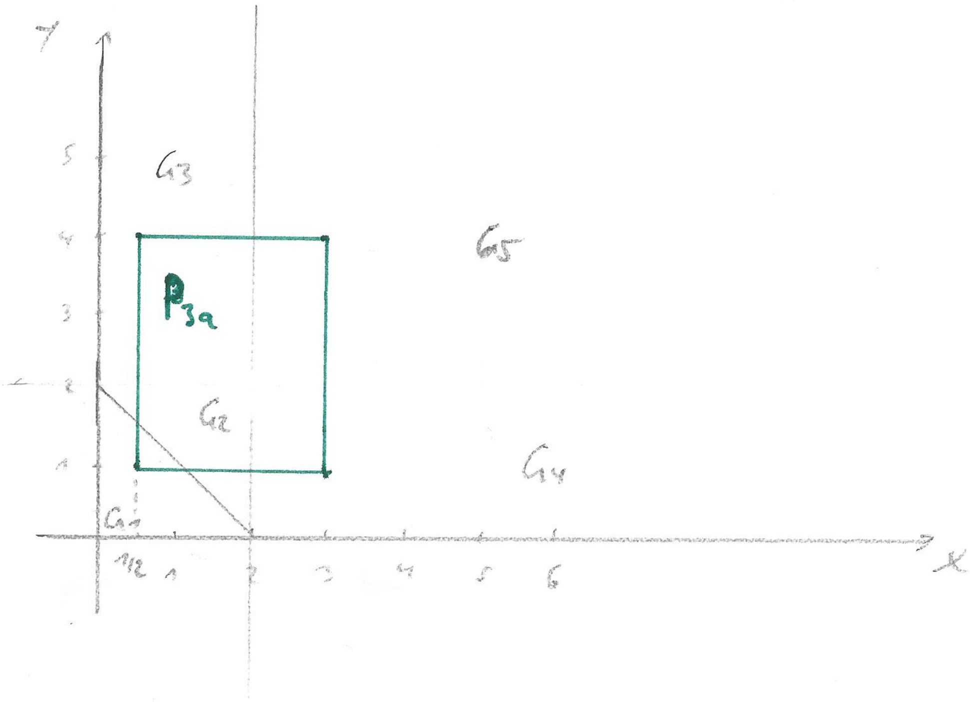
ZA 13.1

(a) Bestimmen Sie die Wahrscheinlichkeit

$p_a = P\left(\tfrac{1}{2} < X \leq 3,\ 1 < Y \leq 4\right)$



$p_a = P(X \leq 3, Y \leq 4) - P(X \leq 3, Y \leq 1) - P(X \leq \tfrac{1}{2}, Y \leq 4)$

$+ P(X \leq \tfrac{1}{2}, Y \leq 1)$

$= F(3,4) - F(3,1) - F(\tfrac{1}{2},4) + F(\tfrac{1}{2},1)$

$\downarrow$ G_5 $\quad\downarrow$ G_4 $\quad\downarrow$ G_3 $\quad\downarrow$ G_1

$= 1 - \frac{1}{16}\cdot 1\,(12 - 1^2) - \frac{1}{16}\cdot\frac{1}{2}\left(12 - \left(\frac{1}{2}\right)^2\right) + \frac{3}{16}\,\frac{1}{2}\cdot 1\,\left(\tfrac{1}{2} + 1\right)$

$= 1 - \frac{11}{16} - \frac{1}{32}\cdot\frac{47}{4} + \frac{3}{32}\cdot\frac{3}{2}$

$= \frac{128}{128} - \frac{88}{128} - \frac{47}{128} + \frac{18}{128} = \frac{11}{128}$

$p_a = \frac{11}{128}$

(a)

ZA 13.1

(b) Bestimmen Sie die Randdichtefunktion $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{d F_X(x)}{dx}$$

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} \frac{1}{16}x(12-x^2), & 0<x<2 \\ 1, & x\geq 2 \\ 0, & \text{sonst} \end{cases}$$ (Randverteilung) "oberen Rand"

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{16}(12-3x^2), & 0<x<2 \\ 0, & \text{sonst} \end{cases}$$

