livres!
- Hubbard West pour les equa diff
- Noether's theorem, Yvette K
- Nother's wonderful theorem
- The annus mirabellus of Einstein
- The Road to Reality, Sir Roger Penrose

Books about Einstein:

- Subtle is the Lord, Abraham Pais [**?**]; biography of Einstein by someone who knew him;

- Einstein's miraculous year: Five papers that changed the face of physics, Penrose & Einstein [?]; English translations of the five papers Einstein published in 1905 while working at the patent office in Bern.
- Quantum: Einstein, Bohr, and the great debate about the nature of reality, Kumar [?]; history of quantum theory from Planck's blackbody radiation to the EPR paradox.

2.2 Groups

- Amenable, a-T-menable, property T, with a diagram
- Mapping class groups
- Profinite groups, locally profinite groups, $Aut(\overline{\mathbb{Q}}/\mathbb{Q})$
- Automorphism of a regular tree, the Grigorchuk group,
- Lamplighter groups, usually
$$\mathbb{Z}_2 \wr \mathbb{Z} = \oplus \mathbb{Z}_2 \rtimes \mathbb{Z}.$$
More generally, wreath products $H \wr \Gamma = H^\Gamma \rtimes \Gamma$.
- $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ generate a free group of finite index in $SL(2, \mathbb{Z})$. The corresponding semi-direct product $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{F}_2$ does not have Haagerup's property ($(\Gamma, \mathbb{Z}^2)$ has relative property (T)).

- Cayley graphs: finite groups, symmetric groups, $\mathbb{Z}^2$, $\mathbb{F}_2$, $\mathbb{Z}$ with original generating sets. $B(1, 2)$. Lamplighter groups: meta-abelian without finite presentation.

  $SL(2, \mathbb{Z})$ has presentation
$$< x, y \mid x^4 = 1, x^2 = y^3 > \quad p, q \geq 1,$$
and in this presentation, the quotient by $< x^2 >$ is isomorphic to
$$PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$
This gives a way to draw their Cayley graph easily.

- Baumslag-Solitar monster
$$BS_{p,q} =< a, b \mid ab^p a^{-1} = b^q >$$
are nonhopfian when p and q are coprime and at least 2. $BS_{p,q}$ is the Higman-Neumann-Neumann extension $HNN(\mathbb{Z}, p\mathbb{Z}, \theta)$ where $\theta(p) = q$:
$$BS_{p,q} < Aut(T_{p+q}) \quad \text{where } T_{p+q} \text{ homogeneous tree of degree } p + q.$$

On the other hand, one has a non injective morphism $BS_{p,q} \to Aff(\mathbb{R}); a \mapsto \frac{qx}{p}; b \mapsto x+1$. The diagonal morphism $BS_{p,pq} \to Aut(T_{p+q}) \times Aff(\mathbb{R})$ has discrete image and $BS_{p,q}$ has Haagerup's property (because both $Aut(T_{p+q})$ and $Aff(\mathbb{R})$ have it.

- Infinite torsion questions: subgroups of $GL(n, \mathbb{Z}[\frac{1}{p}])$.

- Tarski monsters: p a prime, then every x generates a cyclic subgroup of order p, and the set of x together with any element not contained in this cyclic subgroup subgroup generates Γ.

Tessera and Arhantseva showed that there exists a group which is a split extension of two groups that are coarsely embeddable into Hilberrt space, and that does not admit such an embedding.

- **Amenability** Abelian, Compact, extension of such (Elementary amenable), Grigorchuk group: amenable but not elementary amenable (first example of finitely generated group with intermediate growth, i.e. faster than polynomial but subexponential). Every group with subexponential growth (equivalent to virtually nilpotent by Gromov's Polynomial growth theorem). When discrete, Γ is amenable iff $C_r^*(\Gamma)$ is nuclear.

  In terms of CP functions? Γ is amenable iff there exists a net of commpactly supported continuous positive definite functions converging pointwise to 1.

- **Haagerup's property** Introduced by Haagerup on his work on the Free groups. Incidentatly, $\mathbb{F}_2$ is not amenable but has Haagerup's property. Stability by closed subgroups so $\mathbb{F}_n$ and the free group with countably many generators. Equivalent to Gromov's a-T-menability and property FH in the locally compact case. Every such group satisfies the Baum-Connes conjecture with coefficients, and is K-amenable, i.e.
$$\lambda \in KK_0(C_{max}(\Gamma), C_r^*(\Gamma))$$
is invertible. Same for $SL(2,\mathbb{Z})$. Amenable groups, Coxeter groups, Groups acting metrically properly on trees or spaces with walls. $SU(n,1)$ and $SO(n,1)$: $g \mapsto d(gx_0, x_0)$ is conditionally negative and definite, where d is the hyperbolic distance and x_0 any point in real or complex projective space. Baumslag-Solitar 's groups $BS_{p,q}$.

  In terms of CP functions? G has Haagerup's property iff there exists a continuous proper conditionally negative definite function $G \to \mathbb{R}_+$, iff there exists a sequence of continuous normalized positive definite functions converging uniformly on compact subsets of G.

- **Property T** Any compact group. $SL(n,\mathbb{Z})$ for $n \geq 3$. Simple real Lie groups with real rank ≥ 2 and their lattices: $SL(n,\mathbb{R})$, $n \geq 3$; $SO(p,q)$, $p > q \geq 2$; $SO(p,p)$,

$p \geq 3$. Simple algebraic groups of rank ≥ 2 over a local field. $Sp(n,1)$, $n \geq 2$ which is a simple real Lie group of real rank 1, and its lattices, which are dicrete countable hyperbolic groups. $Aut(\mathbb{F}_5)$. Mapping class groups are supposed to have property (T), but the proof is still not clear and contains gaps.

$$\text{Property (T)} \ + \ \text{Haagerup} \ = \ \text{Compact}.$$

In terms of CP functions? Γ has (T) iff every sequence of continuous normalized positive definite functions that converges uniformly on compact subsets to 1 converges uniformly to 1. Or iff every continuous conditionally negative definite function on Γ is bounded.

- **Asymptotic dimension** asdim $|\Gamma| = \dim_{nuc}(C_u^*(\Gamma))$. $\mathbb{Z}^n$ of asymptotic dimension n. Asymptotic dimension of a tree is one: $asdim(\mathbb{F}_n) = 1$. Hyperbolic groups (trees from far away) are of finite asymptotic dimension (which can be arbitrarily large). Finitely generated solvable goups such that the abelian quotients are finitely generated have finite asymptotic dimension. Example of such: the group

$$Sol = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \quad \text{where} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\{e\} < \mathbb{Z} < \mathbb{Z}^2 < Sol \quad \text{ with } Sol/\mathbb{Z}^2 \cong \mathbb{Z}.$$

Every almost connected Lie group has finite asymptotic dimension, and any of their discrete subgroup. For instance $SL(n,\mathbb{Z})$ for every n. Mapping class groups have finite asymptotic dimension.

All finite asymtotic dimension groups satisfy the Novikov conjecture.

The groups $\mathbb{Z}^{(\infty)} = \bigoplus_{j=0}^{\infty} \mathbb{Z}$ with $d(x,y) = \sum j|x_j - y_j|$ and $\mathbb{Z} \wr \mathbb{Z}$ have infinite asymptotic dimension.

