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STABILITY AND FINITENESS PROPERTIES OF MEDIAL AXIS AND SKELETON

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ABSTRACT. The medial axis is a geometric object associated with any bounded open set in $\mathbb{R}^n$ which has various applications in computer science. We study it from a mathematical point of view. We give some results about its geometrical structure when the open set is subanalytic and we prove that it is stable under $\mathcal{C}^2$-perturbations when the open set is bounded by a hypersurface with positive local feature size.

1. INTRODUCTION

Medial axes of bounded open subsets of $\mathbb{R}^n$ are natural objects widely used in engineering and computer science. For a bounded subset Ω of $\mathbb{R}^n$, the *medial axis* $\mathcal{M}_\Omega$ is the set of points in Ω which have at least two closest points on the boundary $\partial\Omega$ of Ω (see Fig. 1). This concept first introduced by Blum [9], along with some of its variants such as skeleton and cut locus, has been extensively studied in various fields—for example, vision or surface reconstruction—from an algorithmic point of view. However there are few results about its mathematical properties in general. In this paper we investigate:

1. the geometric structure of medial axes of the class of semi- and subanalytic open subsets of $\mathbb{R}^n$;
2. the stability of medial axes of bounded open subsets with $\mathcal{C}^2$-boundary or, more generally, with so-called positive local feature size.

In [13], the authors give a detailed description of the medial axes of open subsets Ω in $\mathbb{R}^2$ whose boundary is a piecewise analytic curve. They prove that $\mathcal{M}_\Omega$ is a finite piecewise analytic graph which has the homotopy type of Ω. Very few theoretical results are known in higher dimensions. For example, it has recently been proved that $\mathcal{M}_\Omega$ has the homotopy type of Ω for

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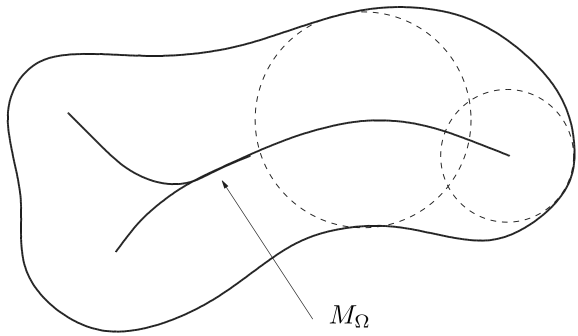

Fig. 1. The medial axis

any bounded open set $\Omega \in \mathbb{R}^n$ [25]. One of the aims of this paper is to extend results of [13] to dimensions greater than two. There exists a natural generalization of the notion of piecewise analytic curve in any dimension: the notion of semi- or subanalytic set. Roughly speaking, a semianalytic subset of $\mathbb{R}^n$ is a set defined by analytic equations and inequalities. Subanalytic sets are defined as images of semianalytic sets under proper linear projections. Subanalytic geometry can be considered as a generalization of real semialgebraic geometry. It was initially developed by S. Lojasiewicz [26, 27] and the geometric structure of subanalytic sets is now well understood. Using arguments and techniques of this theory we prove that for any subanalytic bounded open set $\Omega \subset \mathbb{R}^n$, $\mathcal{M}_\Omega$ is subanalytic (Theorem 2.1). (Note that in [30], Tamm has also used this theory to establish the subanalyticity of the cut-locus in the sense of Riemannian geometry [31]. Medial axis and skeleton are, of course, related to this notion.) Various finiteness properties of $\mathcal{M}_\Omega$ are derived from this result. In particular, $\mathcal{M}_\Omega$ admits an analytic stratification. For applications, finiteness properties of medial axes are involved in the study of the complexity of Voronoi diagrams and Delaunay triangulations. The case of open subsets of $\mathbb{R}^3$ bounded by a compact surface are of particular interest (see, e.g., [1–3, 19]). From a practical point of view, it is not really restrictive to consider semianalytic open sets: in most applications, geometrical objects are represented by equations and inequalities involving analytic functions (polynomials, rational functions, splines, NURBS, etc.). Moreover, it is not possible to obtain such finiteness results for general open sets (even if the boundary is a $\mathcal{C}^\infty$-manifold): it is an easy exercise to construct a $\mathcal{C}^\infty$-simple curve in $\mathbb{R}^2$ which bounds an open subset whose medial axis is a graph with an infinite number of vertices (see, e.g., [13]).

