Notes

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1 Application d'assemblage quantitative

1.1 Longueur, propagation et suite exacte à six termes.

Dans la suite, $\mathcal{G}$ dénote un groupoïde localement compact de base $\mathcal{G}^{(0)} = X$ et muni d'un système de Haar $(\lambda^x)_{x\in X}$. On rappelle qu'une suite exacte courte est dite semi-scindée si elle admet une section complètement positive. Par équivalence de Morita, on entend les isomorphismes de groupes

$$\mathcal{M}_A^{\epsilon,r} : K_*^{\epsilon,r}(A) \to K_*^{\epsilon,r}(A\otimes\mathbb{K}) \quad ,\forall r\geq 0, \forall\epsilon\in(0,\frac{1}{4})$$

induit par $A\to A\otimes\mathbb{K}; x\mapsto x\otimes e$, où e est n'importe quel projecteur de rang 1, A une C^*-algèbre, et $\mathbb{K}$ l'idéal des opérateurs compacts sur l'espace de Hilbert séparable.

On suppose que le groupoïde est muni d'un système de longueur l, i.e. d'une famille d'applications $(l^x)_{x\in X}$ définies sur les fibres $\mathcal{G}^x$ à valeurs dans $\mathbb{R}_+$, telles que

$$\begin{array}{l} l^x(e_x) = 0 \\ l^{r(\gamma)}(\gamma) = l^{s(\gamma)}(\gamma^{-1}) \\ l^x(\gamma_1^{-1}\gamma_2) \leq l^x(\gamma_1) + l^x(\gamma_2) \end{array}$$

ce qui permet de définir une filtration sur les algèbres produits croisés associées au groupoïde par :

$$(A\rtimes\mathcal{G})_r = \{f\in C_c(\mathcal{G},A) : \text{ supp } f\subset \cup_{x\in X} B_x(e_x,r)\}$$

où $B_x(e_x,r)$ dénote la boule $\{\gamma\in\mathcal{G} : l^{r(\gamma)}\leq r\}$. Ici, $\rtimes$ peut être à loisir $\rtimes_{red}$ ou $\rtimes_{max}$. On rappelle que $A\rtimes\mathcal{G}$ est fonctoriel en A, de la catégorie des $\mathcal{G}$-C^*-algèbres avec $*$-morphismes équivariants, dans celle des C^*-algèbres avec $*$-moprhismes. Pour $\phi : A\to B$ un $*$-morphisme, on note $\phi_\mathcal{G} : A\rtimes\mathcal{G}\to B\rtimes\mathcal{G}$ le morphisme induit.

Lemme 1. On suppose le groupoïde $\mathcal{G}$ localement compact, muni d'un système de Haar et d'une longueur l.

Soit $\quad 0\longrightarrow J\xrightarrow{\phi} A\xrightarrow{\psi} A/J\longrightarrow 0\quad$ une suite exacte semi-scindée de $\mathcal{G}$-C^*-algèbres.

Alors la suite $\quad 0\longrightarrow J\rtimes\mathcal{G}\xrightarrow{\phi_\mathcal{G}} A\rtimes\mathcal{G}\xrightarrow{\psi_\mathcal{G}} A/J\rtimes\mathcal{G}\longrightarrow 0\quad$ est une suite exacte semi-scindée et filtrée.

Preuve 1. Par fonctorialité, $\psi_\mathcal{G}\circ\phi_\mathcal{G} = 0$. Soit $f\in C_c(G,A)$ telle que $\phi_\mathcal{G}(f) = 0$, i.e. $\phi(f(\gamma)) = 0, \forall\gamma\in\mathcal{G}$. Alors, pour chaque γ, il existe un unique $g(\gamma)\in A'$ telle que $\Psi(g(\gamma)) = f(\gamma)$. De plus, $\gamma\mapsto g(\gamma)$ est continue, de support contenu dans celui de f, donc $g\in C_c(\mathcal{G},A')$ et $\phi_\mathcal{G}(g) = f$.

Si maintenant $a\in A\rtimes\mathcal{G}$, il existe une suite de fonctions $f_n\in C_c(\mathcal{G},A)$ telle que $||a-f_n||\to 0$. Mais $f_n - s_\mathcal{G}\circ\psi_\mathcal{G}(f_n)$ est annulée par $\psi_\mathcal{G}$, donc il existe $g_n\in C_c(\mathcal{G},J)$ telle que $\phi_\mathcal{G}(g_n) = f_n - s_\mathcal{G}\circ\psi_\mathcal{G}(f_n)$. Alors

$$\lim_n ||a-\phi_\mathcal{G}(g_n)|| = 0$$

et comme Im $\phi_{\mathcal{G}}$ est fermé, Ker $\psi_G =$ Im $\phi_{\mathcal{G}}$.
La suite est bien scindée par $\sigma_{\mathcal{G}}$, et les morphismes respecte la propagation, elle est donc filtrée.

□

La proposition suivante découle du lemme.