- **FDC** $\mathbb{Z}^{(\infty)}$ has FDC and infinite asymptotic dimension, but is not finitely generated. The following subgroup of $SL(2,\mathbb{R})$ has FDC, infinite asymptotic dimension and is finitely generated:

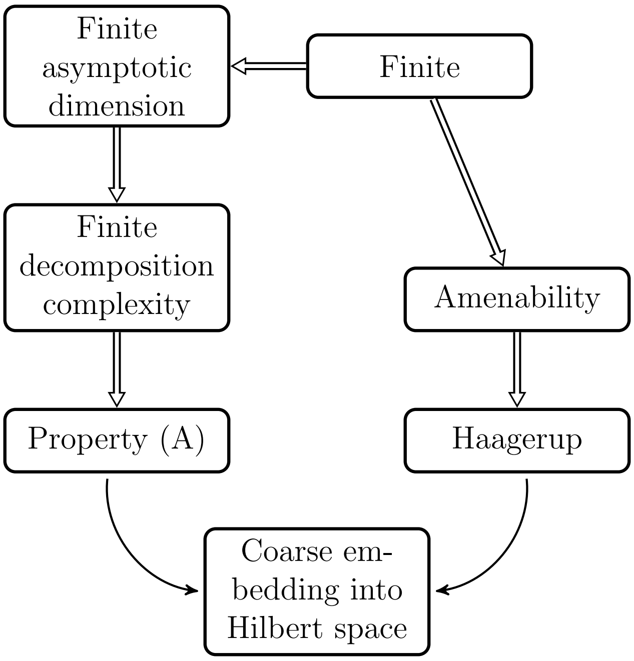
$$G = \left\{ \begin{pmatrix} \pi^n & P(\pi) \\ 0 & \pi^{-n} \end{pmatrix} | n \in \mathbb{Z}, P \text{ Laurent polynomial with integer coefficients} \right\},$$

with $\left\{ \begin{pmatrix} 1 & P(\pi) \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z} \wr \mathbb{Z}$ as a subgroup (so infinite dimension). Any countable subgroup of $GL(n,R)$ for R a commutative ring has FDC, countable subgroups of almost connected Lie groups, elementary amenable, finite asymptotic dimension and hyperbolic, all have FDC.

- **Property A** $|\Gamma|$ has property (A) iff $C_u^*(\Gamma)$ is nuclear (Ozawa, but Guentner-Kaminker...) iff $\beta\Gamma \rtimes \Gamma$ is amenable. Non-equivariant version of Haagerup's property. All FDC groups have (A).

- **Coarsely embeddable into Hilbert space** $|\Gamma|$ coarsely embeds iff $\beta\Gamma \rtimes \Gamma$ is a-T-menable.

Other properties: hyperbolicity in Gromov's sense, K-amenability, polya-$\mathcal{P}$ (polyabelian = solvable?, polycyclic,..), virtually abelian or nilpotent, Rapid decay property,... Exactness: Gromov's monsters are the only groups known not to be exact. C^*-simplicity: nonabelian Free groups,



Stability:

	Amenablitiy	**Haagerup**	**(T)**	**Baum-Connes**
Product	Yes			
Subgroups	Yes	No Closed yes	No, $\mathbb{Z} < SL(3,\mathbb{Z})$ Finite index yes	
Quotients	Yes		Yes	
Extensions	Yes			
Direct limits	Yes			
Free products	No, $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$			
HNN extensions				
Free amalgamated products				

	Finite Asymptotic dimension	**FDC**	**(A)**
Product	Yes	Yes	
Subgroups	Yes	Yes	
Quotients			By amenable subgroups
Extensions	Yes	Yes	Yes
Direct limits			Yes
Direct unions	No, $\mathbb{Z}^{(\infty)}$	Yes	
Free products			Yes
HNN extensions	Yes	Yes	
Free amalgamated products	Yes	Yes	

Other group like objects, but with less properties.

2.2.1 Groupoids

- The *coarse groupoid* $G(X)$: étale (even ample) with totally disconnected basis βX. Dynamical asymptotic dimension of asymptotic dimension of X. Amenable iff X has property A. A-T-menable iff X coarsely embeds into Hilbert space.
- HLS groupoid associated to a sequence of finite metric spaces X_n equipped with maps $X_n \to \Gamma$ to a finitely generated group $\Gamma = \langle S \rangle$.
- Groupoids of germs of semigroup of partial homeomorphisms acting on a topological space
- Full topological groups of an ample second-countable groupoid with compact base space
- Tillings groupoids ($\Omega \rtimes G$, usually amenable)
- Holonomy groupoids of a foliation
- Action groupoids $X \rtimes \Gamma$, principal bundles groupoids $P \times_G P$, where $P \to X$ is a G-bundle
- Equivalence relation groupoids

If G is étale, G is amenable iff $C^*_r(G)$ is nuclear. Amenability implies that the full C^*-algebra and the reduced coincides, but the converse is false by a result of Willett [?].

2.2.2 Quantum groups

One of the most useful ideas used by quantum groups theorists is to try and adapt concepts from geometric group theory in their setting. We could think of it as a way to algebraize notions like amenability, a-T-menability, etc. In the case of a discrete group Γ, it is known [?] that

$$asdim(\Gamma) = dim_{nuc}(l^\infty(\Gamma) \rtimes_r \Gamma).$$

By analogy, define

$$asdim(\hat{G}) = dim_{nuc}(l^\infty(\hat{G}) \rtimes_r \hat{G})$$

for a discrete quantum group $(\hat{G}, \hat{\Delta})$. Here

$$l^\infty(\hat{G}) = \prod_{x \in Irr(G)} B(H_x)$$

is naturally a $\hat{G}$-algebra. (Describe explicitely the action and give the example of $\hat{SU}(2)$.)

Remark post-discussion with Mehrdad: In general, $l^\infty(\hat{G}) \rtimes_r \hat{G}$ is not exact so its nuclear dimension is not finite. Such a definition is thus hopeless.

The natural filtration of any crossed-product of a $\hat{G}$-algebra by $\hat{G}$ is given by the *coarse structure* $\mathcal{E}_G$ of the finite dimensional symmetric representations of the compact dual G.

This suggests that the *coarse geometry* of the discrete quantum group $\hat{G}$ is encoded in $\mathcal{E}_G$. Indeed, the first thing one can do is to define a notion of S-separation for $x, y \in Irr(G)$ and $S \subseteq Irr(G)$:

$$(x, y) \in \Delta_S \text{ iff } \Delta(p_x)(p_y \otimes p_S) \neq 0.$$

- is it true that $asdim(\hat{G}) = d$ iff for every $R \in \mathcal{E}_G$, there exists a partition

$$Irr(G) = U_0 \coprod U_1 \coprod \ ... \ \coprod U_d$$

  such that each U_i is a disjoint union $\coprod_j U_{ij}$ of uniformly bounded subsets R-separated:

  1. there exists $S \in \mathcal{E}_G$ such that $(x, y) \in \Delta_S$ for every $x, y \in U_{ij}$,
  2. if $x \in U_{ij}$ and $y \in U_{ik}$, $j \neq k$, then $(x, y) \neq \Delta_R$.

- in the presence of finite asymptotic dimension, do we get a controlled Mayer-Vietoris pair?

$$\prod_{x \in U_i} B(H_x) \rtimes_r \hat{G}$$

- Define an assembly map for $l^\infty(\hat{G}) \rtimes_r \hat{G}$, and try to prove it is an isomorphism with *controlled cutting and pasting* techniques.

2.3 C^*-algebras

How to construct C^*-algebras?

- Finitely generated/presented C^*-algebras? You have to get *bounded relations.* (See Loring's book, *Lifiting solutions to perturbing problems in C^*-algebras* [?].)

- Basic blocks: commutative $C_0(X)$, finite dimensional or matrix blocks $\oplus M_{d_k}(\mathbb{C})$, $B(H)$ and its essential ideal $\mathfrak{K}(H)$, the Calkin quotient $Q(H)$,

- Sum and tensor products.

- Convolution algebras.

- Crossed product by an action by automorphisms by a group-like object: groups, groupoids, semi-groups, quantum groups. Stress that it is a kind of semi-direct product in the category of C^*-algebras: for instance, $A \rtimes_r \Gamma$ can be defined as a particular completion of the algebraic tensor product
$$A \otimes_{alg} C_r^*(\Gamma)$$
where the product is not the usual one, but twisted by the action of Γ on A, or, it is the same, the coaction of $C_r^*(\Gamma)$ on A.

Example of C^*-algebras:

- CAR algebra $C^*\langle a_i, a_i a_j + a_j a_i = \delta_{ij}\rangle$ or $\bigotimes M_2(\mathbb{C})$ or
$$\varinjlim \left\{ \begin{array}{rcl} M_{2^n}(\mathbb{C}) & \to & M_{2^{n+1}}(\mathbb{C}) \\ a & \mapsto & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{array} \right.$$

Class of C^*-algebras.

