STABILITY OF CONTROL SYSTEMS

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STABILITY

- The stability is a measure of the system’s sensitivity to changes in its parameters.
- Only stable systems are useful and desirable.
- The location of poles and zeros can be used as basis for the stability of the system.

TRANSFER FUNCTION T(s)

- THE TRANSFER FUNCTION IS THE RATIO OF THE LAPLACE TRANSFORM OF OUTPUT TO THE LAPLACE TRANSFORM OF THE INPUT.

$$T(s) = \frac{P(s)}{Q(s)}$$

ZEROS AND POLES

ZEROS

- Zeros of the function are the roots of the polynomial P(s) in which T(s) will be zero

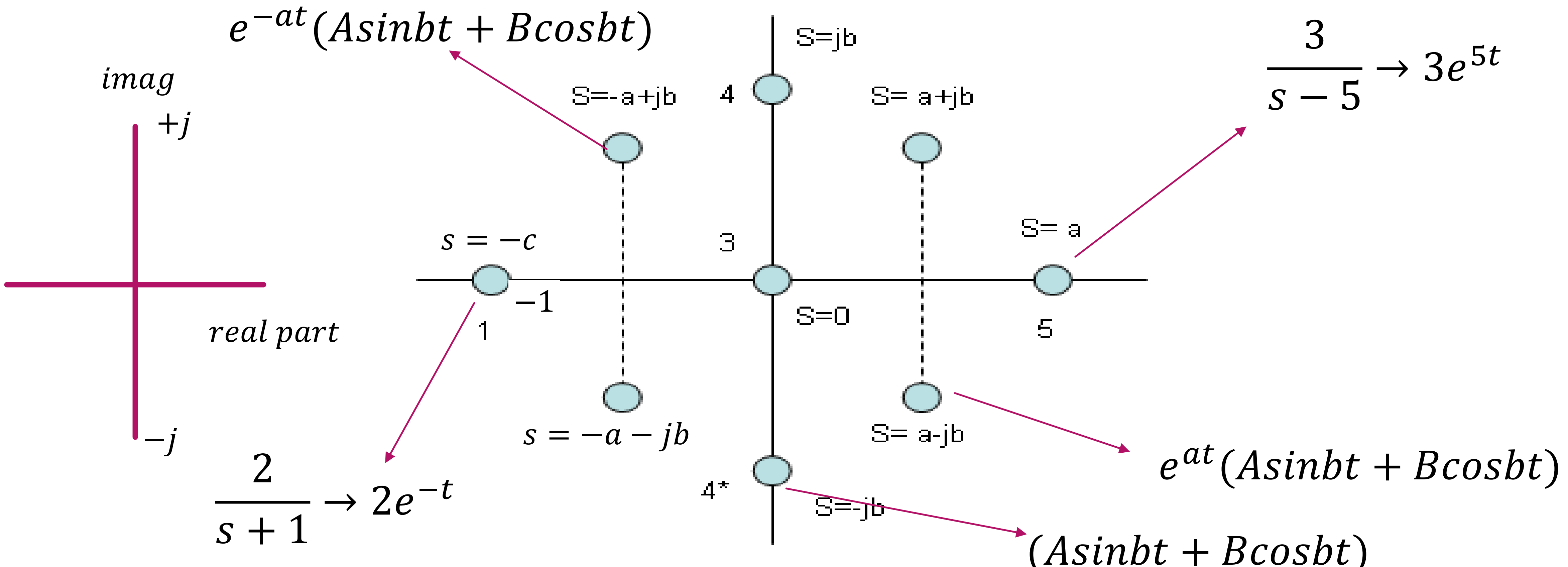
POLES

- Poles are the roots of Q(s) in which T(s) will become infinite.

GENERALIZATIONS

- When all the roots of Q(s) are found in the left hand side of the s-plane, the system response will decrease to zero as time approaches infinity.
- If one or more roots of Q(s) are on the imaginary axis of the s-plane, the response will be undamped sinusoidal function.
- If one or more roots are located on the right hand side of the s-plane, the response will be increasing exponentially with time.