A problem arising when one studies medial axis is that it does not vary continuously under (small) perturbations of the open set (see, e.g., [4, 14,

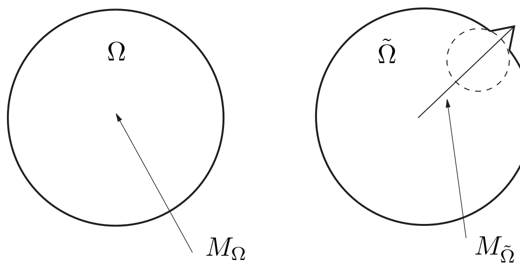

Fig. 2. Perturbation of the medial axis

29]). More precisely, a small $\mathcal{C}^0$-perturbation of Ω can lead to huge variations of $\mathcal{M}_\Omega$ as shown in Fig. 2. This fact gives rise to many problems for practical applications because two close objects (for Hausdorff distance) may have very different medial axes. Therefore, a natural question is as follows: Under which regularity conditions does the medial axis vary continuously with the open set? We prove that if $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth $\mathcal{C}^2$-boundary, then a small $\mathcal{C}^2$-perturbation of Ω induces a small perturbation of $\mathcal{M}_\Omega$ for Hausdorff topology (Theorem 3.2). A precise definition of $\mathcal{C}^2$-perturbation is given in Sec. 3. Intuitively, our result means that if one makes a small change on $\partial\Omega$, on its tangent space, and on its curvature, then one induces a small change on its medial axis. Note that such a result probably does not provide a solution for practical problems because usually data describing geometric objects are given with noise which is clearly not $\mathcal{C}^2$. Nevertheless, our theorem makes precise and rigorous the intuition that "huge variations of medial axes are implied by huge variations of the curvature." The stability under $\mathcal{C}^2$-perturbations is first established for spheres using certain properties of convex sets. The general case easily follows. Finally, a bounded open subset of $\mathbb{R}^n$ has *positive local feature size* if the Hausdorff distance between its boundary and its medial axis is positive. The result of stability can be generalized to $\mathcal{C}^2$-perturbations of open sets with positive local feature size (Theorem 3.3).

The paper is organized as follows: in order to make it more comprehensible for nonspecialists, we present a short introduction to real analytic geometry in the appendix (Sec. 4). Section 2 is devoted to the study of the medial axes and skeletons of subanalytic open sets. Precise definitions of medial axis and skeleton are given there. Stability of medial axis under $\mathcal{C}^2$-perturbations is established in Sec. 3.

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2. Finiteness properties of medial axes and skeletons of semianalytic domains

In this section, we apply results and concepts of real analytic geometry to the study of two geometrical objects arising from problems of shape reconstruction: the medial axis and the skeleton of an open domain of $\mathbb{R}^n$.

2.1. Medial axes and skeletons of semianalytic domains. In the sequel, Ω will denote a connected relatively compact semianalytic open subset of $\mathbb{R}^n$. Note that, from classical properties of semianalytic sets (see Appendix), the boundary $\partial\Omega$ of Ω is semianalytic and admits analytic stratification. If $\partial\Omega$ is a topological manifold, we say that $\partial\Omega$ is a *piecewise analytic manifold*. In particular, if $n = 3$, $\partial\Omega$ is a piecewise analytic surface. It may seem quite restrictive to consider only such domains. In fact, it is also possible to consider more general class of domains like subanalytic domains, but the assumption on the semianalyticity of domains is sufficiently general in many applications. On the other hand, it is not possible to prove finiteness results for general domains, even if the boundary is a smooth $\mathcal{C}^\infty$-manifold (see, e.g., [13]).

Let us denote by $B_r(p)$ the closed ball of radius $r \geq 0$ centered at $p \in \mathbb{R}^n$.

For any point $x \in \Omega$, we denote by $\mathcal{B}(x)$ the set of closest boundary points:

$$\mathcal{B}(x) = \{y \in \Omega^c \mid d(x, y) = d(x, \Omega^c)\}.$$

Note that $\mathcal{B}(x)$ is not empty since Ω^c is closed and Ω is bounded.

Definition 2.1 (Medial axis)**.** The *medial axis* of the open set Ω is the set $\mathcal{M}_\Omega$ of points $x \in \Omega$ which have at least two closest boundary points:

$$\mathcal{M}_\Omega = \{x \in \Omega \mid \mathrm{Card}(\mathcal{B}(x)) \geq 2\}.$$

We will denote the medial axis by $\mathcal{M}$ if there is no ambiguity.

Example 2.1. The medial axis of an open (Euclidean) ball is reduced to its center, the medial axis of an ellipse is a segment and the medial axis of a rectangle is shown in Fig. 3.

Definition 2.2. The *core* $\mathcal{C}(\Omega)$ of Ω is the set of closed balls $B_r(p)$ satisfying the following property: $B_r(p) \subseteq \overline{\Omega}$ and if $B_s(q)$ is such that $B_r(p) \subseteq B_s(q) \subseteq \overline{\Omega}$, then $B_r(p) = B_s(q)$.

In other words, the core of Ω is the set of maximal balls inscribed in Ω. A ball which is in $\mathcal{C}(\Omega)$ is called a *maximal ball* and its boundary is called a *maximal sphere*. The *skeleton* is then the set of centers of maximal balls.