- Nuclear C^*-algebras. Finite dimensional, commutative, AF-algebras are nuclear. $C_r^*(\Gamma)$ is nuclear iff Γ is amenable (true for a discrete group, and more generally an étale groupoid). $C_u^*(X)$ is nuclear iff X has Yu's property A. So $C_r^*(\mathbb{F}_2)$ is not nuclear but $l^\infty(\mathbb{F}_2) \rtimes_r \mathbb{F}_2$ is, giving an instance where nuclarity fails to pass to sub-algebras.

- The bootstrap class $\mathcal{B}$. If Γ has Haagerup's property, then $C_r^*(\Gamma)$ is Bootstrap. It is a specialization of a famous result of Tu ([?], lemma 10.6) that if G is a-T-menable, there exists a G-proper C^*-algebra A, KK^G-equivalent to $\mathbb{C}$, such that $A \rtimes_r G$ is Bootstrap.

- The class $\mathcal{N}$ of C^*-algebras A such that the map

$$\alpha_{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

is an isomorphism for every C^*-algebra B such that $K_*(B)$ is a free abelian group. In [?], it is shown that $\mathcal{N}$ contains all of the bootstrap class.

- Exact C^*-algebras. We say that Γ is exact if $C_r^*(\Gamma)$ is exact. It is shown by Ozawa (completing work of Guentner and Kaminker [?]) in [?] that Γ is exact iff $\beta\Gamma \rtimes \Gamma$ is amenable iff $C_u^*(|\Gamma|)$ is nuclear.

- Non exact C^*-algebras:

  - For any integer sequence k_n which tends to ∞ as n goes to ∞,

$$\prod_{n \geq 0} M_{k_n}$$

    is not exact. As a result, for any discrete quantum group $\hat{G}$ which is truely noncommutative, $l^\infty(\hat{G})$ is not exact. So is any of its crossed-product, so that the naive definition of the uniform-Roe algebra

$$l^\infty(\hat{G}) \rtimes_r \hat{G}$$

    is not exact, hence not nuclear.

  - The reduced C^*-algebra $C_r^*(\Gamma)$ of a finitely generated group whose Cayley graph contains expander. Using Ozawa's result [?], one can construct finite dimensional C^*-algebras M_{X_n} such that

$$0 \to C_r^*(\Gamma) \otimes \bigoplus M_{X_n} \to C_r^*(\Gamma) \otimes \prod M_{X_n} \to C_r^*(\Gamma) \otimes \left(\prod M_{X_n} / \bigoplus M_{X_n} \right) \to 0$$

    is not exact in the middle.
    The problem of the existence of such a group is an interesting question, which was stated by Gromov and proved rigorously by several people in the wake of this.

  - If Γ is a discrete group with Kirchberg's approximation property, then Γ is amenable is equivalent to the maximal C^*-algebra $C^*(\Gamma)$ is exact ([?], prop 3.7.11).

    Any residually finite group satisfies Kirchberg's approximation property, hence *any nonamenable residually finite group has a non-exact maximal C^*-algebra.* For instance, $C^*(\mathbb{F}_n)$ and $C^*(SL(n, \mathbb{Z}))$, $n \geq 2$, are not-exact, for the reason that

$$0 \longrightarrow J \otimes_{min} C^*(\Gamma) \longrightarrow C^*(\Gamma) \otimes_{min} C^*(\Gamma) \longrightarrow C_r^*(\Gamma) \otimes_{min} C^*(\Gamma) \longrightarrow 0$$

is not exact, where $J = \ker C^*(\Gamma) \to C^*_r(\Gamma)$.

Recall that Γ has Kirchberg's approximation property if

$$\lambda \times \rho : C^*(\Gamma) \otimes_{alg} C^*(\Gamma) \to B(l^2\Gamma)$$

is min-continuous, i.e. extends to $C^*(\Gamma) \otimes_{min} C^*(\Gamma)$.

One can define analog of approximation properties in the setting of K-theory.

- A is K-nuclear if the class of the natural map
$$p_{A,B} : A \otimes_{max} B \to A \otimes_{min} B$$
is invertible as an element of $KK(A \otimes_{max} B, A \otimes_{min} B)$.

- G is K-amenable if the class of the regular representation
$$\lambda_G : C^*_{max}(G) \to C^*_r(G)$$
is invertible as an element of $KK(C^*_{max}(G), C^*_r(G))$.

For instance, Skandalis proves in [**?**] that, if Λ is an infinite hyperbolic property T group, then $C^*_r(\Lambda)$ is not K-nuclear. In particular, it is not KK-equivalent to a nuclear C^*-algebra, and cannot be Bootstrap. This completely renders proving the Baum-Connes conjecture by mean of Dirac-Dual-Dirac method hopeless. An example of such a group is given by any lattice in $Sp(n, 1)$ for instance. (higher rank algebraic semisimple groups?)

After developing a restriction principle for groupoids, a natural question was to find a C^*-algebra coming from a groupoid crossed-product that we were able to prove that it satisfied the Künneth formula, while still not being a consequence of previous results. One could have started with the so called HLS groupoid $G_{\mathcal{N}}(\Gamma)$ associated to a residually finite finitely generated group Γ and a nested sequence of decreasing normal sugroups of finite index $\mathcal{N}$.

One always has the following exact sequence of $*$-algebras

$$0 \to \oplus\mathbb{C}[\Gamma_n] \to C_c(G) \to \mathbb{C}[\Gamma] \to 0$$

which induces the following exact sequence of C^*-algebras

$$0 \to \oplus\mathbb{C}[\Gamma_n] \to C^*_r(G) \to C^*_{\mathcal{N}}(\Gamma) \to 0$$

where $C^*_{\mathcal{N}}(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ w.r.t. to the norm

$$||x||_{\mathcal{N}} = \sup_{N \in \mathcal{N}} ||\lambda_N(x)|| \quad x \in \mathbb{C}[\Gamma]$$

induced by the quasi-regular representations $\lambda_N : C^*_{max}(\Gamma) \to \mathcal{L}(l^2(\Gamma/N))$.

Now this exact sequence intertwines the Baum-Connes assembly maps, and the Baum-Connes conjecture for $G_{\mathcal{N}}(\Gamma)$ is equivalent to $\mu_{\Gamma,\mathcal{N}}$ being an isomorphism.

- If $\Gamma = \mathbb{F}_2$ and
$$N_n = \cap ker\phi$$
for ϕ running accross all group homomorphisms from Γ to a finite group of cardinality less than n, then $C^*_{\mathcal{N}}(\Gamma) \cong C^*_{max}(\Gamma)$ and G satifies the Baum-Connes conjecture, is ample and satisfies the restriction condition. So we get that $C^*_r(G)$ satisfies the Künneth formula. It is still a result that one can get using the fact that Γ being a-T-menable, it is K-amenable. Hence $C^*_{max}(\Gamma)$ and $C^*_r(\Gamma)$ are KK-equivalent and bootstrap, so that $C^*_r(G)$ also is by extension stability of bootstrapness. A remark of R. Willett is worth mentioning: $\mathbb{F}_2$ being the fundamental group of the wedge of two circles, it is KK-equivalent to $C(\mathbb{S}^1 \wedge \mathbb{S}^1)$.

- One can artificially try to get rid of bootstrapiness by spatially tensoring this exact sequence by $C^*_r(\Lambda)$ for a infinite hyperbolic property T group. One then get the extension
$$0 \to \oplus \mathbb{C}[\Gamma_n] \otimes_{min} C^*_r(\Lambda) \to C^*_r(G \times \Lambda) \to C^*_{\mathcal{N}}(\Gamma) \otimes_{min} C^*_r(\Lambda) \to 0.$$
The restriction principle applies for the groupoid $G_{\mathcal{N}}(\Gamma) \times \Lambda$, and induces that its reduced C^*-algebra satisfies the Künneth formula. But then again, one can deduce this from a previous result, namely the restriction principle for groups. Indeed, apply it to Λ with coefficient on the trivial bootstrap Λ-algebra $C^*_r(G)$.