POSSIBLE LOCATION OF POLES



Location of Poles in s-Plane

Relationship of the Response to the Location of the Poles

Position of Poles	Form of Response	Characteristic
1	Ae^{-at}	Damped exponential
2-2*	$Ae^{-at}\sin(bt + c)$	Exponentially damped sinusoidal signal
3	A	Constant
4-4*	$A\sin(bt + c)$	Pure sinusoidal signal
5	Ae^{at}	Increasing exponential
6-6*	$Ae^{at}\sin(bt + c)$	Exponentially increasing sinusoid

COEFFICIENT TESTS FOR STABILITY

- FOR FIRST AND SECOND ORDER SYSTEMS, STABILITY IS DETERMINED BY INSPECTION OF THE CHARACTERISTIC POLYNOMIAL Q(s)

- A FIRST OR SECOND ORDER POLYNOMIAL HAS ALL ROOTS IN THE LEFT HALF OF THE S-PLANE (LHP) IF AND ONLY IF ALL COEFFICIENTS HAVE THE SAME ALGEBRAIC SIGN

- COEEFICIENTS FOR HIGHER ORDER SYSTEMS MAY OR MAY NOT YIELD INFORMATION REGARDING THE STABILITY

SUMMARY ON COEFFICIENT TEST

	PROPERTIES OF THE POLYNOMIAL COEFFICIENTS	CONCLUSION ABOUT ROOTS FROM THE COEFFICIENT TEST
$as^4 + bs^3 + cs^2 + ds + e$	SAME ALGEBRAIC SIGN, NON-ZERO	NO DIRECT INFORMATION ABOUT THE ROOTS, FURTHER TESTING REQUIRED
$as^4 + bs^3 - ds + e$	DIFFERING ALGEBRAIC SIGNS (SOME MAY BE ZERO)	AT LEAST ONE RHP ROOT, POSSIBILITY OF IMAGINARY AXIS ROOTS
$as^4 + cs^2 + ds + e$	SAME SIGN AND ONE OR MORE COEFFICIENTS ZERO	IMAGINARY AXIS OR RHP OR BOTH

Illustrations of Coefficient Test:

$-0.1667 \pm j1.818$

$3s^2 + s + 10$	STABLE
$-2, 1.6667$	
$3s^2 + s - 10$	UNSTABLE
$7s^6 + 5s^4 - 3s^3 - 2s^2 + s + 10$	ONE OR MORE RHP ROOTS
$8s^5 + 6s^4 + 3s^3 + 2s^2 + 7s + 10$	NO DEFINITE INFORMATION ABOUT POSSIBLE ROOT LOCATIONS

ROUTH-HURWITZ TESTING

- A NUMERICAL PROCEDURE USED TO DETERMINE THE NUMBER OF ROOTS LOCATED ON THE RHP AND ON THE IMAGINARY AXIS

Routh's Stability Criterion

$$\begin{matrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ 3 & 5 & -2 \end{matrix}$$

$$T(s) = \frac{P(s)}{b_v s^v + b_{v-1} s^{v-1} + \ldots\ldots\ldots + b_1 s + b_0}$$

$$Q(s) = b_v s^v + b_{v-1} s^{v-1} + b_{v-2} s^{v-2} + \ldots + b_1 s + b_0 = 0$$

(CHARACTERISTIC EQUATION)

$b's - constant\ coefficients$

CREATING THE ROUTHIAN ARRAY

$$
\begin{array}{l|llll}
s^{v} & b_{v} & b_{v-2} & b_{v-4} \ldots\ldots \\
s^{v-1} & b_{v-1} & b_{v-3} & b_{v-5} \ldots\ldots \\
s^{v-2} & c_1 & c_2 & c_3 \ldots\ldots \\
 & & & \\
 & & & \\
s^{1} & & & \\
 & & &
\end{array}
$$

$$
c_1 = \frac{-\begin{vmatrix} b_v & b_{v-2} \\ b_{v-1} & b_{v-3} \end{vmatrix}}{b_{v-1}}
$$

$$
c_2 = \frac{-\begin{vmatrix} b_v & b_{v-4} \\ b_{v-1} & b_{v-5} \end{vmatrix}}{b_{v-1}}
$$