Definition 2.3. The *skeleton* $\mathcal{S}_\Omega$ of Ω is the set

$$\mathcal{S}_\Omega = \{p \in \overline{\Omega} \mid B_r(p) \in \mathcal{C}(\Omega)\}.$$

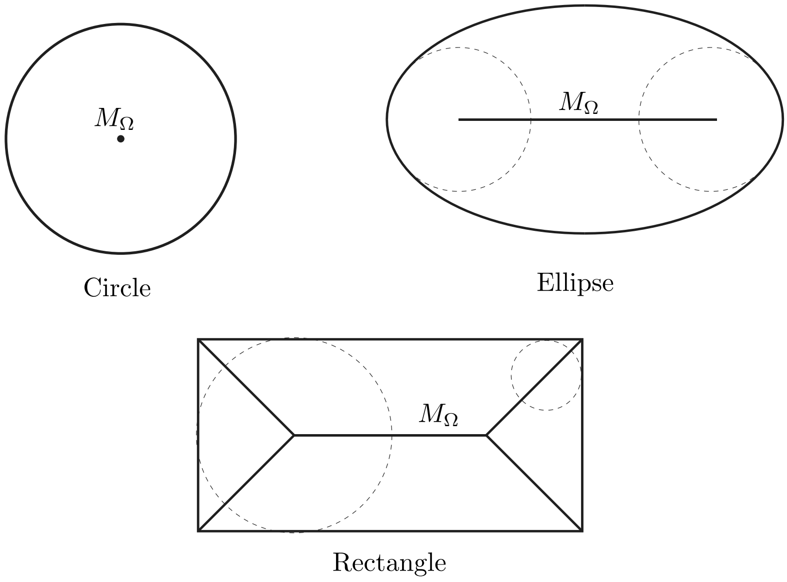

Fig. 3. Examples of medial axes

Example 2.2. As was proved in [13], the skeleton $\mathcal{S}_\Omega$ of a compact domain $\overline{\Omega}$ in $\mathbb{R}^2$ with piecewise analytic boundary is a strong deformation retract of $\overline{\Omega}$. In particular, it is compact. We provide here an example which shows that this fact becomes false in 3-dimensional situations.

Let $\overline{\Omega}$ be the following compact subset of $\mathbb{R}^3$:

$$\overline{\Omega} = B_1 \cup C \cup B_2,$$

where B_1 is the closed unit ball centered at the origin, C is the cylinder $\{(x, y, z) \mid 0 \leq x \leq 2, y^2 + z^2 \leq 1\}$, and B_2 is the closed ball of radius 2 centered at the point $(2, 0, 1)$ (see Fig. 4). Therefore, $\overline{\Omega}$ is a compact semialgebraic subset of $\mathbb{R}^3$ and $\partial\Omega$ is a compact semialgebraic surface. Since the spheres centered at the points $(x, 0, 0)$, $x \in [0, 2[$, are osculating spheres inscribed in $\overline{\Omega}$, the segment $[0, 2[\times\{0\} \times \{0\}$ is a part of the skeleton of Ω. Therefore, the point $m = (2, 0, 0)$ is in $\overline{\mathcal{S}_\Omega}$. The ball $B_1(m)$ is the maximal ball centered at m and contained in $\overline{\Omega}$. But it is not a maximal ball of $\overline{\Omega}$ because it is strictly contained in $B_2 \subset \Omega$. Hence $m \notin \mathcal{S}_\Omega$ and $\mathcal{S}_\Omega$ is not closed.

At last, we mention the notion of *cut locus* which is the closure of the medial axis. Hence we have

$$\mathcal{M}_\Omega \subseteq \mathcal{S}_\Omega \subseteq \overline{\mathcal{M}_\Omega}.$$

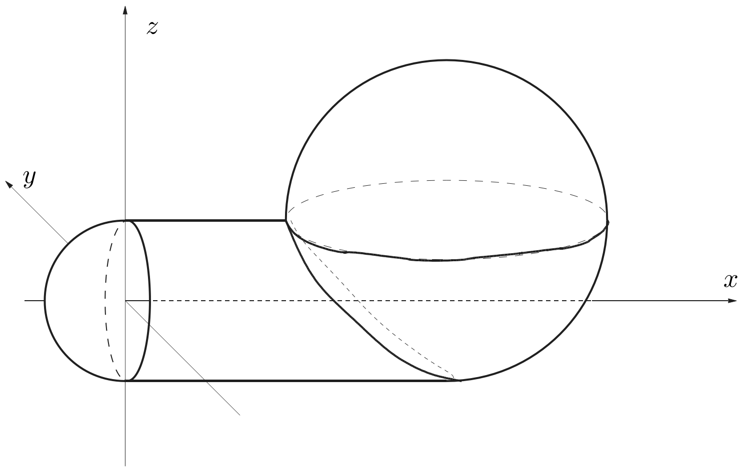

Fig. 4. An open set with noncompact skeleton

It is an easy exercise to find explicit examples where these inclusions are strict (see, e.g., [13]).

Remark 2.1. Note that there exist different terminologies in the literature. Sometimes medial axes are called skeletons and skeleton are called medial axes. We adopt here the (more or less) commonly admitted terminology in computer science.

One of the main results of this paper is the following theorem. We will deduce all finiteness properties from it.

