Introduction to Ramsey Theory

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Beginning of Ramsey Theory

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What is Ramsey Theory?

Ramsey theory is the study of the preservation of properties under set partitions. In other words, given a particular set S has a property P, is it true that whenever S is partitioned into finitely many subsets, one of the subsets must also have property P?

Illustration of Ramsey Theory

Example 1.1

Obviously, the equation $x + y = z$ has a solution in the set of $\mathbb{Z}^+$; for example, $x = 1, y = 4, z = 5$ is one solution. Here's the question:

- Is it true that whenever the set of positive integers is partitioned into a finite number of sets $S_1, S_2, \ldots, S_r$, then at least one of these sets will contain a solution to $x + y = z$?

The answer turns out be yes, as we shall see later in this slides.

Historical Context

The Party Problem

- At a party of six people, there must exist either three people mutually know each other or three people mutually do not know each other.

Proof Sketch

To clarify the problem we will make some assumptions:

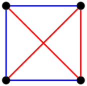
- Every pair of people at the party is a pair of friends or strangers (not both)
- The stranger and friend relationships are symmetrical

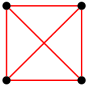
We can model the relationships at our party with graph theory

The Party Problem

Proof Sketch

To solve our problem, we will represent each person at our party as a vertex on our graph . We will place a red edge between every pair of friends and a blue edge between every pair of strangers.





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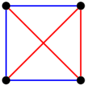
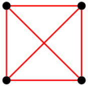
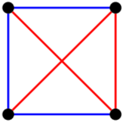


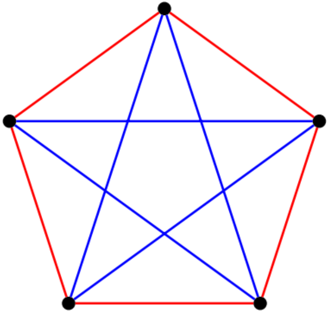


The following graph shows us that inviting 4 people is not enough to guarantee a group of 3 mutual friends or a group of 3 mutual strangers. What about 5 people?



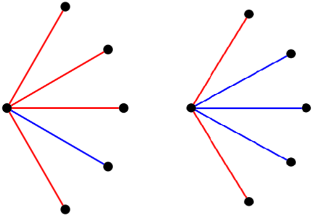
The Party Problem

Five is also not enough people.



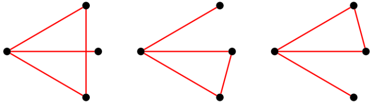
Party Problem

But 6 is enough people, and we can prove it. Isolate one person. No matter how we color the edges there are always at least 3 edges that are the same color.



Party Problem

Suppose these three edges are red. Then if any of the edges between those three people are also red, we have the group of 3 friends that we wanted.



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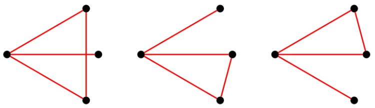
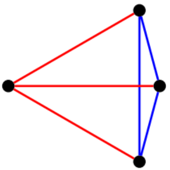


The only other case is that all three of these edges are blue, which gives us a group of three strangers.



Party Problem

- Combinatoric study the generalization of this problem in a field known as Ramsey Theory.
- Let the Ramsey number $R(m, n)$ be the minimum number k such that when the edges of the complete graph on k vertices are colored with red and blue, there will always either be a red clique of size m or a blue clique of size n.

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Example 1.2

$R(3,3) = 6,\ R(3,4) = 9,\ R(4,4) = 18,\ R(4,5) = 25,\ 43 \leq R(5,5) \leq 48.$

Introduction to Formulation

Definition 1.1

An *edge-coloring* of a graph is an assignment of a color to each edge of the graph. A graph which has been edge-colored is called a *monochromatic graph* if all its edges are the same color.

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Observation

- We may now express the solution of the party problem in graph theoretical language. It says that for every 2-coloring, using the colors red and blue, of the edges of K_6 there must be either a red K_3 or a blue K_3.
- Furthermore, that there exists a 2-coloring of the edges of K_5 that fails to have this property.

Introduction

Ramsey Theorem for Two Colors (1930)

Let $k, l \geq 2$. There exists a least positive integer $R = R(k, l)$ such that every edge coloring of K_R, with the colors red and blue, admits either a red K_k subgraph or a blue K_l subgraph.

