Notes

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1 Groupoïds

1.1 Definitions

Définition 1. A groupoid is a small category whose arrows are all invertible. More concretly, it is the data of a set G together with a set of units $G^{(0)}$ and two maps $r, s : G \to G^{(0)}$. We can compose two arrows when the range of the first agrees with the source of the second. If we denote, for $x \in G^{(0)}$, $G_x = \{\gamma \in G : s(\gamma) = x\}$ and $G^x = \{\gamma \in G : r(\gamma) = x\}$, this can be rephrase as the existence of a family of maps

$$\left\{ \begin{array}{ccc} G_x \times G^x & \to & G \\ (\gamma, \gamma') & \mapsto & \gamma\gamma' \end{array} \right. , \forall x \in G^{(0)}.$$

An automorphism of a groupoid is just an endofunctor which is invertible.

Depending on the situation, we will require these to be topological spaces with continuous maps, manifolds with smooth functions, etc. In these cases, we will talk about topological or smooth groupoïds. For now on, L_γ denotes the left translation $G^{s(\gamma)} \to G^{r(\gamma)}; \gamma' \mapsto \gamma\gamma'$, and $X = G^{(0)}$ is the set of units.

Définition 2. A Haar system $\lambda = (\lambda^x)_{x \in G^{(0)}}$ is a family of borelian measures λ^x with support G^x such that :

1. for all continuous function with compact support $f \in C_c(G)$, the map $x \mapsto \int_{G^x} f d\lambda^x$ is continuous.
2. λ is left-invariant w.r.t G, i.e. $L_{\gamma,*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)} \forall \gamma \in G$ or

$$\int_{G^{s(\gamma)}} f(\gamma\gamma') d^{s(\gamma)}\gamma' = \int_{G^{r(\gamma)}} f(\gamma') d^{r(\gamma)}\gamma'.$$

From $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$, we deduce

$$\int_{G^{s(\alpha^{-1}(\gamma))}} f(\gamma\alpha(\gamma')) d\gamma' = \int_{G^{r(\gamma)}} f(\gamma') \frac{1}{\rho(\alpha^{-1}(\gamma^{-1}\gamma'))} d\gamma'$$

$$\int_{G^{s(\alpha^{-1}(\gamma))}} f(\alpha(\alpha^{-1}(\gamma)\gamma')) = \int_{G^{r(\gamma)}} f(\gamma') \frac{1}{\rho(\alpha^{-1}(\gamma'))} d\gamma'$$

and $\rho(\gamma^{-1}\gamma') = \rho(\gamma')$. In particular, ρ is constant on G_x, for all $x \in X$.

Définition 3. An automorphism α of G preserves a Haar system λ if, for each $x \in X$, $\alpha_*\lambda^x$ is absolutely continuous w.r.t $\lambda^{\alpha(x)}$ and there exists a continuous function $\rho_\alpha : G \to \mathbb{R}^+$ such that ρ_α restricted to $G^{\alpha(x)}$ is the Radon-Nikodym derivative $\frac{d\alpha_*\lambda^x}{d\lambda^{\alpha(x)}}$.

Définition 4. Given an automorphism α of a groupoid G, we can form the suspension groupoid relative to α as follow. It is the groupoid with arrows

$$G_\alpha = G \times \mathbb{R}/\sim \quad \text{where } (\gamma, t) \sim (\alpha(\gamma), t-1)$$

and units

$$X_\alpha = X \times \mathbb{R}/\sim \quad \text{where } (x, t) \sim (\alpha_X(x), t-1).$$

If $[\gamma, t]$ and $[x, t]$ denote the equivalence classes in G_α and X_α respectively, then the source and the range map are given by

$$s([\gamma, t])] = [s(\gamma), t] \quad \text{and} \quad r([\gamma, t])] = [r(\gamma), t].$$

The composition is $[\gamma, t][\gamma', t] = [\gamma\gamma', t]$.