Zusatzaufgabe 13.1

Verbundverteilungsfunktion:

$$F_{X,Y}(x,y) = \begin{cases} \frac{3}{16}\cdot x\cdot y\cdot(x+y) & ,(x,y)\in G_1 \\ \frac{1}{16}(12x+12y-x^3-y^3-16) & ,(x,y)\in G_2 \\ \frac{1}{16}\cdot x\cdot(12-x^2) & ,(x,y)\in G_3 \\ \frac{1}{16}\cdot y\cdot(12-y^2) & ,(x,y)\in G_4 \\ 1 & ,(x,y)\in G_5 \\ 0 & ,\text{sonst} \end{cases}$$

$y \leq 2-x$

$G_1 = \{(x,y)\in \mathbb{R}\times\mathbb{R} \mid 0\leq x, 0\leq y, x+y \leq 2\}$

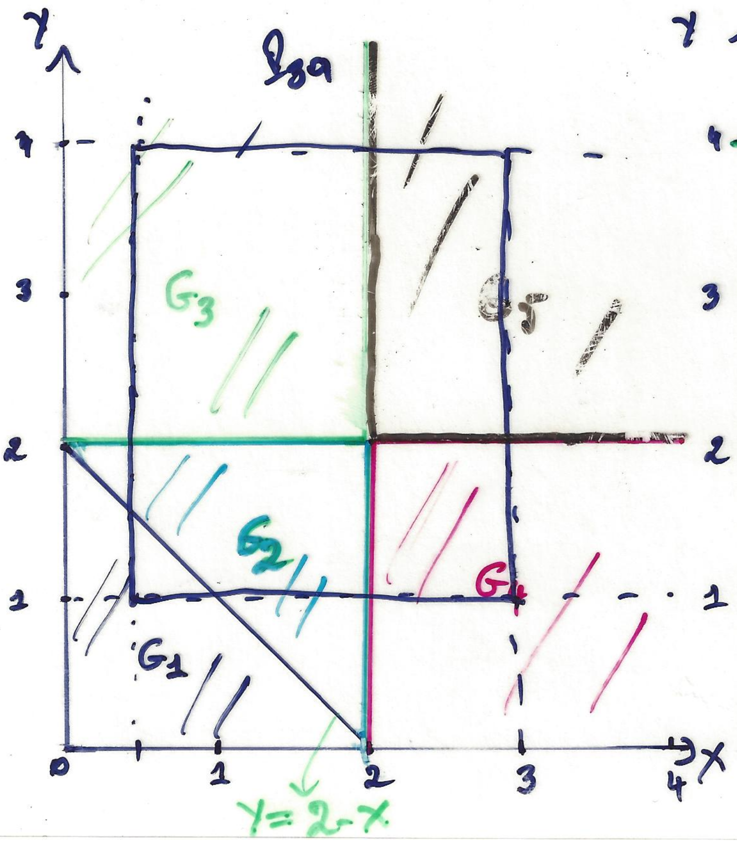
$y > 2-x$

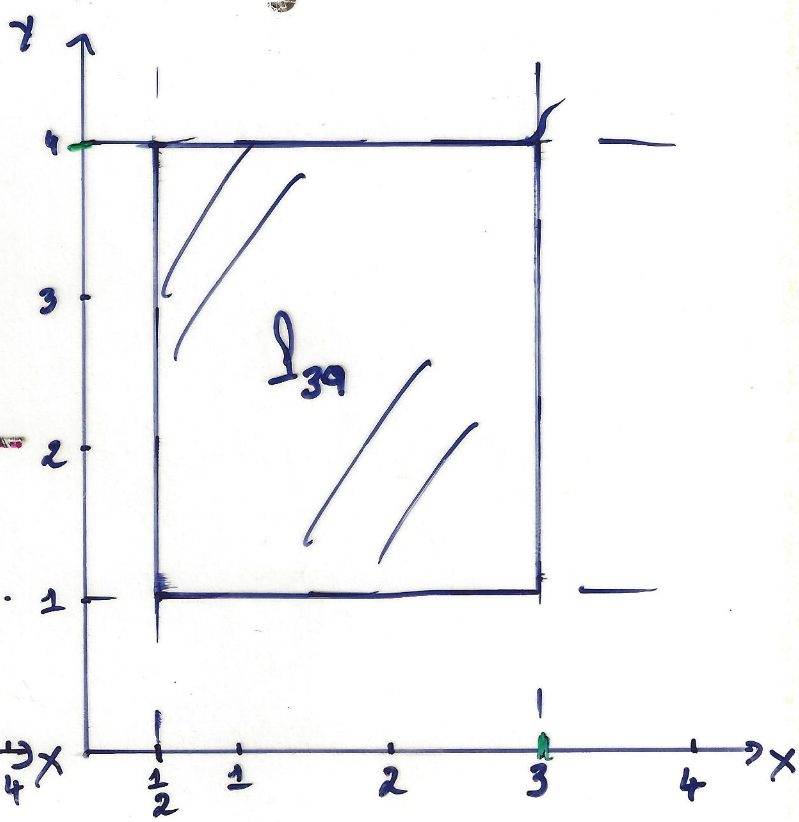
$G_2 = \{(x,y)\in \mathbb{R}\times\mathbb{R} \mid x\leq 2, y\leq 2, x+y > 2\}$