Theorem 2.1. *The sets $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$ are subanalytic.*

Proof. Let us begin with the subanalyticity of $\mathcal{S}_\Omega$. We define

$$BI(\Omega) = \{(q,s) \in \overline{\Omega} \times \mathbb{R} \mid d_{\partial\Omega}(q) \leq s\}$$

where $d_{\partial\Omega}$ denotes the distance function to the compact set $\partial\Omega$. This set $BI(\Omega)$ is the set of points (q,s) such that the closed ball of radius s centered at q is contained in $\overline{\Omega}$. Recall that a closed ball $B_r(p)$ of $\mathcal{C}(\Omega)$ is such that $B_r(p)$ is contained in $\overline{\Omega}$ and if there exists $(q,s) \in \overline{\Omega} \times \mathbb{R}$ such that $B_r(p) \subseteq B_s(q) \subseteq \overline{\Omega}$, then $B_r(p) = B_s(q)$. The first condition is equivalent to the following one:

$$d_{\partial\Omega}(p) \leq r,$$

i.e., $(p,r) \in BI(\Omega)$, and the second condition is equivalent to

$$\forall (q,s) \in BI(\Omega) \text{ such that } (q,s) \neq (p,r),\ \exists x \in \overline{\Omega},\ d(x,p) \leq r,\ d(x,q) > s.$$

Now let us define the following set:

$$\mathcal{ST} = \{(p,r) \in \overline{\Omega} \times \mathbb{R} \mid B_r(p) \in \mathcal{C}(\Omega)\}.$$

This is the set of pairs (center, radius) of maximal balls and we have:

$$\mathcal{ST} = \{(p,r) \in BI(\Omega) \mid \ \forall (q,s) \in BI(\Omega) \text{ such that } (q,s) \neq (p,r), \\ \exists x \in \overline{\Omega}, \ \ d(x,p) \leq r, \ d(x,q) > s\}.$$

Now we define the following set:

$$A = \{((p,r),(q,s),x) \in BI(\Omega) \times BI(\Omega) \times \overline{\Omega} \mid \\ (q,s) \neq (p,r), \ d(x,p) \leq r, \ d(x,q) > s\};$$

$\overline{\Omega}$ and $\partial\Omega$ are compact semianalytic sets, the Euclidean distance function is a semianalytic function (even semialgebraic) and according to Lemma 4.1, the function $d_{\partial\Omega}$ is a subanalytic function. Hence the set $BI(\Omega)$ is a subanalytic subset of $\overline{\Omega} \times \mathbb{R}$. For the same reasons and since the subanalyticity is stable under Cartesian product, the set A is also subanalytic. Let

$$\pi_1((p,r),(q,s),x) = ((p,r),(q,s)), \quad \pi_2((p,r),(q,s)) = (p,r)$$

be the canonical projections. Then we obtain

$$\pi_1(A) = \{((p,r),(q,s)) \mid \ (q,s) \neq (p,r), \ \exists x \in \overline{\Omega}, \ d(x,p) \leq r, \ d(x,q) > s\}.$$

Let $\Delta = \{((q,s),(q,s)) \in BI(\Omega) \times BI(\Omega)\}$ be the diagonal of $BI(\Omega)$. Thus the complement of $\pi_1(A)$ minus Δ is equal to

$$\pi_1(A)^c \setminus \Delta = \{((p,r),(q,s)) \mid \forall x \in \overline{\Omega}, \ d(x,p) > r \text{ or } d(x,q) \leq s\}.$$

Taking the projection π_2 we obtain

$$\pi_2(\pi_1(A)^c \setminus \Delta) = \{(p,r) \in BI(\Omega) \mid \exists (q,s) \, \forall x \in \overline{\Omega}, d(x,p) > r \text{ or } d(x,q) \leq s\}.$$

It suffices now to take the complement of the above set one more time to obtain

$$\mathcal{ST} = \left(\pi_2(\pi_1(A)^c \setminus \Delta)\right)^c.$$

Hence $\mathcal{ST}$ is the image of the subanalytic set A under finitely many linear projections and complementations and, therefore, it is a subanalytic set. Since $\mathcal{S}_\Omega$ is the image of $\mathcal{ST}$ under a linear projection, it is subanalytic.

The medial axis $\mathcal{M}_\Omega$ is the set of points $x \in \Omega$ such that $\mathcal{B}(x)$ contains at least 2 points. Introduce the set

$$A = \{(x,z,z') \in \Omega \times \Omega^c \times \Omega^c \mid z \neq z' \text{ and } d(x,z) = d(x,z') = d(x,\Omega^c)\},$$

which is clearly subanalytic. If $\pi : \Omega \times \Omega^c \times \Omega^c \to \Omega$ denotes the canonical projection, then $\mathcal{M}_\Omega = \pi(A)$. Therefore, $\mathcal{M}_\Omega$ is subanalytic. □

Let us note that the above proof is not particular for the subanalytic situation. Indeed, it works in exactly the same way for a set Ω which belongs to some *o-minimal structure*. We will not recall here any part of this theory (for an introduction, see [18]). Let us just mention the following consequence of the proof of Theorem 2.1.