Proposition 1. Il existe une paire de contrôle (λ, h) telle que pour toute extension semi-scindée de $\mathcal{G}$-C^*-algèbres

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\psi} A/J \longrightarrow 0 \quad ,$$

les diagrammes suivant soient commutatifs et exacts :

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_r \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_r \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_r \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_r \mathcal{G}) \end{array} \quad ,$$

$$\begin{array}{ccccc} \hat{K}_0(J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_0(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_0(A/J \rtimes_{max} \mathcal{G}) \\ \uparrow & & & & \downarrow \\ \hat{K}_1(A/J \rtimes_{max} \mathcal{G}) & \xrightarrow{\phi_{\mathcal{G},*}} & \hat{K}_1(A \rtimes_{max} \mathcal{G}) & \xrightarrow{\psi_{\mathcal{G},*}} & \hat{K}_1(J \rtimes_{max} \mathcal{G}) \end{array} \quad .$$

1.2 Transformée de Kasparov

Soient A et B deux $\mathcal{G}$-C^*-algèbres. On notera H un espace de Hilbert séparable, $l^2(\mathbb{Z})$ par exemple, et $H_{\mathcal{G}} = H \otimes L^2(\mathcal{G}, \lambda)$. Le module hilbertien standard pour B est noté $H_B = H_G \otimes B$, et la C^*-algèbre des opérateurs compacts pour H_B est noté $K_B = K \otimes L^2(\mathcal{G}, \lambda) \otimes B$.

Tout K-cycle $z \in KK^{\mathcal{G}}(A, B)$ peut être représenté comme la classe d'un triplet (H_B, π, T) où :

- $\pi : A \to \mathcal{L}_B(H_B)$ est une $*$-représentation de A sur H_B.
- $T \in \mathcal{L}_B(H_B)$ est un opérateur autoadjoint
- π et T vérifient les conditions de K-cycles, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - id_{H_B})$ et $\pi(a)(\gamma.T - T)$ sont des opérateurs compacts, pour tout $a \in A$, $\gamma \in \mathcal{G}$.

On peut définir $T_{\mathcal{G}} = T \otimes id_{B \rtimes_r \mathcal{G}} \in \mathcal{L}_B(H_B \otimes (B \rtimes_r \mathcal{G})) \simeq \mathcal{L}_B(H_{B \rtimes_r \mathcal{G}})$, et $\pi_G : A \rtimes_r \mathcal{G} \to \text{Ł}_B(H_{B \rtimes_r \mathcal{G}})$. Le K-cycle $(H_{B \rtimes_r \mathcal{G}}, \pi_{\mathcal{G}}, T_{\mathcal{G}})$ représente l'élément $j_{red,\mathcal{G}}(z) \in KK(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$. Nous allons construire un morphisme contrôlé associé à z,

$$J_{red,\mathcal{G}}(z) : \hat{K}_(A \rtimes_r \mathcal{G}) \to \hat{K}(B \rtimes \mathcal{G}),$$

qui induit la multiplication à droite par $j_{red,\mathcal{G}}(z)$ en K-théorie.

1.2.1 Cas impair

Considérons tout d'abord le cas où $z \in KK_1(A,B)$. Soit (H_B, π, T) un K-cycle représentant z. On pose $P = \frac{1+T}{2}$ et $P_{\mathcal{G}} = P \otimes id_{B \rtimes_r \mathcal{G}}$. Définissons

$$E_r^{(\pi,T)} := \{(x, P_{\mathcal{G}}\pi_{\mathcal{G}}(x)P_{\mathcal{G}} + y) : x \in (A \rtimes_r \mathcal{G})_r, y \in K \otimes (B \rtimes_r \mathcal{G})_r\}$$

ce qui nous donne une extension de C^*-algèbres filtrées

$$0 \longrightarrow K_{B \rtimes_r \mathcal{G}} \longrightarrow E^{(\pi,T)} \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

et semi-scindée par $s : \left\{ \begin{array}{lll} A \rtimes_r \mathcal{G} & \rightarrow & E^{(\pi,T)} \\ x & \mapsto & (x, P_{\mathcal{G}}\pi_{\mathcal{G}}(x)P_{\mathcal{G}}) \end{array} \right.$.

Montrons que le bord de cette extension ne dépend que de la classe z du K-cycle. Soit donc $(H_B, \pi_j, T_j), j = 0, 1$ deux K-cycles homotopes via $(H_{B[0,1]}, \pi, T)$. On note e_t l'évaluation en $t \in [0,1]$ pour un élément de $B[0,1]$. L'application

$$\phi : \left\{ \begin{array}{lll} E^{(\pi,T)} & \rightarrow & E^{(\pi_t,T_t)} \\ (x,y) & \mapsto & (x, y_t) \end{array} \right.$$

vérifie $\phi(K_{B[0,1] \rtimes_r \mathcal{G}}) \subset K_{B \rtimes_r \mathcal{G}})$ et fait commuter le diagramme

$$\begin{tikzcd}
0 \arrow[r] & K_{B[0,1] \rtimes_r \mathcal{G}} \arrow[r] \arrow[d, "\phi_{|K_{B[0,1] \rtimes_r \mathcal{G}}}"] & E^{(\pi,T)} \arrow[r] \arrow[d, "\phi"] & A \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & K_{B \rtimes_r \mathcal{G}} \arrow[r] & E^{(\pi_t,T_t)} \arrow[r] & A \rtimes_r \mathcal{G} \arrow[r] & 0
\end{tikzcd}.$$

D'après la remarque 3.7 de [6], on a donc la relation suivante sur les bords contrôlés

$$D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_t,T_t)}} = \phi_* \circ D_{K_{B[0,1] \rtimes_r \mathcal{G}}, E^{(\pi,T)}}.$$

Comme $id \otimes e_t$ établit une homotopie entre $id \otimes e_0$ et $id \otimes e_1$, et que si 2 $*$-morphismes sont homotopes, alors ils sont égaux en K-théorie contrôlée, on obtient que

$$D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_0,T_0)}} = D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi_1,T_1)}}$$

et le bord de l'extension $E^{(\pi,T)}$ ne dépend que de z.