$G_3 = \{(x,y)\in \mathbb{R}\times\mathbb{R} \mid 0\leq x\leq 2, y>2\}$

$G_4 = \{(x,y)\in \mathbb{R}\times\mathbb{R} \mid x>2, 0\leq y\leq 2\}$

$G_5 = \{(x,y)\in \mathbb{R}\times\mathbb{R} \mid x>2, y>2\}$





a-) $P_{3a} = P\left(\frac{1}{2} < X \le 3,\ 1 < Y \le 4\right)$?

$\frac{1}{2} = x_1$, $3 = x_2$, $1 = y_1$, $4 = y_2$

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy$$

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2) = P(X \le x_2, Y \le y_2) - P(X \le x_1, Y \le y_2)$$
(13.13) S. 147
$$- P(X \le x_2, Y \le y_1) + P(X \le x_1, Y \le y_1)$$

$$= F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$

(13.14) S. 147

$$\Rightarrow P_{3a} = F_{XY}(3,4) - F_{XY}\left(\tfrac{1}{2},4\right) - F_{XY}(3,1) + F_{XY}\left(\tfrac{1}{2},1\right)$$

$F_{XY}(3,4)$: $x > 2$, $y > 2$ $\Downarrow$ F_5

$F_{XY}\left(\tfrac{1}{2},4\right)$: $0 \le x \le 2$, $y > 2$ $\Downarrow$ F_3

$F_{XY}(3,1)$: $0 \le y \le 2$, $x > 2$ $\Downarrow$ G_4

$F_{XY}\left(\tfrac{1}{2},1\right)$: $x, y \le 2$, $x \ge 0$, $y \ge 0$ $\Downarrow$ G_2

$$= 1 - \frac{1}{16}\cdot\frac{1}{2}\cdot\left(12 - \frac{1}{4}\right) - \frac{1}{16}\cdot 1\cdot(12 - 1) + \frac{3}{16}\cdot\frac{1}{2}\cdot 1\cdot\left(\frac{1}{2}+1\right)$$

$$= \frac{128}{128} - \frac{47}{128} - \frac{88}{128} + \frac{18}{128} = \frac{11}{128} = 8{,}59\,\%$$

$$\underline{\underline{P_{3a} = 8{,}59\,\%}}$$

b-) Randdichtefunktion $f_X(x) = \dfrac{dF_X(x)}{dx} = \displaystyle\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

(13.17) S. 149

$$F_X(x) = F_{XY}(x,\infty) = \begin{cases} 0 & ,\ x < 0 \\ \frac{1}{16}\cdot x\cdot(12 - x^2) & ,\ 0 \le x \le 2 \\ 1 & ,\ x > 2 \end{cases}$$

(13.15) S. 149 $\quad y = \infty$

$$F_X(x) = P(X \le x) = P(X \le x,\ Y \le \infty) = F_{XY}(x,\infty)$$
$$= \int_{-\infty}^{x} f(z)\,dz$$

$$f_X(x) = \begin{cases} 0 & , x < 0 \\ \frac{3}{4} - \frac{3}{16}x^2 & , 0 \le x \le 2 \\ 0 & , x > 2 \end{cases} = \begin{cases} \frac{3}{4} - \frac{3}{16}x^2 & , 0 \le x \le 2 \\ 0 & \text{sonst} \end{cases}$$

$$\boxed{f_X(x) = \begin{cases} \frac{3}{4} - \frac{3}{16} \cdot x^2 & , 0 \le x \le 2 \\ 0 & , \text{sonst} \end{cases}}$$

a) $P_a = P\left(\frac{1}{2} < X \le 3 \;,\; 1 < Y \le 4\right)$

$$P_a = P(X \le 3, Y \le 4) - P(X \le 3, Y \le 1) - P(X \le \tfrac{1}{2}, Y \le 4) + P(X \le \tfrac{1}{2}, Y \le 1)$$

$$= \underbrace{F(3,4)}_{\Downarrow\, G_5} - \underbrace{F(3,1)}_{\Downarrow\, G_4} - \underbrace{F(\tfrac{1}{2},4)}_{\Downarrow\, G_3} + \underbrace{F(\tfrac{1}{2},1)}_{\Downarrow\, G_1}$$

