RESEARCH STATEMENT

CHRISTOPHE DETHIER

1. Broad Descriptions

At the beginning of the twentieth century, Thue proved in [10] that the equation

$$F(x, y) = m,$$

where m is an nonzero integer and F is an irreducible binary form of degree at least three, has finitely many solutions in pairs of integers (x, y). Such equations are called Thue equations. Results over the subsequent century fall into two categories. On the one hand, one might give a bound on the number of solutions. On the other hand, one might give a finite range of possibilities for x and y, thus reducing the problem to checking a finite list of possibilities for solutions. The first type, depending on the method used, can lead to a better understanding of the solutions, while the second type can lead to better computational methods for solving such equations.

In my research I've considered a simpler problem by restricting the shape of F. I have only considered diagonalizable forms. A diagonalizable form with integer coefficients which has a presentation as

$$F(x, y) = (\alpha x + \beta y)^n - (\gamma x + \delta y)^n$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfy $\alpha\delta - \beta\gamma \neq 0$. The reasoning behind this shape is to consider forms similar to diagonal forms, those with shape

$$F(x, y) = ax^n - by^n,$$

while still considering a family of forms which is closed under linear substitution (which diagonal forms are not). Thue equations with diagonal forms have been shown to have very few solutions, see [4] for example.

The typical methods in this subject may be applied more simply to the family of diagonalizable forms and will yield stronger results than for general F. The two main methods correspond to the two types of results one may obtain.

The first method, the Thue-Siegel method, allows one to give an upper bound on the number of solutions to a Thue equation. As the name suggests, this method was originated by Thue in [11], and further developed by Siegel in [9]. This method requires one show that the solutions, in a sense, repel each other (This is called a gap principle.) Simultaneously, the solutions also serve as approximations to roots of $F(x, 1)$. These two restrictions can be combined to give an upper bound on the number of solutions. In my first paper I applied this method to the family of diagonalizable forms of degree four with negative discriminant. The case when the degree is four with positive discriminant is handled in [1], while the case of degree greater than four is handled in [3].

The second method uses Baker's theory of linear forms in logarithms. This has the advantage of allowing one to solve a Thue equation by explicit computation. I am currently in the early stages of applying this theory to the family of diagonalizable forms.

I also have two side projects, one close to completed and one in early stages. Both concern the family of diagonalizable forms. The first provides a reduction theory for diagonalizable forms. It is a classical result of invariant theory that there are finitely many forms up to linear substitution once all of the invariants have been specified. An invariant is a homogeneous polynomial in the coefficients of a form which is invariant under linear substitution - like the discriminant. Furthermore, Hilbert showed that the ring of invariants for forms in a particular degree is finitely generated. A reduction theory is a method of explicitly producing a finite list of polynomials given a specified values for the invariants.

Unfortunately, a reduction theory for general polynomials is very difficult (to my knowledge, explicit algorithms only exist in degree four and lower). In higher degrees, the invariants become very complicated and unwieldy. As early as degree five one of the generators of the ring of invariants is a polynomial with 848 terms and very large coefficients. To my knowledge, explicit forms of the generators of the invariant ring are not known for degrees 11 or greater than 12.

Diagonalizable forms, once again, are a much simpler family of forms. The value of every invariant of a diagonalizable form is determined by its discriminant. In addition, diagonalizable forms, in a sense, correspond to quadratic forms. One may thus utilize the reduction theory of quadratic forms to provide a reduction theory for diagonalizable forms in high degree.

The main purpose of producing a reduction theory is to assist in computations. For example, in my first paper, the Thue-Siegel method itself provides upper bounds on the number of solutions to a Thue equation when the discriminant is large enough. When the discriminant is small one must check explicitly that the corresponding Thue equation has few solutions. Fortunately, PARI can be used to efficiently solve the Thue equations in question. Before doing this, however, one must produce an exhaustive list of forms to check. Fortunately, as that paper was concerned with forms of degree four, an efficient reduction theory is known. In higher degrees, these computations could be done using the algorithm given in my second paper. This paper also includes other results which are useful in the computational study of diagonalizable forms, the first is a simple method to check whether a general form is diagonalizable (it was already known that this condition was necessary, but not that it was sufficient). The second is a severe restriction on the discriminant of a diagonalizable form.