Définition 1. La transformée de Kasparov contrôlée d'un élément $z \in KK_1^{\mathcal{G}}(A,B)$ est définie comme la composition

$$J_{red,\mathcal{G}}(z) = \mathcal{M}^{-1}_{B \rtimes_r \mathcal{G}} \circ D_{K_{B \rtimes_r \mathcal{G}}, E^{(\pi,T)}}.$$

Proposition 2. Soient A et B deux $\mathcal{G}$-C^*-algèbres. Pour tout élément $z \in KK_1^{\mathcal{G}}(A,B)$, il existe un morphisme contrôlé

$$J_{red,\mathcal{G}}(z) : \hat{K}_*(A \rtimes_r \mathcal{G}) \rightarrow \hat{K}_{*+1}(B \rtimes_r \mathcal{G})$$

tel que :

(i) $J_{red,\mathcal{G}}(z)$ induit en K-théorie la multiplication à droite par $j_{red,\mathcal{G}}(z)$;

(ii) $J_{red,\mathcal{G}}$ est additif, i.e.

$$J_{red,\mathcal{G}}(z+z') = J_{red,\mathcal{G}}(z) + J_{red,\mathcal{G}}(z').$$

(iii) Pour tout $\mathcal{G}$-morphisme $f : A_1 \to A_2$,

$$J_{red,\mathcal{G}}(f^*(z)) = J_{red,\mathcal{G}}(z) \circ f_{\mathcal{G},red,*}$$

pour tout $z \in KK_1^G(A_2, B)$.

Preuve 2. (i) Le bord $[\delta_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}}] \in KK_1(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$ associé à l'extension $E^{(\pi,T)}$ induit par définition, modulo équivalence de Morita, l'application $j_{red,\mathcal{G}}$, ce qui assure directement ce point.

(ii) Si z, z' sont deux eléments de $KK_1^G(A, B)$, repésentés par des K-cycles (H_B, π_j, T_j), et si l'on note (H_B, π, T) un K-cycle représentant la somme $z + z'$, alors $E^{(\pi,T)}$ est naturellement isomorphe à l'extension somme des $E_j := E^{(\pi_j,T_j)}$

$$0 \longrightarrow K_{B\rtimes_r\mathcal{G}} \longrightarrow D \longrightarrow A \rtimes_r \mathcal{G} \longrightarrow 0$$

où

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K(E_j, E_i) \right\}.$$

Par naturalité du bord contrôlé [6], le bord de la somme de deux extensions est la somme des bords de chaque extension, d'où le résultat.

(iii) Soit $z \in KK_1^{\mathcal{G}}(A_2, B)$, représenté par un cycle (H_B, π, T). Un représentant de $f^*(z)$ est $(H_B, f^*\pi, T)$ avec bien sûr $f^*\pi = \pi \circ f$. L'application

$$\phi : \left\{ \begin{array}{lcl} E^{f^*(\pi,T)} & \to & E^{(\pi,T)} \\ (x, P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) & \to & (f_{\mathcal{G}}(x), P_{\mathcal{G}}(f^*\pi)(x)P_{\mathcal{G}} + y) \end{array} \right.$$

vérifie

- $\phi(K_{B\rtimes_r\mathcal{G}}) \subset K_{B\rtimes_r\mathcal{G}}$, et s'insère dans le diagramme

$$\begin{tikzcd} 0 \arrow[r] & K_{B\rtimes_r\mathcal{G}} \arrow[r] \arrow[d, "="] & E^{f^*(\pi,T)} \arrow[r] \arrow[d, "\phi"] & A_1 \rtimes_r \mathcal{G} \arrow[r] \arrow[d, "f_{\mathcal{G}}"] & 0 \\ 0 \arrow[r] & K_{B\rtimes_r\mathcal{G}} \arrow[r] & E^{(\pi,T)} \arrow[r] & A_r \rtimes_r \mathcal{G} \arrow[r] & 0 \end{tikzcd}.$$

- Elle entrelace les sections de ces deux extensions.

La remarque 3.7 de [6] assure donc que

$$D_{K_{B\rtimes_r\mathcal{G}},E^{f^*(\pi,T)}} = D_{K_{B\rtimes_r\mathcal{G}},E^{(\pi,T)}} \circ f_{\mathcal{G},*}$$

, et l'assertion est claire en composant par $\mathcal{M}^{-1}_{B\rtimes_r\mathcal{G}}$.

1.3 Transformée de Kasparov, ancienne version

Si l'élément $z \in KK^{\mathcal{G}}(A,B)$ est représenté par un cycle (H,π,T), nous allons définir sa transformée de Kasparov $J_{\mathcal{G}}(z) \in KK(A \rtimes_r \mathcal{G}, B \rtimes_r \mathcal{G})$.