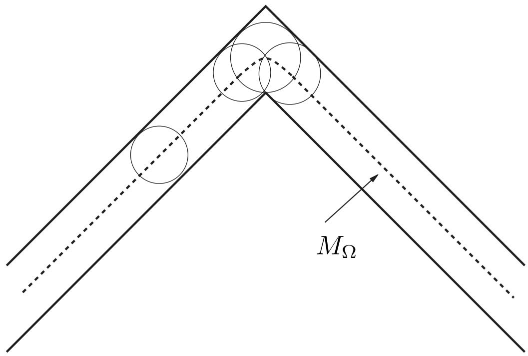

Fig. 5. Medial axis of piecewise linear sets is not piecewise linear

Proposition 2.1. *If Ω is semialgebraic, then $\mathcal{S}_\Omega$ and $\mathcal{M}_\Omega$ are also semialgebraic.*

As a direct consequence of Theorems 2.1 and 4.2, we mention the following result which is close to some result appearing in [13].

Proposition 2.2. *Let Ω be a relatively compact semianalytic open subset of $\mathbb{R}^2$, then both $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$ are bounded semianalytic curves.*

Remark 2.2. We complete this section with an other reference to [13]. In this work, the authors define a set called *medial axis tranform*. It is as follows:

$$MAT_\Omega = \{(p, r) \in \Omega \times (\mathbb{R}_+ \cup \{0\}) \mid B_r(p) \in \mathcal{C}(\Omega)\}.$$

It corresponds to our set $\mathcal{ST}$ in the proof of Theorem 2.1 and, therefore, it is also subanalytic. The finiteness results of [13] for the topological properties of MAT_Ω are thus consequences of Theorem 2.1 and the topological properties of subanalytic sets.

Remark 2.3. The previous result remains clearly true in semialgebraic setting: the medial axis of a semialgebraic set is semialgebraic. Nevertheless it does not seem easy to find a smaller class of sets such that the medial axes remain in the same class. For example, medial axes of piecewise linear sets are not piecewise linear: the medial axis of the polygonal line in Fig. 5 contains a piece of parabola. More generally, the medial axis of a polygonal domain in the plane (resp. of a polyhedron) is piecewise conic (resp. piecewise quadratic) and its effective computation is usually not an easy task (see [16]).

2.2. **Finiteness properties.** From subanalyticity of medial axis and skeleton (Theorem 2.1) and from finiteness properties of subanalytic sets (Sec. 4), we deduce several properties of $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$.

Corollary 2.1. *Let Ω be as in the previous section.*

1. *The sets $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$ are stratifiable of the same dimension $d \leq \dim(\partial\Omega)$ and have finite d-volumes. Moreover, if the number of maximal balls which intersect $\partial\Omega$ in an infinite number of points is finite, then $d = \dim(\partial\Omega)$.*
2. *If $\Omega \subset \mathbb{R}^3$, the boundaries $\partial\mathcal{M}_\Omega$ and $\partial\mathcal{S}_\Omega$ are finite unions of compact semianalytic curves and points. Hence they have finite length. More generally, in higher dimensions, they are compact subanalytic sets of dimension strictly less than dimension of $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$.*

Proof. This is an immediate consequence of Theorem 2.1 and the subanalytic version of Theorem 4.1. Just note that the *semianalyticity* of $\partial\mathcal{M}_\Omega$ and $\partial\mathcal{S}_\Omega$ is a consequence of Theorem 4.2. □

The following result is also a consequence of the subanalyticity of $\mathcal{S}_\Omega$. It is useful for the study of Delaunay triangulations [1, 2] and gives a positive answer to a question of J.-D. Boissonnat to one of the authors.

Proposition 2.3. *Let $C \subseteq \overline{\mathcal{S}_\Omega}$ be a compact semianalytic curve and $C_{\partial\Omega}$ be the set of intersection points of $\partial\Omega$ with maximal spheres with centers on C. Then $C_{\partial\Omega}$ is a subanalytic set. Moreover, if there is only a finite number of points $p \in C$ such that the intersection of the maximal ball centered at p with $\partial\Omega$ is infinite, then $C_{\partial\Omega}$ is a finite union of semianalytic curves of finite length.*

Proof. The function $d_{\partial\Omega}$ defined by $d_{\partial\Omega} = d(p, \partial\Omega)$ is a continuous subanalytic function (Lemma 4.1). Hence the set

$$X = \{(p, q) \in \partial\Omega \times \overline{\Omega} \mid \|q - p\|^2 = d(p, \partial\Omega)^2\} \cap \partial\Omega \times C$$

is subanalytic. Its projection under $\pi : \partial\Omega \times \overline{\Omega} \to \partial\Omega$ is exactly $C_{\partial\Omega}$. Therefore, $C_{\partial\Omega}$ is a compact subanalytic set of dimension 1. Its length is finite. □

3. Continuous variation of medial axis and skeleton

In this section, we study the variation of medial axis and skeleton under some deformations. Of course, a small $\mathcal{C}^0$-perturbation of the open set Ω can lead to huge variations of $\mathcal{M}_\Omega$ and $\mathcal{S}_\Omega$. The reader may easily be convinced by taking a small $\mathcal{C}^0$-perturbation of an Euclidean ball so that the medial axis jumps from one point to a segment whose length can be at least equal to the radius of the ball. A small $\mathcal{C}^1$-perturbation can also lead to huge variations of $\mathcal{M}_\Omega$ and S_Ω (an example is given at the end of the section). Hence we look for conditions of regularity under which the medial axis and the skeleton vary continuously in the Hausdorff topology. It appears that the $\mathcal{C}^2$-topology on the spaces of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ is the good one to consider.

