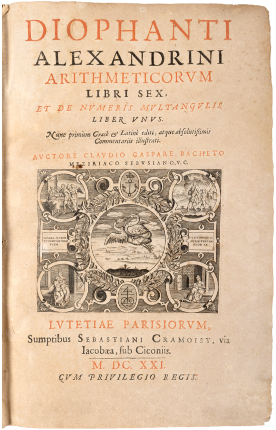
Diophantine m-tuples

Celal Devret

Yildiz Technical University

May - 2024

Diophantus of Alexandria (AD 200∼214-284∼298)



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantine m-tuples

Definition 1.1

A set of m positive integers $\{a_1, \ldots, a_m\}$ is called *a Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Definition 1.2

A set of m non-zero rational numbers $\{a_1, \ldots, a_m\}$ is called *a rational Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Examples

Example 1.1 (Diophantus)

The set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ is rational Diophantine 4-tuple.

$$\frac{1}{16}\frac{33}{16}+1=\frac{289}{256}=\frac{17^2}{16^2} \qquad \frac{1}{16}\frac{17}{4}+1=\frac{81}{64}=\frac{9^2}{8^2}$$
$$\frac{1}{16}\frac{105}{16}+1=\frac{361}{256}=\frac{19^2}{16^2} \qquad \frac{33}{16}\frac{17}{4}+1=\frac{625}{64}=\frac{25^2}{8^2}$$
$$\frac{33}{16}\frac{105}{16}+1=\frac{3721}{256}=\frac{61^2}{16^2} \qquad \frac{17}{4}\frac{105}{16}+1=\frac{1849}{64}=\frac{43^2}{8^2}$$

Examples

Example 1.2 (Fermat)

The set $\{1, 3, 8, 120\}$ is a Diophantine 4-tuple.

$$1 \cdot 3 + 1 = 4 = 2^2 \qquad 1 \cdot 8 + 1 = 9 = 3^2$$

$$1 \cdot 120 + 1 = 121 = 11^2 \qquad 3 \cdot 8 + 1 = 25 = 5^2$$

$$3 \cdot 120 + 1 = 361 = 19^2 \qquad 8 \cdot 120 + 1 = 961 = 31^2$$

Examples

Example 1.3 (Euler)

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- It was proved in 1969, by Baker and Davenport that a fifth positive integer can not be addded to Fermat's Diophantine 4-tuples.

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- If we take $a=1$ and $b=3$, we get Fermat's Diophantine 4-tuple.
- It was proved in 1969, by Baker and Davenport that a fifth positive integer can not be addded to Fermat's Diophantine 4-tuples.
- However, Euler was able to extend Fermat's Diophantine 4-tuples by adding the rational number $\frac{777480}{8288641}$.

Some Conjectures and Theorems

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Theorem 1.2 (Andrej Dujella, 2004)

There are only finitely many Diophantine 5-tuples.

- Finally, in 2019, He, Togbé and Ziegler gave the proof of the Diophantine 5-tuple conjecture.

Connections with Pell's Equations

It is clear that the set $\{1, 3\}$ is a Diophantine 2-tuple. Let's try to extend this set. Let the set $\{1, 3, a\}$ be a Diophantine 3-tuple for some $a \in \mathbb{Z}^+$. Then

$$a + 1 = r^2$$
$$3a + 1 = s^2$$

for some $r, s \in \mathbb{Z}^+$. If we multiple the first equation by 3 and substract from the second equation, we get the following equation

$$s^2 - 3r^2 = -2.$$

Therefore, we must solve the following equation

$$x^2 - 3y^2 = -2.$$

The Equation $x^2 - 3y^2 = -2$

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There exists a solution, since 1 is quadratic residue in modulo 3.
How to find it?

$$x^2 - 3y^2 = -2 \implies x^2 = 3y^2 - 2$$

If $y = 1$, then $x = 1$. The $(x_0, y_0) = (1, 1)$ is a solution.