Tout d'abord, le cas pair. Notons $P_{\mathcal{G}} = P \otimes_B id_{B \times_r \mathcal{G}}$ l'opérateur sur $H \otimes B \times_r \mathcal{G}$ induit par $P = \frac{T + id_{H\otimes B}}{2}$. Si l'on pose

$$\mathcal{E} := \{(x,y) \in A \times_r \mathcal{G} \oplus \mathcal{L}(H \otimes B \times_r \mathcal{G}) : P_{\mathcal{G}} \pi_{\mathcal{G}}(x) P_{\mathcal{G}} = y \ mod \ \mathbb{K} \otimes B \times_r \mathcal{G}\},$$

observons l'extension

$$(E): \quad 0 \longrightarrow \mathbb{K} \otimes B \times_r \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow A \times_r \mathcal{G} \longrightarrow 0 \quad .$$

Toute extension $(Ext): \quad 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ induit une application contrôlée

$$D_{\mathrm{Ext}} = D_{A'}^{A} : \hat{K}(A'') \to \hat{K}(A').$$

Montrons que D_E ne dépend que de la classe de z, et pas de π et T.
A FAIRE

Définition 2. La transformée de Kasparov d'un élément z de $KK^{\mathcal{G}}(A,B)$ est le morphisme contrôlé

$$J_{\mathcal{G}}(z) = M^{-1}_{B \times_r \mathcal{G}} \circ D_E : \hat{K}_*(A \times_r \mathcal{G}) \to \hat{K}_*(B \times_r \mathcal{G}),$$

où (E) est l'extension précédemment décrite.

Ce morphisme $J_{\mathcal{G}} : KK^{\mathcal{G}}(A,B) \to \mathrm{Hom}_0(\hat{K}(A \times_r \mathcal{G}), \hat{K}(B \times_r \mathcal{G}))$ nous permet de définir l'application d'assemblage associée à n'importe quel élément de $\hat{K}(A \times_r \mathcal{G})$ par simple évaluation :

$$Ind_x(z) = J_{\mathcal{G}}(z)(x).$$

La conjecture de Baum-Connes s'intéresse à l'application d'assemblage associé à un certain élément. Dans le cas des groupoïdes, il existe une fonction continue à support compact $h : P_d(\mathcal{G}) \to [0,1]$ telle que

$$\sum_{\gamma \in \mathcal{G}} \gamma(h^2) = 1.$$

Alors $\gamma \to \sum_{h \in \mathcal{G}} h\gamma(h)$ définit un projecteur de $A = C_0(P_d(\mathcal{G})) \times_r \mathcal{G}$ de propagation finie, majorée par une certaine constante s. Comme les fonctions h admissibles forment un ensemble convexe, la classe de $[e_h, 0] \in K_0^{\epsilon, r(A)}$ ne dépend pas de la fonction h choisie, et l'application d'assemblage de Baum-Connes est définie par l'évaluation en cette classe.

2 Géométrie asymptotique

Pour tout $z \in KK^{\Gamma}(A,B)$, on peut construire une application de descente

$$- \otimes \sigma_X(z) : K_*(C^*(X,A)) \to K_*(C^*(X,B)),$$

et l'application d'assemblage asymptotique est simplement l'application de decente prise en un certain élément. On va montrer que l'on peut en fait construire un morphisme contrôlé $\tau_X(z) : \hat{K}_*(C^*(X,A)) \to \hat{K}_*(C^*(X,B))$ qui induit la multiplication à droite par $\sigma_X(z)$.

Pour tout $z \in KK^\Gamma(A,B)$, il existe un morphisme contrôlé

$$\tau_X(z) : K_*(C^*(X,A)) \to K_*(C^*(X,B)).$$

Rappelons que le morphisme de groupoïdes $\quad \iota : \{e_x\} \hookrightarrow \Gamma \quad$ induit un isomorphisme de groupes abéliens

$$\iota^* : KK_*^\Gamma(\tilde{A}, l^\infty(X, B \otimes \mathbb{K})) \xrightarrow{\simeq} KK_*(A,B) \quad ,$$

où $\tilde{A}$ est la $C(\Gamma)$-algèbre $C_0(P_d(\Gamma))$, de fibre $A = \tilde{A}_x = C_0(P_d(X))$. On dispose de plus d'un $*$-isomorphisme $\Phi_B : C^*(X,B) \to l^\infty(X, B \otimes \mathbb{K}) \rtimes_r \Gamma$ pour toute C^*-algèbre B, qui préserve la filtration, et donc induit un morphisme contrôlé. On peut alors définir, pour $z \in KK(A,B)$, le morphisme contrôlé $\tau_X(z)$ par

$$J_\Gamma(\iota^{*-1}(z)) = \Phi_{\tilde{B}*} \circ \tau_X(z) \circ \Phi_{\tilde{A}*}^{-1},$$

où $\tilde{B} := l^\infty(X, B \otimes \mathbb{K}) \rtimes_r \Gamma$. Montrons que $\tau_X(z)$ induit la multuplication par σ_X en KK-théorie. **A FAIRE**