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Proof

- First observe that $R(k, 2) = k$ for all $k \geq 2$ and $R(l, 2) = l$ for all $l \geq 2$. We will prove by induction on the sum $k + l$. Let $k + l \geq 6$ with $k, l \geq 3$. By induction, we may assume that both $R(k-1, l)$ and $R(k, l-1)$ exist. We claim that $R(k, l) \leq R(k-1, l) + R(k, l-1)$, which will prove the theorem.

Introduction

Proof

- Let $n = R(k-1, l) + R(k, l-1)$ and choose a vertex v from the K_n. Then there are $n-1$ edges from v to the other vertices. Let A be the number of red edges and B the number of blue edges coming out of v. Then, either $A \geq R(k-1, l)$ or $B \geq R(k, l-1)$. If not, then we have $A < R(k-1, l)$ and $B < R(k, l-1)$. So that $A + B \leq n-2$, contradicting the fact that $A + B = n-1$. We may assume that $A \geq R(k-1, l)$.

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- Let V be the set of connected to v by a red edge, thus $|V| \geq R(k-1, l)$. By the induction hypothesis, K_V contains either a red K_{k-1} subgraph or a blue K_l subgraph. If it contains a blue K_l subgraph, we are done. If it contains a red K_{k-1} subgraph, then by connecting v to each vertex of this subgraph we have a red K_k subgraph. $\square$

Ramsey numbers for r-colors

More generally, Ramsey's theorem for two colors can easily be generalized to $r \geq 3$ colors, in which case Ramsey numbers are denoted by $R(k_1, k_2, \ldots, k_r)$.

Three Classical Theorems

Definition 1.2

A k-term arithmetic progression is a sequence of the form
$a, a+d, a+2d, \ldots, a+(k-1)d$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$.

Van der Waerden's Theorem (1927)

For all positive integers k and r, there exist a least positive integer $w(k;r)$ such that for every r-coloring of $[1, w(k;r)]$ there is a monochromatic arithmetic progression of length k.

- For any positive integers r and k there exists a positive integer N such that if the integers $\{1, 2, ..., N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression all of the same color. The smallest such N is the van der Waerden number $w(k;r)$. For example, $w(2;2) = 3$.

Three Classical Theorems

Definition 1.3

Note that, we call the numbers that satisfy Schur's theorem the Schur numbers and denote them by $s(r)$.

Definition 1.4

A triple x, y, z that satisfies $x + y = z$ is called a Schur triple.
The only values known for the Schur numbers are for $r = 1, 2, 3, 4$:
$s(1) = 2, s(2) = 5, s(3) = 14$, and $s(4) = 45$.

Schur's Theorem (1926)

For any $r \geq 1$ there exists a least positive integer $s = s(r)$ such that, for any r-coloring of $[1, s]$, there exists a monochromatic solution to $x + y = z$.

Three Classical Theorems

Definition 1.5

For $r \geq 1$, a linear equation $\mathcal{L}$ is called r-regular if there exists a $n = n(\mathcal{L}; r)$ such that for every coloring of $[1, n]$ there is a monochromatic solution to $\mathcal{L}$. It is called regular if it is r-regular for every $r \geq 1$.

Example 1.2

Using definition, Schur's theorem can be stated as "the equation $x + y = z$ is regular."

Rado's Theorem for Single Linear Equation (1933)

Let $\mathcal{L}$ represent the linear equation

$$c_1 x_1 + \cdots + c_n x_n = 0$$

where each c_i is nonzero integer. Then $\mathcal{L}$ is regular if and only if some nonempty subset of the c_i's sums to 0.

Proof of the Schur's Theorem

Proof

- By Ramsey's theorem there exists an integer $n = R(3; r)$ such that for any r-coloring of K_n there is a monochromatic triangle. We will use a special coloring as follows. Number the vertices of K_n by $1, 2, \ldots, n$. Next, arbitrarily partition $\{1, 2, \ldots, n-1\}$ into r sets. In other words, randomly place each $x \in \{1, 2, \ldots, n-1\}$ into exactly one of the r sets. These will be correspond to r colors. Color the edge that connects i and j according to the set of which $|j - i|$ is a member. By Ramsey's theorem a monochromatic triangle must exist. Let the vertices of this monochromatic triangle be $a < b < c$. Hence, $b - a, c - a$ and $c - b$ are all the same color. To finish the proof, let $x = b - a, y = c - b$ and $z = c - a$ and notice that $x + y = z$. $\square$

Open Question

Computational Complexity

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens.
—Joel Spencer

Results

- In 1989, Exoo showed that $R(5,5) \geq 43$.
- In 2018, V. Angeltveit and B.D. Mckay showed that $R(5,5) \leq 48$.

Thank you!

Welcome any question.

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