Lemme 1. If $\rho_\alpha \circ \alpha = \rho_\alpha$, then the suspension groupoid G_α admits a Haar system λ_α, given by

$$\lambda^{[x,t]}(f) = \int_{G^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma).$$

Preuve 1. We shall first demonstrate that this definition does make sense, i.e. that it is independent of the representant of the class $[x, t]$.

$$\begin{aligned}
\lambda^{[x,t]}(f) &= \int_{G^x} \rho(\alpha(\gamma))^{-t} f([\gamma, t]) d^x\gamma \\
&= \int_{G^{\alpha(x)}} \rho(\gamma)^{-t} f([\alpha^{-1}(\gamma), t]) \frac{d^{\alpha(x)}\gamma}{\rho(\gamma)} \\
&= \int_{G^{\alpha(x)}} \rho(\gamma)^{-t+1} f([\gamma, t-1]) d^{\alpha(x)}\gamma \qquad = \lambda^{[\alpha(x), t-1]}(f).
\end{aligned}$$

As the continuity is clear, we can conclude by showing the left-invariance.

$$\begin{aligned}
\int_{G_\alpha^{[s(\gamma),t]}} f([\gamma\gamma', t]) d^{[s(\gamma),t]}[\gamma', t] &= \int_{G^{s(\gamma)}} \rho^{-t}(\gamma') f([\gamma\gamma', t]) d^{s(\gamma)}\gamma' \\
&= \int_{G^{r(\gamma)}} \rho^{-t}(\gamma^{-1}\gamma') f([\gamma', t]) d^{r(\gamma)}\gamma' \\
&= \int_{G^{r(\gamma)}} \rho^{-t}(\gamma') f([\gamma', t]) d^{r(\gamma)}\gamma'
\end{aligned}$$

The last equality follows from the fact that ρ is constant on G_x, for all $x \in X$, and then

$$\int_{G_\alpha^{[s(\gamma),t]}} f([\gamma\gamma', t]) d^{[s(\gamma),t]}[\gamma', t] = \int_{G_\alpha^{[r(\gamma),t]}} f([\gamma', t]) d^{[r(\gamma),t]}[\gamma', t].$$

□

1.2 Principal *étale* groupoids

In this section, we are interested in locally compact groupoids. The maps $r, s : G \to X$, the composition and inverse maps are continuous.

Définition 5. A groupoid is said to be *étale* if $r : G \to X$ is a local homeomorphism.
It is principal if the product map $s \times r : G \to X \times X$ is one-to-one.

Let $x \in X$ and $\gamma \in G^x$. If G is *étale*,there exists a neighborhood U of γ such that $r_{|U}$ is a homeomorphism. So $G^x \cap U = \{\gamma\}$ is open in G^x. That show that the fibers G^x are discrete for all $x \in X$.

Proposition 1. If G is a principal étale groupoid, the fibers G^x are discrete for all $x \in X$ and the only Haar systems are the multiple of the counting measure on the fibers.

Preuve 2. If λ is a non-zero Haar system and G is principal, λ^x is a measure on the discrete space G^x, which entails that there exists a $\gamma \in G^x$ such that $\lambda(\gamma) > 0$. By left-invariance,

$$\lambda^{r(\gamma')}\{\gamma'\gamma\} = \lambda\{\gamma\} > 0.$$

Replacing $\gamma' = \gamma^{-1}$ in this relation, we have $\lambda^x\{x\} > 0$, which we can suppose equal to 1. The left invariane assures then that

$$\lambda^x\{\gamma\} = 1 \quad \forall \gamma \in G^x.$$

$\square$

2 Stone-Cech compactification

Let X be a topological space. The Stone-Cech compactification of X, denoted by βX, is defined as the compact Hausdorff space, unique up to homeomorphism, together with a $C_b(X)$-embedding $\phi_X : X \to \beta X$ such that, for any continuous map $f : X \to K$ in a compact space K, there exists a unique continuous map $\tilde{f} : \beta X \to K$ that makes the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"] & K \\ \beta X \arrow[ur, "\tilde{f}"] & \end{tikzcd}.$$