3 English Notes

In a serie of papers, H. Oyono-Oyono and G. Yu have defined a controlled version of operator K-theory, [8] [6] [7], that allows them to define a quantitative and local assembly map, and a related Baum-Connes Conjecture. The aim of this work is to extend their work to the realm of groupoids. In a second part, we shall see how this setting can help us understand better the relation between the coarse-Baum-Connes conjecture and the Baum-Connes Conjecture with coefficient for groupoids. Indeed, in [20] and [18], G. Skandalis, J-L.Tu and G. Yu proved that there is a commutative diagramm

$$\begin{array}{ccc} K_*(X) & \xrightarrow{A} & K_*(C^*X) \\ \downarrow \simeq & & \downarrow \simeq \\ K_*(\mathcal{E}\Gamma) & \xrightarrow{\mu_r} & K_*(C_r\Gamma) \end{array}$$

where :

- the left sides are K-homology of certain spaces, the left sides are K-theory of certain C^*-algebras,
- the first line is the coarse assembly map associated to a coarse space X, the second being the assembly map for groupoids associated to the coarse groupoid of X, namely $\Gamma = G(X)$,
- the vertical arrows are isomorphisms on the level of KK-groups. The left one will be studied later, the right one derives from an isomomorphism on the level of the C^*-algebras. Indeed, it has been shown that the Roe algebra is $*$-isomorphic to the crossed product of $l^\infty(X)$ by Γ : $C^*(X) \simeq l^\infty(X) \times \Gamma$.

We shall see that this relation is already true "locally" and factorises through quantitative K-theory. Locally here means that we can factorize the assembly map through $KK(C_0(P_d(X), B)$. Indeed, the K-homology can be expressed as an inductive limit

$$K_*(X, B) = \varinjlim_{d \to \infty} KK(C_0(P_d(X)), B) \quad \text{and}$$

hence the local.

The local quantitative assembly maps defined in [6] are of the form

$$KK^F(C_0(P_d(F)), B) \to K_*^{\epsilon,r}(B \times_r \Gamma)$$

where F is a finite group.

We will be using the bivariant functor KK^Γ introduced by P-Y. Le Gall in his thesis [4], which is a generalization of Kasparov's bifunctor for the case where Γ is a Hausdorff locally compact groupoid with Haar system. As for the KK-theory for Banach algebras introduced by V. Lafforgue [10], there is no (not yet ?) a Kasparox product in quantitative K-theory. The crucial point to define an assembly map is the existence of a morphism

$$\hat{J} : KK^\Gamma(A, B) \to Hom^*(\hat{K}_*(A), \hat{K}_*(B))$$

which allows us, for every element $x \in K_*(A)$, to consider the related index

$$Ind_x \begin{cases} KK_*(A,B) & \to & K_*(B) \\ z & \mapsto & \hat{J}(z)(x) \end{cases}$$

to construct $\hat{J}$, we will mimic the construction for $\mathcal{J}$ in [6], which gives the right morphism when considering finite groups. The starting point is the following

Lemme 2. Let Γ be a Hausdorff locally compact groupoid with Haar system. Then, if A is a $\Gamma - C^*$-algebra, forming the reduced (and maximal) crossed product $A \times_r \Gamma$ is functorial in A. Moreover, if Γ is *étale*, then it preserves short semi-split filtered exact sequences of $\Gamma - C^*$-algebras, meaning that if

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is a semi-split filtered exact sequence of $\Gamma - C^*$-algebras, then

$$0 \longrightarrow A' \times_r \Gamma \longrightarrow A \times_r \Gamma \longrightarrow A'' \times_r \Gamma \longrightarrow 0$$

is too.

Preuve 3. Let

$$0 \longrightarrow A' \xrightarrow{\Psi} A \xrightarrow{\Phi} A'' \longrightarrow 0$$

be a semi-split filtered exact sequence of $\Gamma - C^*$-algebras.

Let $f \in C_c(\Gamma, A)$ such that $\Phi_\Gamma(f) = 0$. For every $\gamma \in \Gamma$, there exists a unique $g(\gamma) \in A'$ such that $\Psi(g(\gamma)) = f(\gamma)$. But, as Ψ and Φ commute with the action of Γ, we have $g(\gamma'^{-1}\gamma) = \gamma'.g(\gamma)$, so that g is continuous.

Let $a \in A \times \Gamma$ such that $\Phi_\Gamma(a) = 0$. We can approximate a by a sequence of $f_n \in C_c(\Gamma, A)$. But, with the preceeding work on compactly supported continuous functions, there exist a sequence $g_n \in C_c(\Gamma, A')$ such that :

$$f_n = \Psi(g_n) + \sigma \circ \Phi(f_n).$$

Moreover, Im Ψ is a closed ideal in A, and $\lim ||f_n - \Psi(g_n)|| = 0$ so that $a \in$ Im Ψ : the sequence is exact in the middle. □

3.1

Let X be a discrete metric space with bounded geometry and Γ its coarse groupoid. $\mathbb{C}[X]$ is the algebra of locally compacts operators with finite propagation on $l^2(X) \otimes H$. We denote by $\tilde{A}$ the $C(\Gamma)$-algebra $C_0(P_d(\Gamma))$, and by A the C^*-algebra $C_0(P_d(X))$. As $P_d(\Gamma)$ is equivariantly homeomorphic to $\beta X \times P_d(X)$, the fiber $\tilde{A}_x$ is isomorphic to A as a C^*-algebra.

