Given sequence S as $s_0 = 8, s_1 = 25, s_n = s_{n-2} + s_{n-1}, n \in Z^{\geq 0}$

I will now solve for the explicit formula (which would be useful in proofs later).

We know that $s_n = s_{n-1} + s_{n-2},\ s_n = \alpha {r_1}^n + \beta {r_2}^n$ where $r_1 = \left(\frac{1+\sqrt{5}}{2}\right)$ and $r_2 = \left(\frac{1-\sqrt{5}}{2}\right)$. Thus…

$$s_0 = 8 = \alpha {r_1}^8 + \beta {r_2}^8$$

$$\alpha = \frac{8 - \beta {r_2}^8}{{r_1}^8}$$

$$s_1 = 25 = \alpha {r_1}^{25} + \beta {r_2}^{25}$$

$$25 = \left(\frac{8 - \beta {r_2}^8}{{r_1}^8}\right) {r_1}^{25} + \beta {r_2}^{25}$$

$$25 = 8{r_1}^{17} - \beta {r_2}^8 {r_1}^{17} + \beta {r_2}^{25}$$

$$25 - 8{r_1}^{17} = \beta(-{r_2}^8 {r_1}^{17} + {r_2}^{25})$$

$$\beta = -\frac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^8 - {r_2}^{25}}$$

$$\alpha = \frac{8 + \left(\frac{25 - 8{r_1}^{17}}{{r_2}^8 {r_1}^{17} - {r_2}^{25}}\right){r_2}^8}{{r_1}^8}$$

$$\alpha = \frac{8 + \left(\frac{25 - 8{r_1}^{17}}{{r_1}^{17} - {r_2}^{17}}\right)}{{r_1}^8}$$

$$\alpha = \frac{\left(\frac{25 - 8{r_2}^{17}}{{r_1}^{17} - {r_2}^{17}}\right)}{{r_1}^8}$$

$$\alpha = \frac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^8 {r_2}^{17}}$$

Thus…

$$s_n = \frac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^8 {r_2}^{17}}({r_1}^n) - \frac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^8 - {r_2}^{25}}({r_2}^n)$$

PLEASE SEE NEXT PAGE ->

Prove that $\forall n \in Z^{\geq 0} \mid P(n) = S_n, \forall a \in Z^{\geq 1}, P(3a)\ is\ even\ \land P(b)\ is\ odd, b \neq 3a, \forall a \in Z$
(Colloquially, starting at S_2, the pattern onwards is odd-even-odd infinitely.)

Basis Step: prove $P(a), P(b), a = 1, b = 2$

1	$\forall n \in Z^{\geq 0} \mid P(n) = S_n$ $P(3)\ is\ even\ \land P(2)\ is\ odd$	To be proved
2	$P(3) = 2m, \exists m \in Z$	Definition of even
3	$P(3) = \frac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^8{r_2}^{17}}({r_1}^3) - \frac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^8 - {r_2}^{25}}({r_2}^3)$ … $P(3) = 58$	Algebra
4	$58 = 2(29)$	Algebra
5	$\therefore P(3)\ is\ even$	As shown
6	$P(2) = 2m + 1, \exists m \in Z$ $s.t. b \neq 3a, \forall a \in Z$	Definition of odd, noting that 2 is not a multiple of 3a
7	$P(2) = \frac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^8{r_2}^{17}}({r_1}^2) - \frac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^8 - {r_2}^{25}}({r_2}^2)$ … $P(2) = 33$	Algebra
8	$33 = 2(16) + 1$	Algebra
9	$\therefore P(2)\ is\ odd$	As shown
10	$\therefore P(3)\ is\ even\ \land P(2)\ is\ odd$	As shown
11	$\therefore P(a), P(b), a = 1, b = 2$	As shown

PLEASE SEE NEXT PAGE ->

Inductive Step: prove

1. $\forall k \in Z^{\geq 1}, P(3k) \rightarrow P(3(k+1))$
   a. $P(3k)$ is inductive hypothesis
   b. $P(3(k+1))$ is general successor with property P
2. $\forall k \in Z^{\geq 1}, P(b) \rightarrow P(b+1),\ b \neq 3k$
   a. $P(b)$ is inductive hypothesis
   b. $P(b+1)$ is general successor with property P

12	$\forall k \in Z^{\geq 1}, P(3k) \rightarrow P(3(k+1))$	To be proved
13	$\exists P\big(3(k+1)\big) = P(3k+3)$	Instantiate by closure
14	$P(3k+3) = P(3k+1) + P(3k+2)$	Substitute into inductive hypothesis
15	$P(3k+1) + P(3k+2) = \dfrac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^{8}{r_2}^{17}}\left({r_1}^{3k+3}\right) - \dfrac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^{8} - {r_2}^{25}}\left({r_2}^{3k+3}\right)$	Tedious algebra
16	$P\big(3(k+1)\big) = \dfrac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^{8}{r_2}^{17}}\left({r_1}^{3k+3}\right) - \dfrac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^{8} - {r_2}^{25}}\left({r_2}^{3k+3}\right)$	Definition of $P\big(3(k+1)\big)$
17	$\therefore \forall k \in Z^{\geq 1}, P(3k) \rightarrow P(3(k+1))$	By direct proof
18	$\forall k \in Z^{\geq 1}, P(b) \rightarrow P(b+1), b \neq 3k$	To be proved
19	$\exists\ P(b+1)$	Instantiate by closure
20	$P(b+1) = P(b) + P(b-2)$	Substitute into inductive hypothesis, noting that $b - 1 = 3k$ which is not allowed
21	$P(b) + P(b-2) = \dfrac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^{8}{r_2}^{17}}\left({r_1}^{b+1}\right) - \dfrac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^{8} - {r_2}^{25}}\left({r_2}^{b+1}\right)$	Tedious algebra
22	$P(b+1) = \dfrac{25 - 8{r_2}^{17}}{{r_1}^{25} - {r_1}^{8}{r_2}^{17}}\left({r_1}^{b+1}\right) - \dfrac{25 - 8{r_1}^{17}}{{r_1}^{17}{r_2}^{8} - {r_2}^{25}}\left({r_2}^{b+1}\right)$	Definition of $P(b+1)$
23	$\therefore \forall k \in Z^{\geq 1}, P(b) \rightarrow P(b+1), b \neq 3k$	By direct proof
24	$P(a), P(b), a = 1, b = 2\ \wedge$ $\forall k \in Z^{\geq 1}, P(3k) \rightarrow P\big(3(k+1)\big) \wedge$ $\forall k \in Z^{\geq 1}, P(b) \rightarrow P(b+1), b \neq 3k$	As shown 11, 17, 23
25	$\forall n\ \in Z^{\geq 0} \mid P(n) =\ S_n$ $\forall a \in Z^{\geq 1}, P(3a)\ is\ even\ \wedge P(b)\ is\ odd, b \neq 3a, \forall a \in Z$	By mathematical induction