THE NUMBER OF RHP

- *THE NUMBER OF RHP ROOTS OF THE POLYNOMIAL IS EQUAL TO THE NUMBER OF ALGEBRAIC SIGN CHANGES IN THE LEFT COLUMN OF NUMBERS GOING FROM TOP TO BOTTOM*

$$Q(s) = s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10$$

Laying out the coefficients:

1	3	16
1	9	10

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 16 \\ s^4 & 1 & 9 & 10 \\ s^3 & -6 & 6 & 0 \\ s^2 & 10 & 10 & 0 \\ s^1 & 12 & & \\ s^0 & 10 & & \end{array}$$

2 SIGN CHANGES

2 SIGN CHANGES = 2 RHP

$$-\frac{\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}}{1} = -6 \qquad -\frac{[1(9)-1(3)]}{1}$$

$$-\frac{\begin{vmatrix} 1 & 16 \\ 1 & 10 \end{vmatrix}}{1} = 6 \qquad -\frac{[1(10)-1(16)]}{1}$$

$$-\frac{\begin{vmatrix} 1 & 0 \\ -6 & 0 \end{vmatrix}}{-6} = 0$$

$$-\frac{\begin{vmatrix} 1 & 9 \\ -6 & 6 \end{vmatrix}}{-6} = 10 \qquad -\frac{\begin{vmatrix} 1 & 10 \\ -6 & 0 \end{vmatrix}}{-6} = 10$$

$$-\frac{\begin{vmatrix} -6 & 6 \\ 10 & 10 \end{vmatrix}}{10} = 12 \qquad -\frac{\begin{vmatrix} -6 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$$

$$-\frac{\begin{vmatrix} 10 & 10 \\ 12 & 0 \end{vmatrix}}{12} = 10$$

$$Q(s) = s^4 + 2s^3 + 3s^2 + 4s + 1$$

s^4	1	3	1
s^3	2	4	
s^2	1	1	
s^1	2		
s^0	1		

$$\frac{-\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}{2} = -\frac{[1(4) - 2(3)]}{2} = 1$$

$$\frac{-\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{2} = -\frac{[1(0) - 2(1)]}{2} = 1$$

$$\frac{-\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}}{1} = -\frac{[2(1) - 1(4)]}{1} = 2$$

$$\frac{-\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{2} = -\frac{[1(0) - 2(1)]}{2} = 1$$

NO SIGN CHANGES MEANS NO RHP

$$Q(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

$$\begin{array}{c|ccc}
s^4 & 1 & 2 & 3 \\
s^3 & 1 & 2 & \\
s^2 & 0 & 3 & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

$$-\frac{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}{1} = 0 \qquad -\frac{\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}}{1} = 3$$

$$-\frac{\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}}{0}$$

WILL RESULT TO DIVISION BY ZERO

When the left column becomes zero, we will need to assume any small number that would replace the zero. This is to avoid any division by zero.

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 3 \\ s^3 & 1 & 2 & \\ s^2 & \varepsilon & 3 & \\ s^1 & \dfrac{2\varepsilon - 3}{\varepsilon} & & \\ s^0 & 3 & & \end{array}$$

$WHEN: \varepsilon \to 0:$

$$\frac{2\varepsilon - 3}{\varepsilon} \to -\infty$$

0

$e.g. \varepsilon = 0.001$

$$\frac{-\begin{vmatrix} 1 & 2 \\ \varepsilon & 3 \end{vmatrix}}{\varepsilon} = \frac{-(3 - 2\varepsilon)}{\varepsilon} = \frac{2\varepsilon - 3}{\varepsilon}$$

$$\frac{-\begin{vmatrix} \varepsilon & 3 \\ \frac{2\varepsilon - 3}{\varepsilon} & 0 \end{vmatrix}}{\frac{2\varepsilon - 3}{\varepsilon}} = \frac{-(0 - 3\cancel{\frac{2\varepsilon - 3}{\varepsilon}})}{\cancel{\frac{2\varepsilon - 3}{\varepsilon}}}$$

$$\begin{array}{c|c} s^4 & 1 \\ s^3 & 1 \\ s^2 & 0+ \\ s^1 & -\infty \\ s^0 & 3 \end{array}$$

2 SIGN CHANGES

$2\ RHP$

Another approach that can be made when the left column becomes zero is to multiply the characteristic equation by a known root.