The universal property of the Stone-Cech compactified makes it a functor from the category of topological spaces to the category of compact Hausdorff spaces. Indeed, it is general property that, if we are given two categories C and C' and a functor $\phi : C \to$, such that for every functor $F : C \to C'$, there exists

Let X be a compact Hausdorff space. Then the maximal ideals of $C(X)$ are in a one-to-one correspondence with the points of X. Explicitely, to a point $p \in X$ corresponds the maximal ideal

$$\mathfrak{M}_p = \{f \in C(X) : f(p) = 0\}.$$

If one endorses the spaces of maximal ideals of $C(X)$ with the Stone topology, this correspondence $p \mapsto \mathfrak{M}_p$ is actually a homeomorphism. Now, it is a theorem that when X is just locally compact, $C(\beta X)$ and $C_b(X)$ are homeomorphic, and then we have that $\beta X \simeq \mathfrak{M}(C(\beta X)) \simeq \mathfrak{M}(C_b(X))$. This amounts saying that we can see βX as the spectra of $C_b(X)$, for all locally compact spaces.

3 Asymptotic dimension

Définition 6. Let X be a metric space.
The multiplicity of a cover $\mathcal{U}$ of X is the largest number $n \in \mathbb{N}$ such that every point $x \in X$ is contained in at most n elements of $\mathcal{U}$.
If $R > 0$, the R-multiplicity is the defined as the multiplicity restricted on covers uniformly bounded by R.
The asymptotic dimension of x is the smallest natural integer $n \in \mathbb{N}$ such that, for all $R > 0$, there exists a uniformly bounded cover $\{U_j\}$ with R-multiplicity $n + 1$.
Such a space is said to have finite asymptotic dimension if this number, denoted $\dim_\infty X$, is bounded.

4 Assembly maps for groupoids and for coarse spaces

4.1 The case of a finitely generated group

Let Γ be a discrete finitely generated group. The word length provides a structure of metric space, of which the class up to coarse equivalence is independent of the set of generators.

Denoting $C^*(\Gamma)$ the Roe algebra, i.e. the C^*-algebra generated by locally compact operators on $l^2(\Gamma) \otimes H$ with finite propagation, we can show that

$$C_u^*(\Gamma) = l^\infty(\Gamma) \times_\alpha \Gamma$$

$$C^*(\Gamma) = l^\infty(\Gamma, \mathfrak{K}(H)) \times_\alpha \Gamma.$$

Here $\alpha \in Aut(A)$ is the automorphism encoding the left action of Γ on A :

$$\alpha_\gamma(a) = s \to a_{s\gamma^{-1}}.$$

Let S_γ be the operator acting on $l^2(\Gamma)$ as

$$(S_\gamma \eta)_s = \eta_{s\gamma^{-1}}.$$

We see $l^\infty(\Gamma)$ as an algebra of operator, acting by left multiplication on $l^2(\Gamma)$. Then

$$S_\gamma a S_\gamma^* = \alpha_\gamma(a),$$

for any $a \in l^\infty(\Gamma)$ and $\gamma \in \Gamma$. The algebra $C^*(\Gamma)$ is generated by finite sums of the form

$$\sum_\gamma a_\gamma S_\gamma$$

which are of finite propagation $\max_\gamma\{l(\gamma) : a_\gamma \neq 0\}$ and locally compact.

4.2 Relation between the coarse and the groupoid assembly maps

We have to show that there is an isomorphism

$$KX_*(X) \to KK_*^{top}(G(X), l^\infty(X, \mathfrak{K})).$$

Let us recall that the Stone-Cech compactification of our coarse groupoid $\Gamma = G(X)$ identified itself to the spectrum of the bounded continuous functions over X, which is discrete. We have

$$C(\beta X) \simeq l^\infty(X)$$

and we can think of $C(\beta X)$-algebras as $l^\infty(X)$-algebras.