3.1. $\mathcal{C}^r$**-Topology.** Let us briefly introduce a topology on the function spaces $\mathcal{C}^r(U,V)$ of $\mathcal{C}^r$-mappings from U to V where U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For more information and more general settings, see [22].

Let $f \in \mathcal{C}^r(U,V)$, $K \subset U$ be a compact set such that $f(K) \subset V$, and $\varepsilon > 0$. We define a basic neighborhood of f in $\mathcal{C}^r(U,V)$ to be the set $\mathcal{N}^r(f,K,\varepsilon)$ of mappings $g \in \mathcal{C}^r(U,V)$ such that $g(K) \subset V$ and, for all $x \in K$ and all $k = 0,\dots,r$:

$$\|D^k(f)(x) - D^k(g)(x)\| < \varepsilon,$$

where $D^k(f)$ denotes the kth derivative of f. The *weak* $\mathcal{C}^r$*-topology* on $\mathcal{C}^r(U,V)$ is generated by these sets. A neighborhood of f in this topology is thus a set which contains the intersection of a finite number of sets of the type $\mathcal{N}^r(f,K,\varepsilon)$.

In the sequel, we will actually consider only the case where $n = m$ and U and V are relatively compact. In such a situation, we consider compact sets K and K' such that $U \subset \operatorname{int}(K) \subset K$ and $V \subset \operatorname{int}(K') \subset K'$ and the space of mappings $\mathcal{C}^r(K,K')$. We also must restrict ourselves to diffeomorphisms which are close to the identity. We thus give the following notation: let $D^r(K,\varepsilon)$ be the set of mappings $g \in \mathcal{C}^r(K,K)$ such that g is a $\mathcal{C}^r$-diffeomorphism from $\operatorname{int}(K)$ to $\operatorname{int}(K)$ and g and g^{-1} belong to the neighborhood $\mathcal{N}^r(\mathrm{Id},K,\varepsilon)$ of the identity mapping in the weak topology.

3.2. **Continuous variation of medial axis and skeleton of ball.** We begin with the study of the effects of $\mathcal{C}^2$-perturbation on Euclidean balls. Results for balls are used to give a general result in the next section. In what follows, let K be a compact subset of $\mathbb{R}^n$.

Definition 3.1. Let $\varepsilon > 0$. One says that $\Omega' \in \operatorname{int}(K)$ is an ε small $\mathcal{C}^2$-perturbation of $\Omega \in \operatorname{int}(K)$ if there exists a diffeomorphism $f \in D^2(K,\varepsilon)$ such that $\Omega' = f(\Omega)$.

We start from the following general lemma of differential geometry.

Lemma 3.1. *Let $B_0 \subset \operatorname{int}(K)$ be an Euclidean ball of radius $R_0 > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in D^2(K,\delta)$, if $V = f(B_0)$, then at each point $m \in \partial V$, the principal curvatures K_i, $i = 1,\dots,n-1$, of ∂V satisfy the following inequality*:

$$\left|\frac{1}{K_i} - R_0\right| \le \varepsilon.$$

Proof. Since the function f is a diffeomorphism, $\Sigma = \partial V$ is a smooth hypersurface. The problem is local on Σ, therefore, let us consider a point $m \in \Sigma$ and let $m_0 = f^{-1}(m)$. Let φ be a local parametrization of B_0 in a neighborhood of m_0 such that the matrix of the second fundamental form II_0 of B_0 in the coordinates $(u_1, \cdots, u_{n-1})$ has the form

$$\mathrm{II}_0(0, \cdots, 0) = \mathrm{diag}\left(\frac{1}{R_0}, \cdots, \frac{1}{R_0}\right).$$

Note that $f \circ \varphi$ is a local parametrization of Σ in a neighborhood of m and denote by $\mathrm{II}(u_1, \cdots, u_{n-1})$ the matrix of the second fundamental form of Σ. Principal curvatures of Σ are eigenvalues of II on Σ. Thus, given $\varepsilon > 0$, there exists $\alpha > 0$ such that if

$$\|\mathrm{II}(u_1, \cdots, u_{n-1}) - \mathrm{II}_0(u_1, \cdots, u_{n-1})\| < \alpha,$$

then the eigenvalues of $\mathrm{II}(u_1, \cdots, u_{n-1})$ satisfy the required inequalities. To conclude the proof recall that the expression of the second fundamental form of a hypersurface in $\mathbb{R}^n$ involves only first and second derivatives of a local parametrization (see, e.g., [12, 21]). Therefore, there exists $\delta > 0$ such that $f \in D^2(K, \delta)$ implies $\|\mathrm{II}(u_1, \cdots, u_{n-1}) - \mathrm{II}_0(u_1, \cdots, u_{n-1})\| < \alpha$. ☐

The following lemma is a consequence of a particular case of a general result of Hadamard [21, p. 225].