- Bekka shows that ??

2.4 Useful constructions in KK-theory

This section tries to compile interesting constructions in bivariant KK-theory that can be applied for the study of apporximation properties of C^*-algebras.

Extensions, boundaries and KK_1

Kasparov-Stinespring and $E^{(\pi,T)}$.
Toeplitz and suspension..

Mapping cone and double cone constructions

The mapping cone of a $*$-homomorphism $\phi : A \to B$ is the C^*-algebra
$$C_\phi = \{(a, f) \in A \oplus B[0, 1] \mid f(0) = 0 \text{ and } f(1) = \phi(a)\}.$$
The mapping cone naturally fits in the short exact sequence
$$s : \ 0 \longrightarrow SB \longrightarrow C_\phi \longrightarrow A \longrightarrow 0$$
and the boundary map of this sequence coincides with ϕ_* modulo suspension, i.e.
$$\partial_{SB,CB} \otimes \partial_s = \phi_*.$$

(This remains true for any homology or cohomology theory.)

Given a sequence

$$s:\ 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

with zero composition, consider the natural inclusion $\gamma : A \to C_\phi$. We call C_γ the *double cone* of s and denote it by $C(s)$. Notice that $C(A \otimes C(s)) = A \otimes C(s)$.

Proposition 2.4.1 (see [?] rk 4.3 and [?] section 1)**.** The sequence $K(A) \to K(B) \to K(C)$ is exact iff $K(C(s)) = 0$.

For the proof, use that $K(C(s)) = 0$ iff γ_* is an isomorphism. This property is useful to show failure of K-exactness.

Corollary 2.4.2. If A is in $\mathcal{N}$ (satisfies the Künneth formula and separable...), then A is K-exact.

Proof. If A is not K-exact, there exists a short exact sequence s such that $A \otimes s$ is not exact in K-theory. Then $K(A \otimes C(s)) \neq 0$ while $K(C(s)) = 0$, which prevents A from satisfying the Künneth formula. □

Geometric injective and projective resolutions

2.5 Baum-Connes

- Compact groups, or better: proper groupoids: Green-Julg.

- Proof for the integer group $\mathbb{Z}$:

  A model for $\underline{E}\Gamma$ is the space of finitely supported probabilty measures. For $\mathbb{Z}$, the barycenter map

  $$\left\{ \begin{array}{ccc} \underline{E}\mathbb{Z} & \to & \mathbb{R} \\ \sum_n p_n \delta_n & \mapsto & \sum_n p_n n \end{array} \right.$$

  is an equivariant continuous map homotopic to the identity.

  Prove the isomorphism

  $$RK_*^{\mathbb{Z}}(\mathbb{R}, B) \cong RK_*(\mathbb{S}^1, B),$$

  under which the assembly map sends the Toeplitz extension, which is a generator of the right side, to a generator of the K-theory group of $C^*(\mathbb{Z})$.

- For the free group on two elements $\mathbb{F}_2$, take $\mathbb{F}_2$'s Cayley graph T as a model for $\underline{E}\mathbb{F}_2 = E\mathbb{F}_2$, and $B\mathbb{F}_2$ is the wedge of two circles,

  $$RK_*^{\mathbb{F}_2}(T, B) \cong RK_*(\mathbb{S}^1 \wedge \mathbb{S}^1, B),$$

  and then?

- Connes-Kasparov: proof by representation theory (Wasserman, etc)

- Kasparov's Conspectus: towards Higson-Kasparov paper and the proof for Haagerup (J-L. Tu's general version in KK-theory, plus the beautiful result that aTmenability implies bootstrap)

- Ideas from Coarse geometry, and Yu and Roe's work, SkandalisTuYu etc.

- A γ-element is a class $\gamma \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$ such that there exists a $\underline{E}\Gamma \rtimes \Gamma$-algebra A and elements
$$\eta \in KK^{\Gamma}(\mathbb{C}, A) \quad \text{and} \quad D \in KK^{\Gamma}(A, \mathbb{C})$$
such that $\gamma = \eta \otimes_A D$ and $p^*(\gamma) = 1$ in $KK^{\Gamma}(C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma))$. See [?].

  Then γ and $D \otimes \eta$ are projections, and γ is unique.

  Theorem 2.5.1. If Γ has a γ-element, then $K^{top}(\underline{E}\Gamma, B)$ identifies with
$$K((A \otimes B) \rtimes_r \Gamma)p$$
where $p = j_\Gamma(\Sigma_{\underline{E}\Gamma,B}(\gamma))$ and the assembly maps $\mu_{r,\Gamma}$ and $\mu_{max,\Gamma}$ are injective. Moreover if
$$j_\Gamma(\gamma)_* : K(B \rtimes \Gamma) \to K(B \rtimes \Gamma)$$
is the identity, $\mu_{max,\Gamma}$ is an isomorphism. If $\gamma = 1$, then $\lambda_* \in KK(C_{max}(\Gamma), C_r^*(\Gamma))$ is invertible and $\mu_{r,\Gamma}$ and $\mu_{max,\Gamma}$ are isomorphisms.

- Rubén asked:

  Do you know a group satisfying Baum-Connes but which doesn't have a γ-element equal to 1? Do you know a group which is not K-amenable?

  Answer: Any non compact group having property T cannot have $\gamma = 1$, because the class of the Kazhdan projection is not zero in $K_0(C^*_{max}(\Gamma))$ but is in $K_0(C^*_r(\Gamma))$. For any infinite hyperbolic group Γ having T, its reduced C^*-algebra $C^*_r(\Gamma)$ is not K-nuclear ([?]), so any lattice in $Sp(n, 1)$ works out. For instance: $Sp_{n,1}(\mathbb{Z})$.

  Do you know when Γ is amenable is equivalent to its maximal C^*-algebra being exact? When does $C^*_r(\Gamma)$ is exact implies $C^*(Gamma)$ is exact.

  Answer: When Γ has Kirchberg's approximation property, i.e. $\lambda \otimes \rho$ extends to $C^*(\Gamma) \otimes_{min} C^*(\Gamma)$, then Γ is amenable iff $C^*(\Gamma)$ is exact.

- Direct splitting method (Nishikawa 2018 [?]):

Definition 2.5.2. A Kasparov cycle $(H, T) \in E^{\Gamma}(\mathbb{C}, \mathbb{C})$ has property (γ) if there exists a non-degenerate representation of the Γ-algebra $(C_0(\underline{E}\Gamma), \alpha)$,

$$\pi : C_0(\underline{E}\Gamma) \to \mathcal{L}(H),$$

such that

$$\gamma \mapsto [\alpha_\gamma(\phi), T] \in C_0(\Gamma, \mathfrak{K}(H)) \quad \forall \phi \in C_0(\underline{E}\Gamma)$$

and

$$\int_\Gamma \alpha_\gamma(c^{\frac{1}{2}}) T \alpha_\gamma(c^{\frac{1}{2}}) d\mu_\Gamma - T \in \mathfrak{K}(H),$$

for some cutoff function c on Γ and Haar measure μ_Γ. (integral in the strong topology)

If such a pair (H, T) and π is given, define:

- the Γ-equivariant Hilbert A-module $\tilde{H} = H \otimes l^2(\Gamma) \otimes A$,
- the Fredholm operator $(\tilde{T})_{\gamma\gamma} = \gamma T \gamma^*$,
- the representation $\tilde{\pi} = \pi \otimes \rho_{\Gamma,A}$, where $\rho_{\Gamma,A}$ is the right regular representation on $l^2(\Gamma) \otimes A$.