The left handside $KX_*(X)$ is defined as the limit of the directed groups

$$KK_*(C_0(P_E(X), \mathbb{C})$$

when E is an entourage of X. Here $P_E(X)$ denotes the Rips complex defined by the entourage E, which is the set of simplexes $[x_0, ..., x_n]$ such that $(x_i, x_j) \in E$.

Now the classifying space $\mathcal{E}\Gamma$ of the groupoid $G(X)$ is unique up to homotopy, and can be realised by the space of measures μ on $G(X)$ which satisfied $s^*\mu$ is a Dirac measure on $G^{(0)} = \beta X$, and $\frac{1}{2} < |\mu| \leq 1$. Saying that $s^*\mu$ is a Dirac measure is the same as demanding μ to be supported in a fiber Γ_x for some $x \in X$. The abelian group $KK_*^{top}(G(X), l^\infty(X, \mathfrak{K}))$ is defined as the inductive limit of

$$KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$$

when Y is a Γ-compact space of $\mathcal{E}\Gamma$.

Let E be an entourage of X. A Fredholm module (H, ϕ, F) in $E(C_0(P_E(X)), \mathbb{C})$ is defined by a Hilbert space H, a $*$-homomorphism $\phi : C_0(P_E(X)) \to \mathcal{L}(H)$ and an operator F satisfying all definitions.
We can form the $l^\infty(X, \mathfrak{K})$-module $\mathcal{E} = H \otimes_{\mathbb{C}} l^2(\Gamma, \mathfrak{K}))$, and extend ϕ into $\phi \otimes id : C_0(P_E(X)) \to \mathcal{L}(H \otimes l^2(\Gamma, \mathfrak{K})$. We do the same with $F : \hat{F} := F \otimes id$. Then, as $P_E(X)$ identifies itself as a G-compact of $\mathcal{E}G$, $(\mathcal{E}, \phi \otimes id, F \otimes id)$ defines an element of $KK_{G(X)}(C_0(Y), l^\infty(X, \mathfrak{K}))$.

5 Correspondance between the coarse K-homology of a space and the one of its coarse groupoid

The aim of this section is to give a proof of a result of [12], in which it is stated that the following diagramm commutes :

$$\begin{array}{ccc} KX_*(X,B) & \xrightarrow{A} & K_*(C^*X,B) \\ \downarrow \simeq & & \downarrow \simeq \\ K_*(G(X), l^\infty(X,B)) & \xrightarrow{\mu} & K_*(C_r(G(X)),B). \end{array}$$

The vertical arrow from the left comes from an isomorphism at the C^*-algebraic level, as

$$C^*(X) \simeq l^\infty(X) \times G(X).$$

The rest of this section is devoted to describe the vertical arrow from the right in the langage of Kasparov KK-theory, i.e.

$$\varinjlim_{d} KK(C_0(P_d(X)), B) \to \varinjlim_{Y \subset \mathcal{E}G(X)} KK(C_0(Y), B),$$

were the inductive limite on the right is taken among the proper $G(X)$-compact subsets Y of the universal classifying space for proper actions of $G(X)$.

Recall from [11] that we can take for $\mathcal{E}G(X)$ the space $\mathfrak{M}$ of positive measures μ on $G(X)$ satisfying :

- $\frac{1}{2} < \mu(G(X)) \leq 1$,
- $s^*\mu$ is a Dirac measure, i.e. its support consists of arrows of $G(X)$ that all source from the same base point of βX.

If $\mathfrak{M}_d$ denotes the space of measures μ of $\mathfrak{M}$ such that :

— μ is a probability measure

— for all γ and γ' in the support of μ, $\gamma'\gamma^{-1}$ is d-controlled, i.e. $d(r(\gamma), r(\gamma')) \leq d$,

then $\mathfrak{M} = \varinjlim \mathfrak{M}_d$.