Ex. Q(s) is multiplied by (s + 1).

$$(s^4+s^3+2s^2+2s+3)(s+1) = s^5+2s^4+3s^3+4s^2+5s+3$$

s^5	1	3	5
s^4	2	4	3
s^3	1	7/2	
s^2	-3	3	
s^1	4.5		
s^0	3		

2 SIGN CHANGES

$2\ RHP$

$$\frac{-\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}{2} = 1$$

$$\frac{-\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}}{2} = 7/2$$

$$\frac{-\begin{vmatrix} 2 & 4 \\ 1 & 7/2 \end{vmatrix}}{1} = -(7-4)/1=-3$$

$$\frac{-\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}}{1} = -(0-3)/1= 3$$

$$\frac{-\begin{vmatrix} 1 & 7/2 \\ -3 & 3 \end{vmatrix}}{-3} = \frac{-(3-\left(-\frac{21}{2}\right))}{-3} = 4.5$$

$$\frac{-\begin{vmatrix} -3 & 3 \\ 4.5 & 0 \end{vmatrix}}{4.5} = 3$$

$= 3$

Theorem 1: Division of a Row. The coefficients of any row maybe multiplied or divided by a positive number without changing the signs of the first column. The labor of evaluating the coefficients in Routh's array can be reduced by multiplying or dividing any row by a constant. This may result in reducing the size of the coefficients and therefore simplifying the evaluation of the remaining coefficients.

$$Q(s) = s^6 + 3s^5 + 2s^4 + 9s^3 + 5s^2 + 12s + 20$$

$$\frac{-\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{1} = -1 \qquad \frac{-\begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix}}{1} = 1 \qquad \frac{-\begin{vmatrix} 1 & 20 \\ 1 & 0 \end{vmatrix}}{1} = 20$$

The Routhian array is

$$\begin{array}{c|cccc}
s^6 & 1 & 2 & 5 & 20 \\
s^5 & \cancel{3} & \cancel{9} & \cancel{12} & \\
 & 1 & 3 & 4 & \text{(after dividing by 3)} \\
s^4 & -1 & 1 & 20 & \\
s^3 & 4 & 24 & & \\
 & 1 & 6 & & \text{(after dividing by 4)} \\
s^2 & 7 & 20 & & \\
s^1 & \frac{22}{7} & & & \text{(after multiplying by 7)} \\
s^0 & 20 & 2RHP & &
\end{array}$$

$$\frac{-\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}}{-1} = 4 \qquad \frac{-\begin{vmatrix} 1 & 4 \\ -1 & 20 \end{vmatrix}}{-1} = 24$$

$$\frac{-\begin{vmatrix} -1 & 1 \\ 1 & 6 \end{vmatrix}}{1} = 7 \qquad \frac{-\begin{vmatrix} -1 & 20 \\ 1 & 0 \end{vmatrix}}{1} = 20$$

$$\frac{-\begin{vmatrix} 1 & 6 \\ 7 & 20 \end{vmatrix}}{7} = \frac{22}{7} \qquad \frac{-\begin{vmatrix} 7 & 20 \\ \frac{22}{7} & 0 \end{vmatrix}}{\frac{22}{7}} = 20$$

Theorem 2: Zero Coefficient in the First Column. When the first term in a row is zero but not all the other terms are zero, the following methods can be used:

1. Substitute s=1/x in the original equation; then solve for the roots of x with positive real parts. The number of roots x with positive real parts will be the same as the number of s roots with positive real parts.
2. Multiply the original polynomial by the factor (s+1), which introduces an additional negative root. Then form the Routhian array for the new polynomial.