Then $(\tilde{H}, \tilde{\pi}, \tilde{T})$ defines a class in

$$\gamma \in KK_0(C_0(\underline{E}\Gamma) \otimes (A \rtimes_r \Gamma), A)$$

and the splitting map is defined as

$$\nu_{\Gamma,A} : \left\{ \begin{array}{rcl} K_*(A \rtimes_r \Gamma) & \to & KK_*^{\Gamma}(C_0(\underline{E}\Gamma), A) \\ z & \mapsto & \tau_{C_0(\underline{E}\Gamma)}(z) \otimes \gamma \end{array} \right.$$

It is functorial in A w.r.t. Γ-equivariant $*$-homomorphisms. The main result is the following:

Theorem 2.5.3. The composition $\mu_{\Gamma,A} \circ \nu_{\Gamma,A}$ coincides the endomorphism of $K_*(A \rtimes_r \Gamma)$ induced by (H, T).

2.6 GPOTS & NCGOA 2018

2.6.1 Arnaud Brothier: some representations of the Thompson group

2.6.2 Piotr Nowak: Property T for $Out(\mathbb{F}_n)$

2.6.3 Wilhem Winter: Relative nuclear dimension

2.6.4 Rufus Willett: Exactness and exotic crossed-product

2.7 Coarse geometry & dynamics

2.8 Langlands

A modular form of weight k is a section of

$$\Lambda^{k+2}T^*M.$$

The projective space of the $\mathbb{N}$-graded algebra

$$A = \bigoplus \Lambda^{k+2}T^*M$$

is the compactification of the modular curve

$$\mathbb{P}(A) \cong \tilde{\mathcal{C}}.$$

If $F = \mathbb{Q}$ and $G = GL_2$, the finite part of the adele

$$\mathbb{A}_f = \prod_{\text{finite places}} F_\nu = \prod_{p\in\mathcal{P}} \mathbb{Q}_p$$

is ?? and $G(\hat{\mathbb{Z}})$ is the maximal compact of $G(\mathbb{A}_f)$ with $G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$ being two copies of the upper half plane $\mathbb{H}$, and $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ is the modular curve.

Is the right part $G(\mathbb{A}_f)/G(K)$ is isomorphic to the inductive limit $G(\mathbb{Z}/p^k\mathbb{Z})$?

Yes if $G = GL_1$:

$$\varinjlim \mathbb{Z}/p^k\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p.$$

2.9 Haagerup property, cocycles and the mapping class group

If Σ is a closed oriented connected surface (with marked points), we denote by $Mod(\Sigma)$ its so-called mapping class group.

In [?] are used bounded representations of the mapping class group parametrized by a complex number $z \in \mathbb{D}$:

$$\pi_z : \Gamma \to \mathcal{L}(H).$$

Here, H is the Hilbert space obtained as the free Hilbert space generated by multicurves having a finite number of intersections with a fixed triangulation τ of Σ.

2.10 Hawaii

2.10.1 HLS groupoids

Let $(\Gamma, \mathcal{N})$ be an *approximated group* and $G_{\mathcal{N}}$ its associated HLS groupoid. Then:

- $G_{\mathcal{N}}$ is amenable iff Γ is amenable,

- if $G_{\mathcal{N}}$ is a-T-menable, then Γ is a-T-menable. The converse doesn't hold: in [**?**], the authors construct an approximated pair $(\mathbb{F}_2, \mathbb{N})$ such that the assembly map $\mu_{G_{\mathcal{N}},r}$ is not surjective, even if $\mathbb{F}_2$ is a-t-menable.

- $G_{\mathcal{N}}$ has T iff Γ has T,

- the algebraic exact sequence
$$0 \longrightarrow \bigoplus_n \mathbb{C}[\Gamma_n] \longrightarrow C_c(G_{\mathcal{N}}) \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$
extends to
$$0 \longrightarrow \bigoplus_n C_r^*(\Gamma_n) \longrightarrow C_r^*(G_{\mathcal{N}}) \longrightarrow C_{r,\infty}^*(\Gamma) \longrightarrow 0\ ,$$
where the right side algebra is the completion of $\mathbb{C}[\Gamma]$ w.r.t. the norm
$$||x||_{r,\infty} = \sup\{||y||_r : q(y) = x\} \quad \forall x \in \mathbb{C}[\Gamma].$$
This is not an exotic crossed product functor, but one can still define an assembly map $\mu_{\Gamma,r,\infty}$ as the composition of $\mu_{\Gamma,max}$ with the induced at the level of K-theory of the quotient map $C_{max}^*(\Gamma) \to C_{r,\infty}^*(\Gamma)$. This exact sequence and the one induced by the decomposition of $G^0 = \overline{\mathbb{N}}$ is $\mathbb{N}$ and ∞ intertwines the assembly maps so that the next point follows:

- $G_{\mathcal{N}}$ satisfies BC iff Γ satifies BC for $\mu_{\Gamma,r,\infty}$.

- If Γ has T, then if μ_Γ is injective (which is the case for all closed subgroups of connected Lie groups), then $\mu_{\mathcal{G}_{\mathcal{N}}}$ fails to be surjective.

- **Congruence subgroup property.** If Γ has c.s.p. , then the assembly map fails to be surjective for any HLS groupoid $G_{\mathcal{N}}(\Gamma)$. If one can find such a groupoid which is a-T-menable for $SO(n,1)$, then this would imply Serre's c.s.p. conjecture: Any lattice in $SO(n,1)$ does not have c.s.p.

A useful fact from [**?**]:
$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$
is exact implies that the cone C_γ of the natural inclusion $\gamma : J \to C_\beta$ has vanishin K-groups:
$$K_*(C_\gamma) = 0.$$

2.10.2 Visit to PennState, September 18th to 21st 2018

Michael Francis:

- Dixmier-Malliavin theorem: for every Lie group G,

$$C_c^\infty(G) * C_c^\infty(G) = C_c^\infty(G).$$

  The idea is to decompose, in the real case, the Dirac mass at 0 as a derivative of $\delta_0 = g^{(n)}$ for some $g \in C^{n-2}(\mathbb{R})$, but this doesn't quite do the job, so that they show that there exists $g \in C_c^\infty(G)$ and a_n going to 0 as fast as needed so that

$$\delta_0 = \sum a_k g^{(k)}.$$

  The result follows from $f = f * \delta = (\sum(-1)^k a_k f^{(k)}) * g$. Michael extended this result to Lie groupoids.

- a remak of John Roe in his lectures on Coarse Geometry, that there exists a Svarc-Milnor theorem for foliations.

Sarah Browne:

- Advice to read a book Nate Brown gave her, *Lifting solutions to perturbing problems in C^*-algebras*by Terry A. Loring (Fields Institute Monographs). In here canbefind te definition o semi projective C^*-algebras.

2.10.3 Visit to Texas A&M, February 12th to 14th 2019, and University of Houston February 15th

• If for every r, X is 2-decomposable w.r.t. a family of uniformly embeddable spaces (into Hilbert space), then X is itself CEH. This contradicts the (false) example I gave in the preprint with Christian of a group whose uniform Roe algebra satisfies the Künneth formula. For this I used a split extension of two CEH groups, which is not itself CEH built by Arzhantseva and Tessera ([**?**]). A misuse of the fibering theorem made me believe that this group was 2-decomposable w.r.t. a uniformly CEH family.

The main ingredients for this result are the following, and are all contained in a paper of Dadarlat and Guentner (see [**?**]). Let us fix some notation: X is a discrete metric space, and $\mathcal{U}$ is a cover, by which we mean a collection of subset of X whose union form X. We say that $\mathcal{U}$

- has Lebesgue number at least L ($Leb(\mathcal{U}) \geq L$) if any ball of radius L is completely contained in some $U \in \mathcal{U}$,
- has R-multiplicity less than k ($R - mult(\mathcal{U}) \geq R$) if any ball of radius R interesect at most k elements of $\mathcal{U}$. If ignored, R is zero.
- is R-separated if any two elements of $\mathcal{U}$ a at least R-apart,
- is (k, R)-separated if $\mathcal{U}$ admits a partition into $k+1$ families which are R-separated.

By a partition of unity ϕ subordinated to $\mathcal{U}$, we mean a collection of function $\{\phi_U\}_{U\in\mathcal{U}}$, each $\phi_U : X \to [0,1]$ being zero outside of U, and such that $\sum_U \phi_U(x) = 1$ for every $x \in X$. We say $Lip(\phi_U) \leq C$ is there exist $\delta > 0$ such that if $d(x,y) < \delta$, $|f(x) - f(y)| < C$. And $Lip_{l^1(\mathcal{U})}(\phi) < C$ if there exists δ such that if $d(x,y) < \delta$, $\|\phi(x) - \phi(y)\|_{l^1(\mathcal{U})} = \sum_U |\phi_U(x) - \phi_U(y)| < C$.