The second side project, still in early stages, involves positive definite diagonalizable forms. The main purpose here is to adapt an interesting result from [8] to the family of diagonalizable forms. In this paper, it is shown that the Thue equation

$$\Phi_k(x, y) = m$$

has finitely many solutions in integers k, x, and y, where Φ_k refers to the cyclotomic form of degree k (the homogenization of the cyclotomic polynomial of degree k). This is a departure from typical results as it demonstrates a finite number of solutions across an infinite family of Thue equations. Cyclotomic forms and positive definite diagonalizable forms are similar. The roots of a cyclotomic form are roots of unity, while the roots of a diagonalizable form are the roots of unity up to a Möbius transformation. Perhaps one might be able to use results like this to show something interesting about diagonalizable forms with small Mahler measure, as those are closest to cyclotomic forms.

A key step in [8] is a uniform lower bound for the value of a cyclotomic form. My hope is that diagonalizable forms admit a similar lower bound. Based on computational evidence, quartic forms which are reduced in the sense of [7] seem to have the highest possible lower bound. The definition of reduced in that paper translates to a bound on the leading coefficient of the form. It's possible that such bounds lead to a high lower bound for the value of the form.

In the future I would like to finish applying the usual methods to the family of diagonalizable forms. This includes my plans to apply the theory of linear forms in logarithms, as well as an analysis of positive definite diagonalizable forms.

A lofty goal of mine is to prove that the family of diagonalizable forms satisfies Lehmer's conjecture. Diagonalizable forms are similar to cyclotomic forms, as their roots have similar geometric structure. Hopefully the distinction here is enough to illuminate the difference between a form having roots on the unit circle, and a form having roots close to the unit circle. At the very least, it would be an opportunity to familiarize myself with the methods and results in this field.

I would also like to continue writing algorithms for use in Diophantine approximation and implementing them in Sage. Although I don't have any formal training in computer science, I have found the algorithmic work that I have done thus far to be difficult but rewarding. In this field in particular, there are many methods which don't work for small solutions or small polynomials. Because of this, computational methods can be quite useful for filling in the missing gaps.

2. Detailed Descriptions

At this point, I will discuss the two main results I have shown in more detail. The first result follows from the application of the Thue-Siegel method to the family of diagonalizable quartic forms with negative discriminant. For historical interest, we note that Quartic Thue equations have been studied, notably by Wakabayashi in [12] and [13].

Suppose that

$$F(x,y) = (\alpha x + \beta y)^4 - (\gamma x + \beta y)^4 = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$$

is a diagonalizable quartic form with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and

$$j = \alpha\delta - \beta\gamma \neq 0.$$

This is to ensure that the discriminant of F is nonzero, as

$$\Delta_F = (-1)^{\frac{(n-1)(n-2)}{2}} j^{n(n-1)}$$

I am primarily interested in the number of solutions to the following Thue inequality:

$$F(x,y) \leq h \tag{1}$$

with $x, y \in \mathbb{Z}$. To this end, I consider F as a map on the lattice $\mathbb{Z} \times \mathbb{Z}$. From this perspective, it becomes clear that one may pre-compose F with a change of basis of this $\mathbb{Z} \times \mathbb{Z}$ lattice. That is, for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, define

$$G(x, y) = F(ax + by, cx + dy).$$

Crucially, replacing F with G in (1) does not change the number of integer solutions x, y. We also note that the family of diagonalizable forms is closed under this $\mathrm{SL}_2(\mathbb{Z})$ action.

So suppose that one has a quartic form F with several solutions. We consider the solutions which are "closest" to a particular root of unity. The Thue-Siegel method aims to show that the largest solution of F is large enough to arrive at a contradiction. The size that I'm speaking of here is not the obvious one, for these forms we measure of the size of the solution by the values of $Z(x, y)$ and $\zeta(x, y)$ given below:

$$\begin{aligned} Z(x, y) &= \max\{|\alpha x + \beta y|, |\gamma x + \delta y|\} \\ \zeta(x, y) &= \frac{|F(x, y)|}{Z(x, y)}. \end{aligned}$$

I adapted results from [3] to show that the solutions of (1) satisfy the following gap principle:

Lemma 2.1. *Suppose that the solutions to* (1) *which are closest to a particular root of unity are* $(x_1, y_1), \ldots, (x_k, y_k)$*, ordered by decreasing* ζ*-value. Suppose that there are at least two solutions and* $h < \frac{1}{4}|j|^2$*. Then*

$$Z(x_k, y_k) \geq \frac{|j|^{a_1(k)}}{2^{a_1(k)} h^{a_2(k)}}.$$

The constants $a_1(k)$ *and* $a_2(k)$ *are exponential in* k*.*

As is typical in this method, I used Padé approximants to inductively strengthen this gap principle, eventually arriving at the following:

Lemma 2.2. *Suppose that* k *is a fixed integer, and that* h *satisfies* $h < \frac{1}{4}|j|^2$ *as well as*

$$h \leq \min_{0 \leq i \leq 2} C_i |j|^{E_i}.$$

Then

$$Z(x_k, y_k) \geq \frac{3}{4} 2^{-13n-13} |j|^{-2n-3} h^{-2n-1} Z(x_{k-1}, y_{k-1})^{4n}. \tag{2}$$

The exact description of the constants C_i and E_i is too complicated for this discussion. Under more assumptions on the relative sizes of h and $|j|$ I could show that the right side of (2) goes to ∞ as $n \to \infty$. This is of course a contradiction. Then I performed an involved comparison of all of the required assumptions on h and $|j|$ to reduce these to only one assumption which implied the others. This assumption is stated vaguely in the following theorem.

Theorem 2.3. *Let* F *be a diagonalizable quartic form with negative discriminant, and* k *an integer satisfying* $k \geq 3$*. Under the assumption*

$$h \ll_k 2^{-10/7} |j|^{10/7}$$

the inequality (1) *has at most* $2k$ *solutions.*

I then used computation to produce a result which does not require assumptions on the relative sizes of h and $|j|$. To do this I began with the statement above when $k = 4$. (Based on computations I suspect that this argument with $k = 3$ should be possible, but I didn't have enough computation time to show this.) This statement can be interpreted as "$F(x, y) = 1$ has at most eight solutions if its discriminant is large enough". I then used the algorithm described in [5] to produce all of the diagonalizable forms with negative discriminant which is smaller than the one in my statement. PARI is able to solve the Thue equations in question. According to PARI, none of these equations have more than eight solutions, so we arrive at the following statement:

Theorem 2.4. *Suppose that* F *is a diagonalizable quartic form with negative discriminant. The equation* $|F(x, y)| = 1$ *has at most eight solutions.*

The results of these computations as well as the sage code used can be found on my website.

One can adapt results like this to equations like

$$F(x, y) = m$$

using methods like the reduction argument found in [6]. This is where diagonalizable forms are advantageous; the fact that this family is closed under $\mathrm{SL}_2(\mathbb{Z})$ is key in this reduction. Furthermore, one can take bounds on the number of solutions to such equations and translate them to bounds on the number of integral points on certain families of elliptic curves using methods found in, for example, [2].

My second result is on the computational aspects of diagonalizable forms. I will begin by noting two results which simplify computations. I didn't initially set out to prove these but arrived at them tangentially:

Theorem 2.5. *Suppose that* $F(x, y)$ *is a binary integral form of degree* n*. Then* F *is diagonalizable if and only if the Hessian of* F *is the* $n - 2$ *power of a quadratic form.*

Suppose that F *is diagonalizable. If* m *is even then there is an integer* m *such that* $\Delta_F = nm^{n-1}$*. In addition,* j *must satisfy* $nj^n \in \mathbb{Z}$*. In fact,* m *can be taken to be*

$$m = (-1)^{(n+2)/2} nj^n.$$

If n *is odd, then there is an integer* m *such that* $\Delta_F^2 = n^2 m^{n-1}$*. In addition,* j *must satisfy* $n^2 j^{2n} \in \mathbb{Z}$*. In fact,* m *can be taken to be*

$$m = n^2 j^{2n}.$$

Furthermore, if $n \equiv 1 \mod 4$ *then* $\Delta_F > 0$*. If* $n \equiv 3 \mod 4$ *then* Δ_F *and* D *have opposite signs.*

D hasn't been defined yet, this last statement is included to demonstrate that restrictions on the sign of Δ_F can be given when $n \equiv 3 \mod 4$. These two statements help one recognize which forms are diagonalizable and significantly narrow the range of discriminants one must consider.