The Equation $x^2 - 3y^2 = -2$

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To solve this equation we need to continued fraction of $\sqrt{3}$.

$$\sqrt{3} = [1; \overline{1, 2}], \text{ i.e., } \sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

The Equation $x^2 - 3y^2 = -2$

n	-2	-1	0	1	2	3	4
a_n			1	1	2	1	2
p_n	0	1	1	2	5	7	19
q_n	1	0	1	1	3	4	11

Recall that the integers a_n are called the *cofficients* or *terms* of the continued fraction and the rational numbers p_n/q_n are called the *convergents* of the continued fraction.
The integers p_n and q_n given by

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}.$$

The Equation $x^2 - 3y^2 = -2$

Since the length of the period of the continued fraction expansion of $\sqrt{3}$ is even, then all positive solutions of the equation $x^2 - 3y^2 = 1$ are given by

$$x = p_{2k-1} \qquad y = q_{2k-1} \qquad k = 1, 2, \ldots$$

So, $(u_1, v_1) = (2, 1)$ and $(u_2, v_2) = (7, 4)$ are solution of the equation $x^2 - 3y^2 = 1$.

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We have $(x_0, y_0) = (1, 1)$ solution of the equation $x^2 - 3y^2 = -2$ and $(u_1, v_1) = (2, 1)$ solution of the equation $x^2 - 3y^2 = 1$. That is,

$$x_0^2 - 3y_0^2 = -2$$
$$u_1^2 - 3v_1^2 = 1.$$

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$$x_0^2 - 3y_0^2 = -2$$
$$u_1^2 - 3v_1^2 = 1.$$

Let's multiply of these equations.

$$\begin{aligned}
(x_0^2 - 3y_0^2)(u_1^2 - 3v_1^2) &= -2 \\
x_0^2 u_1^2 - 3x_0^2 v_1^2 - 3y_0^2 u_1^2 + 9y_0^2 v_1^2 &= -2 \\
x_0^2 u_1^2 + 6x_0 u_1 y_0 v_1 + 9y_0^2 v_1^2 - 3(x_0^2 v_1^2 + 2x_0 v_1 y_0 u_1 + y_0^2 u_1^2) &= -2 \\
(x_0 u_1 + 3y_0 v_1)^2 - 3(x_0 v_1 + y_0 u_1)^2 &= -2
\end{aligned}$$

The Equation $x^2 - 3y^2 = -2$

Hence, we get a new solution $(x_0 u_1 + 3y_0 v_1, x_0 v_1 + y_0 u_1) = (5, 3)$ of the equation $x^2 - 3y^2 = -2$.

Hence, we get a new solution $(x_0 u_1 + 3y_0 v_1, x_0 v_1 + y_0 u_1) = (5, 3)$ of the equation $x^2 - 3y^2 = -2$.
Recall that we wanted solve

$$s^2 - 3r^2 = -2.$$

Hence $(s, r) = (5, 3)$ is a solution. So we can find a. Since $a + 1 = r^2$ this implies that $a = 8$.

The Equation $x^2 - 3y^2 = -2$

Hence, we get a new solution $(x_0u_1 + 3y_0v_1, x_0v_1 + y_0u_1) = (5, 3)$ of the equation $x^2 - 3y^2 = -2$.
Recall that we wanted solve

$$s^2 - 3r^2 = -2.$$

Hence $(s, r) = (5, 3)$ is a solution. So we can find a. Since $a + 1 = r^2$ this implies that $a = 8$.
Therefore the set $\{1, 3, 8\}$ is a Diophantine 3-tuple.

More Pell's Equations

Now, let's try to extend the set $\{1, 3, 8\}$. Let the set $\{1, 3, 8, b\}$ be a Diophantine 4-tuple for some $b \in \mathbb{Z}^+$. Then

$$\begin{aligned} b + 1 &= r^2 \\ 3b + 1 &= s^2 \\ 8b + 1 &= t^2 \end{aligned}$$

for some $r, s, t \in \mathbb{Z}^+$. Hence, we get

$$\begin{aligned} s^2 - 3r^2 &= -2 \\ t^2 - 8r^2 &= -7. \end{aligned}$$

Therefore, we must solve the following equations:

$$\begin{aligned} x^2 - 3y^2 &= -2 \\ z^2 - 8y^2 &= -7. \end{aligned}$$

The Equations $x^2 - 3y^2 = -2$ & $z^2 - 8y^2 = -7$

We already know the solutions of the equation $x^2 - 3y^2 = -2$. Let's solve the equation $z^2 - 8y^2 = -7$.

$$z^2 - 8y^2 \equiv z^2 \equiv 1 \equiv -7 \pmod 8$$

So, there exists a solution, since 1 is quadratic residue in modulo 8.

$$z^2 - 8y^2 = -7 \implies z^2 = 8y^2 - 7$$

If $y = 1$, then $z = 1$. So $(z_0, y_0) = (1, 1)$ is a solution of the equation $z^2 - 8y^2 = -7$. Now, we need to solve the equation $z^2 - 8y^2 = 1$ to get a new solution of the equation $z^2 - 8y^2 = -7$. To solve the equation $z^2 - 8y^2 = 1$, we need to continued fraction of $\sqrt{8}$.