Lemme 3. Let B be a C^*-algebra and $x \in X$. The morphism of groupoids $\iota : \{e_x\} \to \Gamma$ induces an isomorphism in KK-theory

$$KK_*^\Gamma(\tilde{A}, l^\infty(X,B)) \xrightarrow{\simeq} KK_*(A,B) \quad .$$

Preuve 4. Let us first check that the fiber over $x \in X$ of $\tilde{B} := l^\infty(X, B)$ is B : easily, if I_x denotes the kernel of the evaluation map $\tilde{B} \to B$ at x, $\tilde{B}_x = \tilde{B}/I_x\tilde{B} \simeq B$. Moreover, as $\tilde{A}_x \simeq A$, functoriality assures that ι induces $\iota^* : KK_*^\Gamma(\tilde{A}, l^\infty(X, B)) \to KK_*(A, B)$.

We can now construct an explicit R inverse for ι^*. Let $z = (\hat{H}_{\tilde{B}}, \Phi, T) \in \mathbb{E}^\Gamma(\tilde{A}, \tilde{B})$ be a K-cycle. By stabilization theorem, we can suppose that the Hilbert $\tilde{B}$ module is in canonical form, that is the standard separable Hilbert module over $\tilde{B}$ with usual grading. We will also suppose that T is Γ-equivariant, which is always true up to compact perturbation. The image of z under R is defined to be its restriction to x, $(\hat{H}_B, \Phi_x, T_x)$. This definition gives immediatly $R \circ \iota^* = 1$. Now let $V : s^*\hat{H}_{\tilde{B}} \to \hat{H}_{\tilde{B}}$ be the unitary implementing the action of Γ on $\hat{H}_{\tilde{B}}$. The equivariance of T can be written as

$$V_\gamma T_{s(\gamma)} V_\gamma^* = T_{r(\gamma)} \quad , \forall \gamma \in \Gamma.$$

If $T' = \iota^* \circ R(T)$, it is the constant operator with fiber $T_x \in \mathcal{L}_B(\hat{H}_B)$ on the trivial Hilbert module $\bigoplus_{y \in X} (H_{\tilde{B}})_x$. Define $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$. It is unitary and intertwines T and T', i.e. the cycles are homotopic and $\iota^* \circ R = 1$.

□

Another result of interest for our study is the expression of the Roe algebras of our discrete space X as crossed products. It was first proven by G. Yu in the case of X being a finitely generated group with the word metric, the general version can be found in the article of G. Skandalis, J-L. Tu and G. Yu. [18]

Lemme 4. Let B be a coefficient C^*-algebra. The uniform Roe algebra of X and the Roe algebra of X are $*$-isomorphic to crossed-product algebras by Γ, respectively :

$$\begin{array}{l} C_u^*(X, B) \simeq l^\infty(X, B) \rtimes_r \Gamma \\ C^*(X, B) \simeq l^\infty(X, \mathbb{K} \otimes B) \rtimes_r \Gamma \end{array}.$$

Preuve 5. If $T \in \mathbb{C}[X]$. For $x, y \in X$, $T_{xy} \in \mathbb{K}$ is the compact operator defined by Riesz representation theorem as

$$\langle T(\delta_x \otimes \xi), \delta_y \otimes \eta \rangle = \langle T_{xy}\xi, \eta \rangle \ , \xi, \eta \in H.$$

Let us recall how one construct the left regular representation of Γ. For $x \in \Gamma^{(0)} = \beta X$, Γ^x has a natural action on $l^2(\Gamma^x)$:

$$(\lambda_x(\gamma)\eta)(\gamma') = \eta(\gamma^{-1}\gamma') \quad , \gamma, \gamma' \in \Gamma^x, \eta \in l^2(\Gamma^x).$$

To any couple $(x, y) \in \beta X \times \beta X$, there exists a unique $\gamma_x^y \in \Gamma$ such that $s(\gamma) = x$ and $r(\gamma) = y$. The map

$$T \to \sum_{x,y} T_{xy} \lambda_x(\gamma_x^y)$$

establishes an $*$-isomorphism between $\mathbb{C}[X]$ and the algebra of finitely supported functions from Γ to $l^\infty(X, \mathbb{K})$, its inverse being the $*$-morphism that associates to a function a the convolution operator :

$$\big(a(\gamma)(\eta \otimes \xi)\big)(x) = (a(\gamma) * \eta)(x)\xi \quad , x \in X, \eta \in l^2(X), \xi \in H.$$

That is it of finite propagation and locally compact follows from the finiteness of the support (it is of propagation the diameter of the support of a !). As this two maps are continuous and their domain are respectivly dense in the Roe algebra and in the crossed product, the morphism extends to the $*$-isomorphism we were looking for. $\square$

This $*$-isomorphism gives an isomorphism at the level of K-theory

$$K_*(C^*(X,B)) \to K_*(l^\infty(X, \mathbb{K}\otimes B) \rtimes_r \Gamma).$$

These lemmas allows us to state a reffinement of a result of G. Skandalis, J-L.Tu and G. Yu [18], namely that the two isomorphisms described above intertwine the (local quantitative ?) assembly maps for the groupoid Γ with coefficients in $l^\infty(X)$ with the coarse assembly map of X. (If local and quantitative :) This result at the level of controlled K-theory on the right side entails that of [18]

Théorème 1. The following diagram

$$\begin{array}{ccc}
A : KK_*(A,B) & \longrightarrow & K_*(C^*(X,B)) \\
\downarrow \simeq & & \downarrow \simeq \\
\mu_r : KK^\Gamma & \longrightarrow & K_*(l^\infty(X, \mathbb{K}\otimes B) \rtimes_r \Gamma)
\end{array}$$

commutes, and the vertical arrows are isomorphisms.

