2019 ADA miniHW 6

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(1) Proof by contradiction: Suppose there exists a root such that the height of the tree $h = \lceil \frac{x}{2} \rceil - n$, for some $n \in \mathbb{N}$ and $n \geq 1$, then the length of longest simple path in the tree can be at most $2 \cdot (\lceil \frac{x}{2} \rceil - n)$. As $2 \cdot \lceil \frac{x}{2} \rceil \leq x + 1$, we have $2 \cdot (\lceil \frac{x}{2} \rceil - n) \leq x + 1 - 2n < x$, contradiction. So any arbitrary root must make the height of the tree $\geq \lceil \frac{x}{2} \rceil$.

(2)

- $path(a, b)$ denotes the path from node a to node b
- $dis(a, b)$ denotes the length of $path(a, b)$

- v is on the longest path: Let $path(u, w)$ be the longest paths that v belongs to, then the height of the tree is $\max(dis(v, u), dis(v, w))$. Suppose the opposite that there exits u' such that the height of the tree is $dis(v, u')$, that is, $dis(v, u') > dis(v, u)$ and $dis(v, u') > dis(v, w)$. In this case, the longest path would have been $path(v', u)$ or $path(v', w)$, contradicting the fact that $path(u, w)$ is the longest path.
- v is not on the longest path: For **any** longest path $path(u, w)$, let v' be the node such that $v' \in path(u, w)$ and v' is the node closest to the root. One can see that the height of the tree is $dis(v, v') + \max(dis(v', u), dis(v', w))$. However, if we instead let v' be the root, then according to the first case, the height is $\max(dis(v', u), dis(v', w))$, which is smaller than the previous number. So we conclude that v must be on the longest path, that is, $v \in S$

(3) Continuing the discussion in (2), the height of the tree $h = \max(dis(v, u), dis(v, w))$. One can easily see that h is minimized when v is the middle vertex on $path(u, w)$, and $h = \max(\lfloor \frac{x}{2} \rfloor, \lceil \frac{x}{2} \rceil) = \lceil \frac{x}{2} \rceil$. This result holds for **any** longest path, so the statement is true.

Reference

none