$$Q(s) = s^4 + s^3 + 2s^2 + 2s + 5$$

The Routhian array is

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 5 \\ s^3 & 1 & 2 & \\ s^2 & 0 & 5 & \end{array}$$

$$\frac{-\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}{1} = 0$$

$$\frac{-\begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}}{1} = 5$$

METHOD 1. Letting $s = 1/x$ and rearranging the polynomial gives

$$= \frac{1}{x^4} + \frac{1}{x^3} + \frac{2}{x^2} + \frac{2}{x} + 5$$

$$Q(x) = 5x^4 + 2x^3 + 2x^2 + x + 1$$

The new Routhian array is

$$\begin{array}{c|ccc} x^4 & 5 & 2 & 1 \\ x^3 & 2 & 1 & \\ x^2 & -1 & 2 & \\ x^1 & 5 & & \\ x^0 & 2 & & \end{array}$$

$2RHP$

METHOD 2.

$$Q_1(s) = Q(s)(s+1) = s^5 + 2s^4 + 3s^3 + 4s^2 + 7s + 5$$

$$\begin{array}{c|ccc}
s^5 & 1 & 3 & 7 \\
s^4 & 2 & 4 & 5 \\
s^3 & 2 & 9 & \\
s^2 & -10 & 10 & \\
s^1 & 11 & & \\
s^0 & 10 & &
\end{array}$$

$2RHP$

Theorem 3: A Zero Row. When all the coefficients of one row are zero, the procedure is as follows:

1. The auxiliary equation can be formed from the preceding row.
2. The Routhian array can be completed by replacing the all-zero row with the coefficients obtained by differentiating the auxiliary equation.
3. The roots of the auxiliary equation are also roots of the original equation. These roots occur in pairs and are the negative of each other. Therefore, these roots may be imaginary (complex conjugates) or real (one positive and one negative), may lie in quadruplets (two pairs of complex- conjugate roots), etc.

$$Q(s) = s^4 + 2s^3 + 11s^2 + 18s + 18 = 0$$

The Routhian array is

$$\begin{array}{c|ccc l}
s^4 & 1 & 11 & 18 & \\
s^3 & \cancel{2} & \cancel{18} & & \\
 & 1 & 9 & & \text{(after dividing by 2)} \\
s^2 & \cancel{2} & \cancel{18} & & \\
 & 1 & 9 & & \\
s^1 & \cancel{0}\ 2 & & & \text{(after dividing by 2)} \\
s^0 & 9 & & &
\end{array}$$

No RHP

The presence of a zero row (the s1 row) indicates that there are roots that are the negatives of each other. The next step is to form the auxiliary equation from the preceding row, which is the s2 row. The highest power of s is s2, and only even powers of s appear.

$$\frac{-\begin{vmatrix} 1 & 9 \\ 2 & 0 \end{vmatrix}}{2} = 9$$

Therefore, the auxililary equation is

$$s^2 + 9 = 0$$

The roots of this equation are

$$s = \pm j3 \qquad A\cos 3t + B\sin 3t$$

These are also roots of the original equation. The presence of imaginary roots indicates that the output includes a sinusoidally oscillating component.

To complete the Routhian array, the auxiliary equation is differentiated and is

$$2s + 0 = 0$$

To complete the Routhian array, the auxiliary equation is differentiated and is

$$2s + 0 = 0$$

The coefficients of this equation are inserted in the s^1 row, and the array is then completed:

$$\begin{array}{c|c} s^1 & 2 \\ s^0 & 9 \end{array}$$

Since there are no changes of sign in the first column, there are no roots with positive real parts.

MEASURING THE SYSTEM PERFORMANCE

- The manipulation of the output signal (or the input signal) in the analysis of any given system is very important.
- if the input to a system is an irregular signal which cannot be represented by simple equation, then what we can do is to convert this into summation of the basic forms of known types of signals.
- if the system is analyze for each of these types of input signals, we can determine how will a given system respond to an irregular input
- From these standard forms, we also have a way of comparing the performance of various systems.