The Equations $x^2 - 3y^2 = -2$ & $z^2 - 8y^2 = -7$

$\sqrt{8} = [2 : \overline{1, 4}]$.

n	-2	-1	0	1	2	3	4
a_n			2	1	4	1	4
p_n	0	1	2	3	14	17	82
q_n	1	0	1	1	5	6	29

Since the length of the period of the continued fraction expansion of $\sqrt{8}$ is even, then all positive solutions of the equation $z^2 - 8y^2 = 1$ are given by

$$x = p_{2k-1} \qquad y = q_{2k-1} \qquad k = 1, 2, \dots$$

So, $(u_1, v_1) = (3, 1)$ and $(u_2, v_2) = (17, 6)$ are solution of the equation $z^2 - 8y^2 = 1$.

The Equations $x^2 - 3y^2 = -2$ & $z^2 - 8y^2 = -7$

If, we try $(z_0, y_0) = (1, 1)$ and $(u_1, v_1) = (3, 1)$ to get a new solution we do not get a solution of the system. So let's try $(z_0, y_0) = (1, 1)$ and $(u_2, v_2) = (17, 6)$. So we have

$$z_0^2 - 8y_0^2 = -7$$
$$u_2^2 - 8v_2^2 = 1$$

Again, let's multiply these equations:

$$(z_0^2 - 8y_0^2)(u_2^2 - 8v_2^2) = -7$$
$$(z_0 u_2 - 8y_0 v_2)^2 - 8(z_0 v_2 - y_0 u_2)^2 = -7$$

Hence, $(z_0 u_2 - 8y_0 v_2, z_0 v_2 - y_0 u_2) = (-31, -11)$ is a new solution of the equation $z^2 - 8y^2 = -7$. Also, $(31, 11)$ is a solution.

The Equations $x^2 - 3y^2 = -2$ & $z^2 - 8y^2 = -7$

Recall that $(19, 11)$ is a solution of the equation $x^2 - 3y^2 = -2$. Moreover, we found $(31, 11)$ is a solution of the equation $z^2 - 8y^2 = -7$. Also recall that we wanted solve

$$s^2 - 3r^2 = -2$$
$$t^2 - 8r^2 = -7.$$

Hence $(s, r) = (19, 11)$ and $(t, r) = (31, 11)$ is a solution of the system. So we can find b. Since $b + 1 = r^2$ this implies that $b = 120$.
Therefore the set $\{1, 3, 8, 120\}$ is a Diophantine 4-tuple.

Generalized Diophantine m-tuples

There are several natural generalizations of the original problem of Diophantus and Fermat. The first of them is the replacement of number 1, in the definition of Diophantine m-tuples, by an arbitrary integer n.

Definition 2.1

Let n be an integer. A set of m positive integers $\{a_1, \ldots, a_m\}$ is said to be have *have the property D(n)* if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Such a set called *a Diophantine m-tuple with the property D(n)* (or $D(n)$-m-tuple, or P_n-set of size m).

Example 2.1

The set $\{11, 13, 52\}$ is a Diophantine 3-tuple with the property $D(53)$.

$$11 \cdot 13 + 53 = 196 = 14^2$$
$$11 \cdot 52 + 53 = 625 = 25^2$$
$$13 \cdot 52 + 53 = 729 = 27^2$$

Examples

Example 2.1

The set $\{11, 13, 52\}$ is a Diophantine 3-tuple with the property $D(53)$.

$$11 \cdot 13 + 53 = 196 = 14^2$$
$$11 \cdot 52 + 53 = 625 = 25^2$$
$$13 \cdot 52 + 53 = 729 = 27^2$$

Example 2.2

The set $\{4, 119, 169\}$ is a Diophantine 3-tuple with the property $D(53)$.

$$4 \cdot 119 + 53 = 529 = 23^2$$
$$4 \cdot 169 + 53 = 729 = 23^2$$
$$119 \cdot 169 + 53 = 20164 = 142^2$$

Thank you!
Welcome any question.

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