4 Assembly maps for groupoids and for coarse spaces

4.1 The case of a finitely generated group

Let Γ be a discrete finitely generated group. The word length provides a structure of metric space, of which the class up to coarse equivalence is independent of the set of generators.

Denoting $C^*(\Gamma)$ the Roe algebra, i.e. the C^*-algebra generated by locally compact operators on $l^2(\Gamma) \otimes H$ with finite propagation, we can show that

$$C_u^*(\Gamma) = l^\infty(\Gamma) \times_\alpha \Gamma$$

$$C^*(\Gamma) = l^\infty(\Gamma, \mathfrak{K}(H)) \times_\alpha \Gamma.$$

Here $\alpha \in Aut(A)$ is the automorphism encoding the left action of Γ on A :

$$\alpha_\gamma(a) = s \to a_{s\gamma^{-1}}.$$

Let S_γ be the operator acting on $l^2(\Gamma)$ as

$$(S_\gamma \eta)_s = \eta_{s\gamma^{-1}}.$$

We see $l^\infty(\Gamma)$ as an algebra of operator, acting by left multiplication on $l^2(\Gamma)$. Then

$$S_\gamma a S_\gamma^* = \alpha_\gamma(a),$$

for any $a \in l^\infty(\Gamma)$ and $\gamma \in \Gamma$. The algebra $C^*(\Gamma)$ is generated by finite sums of the form

$$\sum_\gamma a_\gamma S_\gamma$$

which are of finite propagation $\max_\gamma \{l(\gamma) : a_\gamma \neq 0\}$ and locally compact.

4.2 Relation between the coarse and the groupoid assembly maps

We have to show that there is an isomorphism

$$KX_*(X) \to KK_*^{top}(G(X), l^\infty(X, \mathfrak{K})).$$

Let us recall that the Stone-Cech compactification of our coarse groupoid $\Gamma = G(X)$ identified itself to the spectrum of the bounded continuous functions over X, which is discrete. We have

$$C(\beta X) \simeq l^\infty(X)$$

and we can think of $C(\beta X)$-algebras as $l^\infty(X)$-algebras.

The left handside $KX_*(X)$ is defined as the limit of the directed groups

$$KK_*(C_0(P_E(X), \mathbb{C})$$

when E is an entourage of X. Here $P_E(X)$ denotes the Rips complex defined by the entourage E, which is the set of simplexes $[x_0, ..., x_n]$ such that $(x_i, x_j) \in E$.

Now the classifying space $\mathcal{E}\Gamma$ of the groupoid $G(X)$ is unique up to homotopy, and can be realised by the space of measures μ on $G(X)$ which satisfied $s^*\mu$ is a Dirac measure on $G^{(0)} = \beta X$, and $\frac{1}{2} < |\mu| \leq 1$. Saying that $s^*\mu$ is a Dirac measure is the same as demanding μ to be supported in a fiber Γ_x for some $x \in X$. The abelian group $KK_*^{top}(G(X), l^\infty(X, \mathfrak{K}))$ is defined as the inductive limit of

$$KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$$

when Y is a Γ-compact space of $\mathcal{E}\Gamma$.

Let E be an entourage of X. A Fredholm module (H, ϕ, F) in $E(C_0(P_E(X)), \mathbb{C})$ is defined by a Hilbert space H, a $*$-homomorphism $\phi : C_0(P_E(X)) \to \mathcal{L}(H)$ and an operator F satisfying all definitions.
We can form the $l^\infty(X, \mathfrak{K})$-module $\mathcal{E} = H \otimes_{\mathbb{C}} l^2(\Gamma, \mathfrak{K}))$, and extend ϕ into $\phi \otimes id : C_0(P_E(X)) \to \mathcal{L}(H \otimes l^2(\Gamma, \mathfrak{K})$. We do the same with $F : \hat{F} := F \otimes id$. Then, as $P_E(X)$ identifies itself as a G-compact of $\mathcal{E}G$, $(\mathcal{E}, \phi \otimes id, F \otimes id)$ defines an element of $KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$.

5 Correspondance between the coarse K-homology of a space and the one of its coarse groupoid

The aim of this section is to give a proof of a result of [20], in which it is stated that the following diagramm commutes :

$$\begin{array}{ccc} KX_*(X, B) & \xrightarrow{A} & K_*(C^*X, B) \\ \downarrow \simeq & & \downarrow \simeq \\ K_*(G(X), l^\infty(X, B)) & \xrightarrow{\mu} & K_*(C_r(G(X)), B). \end{array}$$

The vertical arrow from the left comes from an isomorphism at the C^*-algebraic level, as

$$C^*(X) \simeq l^\infty(X) \times G(X).$$

The rest of this section is devoted to describe the vertical arrow from the right in the langage of Kasparov KK-theory, i.e.

$$\varinjlim_{d} KK(C_0(P_d(X)), B) \to \varinjlim_{Y \subset \mathcal{E}G(X)} KK(C_0(Y), B),$$

were the inductive limite on the right is taken among the proper $G(X)$-compact subsets Y of the universal classifying space for proper actions of $G(X)$.