Lemma 3.2. *Let $B_0 \subset \mathrm{int}(K)$ be an Euclidean ball of radius $R_0 > 0$. There exists a constant $\delta(R_0)$ such that for all $f \in D^2(K, \delta(R_0))$, the set $V = f(B_0)$ is convex.*

Proof. We apply Lemma 3.1 together with some Morse theory [28]. Let $\varepsilon > 0$ be such that $\varepsilon < R_0$ and $\delta(R_0) > 0$ be such that conclusion of Lemma 3.1 holds for ε and $\delta(R_0)$. If $f \in D^2(K, \delta(R_0))$, then all the principal curvatures of the boundary of $V = f(B_0)$ remain positive. Assume that V is not convex. Then there exists a contact hyperplane H (i.e., tangent hyperplane) to ∂V which has at least two intersection points with ∂V. Let us consider the affine line Δ, orthogonal to H and passing through the tangency point between H and ∂V. Denote by $h : \partial V \to \Delta$ the projection parallel to H.

We have $b_0(h^{-1}(\alpha)) \geq 2$ for sufficiently small $\alpha > 0$ (b_0 denotes the number of connected components). Hence, if h has no critical point of saddle type, the Morse theory tells us that ∂V must have at least two connected components, a contradiction. Then h has at least one critical point of saddle type. In such a point, at least one of the principal curvatures of ∂V is nonpositive and this also gives a contradiction. ☐

The following remark will be useful to prove the general stability result in the next section.

Remark 3.1. Let $(B_i)_{i \in I}$ be a family of balls whose radii satisfy the inequality $C \leq R_i \leq C'$ for some nonnegative constants $C, C' > 0$. Given $\varepsilon > 0$, one can choose a common $\delta > 0$ such that previous lemmas are satisfied for all balls B_i.

The next statement is known as the Blaschke theorem [8, 11].

Theorem 3.1 (Blaschke). *Let $V \subset \mathbb{R}^n$ be a relatively compact convex open set. Let $\Sigma_0 = \partial V$ and assume that Σ_0 is a $\mathcal{C}^2$-manifold. Assume that the maximum K of the principal curvatures at any point of Σ_0 is finite. Then for all $0 < \varepsilon < 1/K$, the Euclidean ball of radius $R - \varepsilon = 1/K - \varepsilon$ can roll freely on Σ_0 in the interior of V. More precisely, for all $x \in \Sigma_0$, the ball of radius $R - \varepsilon$ which is tangent to Σ_0 at x and whose center points are inside Σ_0 is contained in $\overline{V}$ and has only x as an intersection point with Σ_0.*

The classical proof of this result is based on a reduction to the two-dimensional case and the use of the support set of a convex function (see [23, p. 1055]). We give a direct proof here.

Proof. Let $t \in \mathbb{R}_+$ and let us consider the mappings $\varphi_t : \Sigma_0 \to \mathbb{R}^n$ defined as follows:

$$\varphi_t(x) = x - tN(x),$$

where $N(x)$ is the unitary vector normal to Σ_0 at x, pointing outside V. We denote $\Sigma_t = \varphi_t(\Sigma_0)$. From classical results of differential geometry (see, e.g., [12, 21]), we know that there exists $\varepsilon > 0$ such that if $0 \le t < \varepsilon$, then φ_t is a diffeomorphism between Σ_0 and Σ_t. Actually, we prove that in the convex situation of the theorem, we can choose $\varepsilon = R$.

First, since the differential of φ_t at $x \in \Sigma_0$ is $\mathrm{Id} - tDN(x)$, we can write it in suitable coordinates as follows:

$$D\varphi_t(x) = \mathrm{diag}(1 - tk_1(x), \ldots, 1 - tk_{n-1}(x)), \tag{1}$$

where the $k_i(x)$'s are the principal curvatures of Σ_0 at x. Hence if $tK < 1$, then $D\varphi_t(x)$ has positive determinant for all $x \in \Sigma_0$ which means that φ_t is a local diffeomorphism.