The Rips complex of X, denoted $P_d(X)$, is the topological space of the complexes of diameter less than d, identified with probability measures on X with support of diameter less than d, with the weak topology coming from $C_c(C)$. We will write $[y,t]$ for a point of a simplex defined by barycentric coordinates of k points $y_1, ..., y_k$, ie $\sum t_j \delta_{y_j}$. To such a point $[y,t]$ and an element of the Stone-Cech compactification $w \in \beta X$, we can associate a measure of $\mathfrak{M}_d$ in the following way. As $G(X)$ is a principal and transitive groupoid, there exists only one arrow γ_j such that $s(\gamma_j) = x$ and $r(\gamma_j) = y_j$. To $z = ([y,t], w) = (z_w, w)$, we associate

$$\phi_d(z) = \sum_{j=1,k} t_j \delta_{\gamma_j} \in \mathfrak{M}_d.$$

Proposition 2. The map

$$\phi_d : P_d(X) \times \beta X \to \mathfrak{M}_d$$

is an homeomorphism.

Preuve 3. It is clearly bijective. The bicontinuity comes from the identity :

$$\langle z_w, f \rangle = \langle \phi_d(z), f \circ r \rangle$$

for all $z = (z_w, w) \in P_d(X) \times \beta X$, and $f \in C_c(X)$. □

This homeomorphism ϕ_d gives an $*$-isomorphism at the level of C^*-algebras

$$\Psi_d : C_0(\mathfrak{M}_d) \to C_0(P_d(X) \times \beta X).$$

Let $(\mathcal{E}, \pi, F) \in \mathbb{E}(C_0(P_d(X)), B)$ be an elliptic operator. **A FINIR**

Let $X_0 \subset X_1 \subset ... \subset X_j \subset ...$ the n-skeleton decomposition associated to the simplicial structure of the Rips complex $P_d(X)$, and similarly $\tilde{X}_0 \subset \tilde{X}_1 \subset ... \subset \tilde{X}_j \subset ...$ for $\mathfrak{M}_d$, and

$$Z_j = C_0(X_j) \quad \text{and} \quad \tilde{Z}_j = C_0(\tilde{X}_j).$$

$$Z_{j-1}^j = C_0(X_j - X_{j-1}) \quad \text{and} \quad \tilde{Z}_{j-1}^j = C_0(\tilde{X}_j - \tilde{X}_{j-1}).$$

We will show the isomorphism by a Mayer-Vietoris type argument. By applying the KK-theory to the exact sequence :

$$0 \longrightarrow C_0(X_j - X_{j-1}) \longrightarrow C_0(X_j) \longrightarrow C_0(X_{j-1}) \longrightarrow 0$$

we have a commutative diagramm with exact lines :

$$\begin{tikzcd}
KK_*(Z^j_{j-1},B) \arrow[r, "\delta"] \arrow[d, "\theta^j_{j-1}"] & KK_*(Z_{j-1},B) \arrow[r] \arrow[d, "\theta_{j-1}"] & KK_*(Z_j,B) \arrow[r] \arrow[d, "\theta_j"] & KK_*(Z^j_{j-1},B) \arrow[r, "\delta"] \arrow[d, "\theta^j_{j-1}"] & KK_*(Z_{j-1},B) \arrow[d, "\theta_{j-1}"] \\
KK_*(\tilde{Z}^j_{j-1},B) \arrow[r, "\delta"] & KK_*(\tilde{Z}_{j-1},B) \arrow[r] & KK_*(\tilde{Z}_j,B) \arrow[r] & KK_*(\tilde{Z}^j_{j-1},B) \arrow[r, "\delta"] & KK_*(\tilde{Z}_{j-1},B)
\end{tikzcd}$$

The five lemma assures that if θ_{j-1} and θ^j_{j-1} are isomorphisms, then so is θ_j. Moreover, $X_j - X_{j-1}$ is equivariantly homeomorphic to $\mathring{\sigma}_j \times \Sigma_j$, where $\mathring{\sigma}_j$ denotes the interior of the standard simplex, and Σ_j is the set of centers of j-simplices of X_j. Bott periodicty assures then that, if θ_{j-1} is an isomorphism, then so is θ^j_{j-1}. By induction, proving that θ_0 is an isomorphism concludes the proof.

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