Recall from [19] that we can take for $\mathcal{E}G(X)$ the space $\mathfrak{M}$ of positive measures μ on $G(X)$ satisfying :

- $\frac{1}{2} < \mu(G(X)) \leq 1$,
- $s^*\mu$ is a Dirac measure, i.e. its support consists of arrows of $G(X)$ that all source from the same base point of βX.

If $\mathfrak{M}_d$ denotes the space of measures μ of $\mathfrak{M}$ such that :

— μ is a probability measure

— for all γ and γ' in the support of μ, $\gamma'\gamma^{-1}$ is d-controlled, i.e. $d(r(\gamma), r(\gamma')) \leq d$,

then $\mathfrak{M} = \varinjlim \mathfrak{M}_d$.

The Rips complex of X, denoted $P_d(X)$, is the topological space of the complexes of diameter less than d, identified with probability measures on X with support of diameter less than d, with the weak topology coming from $C_c(C)$. We will write $[y, t]$ for a point of a simplex defined by barycentric coordinates of k points $y_1, ..., y_k$, ie $\sum t_j \delta_{y_j}$. To such a point $[y, t]$ and an element of the Stone-Cech compactification $w \in \beta X$, we can associate a measure of $\mathfrak{M}_d$ in the following way. As $G(X)$ is a principal and transitive groupoid, there exists only one arrow γ_j such that $s(\gamma_j) = x$ and $r(\gamma_j) = y_j$. To $z = ([y, t], w) = (z_w, w)$, we associate

$$\phi_d(z) = \sum_{j=1,k} t_j \delta_{\gamma_j} \in \mathfrak{M}_d.$$

Proposition 3. The map

$$\phi_d : P_d(X) \times \beta X \to \mathfrak{M}_d$$

is an homeomorphism.

Preuve 6. It is clearly bijective. The bicontinuity comes from the identity :

$$\langle z_w, f \rangle = \langle \phi_d(z), f \circ r \rangle$$

for all $z = (z_w, w) \in P_d(X) \times \beta X$, and $f \in C_c(X)$. □

This homeomorphism ϕ_d gives an $*$-isomorphism at the level of C^*-algebras

$$\Psi_d : C_0(\mathfrak{M}_d) \to C_0(P_d(X) \times \beta X).$$

Let $(\mathcal{E}, \pi, F) \in \mathbb{E}(C_0(P_d(X)), B)$ be an elliptic operator. **A FINIR**

Let $X_0 \subset X_1 \subset ... \subset X_j \subset ...$ the n-skeleton decomposition associated to the simplicial structure of the Rips complex $P_d(X)$, and similarly $\tilde{X}_0 \subset \tilde{X}_1 \subset ... \subset \tilde{X}_j \subset ...$ for $\mathfrak{M}_d$, and

$$Z_j = C_0(X_j) \quad \text{and} \quad \tilde{Z}_j = C_0(\tilde{X}_j).$$

$$Z_{j-1}^j = C_0(X_j - X_{j-1}) \quad \text{and} \quad \tilde{Z}_{j-1}^j = C_0(\tilde{X}_j - \tilde{X}_{j-1}).$$

We will show the isomorphism by a Mayer-Vietoris type argument. By applying the KK-theory to the exact sequence :

$$0 \longrightarrow C_0(X_j - X_{j-1}) \longrightarrow C_0(X_j) \longrightarrow C_0(X_{j-1}) \longrightarrow 0$$

we have a commutative diagramm with exact lines :

$$
\begin{tikzcd}
KK_*(Z^j_{j-1},B) \arrow[r, "\delta"] \arrow[d, "\theta^j_{j-1}"] & KK_*(Z_{j-1},B) \arrow[r] \arrow[d, "\theta_{j-1}"] & KK_*(Z_j,B) \arrow[r] \arrow[d, "\theta_j"] & KK_*(Z^j_{j-1},B) \arrow[r, "\delta"] \arrow[d, "\theta^j_{j-1}"] & KK_*(Z_{j-1},B) \arrow[d, "\theta_{j-1}"] \\
KK_*(\tilde{Z}^j_{j-1},B) \arrow[r, "\delta"] & KK_*(\tilde{Z}_{j-1},B) \arrow[r] & KK_*(\tilde{Z}_j,B) \arrow[r] & KK_*(\tilde{Z}^j_{j-1},B) \arrow[r, "\delta"] & KK_*(\tilde{Z}_{j-1},B)
\end{tikzcd}
$$

The five lemma assures that if θ_{j-1} and θ^j_{j-1} are isomorphisms, then so is θ_j. Moreover, $X_j - X_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where $\mathring{\sigma}_j$ denotes the interior of the standard simplex, and Σ_j is the set of centers of j-simplices of X_j. Bott periodicty assures then that, if θ_{j-1} is an isomorphism, then so is θ^j_{j-1}. By induction, proving that θ_0 is an isomorphism concludes the proof.

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