1. if $\mathcal{U}$ is a cover of X with $Leb(\mathcal{U}) \geq L$ and $mult(\mathcal{U}) \leq k$ then

$$\phi_U(x) = \frac{d(x, X-U)}{\sum_{V\in\mathcal{U}} d(x, X-V)}$$

   defines a PDU such that

$$Lip(\phi_U) \leq \frac{2k+3}{L} \text{ and } Lip_{l^1(\mathcal{U})} \leq \frac{(2k+2)(2k+3)}{L}.$$

2. if $\mathcal{U}$ is (k,L)-separated, then $mult(\mathcal{U}) \leq k+1$;

3. if $\mathcal{U}$ is $(k,2R)$-separated, then $R - mult(\mathcal{U}) \leq k+1$;

4. if $L - mult(\mathcal{U}) \leq k+1$, $Leb(\mathcal{U}_L) \geq L$;

5. *Summary:* If $\mathcal{U}$ is $(k,2L)$-separated, then $mult(\mathcal{U}_L) \leq k+1$ and $Leb(\mathcal{U}_L) \geq L$. Also $L - mult(\mathcal{U}) \leq k+1$.

Using [**?**], thm 3.2, the result follows.

• In [**?**], thm 8.3, Willett and Yu constructed a counterexample for the Baum-Connes conjecture *with coefficients*. Let G be a group whose Cayley graph contains an expander X. G does not act on X, but we can enlarge X and set

$$N_\infty(X) = \cup_{R>0} N_R(X),$$

so that G acts on $N_\infty(X)$ and on

$$A = l^\infty(N_\infty(X)) = \overline{\cup_{R>0} l^\infty(N_R(X))}.$$

Then the Baum-Connes assembly map for G with coefficients in A fails to be surjective.

Now can we force this C^*-algebraic coefficients into a group so as to build a counterexample for Baum-Connes without coefficients? For instance, take the unrestricted wreath product

$$\Gamma = \mathbb{Z}_2 \wr_X G = \prod_X \mathbb{Z}_2 \rtimes G.$$

One sees that the reduced C^*-algebra $C^*_r(\Gamma)$ should really look like $A \rtimes_r G$. Indeed, $N = \prod_X \mathbb{Z}_2$ is normal abelian in Γ, so that $C^*_r(\Gamma) \cong C^*_r(N) \rtimes_r G \cong C(\hat{N}) \rtimes_r G$.

2.11 Mayer-Vietoris

2.12 Quantum groups

2.13 Property T

2.14 Number theory

2.15 Fock spaces, CuntzKrieger algebras, and second quantization

2.16 Representations of groupoids

Mettre les references et des rappels sur les champs continus et mesurables d'espaces de Hilbert.

This section is a reminder on the different notions of representations for groupoids that exists. Let us first begin by a reminder on continuous fields of C^*-algebras and Hilbert spaces. All this material can be found in Diximier's book[**?**].

A continuous field of Banach spaces over a topological space X is a pair

$$E = (\{E_x\}_{x\in X}, \Gamma_E)$$

where:

- E_x is a Banach space, with norm denoted $||\ .\ ||_x$,
- Γ_E is a linear subspace of $\prod_{x\in X} E_x$ such that $x \mapsto ||\gamma(x)||_x$ is continuous (or upper semi-continuous according to Lafforgue) for every $\gamma \in \Gamma_E$, and if every $\sigma \in \prod_{x\in X} A_x$ is locally uniformly approximable by sections, i.e. if for every $x \in X$ and $\varepsilon > 0$, there exists $\gamma \in \Gamma_A$ and a neighborhood U of x such that $\sup_{y\in U} ||\sigma - \gamma||_y < \varepsilon$, then $\sigma \in \Gamma_A$,
- $\{\gamma(x)\}_{x\in X}$ is dense in E_x.

The elements of Γ_E are called the continuous sections of E. A continuous section is said to be bounded if

$$\sup_x ||\gamma(x)||_x < \infty.$$

The space of contiuous bounded sections with the sup-norm is a Banach space.

A continuous field of C^*-algebras over a locally compact space X is a pair

$$(\{A_x\}_{x\in X}, \Gamma_A)$$

where:

- A_x is a C^*-algebra,
- $\Gamma_A \subset \prod_{x\in X} A_x$ is a $*$-algebra such that $x \mapsto ||\gamma(x)||$ is continuous for every $\gamma \in \Gamma_A$, and if every $\sigma \in \prod_{x\in X} A_x$ is locally uniformly approximable by sections, i.e. if for every $x \in X$ and $\varepsilon > 0$, there exists $\gamma \in \Gamma_A$ such that $||\sigma - \gamma|| < \varepsilon$, then $\sigma \in \Gamma_A$,
- $\{\gamma(x)\}_{x\in X}$ is dense in A_x.

On the other hand a $C_0(X)$-algebra is a C^*-algebra A endowed with a nondegenerate $*$-homomorphism

$$\phi : C_0(X) \to M(Z(A)).$$

Any field $(\{A_x\}_{x\in X}, \Gamma_A)$ over X defines a $C_0(X)$-algebra

$$C^*(\Gamma_A) := \{\gamma \in \Gamma_A \text{ s.t. } x \mapsto ||\gamma(x)|| \in C_0(X)\}.$$

A $C_0(X)$-algebra A is continuous if $x \mapsto ||a_x||$ is continuous for each $a \in A$. Here, a_x denotes the image of a under the map $A \to A/\phi(I_x)A$,with I_x the ideal of functions vanishing at x.

There is a correspondence between these two notions.

Let G be a locally compact groupoid. Renault defines a representation of G as the following data:

- a measure μ on G^0,
- a measurable field of Hilbert spaces $(\mathcal{H}, \mu)$ over G^0,
- a family of bounded operators $L_g : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ for each $g \in G$ satisfying $L_{g_1}L_{g_2} = L_{g_1g_2}$ for every $(g_1, g_2) \in G^2$, $L_{e_x} = id_{\mathcal{H}_x}$, and

  $$g \mapsto \langle L_g(\xi_{s(g)}), \eta_{r(g)}\rangle$$

  is measurable for every pair of measurable sections.

The main examples are the left and right regular representations, and the trivial one. The left regular representation λ is defined as the family of operators

$$\lambda_g : \left\{ \begin{array}{rcl} L^2(G^x, \lambda^x) & \to & L^2(G^x, \lambda^x) \\ \xi & \mapsto & [\gamma \mapsto \xi_{g^{-1}\gamma}] \end{array} \right.$$

whereas the trivial representation is defined as

$$\tau_g : \left\{ \begin{array}{rcl} H_{s(g)} = \mathbb{C} & \to & H_{r(g)} = \mathbb{C} \\ \xi & \mapsto & \xi \end{array} \right.$$

Recall the following: given a $C_0(G^0)$-Hilbert module E, a unitary representation G is a unitary

$$V \in \mathcal{L}_{s^*C_0(G^0)}(s^*E, r^*E)$$

which satisfies $V_1V_2 = \Delta V$, where:

- V_i is the $C_0(G^2)$-operator induced from V by the projection $p_i : G^2 \to G$,

- Δ is the (comultiplication) map $C_0(G) \to M(C_0(G^2))$-operator induced by the multiplication $\Delta : G^2 \to G$.

Fiberwise this gives you a more restrictive class than the representations in the sense of Renault. Indeed, in the case of a trivial groupoid over a locally compact space X, the spectral theorem ensures that any $*$-representation on a Hilbert space H

$$\pi : C_0(X) \to \mathcal{L}(H)$$

disintegrates into a representation in the sense of Renault on a field of Hilbert space. However, our $\{V_g\}$ gives a continuous field of representation over a continuous field of Hilbert space, which is a priori stronger.