Although these results are helpful, they weren't my goal. I noticed in my previous result that it was useful to be able to produce all diagonalizable forms with a given discriminant up to $\mathrm{SL}_2(\mathbb{Z})$ equivalence. In general one would need to use the whole invariant ring, not just the discriminant. For diagonalizable forms the discriminant generates the invariant ring. Such an algorithm would be referred to as a reduction theory for diagonalizable forms. In my previous paper it was very important that the forms were quartic, because an efficient reduction theory for general forms of degree more than four does not appear to be known.

I will now describe the algorithm I produced for this purpose. Suppose that

$$F(x,y) = (\alpha x + \beta y)^n - (\gamma x + \delta y)^n = \alpha_1 (x - \beta_1 y)^n - \gamma_1 (x - \delta_1 y)^n$$

is a diagonalizable form of degree n. We consider the quadratic form

$$(\alpha x + \beta y)(\gamma x + \delta y) = \chi(Ax^2 + Bxy + Cy^2)$$

where $A, B, C \in \mathbb{Z}$ are relatively prime. We define $D = B^2 - 4AC$ and note that $j^2 = \chi^2 D$. Furthermore, from the integrality of the Hessian coefficients, I noted the following. The first holds when n is even, the second when n is odd. The reason for this distinction is to obtain an integer power of D.

$$\begin{aligned} n^2(n-1)^2 \chi^{n-2} D^{(n-2)/2} &= n^2(n-1)^2 j^n \in \mathbb{Z} \\ n^4(n-1)^4 \chi^{2n-2} D^{n-2} &= n^4(n-1)^4 j^{2n} \in \mathbb{Z}. \end{aligned}$$

So suppose we begin with Δ_F. The above theorem can be used to find m (up to sign), which can be used to find j (up to a root of unity). Once one obtains j it can be used to find a range of possibilities for D, as an integer power of D must divide the right side of the above identities. Now a range of possibilities for χ can be found using $j^2 = \chi^2 D$. Using the classical reduction theory of qudratic forms one can produce a finite list of possibilities for the quadratic form $Ax^2 + Bxy + Cy^2$ at which point my task was half completed.

It has been shown that $\alpha_1, -\gamma_1$ and β_1, δ_1 are pairs of conjugates in $\mathbb{Q}(\sqrt{D})$, which is why this alternate presentation of F is useful (see [3] for details). The roots of

$$\alpha_1^{1/n} \gamma_1^{1/n} (x - \beta_1 y)(x - \delta_1 y) = \chi(Ax^2 + Bxy + Cy^2)$$

are β_1 and δ_1, while reading off the leading coefficient tells us the norm of α_1 in $\mathbb{Q}(\sqrt{D})$ (or γ_1 alternatively). It has also been shown that $n(n-1)\sqrt{D}\alpha_1$ is an algebraic integer in $\mathbb{Q}(\sqrt{D})$ (this is shown in [3]). If $D < 0$ then this gives a finite range of possibilities for $n(n-1)\sqrt{D}\alpha_1$, which completes the algorithm. The same applies to the $D > 0$ which are perfect squares.

However, if $D > 0$ then things are more complicated because we can only specify $n(n-1)\sqrt{D}\alpha_1$ up to multiplication by powers of the fundamental unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$. A priori this leads to infinitely many forms. However, many of these are $\mathrm{SL}_2(\mathbb{Z})$-equivalents. I was able to show that multiplying a particular value of α_1 by a certain power of the fundamental unit produced a form which is equivalent. PARI has a function which gives the families of integers with a specific norm in a real quadratic number field. Along with the fundamental unit, one can give a finite range of values which need to be considered for α_1, which completes the algorithm.

At the time of writing this statement I have not yet completed this work. The mathematics is finished, but I would like to implement this algorithm in Sage to produce some examples. I am hopeful of finishing this work by the end of the 2018 calendar year.

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Department of Mathematics, Deady Hall, Eugene, OR, 97403, USA
Email address: `cdethier@uoregon.edu`