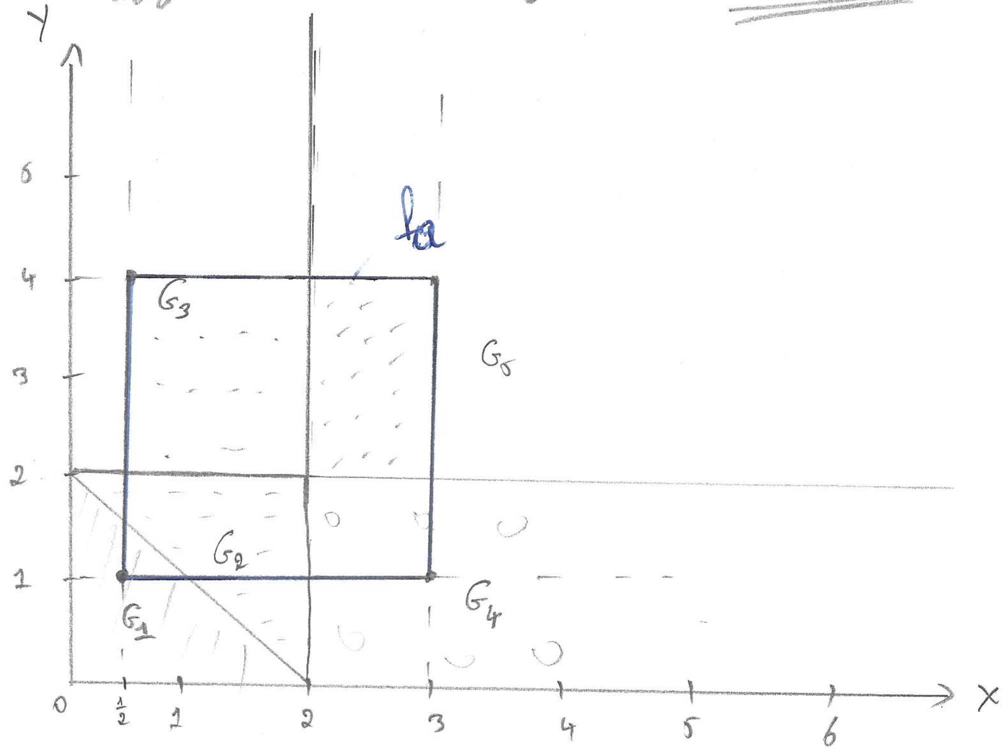
$$= 1 - \frac{1}{16}\cdot 1\cdot(12 - 1^2) - \frac{1}{16}\cdot\frac{1}{2}\left(12 - \left(\frac{1}{2}\right)^2\right) + \frac{3}{16}\cdot\frac{1}{2}\cdot 1\cdot\left(\frac{1}{2}+1\right)$$

$$= 1 - \frac{11}{16} - \frac{1}{32}\left(\frac{48-1}{4}\right) + \frac{3}{32}\left(\frac{3}{2}\right)$$

$$= 1 - \frac{11}{16} - \frac{47}{32\cdot 4} + \frac{9}{32\cdot 2} = \frac{128}{128} - \frac{11\cdot 8}{128} - \frac{47}{128} + \frac{18}{128}$$

$$= \frac{128 - 88 - 47 + 18}{128} = \frac{11}{128}$$

$\underline{\underline{P_a = 11/128}}$



②

b) $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy$ $\qquad f_X(x) = \dfrac{dF_X(x)}{dx}$

$F_X(x) = F_{X,Y}(x, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{XY}(u,v)\, dv\, du$

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} \frac{1}{16} x(12 - x^2) & , \quad 0 \leq x \leq 2 \\ 1 & , \quad x > 2 \\ 0 & , \text{sonst} \end{cases}$$

$$\Rightarrow \quad f_X(x) = \begin{cases} \frac{1}{16}(12 - 3x^2) & , \quad 0 \leq x \leq 2 \\ 0 & , \quad \text{sonst} \end{cases}$$