As in the case of groups, one can try to define a integrated representation, by

$$(V(f)\xi)_x = \int_{g\in G^x} f(g)V_g(\xi_{s(g)})d\lambda^x(g) \quad f \in C_c(G), \xi \in E.$$

This defines a map $C_c(G) \to \mathcal{B}(E)$, where $\mathcal{B}(E)$ denotes the bounded operator of E seen as a Banach space. But this map is not even multiplicative!

Instead, consider G^0 to be discrete, and

$$V(f)_{xy} = \sum_{g\in G^x_y} f(g)V_g,$$

so that

$$(V(f)\xi)_x = \sum_{y\in X} V(f)_{xy}\xi_y = \sum_{g\in G^x} f(g)V_g(\xi_{s(g)}),$$

but this time

$$V(f*g)_{xz} = \sum_{y\in X} V(f)_{xy}V(g)_{yz}.$$

If G^0 is not discrete, suppose there is a measure μ on G^0. Then

$$V(f*g)_{xz} = \int_X V(f)_{xy}V(g)_{yz}d\mu(y).$$

Some facts.

Let G be proper and $V \in \mathcal{L}_{s^*C_0(G^0)}(s^*E, r^*E)$ be a unitary represntations. If $c : G^0 \to [0,1]$ is a cutoff function, any vector $\xi \in E$ can be averaged as to get an invariant one:

$$(\overline{\xi})_x = \int_{G^x} c(x\gamma)V_\gamma(\xi_{s(\gamma)})d\lambda^x(\gamma).$$

A simple computation and $(L_g)^*\lambda^{s(g)} = \lambda^{r(g)}$ gives indeed the $\overline{\xi}$ is G-invariant.
This ensures that any proper groupoid has property T.

2.17 Grothendieck and tensor products, the origin of nuclearity

This section is based on a talk given by Gilles Pisier, and his (exceptionally good) survey article.

Grothendieck started his work in functional analysis. While this is well known, I wanted to write a little post about how his work is important in my field.

Grothendieck did his Licence (his "undergrad") in the south of France, in the city of Montpellier.
If $x = \sum_j \alpha_j \otimes \beta_j$,

$$||x||_\wedge = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_\wedge = \sup\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

and

$$||x||_H = \inf\{||\alpha_j||\ ||\beta_j|| : x = \sum_j \alpha_j \otimes \beta_j\}$$

14 fundamental norms.

Chapter 3

Old notes

3.1 Simple examples for Baum-Connes for groupoids

This is a question asked by Sayan Chakraborty : find a simple example of the Baum-Connes conjecture for groupoids.

We found that one should be able to do actual computations in K-theory, like determining generators of K-group of some known C^*-algebras, and to prove Baum-Connes by hand in some simple examples. The only one we managed to actually do by hand was Baum-Connes for $\mathbb{R}^n$. (Do it !)

The simplest example would be to take the groupoid associated to an action of a group on a topological space $\mathcal{G} = X \rtimes G$. The first thing we want to do is to describe the classifying space for proper actions.

Suppose the groupoid étale equipped with a proper length. A simple model, from J-L. Tu [?], is given by the inductive limite of the spaces

$$Z_d = \{\nu \in \mathcal{M}(\mathcal{G}), s.t. \exists x, \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d, g \in \mathcal{G}^x\}.$$

Indeed, suppose Y is a $\mathcal{G}$-proper cocompact space, then $Y \rtimes \mathcal{G}$ is a proper groupoid, so there exists a cutt-off function $c : Y \to [0,1]$ such that :

$$\sum_{g \in \mathcal{G}^{p(y)}} c(yg) = 1, \forall y \in Y.$$

Now define

$$y \mapsto \sum_{g \in \mathcal{G}^{p(y)}} c(yg)\delta_g$$

which is a $\mathcal{G}$-equivariant continuous map. Moreover Z_d is proper and cocompact, and there exists a d s.t. the map takes its values in it.

Now if $\mathcal{G} = X \rtimes G$, $Z_d \simeq X \times Z'_d$ where $Z_d = \{\nu \in \mathcal{M}(G), s.t. \text{if } g \in \text{supp } \nu \text{ then } l(g) \leq d\}$, so that $KK^{\mathcal{G}}(\Delta, A) \simeq KK^G(\Delta', A)$, where Δ and Δ' are respectively the 0-dimensional part of the equivariant complexes Z_d and Z'_d. This is true because the action of G on Z'_d is proper and cocompact, see lemma 3.6 of [?]. Now a standard Mayer-Vietoris argument (theorem 3.8 [?]) concludes to show that $K^{top}(\mathcal{G}, A) \simeq K^{top}(G, A)$.

As $C^*_r\mathcal{G} = C_0(X) \rtimes_r G$, we see that the Baum-Connes assembly map for $\mathcal{G}$ with coefficients in A is equivalent to

$$K^{top}_*(G, A) \to K_*((A \otimes C_0(X)) \rtimes G).$$

Now we can look for concrete examples.

3.1.1 Non commutative tori

Question : Compute the generators of non-commutative tori. (Sayan did it)

3.1.2 Principal bundle over $U(2)$

This is an example from Olivier Gabriel's talk in Montpellier.

Take the principal bundle $U(2) \to U(2)/\mathbb{T}^2 \simeq \mathbb{S}^2$. You can foliate the fibers by an irrational rotation θ, so that you have an action of $\mathbb{R}$ on $C(U(2))$. Reducing to a complete transversal (take $SU(2)$), the algebra $C(U(2)) \rtimes \mathbb{R}$ turns out to be Morita equivalent to $\underline{A} = C(SU(2)) \rtimes \mathbb{Z}$ (a general result of foliation groupoids I think). $\underline{A}$ can be reduced to $C(\overline{D}) \otimes A_\theta$ and to $Ind_{\mathbb{T}^2}^{U(2)} A_\theta$.

Question : Compute the generators of the K-theory of $\underline{A}$.

3.1.3 Foliations

3.1.4 An example from physics

In Alain Connes' book, we can read the following example.

Take the 2-torus $M = \mathbb{T}^2$. Its fundamental group $\Gamma = \mathbb{Z}^2$ acts on its universal cover $\tilde{M} = \mathbb{R}^2$ by isometries, and the electromagnetic field A gives a two-form w (its curvature) on $\tilde{M}$, so a 2-cocycle on the fundamental groupoid of $\tilde{M}$:

$$w(\tilde{x}, \tilde{y}, \tilde{z}) = e^{2i\pi \int_\Delta \tilde{w}}$$

where Δ a geodesic triangle between the 3 points. It turns out that $H^2(\mathbb{Z}^2, \mathbb{T}^2) = \mathbb{S}^1$, so that $\tilde{w}$ determines a number $\theta \in [0, 1)$, and the twisted reduced algebra of the fundamental groupoid w.r.t. $\tilde{w}$ is equal to $A_\theta = C(\mathbb{T}^2) \rtimes_{r,\theta} \mathbb{Z}^2$. This situation generalizes to general manifold whose fundamental cover are equiped with a line bundle and a conection. We can then associate a 2-cocycle on the fundamental groupoid of $\tilde{M}$ to the curvature of the line bundle.

A question : Does the twisted crossed-product has applications to Yang-Mills theories ?

3.2 Parabolic induction and Hilbert modules

Here is a question formulated by Pierre Julg.

Let G be a real reductive group. For all parabolic subgroup P, there is only one nilpotent normal subgroup N, and the Levi is defined as $P = LN$. The idea of Pierre Julg is to fix first a Levi susgroup L of G. Now there is only a finite numbers of choices for N, so that

$$P(L) = \{N : P = LN \text{ is parabolic}\}$$

is a finite set. The Weyl group $W_L = N_G(L)/L$ acts on it by $w.N = wNw^{-1}$.
Pierre Clare defined a C_r^*L-module $C_r^*(G/N)$, equipped with and action of C_r^*G by compacts operators. He was able to give a nice interpretation of parabolic induction in terms of functors on these modules. Let $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$, where $L = MA$, $\hat{M}_d$ is the discrete dual of M, and $\hat{A} = \mathfrak{a}^*$. Then $\sigma \otimes \tau$ is a représentation of $MA = L$, which we can trivially extend to N to induce it on G. Pierre Clare showed the following fact :

$$Ind_P^G\ H_{\sigma\otimes\tau\otimes 1_N} = C^*(G/N) \otimes_{C_r^*L} H_{\sigma\otimes\tau}.$$

For every $\tilde{w} \in N_G(L)$, the operator $\rho(\tilde{w}) : C_r^*(G/N) \to C_r^*(G/w.N)$ is well defined and gives a morphism

$$Ad\ \rho(\tilde{w}) : \mathfrak{K}_{C_r^*L}(C_r^*(G/N) \to \mathfrak{K}_{C_r^*L}(C_r^*(G/w.N))$$

because C_r^*G is acting on $C^*(G/N)$ by compact operators. This gives a morphism

$$C_r^*G \to \bigoplus_{[L]} \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N)) \right)^{W_L}$$

which Pierre Julg conjectures to be an isomorphism. (This is true but due to very hard work in Harish-Chandra's theory, the aim is to find a relatively easy proof using standard C^*-algebraic tools).

The first step would be to prove that

$$\begin{array}{rcl} C_r^*G & \to & \left(\bigoplus_{N\in P(L)} \mathfrak{K}(C_r^*(G/N))\right)^{W_L} \\ f & \mapsto & (\pi_N(f)) \end{array}$$

is surjective, using Fourier transform and a conjectural formula,

$$\pi_N(F_N^{-1}(T)) = \frac{1}{\#W_L} \sum w.T,$$

for $F_N^{-1}(g) = \mathrm{Tr}_{C_d^*L}\ (T\pi_N(g^{-1}))$.

3.2.1 In $SL(2, \mathbb{R})$

In this case, G acts on the Poincaré disc by homographies, and P can be taken as the stabilizer of a point at infinity, and L stabilizes a geodesic, that is to say two points at infinity, so that

$$P_{1,1} \simeq \{\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}\}, \quad L \simeq \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\}, \quad N \simeq \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}, \quad W_L \simeq \mathbb{Z}_2.$$

Here Julg's point of view applies directly : fixing P amounts to fix a point at infinity, which gives infinite choices for the second point giving the geodesic and L. Now fix two points at infinity, which gives you L. You now only have two choices for P, and the two are exchanges under the action of W_L on the nilpotent groups.

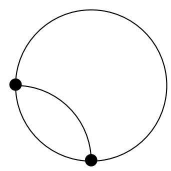


Figure 3.1: Choices for the Levi subgroup

3.3 Universal Coefficient Theorem

Here is a question from Guoliang Yu.

Question : Does a finite nuclear dimensionality condition implies a universal coefficient theorem ?

Let $\mathcal{N}$ be the smallest class of C^*-algebras containing $\mathbb{C}$, closed under countable inductive limits, stable by KK-equivalence and by "2 out of 3" (meaning that in a short exact sequence, whenever 2 of the terms are in $\mathcal{N}$, so is the third). Here is the classical theorem :

Theorem 3.3.1 (Universal Coefficient Theorem)**.** Let A and B be two separable C^*-algebras, where A is in $\mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow KK_*(A,B) \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow 0$$

which is natural in each variable and splits unnaturally.

- The first map ... ??

- The second map is given by the boundary element associated to any impair K-cycle. Namely, if $z \in KK^1(A,B)$, let (H_B, π, T) be a K-cycle representing z, and P the associated projector $P = \frac{1+T}{2}$. Define the pull-back

  $$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K}_B\}$$

  Then the boundary of the folowing extension

  $$0 \longrightarrow \mathfrak{K}_B \longrightarrow E^{(\pi,T)} \longrightarrow A \longrightarrow 0$$

  is given by $\partial = - \otimes z : K_*(A) \longrightarrow K_*(B)$ which depends only on z. The map is just $z \mapsto \partial$

- If $\partial = 0$, then the sequence associated to z splits and we have exact sequences

  $$0 \longrightarrow K_*(B) \longrightarrow K_*\left(E^{(\pi,T)}\right) \longrightarrow K_*(A) \longrightarrow 0$$

  which gives an element of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

3.3.1 Other questions

Now here are some problems that were not resolved during the lectures given by G. Yu during the week.

The first is the classical lemma from Miscenko and Kasparov.

Proposition 3.3.2. Let G be a locally compact group that acts properly and isometrically on a simply connected non positively curved manifold M. Then

$$K^{top}(G) \xrightarrow{\mu} K(C_r^*G) \xrightarrow{\beta} K(C_0(M) \rtimes_r G)$$

is an isomorphism. In particular, the Strong Novikov Conjecture holds for G.

The original point being that G. Yu can prove this (how ?) without usig the heavy machinery of the Dirac Dual-Dirac method, nor anything related to KK^G-theory. The proof is just using cutting and pasting (according to Yu).

The second is of the same type.

Proposition 3.3.3. Let G be a discrete group coarsely embeddable into a Hilbert space, then the Strong Novikov conjecture hold for G.

The usual proof was given by G. Yu himself, relying here again on a Dirac Dual-Dirac method, and a kind of controlled cutting and pasting. Here he presented the idea of the proof, the point not being clear for me was the path to show that

$$K(P_d(G_0)) \sim \prod K(P_d(X_{2k})) \xrightarrow{\mu} \prod K(C^* P_d(X_{2k})) \xrightarrow{\beta} K\left(C^*(P_d(X_{2k}), C(\mathbb{R}^{m_k}))\right)$$

is an isomorphism.

Here are some details : first decompose $G = G_0 \cup G_1$ into two subspaces, which are not necesseraly subgroups, such that each is a R-disjoint union of bounded subsets (in fact finite since G is of bounded geometry) :

$$G_0 = \cup X_{2k}, \quad \text{and} \quad G_1 = \cup X_{2k+1}.$$

Now define $\prod^R C^*(P_d(X_{2k}) = \{(T_{2k})_k : T_{2k} \in C^*(P_d(X_{2k}), prop(T_{2k}) \leq R\}$, so that $C^*(P_d(X_{2k})) \simeq F_{2k} \otimes \mathfrak{K}$, and each X_{2k} corasely embedds into some $\mathbb{R}^{m_k}$. The isomorphism of $\beta \circ \mu$ implies the injectivity of μ, and by cutting and pasting, μ can be shown to be injective for G so that Novikov is satisfied.

3.4 Funky questions, ideas of talks

3.4.1 Expanders

Here are some interesting questions I had after a talk on expanders.

Plan of the talk

I first gave a motivation for considering expanders. Namely, we are interested in the following network theory problem : can we construct a network as big as we want, such that the cost is controlled and which is not subject to easy failure ?

Building a network as big as we want means we want to consider a family of graphs $X_j = (V_j, E_j)$ such that $|V_j| \to +\infty$, and controlling the cost means that $deg(X_j) < k$ for all j. But what does "not easily subject to failure" means ? For this, I want to explain why we should ask our family to stay well connected and why the second value of the discrete Laplacian is a good way to measure that.

The idea is to relate the Laplacian to the uniform random walk on the graph, and to show that $\lambda_1(X)$ controlls the speed of convergence of the uniform random walk to the stationary measure which is the uniform probability on the graph, given by $\nu(x) = C.deg(x)$.

A family of graphs satifying the previous conditions and such that $\lambda_1(X_j) > c > 0$ is called an expander. If time allows, one can then elaborate on metric properties of this type of graphs. The impossibility to embed them coarsely into any separable Hilbert space, and the relations to the Baum-Connes conjecture are close to my work.

Questions

- Paolo Pigato : What is the dynamic at the limit ?
- Anne Briquet : Is $\lambda_1(X)$ such a good way to measure the connectedness of a graph, if you consider the phenomenon of cuttoff for finite Markov Chains.

3.4.2 Ideas of funky talks

- What is the relation between the Fourier transform and quantum groups ?
- What is the relation between the Runge Kutta methode and renormalization in QFT ?
- What is the relation between Brownian motion and second quantization ?