Second, let us compute the 2nd fundamental form of Σ_t. The Gauss mapping $N : \Sigma_t \to S^{n-1}$ associates the unitary vector $N(y)$ normal to Σ_t at y with any $y \in \Sigma_t$. But for $t < \varepsilon$, we have $y = \varphi_t(x)$ with $x \in \Sigma_0$ and, moreover, $N(x) = N(y) = N(\varphi_t(x))$. Differentiating this last equality, we obtain:

$$DN(x) = DN(\varphi_t(x)) \circ D\varphi_t(x).$$

Therefore,

$$DN(y) = DN(x) \circ (D\varphi_t(x))^{-1} = \mathrm{diag}\left(\frac{k_1}{1 - tk_1}, \ldots, \frac{k_{n-1}}{1 - tk_{n-1}}\right).$$

If $tK < 1$, then the principal curvatures of Σ_t at each point remain strictly positive. From Lemma 3.2 we deduce that Σ_t is convex.

Now assume that there exists $t_0 > 0$ such that $t_0 K < 1$, for all $t < t_0$, the mapping φ_t is a diffeomorphism, and φ_{t_0} is not bijective. This means that there exist $x, y \in \Sigma_0$, $x \neq y$, such that $\varphi_{t_0}(x) = \varphi_{t_0}(y) = z \in \Sigma_{t_0}$. Since Σ_0 is (strictly) convex (its Gauss curvature is nowhere zero), its corresponding

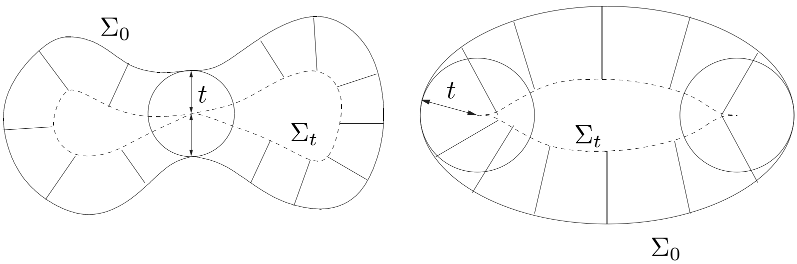

Fig. 6. Σ_t in nonconvex and convex case

Gauss mapping is bijective and onto. Thus we have $N(x) \neq N(y)$. This implies that the Gauss mapping $N : \Sigma_{t_0} \to S^{n-1}$ is singular at z. Hence the mapping φ_{t_0} must have z as a singular value. Since $t_0 < 1/K$, this contradicts (1). It follows that φ_t is a diffeomorphism for all $t < 1/K$ and Σ_t is strictly convex.

To complete the proof, it suffices to note that for all $0 < \varepsilon < 1/K$, since the mapping $\varphi_{1/K-\varepsilon}$ is a diffeomorphism, the ball of radius $1/K - \varepsilon$ which is tangent to Σ_0 at a point x has only x as an intersection point with Σ_0. $\square$

From the Blaschke theorem and the previous lemma, we can state the following result on the continuous variation of the medial axis of balls.

Proposition 3.1. *Let $B_0 \subset \mathrm{int}(K)$ be an Euclidean ball of radius $R_0 > 0$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in D^2(K, \delta)$ we have*:

$$d(\mathcal{M}_{B_0}, \mathcal{M}_{f(B_0)}) \leq \varepsilon,$$

where $d(\cdot, \cdot)$ denotes the Hausdorff distance between subsets of $\mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ be fixed. By Lemma 3.1, there exists $\delta > 0$ such that if $f \in D^2(K, \delta)$, all the principal curvatures of $V = f(B_0)$ satisfy the inequality

$$\left| \frac{1}{k} - R_0 \right| \leq \frac{\varepsilon}{2}.$$

We also may choose δ so that

$$\delta < \max(\varepsilon, \varepsilon/(2R_0)). \tag{2}$$

Let x_0 be the center of B_0 and y_0 be a point of $\mathcal{M}_V$. We will estimate the distance $d(x_0, y_0)$. To do this, we consider $x \in \partial B_0$ and $y = f(x)$ such that y_0 is the center of the maximal ball inscribed in V and tangent to ∂V at y. Since f is $\mathcal{C}^2$-close to the identity, up to a reduction of δ, we can assume that $\tan((x - x_0), (y - y_0)) \leq \delta$ (the tangent spaces to ∂B_0 and ∂V are δ-close). By Lemma 3.2 and the Blaschke theorem, the set V is convex and contains the ball of radius $R_0 - \varepsilon$ which is tangent to ∂V at y. This means that

$$d(y, y_0) \geq R_0 - \varepsilon.$$

If we apply the same argument to the mapping f^{-1}, we see that if the inequality $d(y, y_0) > R_0 + \varepsilon$ holds, then the ball B_0 is contained in a ball of radius $R_0 + \varepsilon/2$ inscribed in $f^{-1}(V)$ which is a contradiction. Hence we obtain:

$$R_0 - \varepsilon \leq d(y, y_0) \leq R_0 + \varepsilon.$$

Consider now $z_0 \in [y, y_0[$ such that $d(z_0, y) = R_0$ and the point x'_0 such that $[y, x'_0[$ is parallel to $[x, x_0[$ and $d(y, x'_0) = R_0$. This means in particular that $d(x_0, x'_0) \leq \delta$. A straightforward computation gives

$$d(z_0, x'_0) \leq 2\delta R_0,$$

from which we obtain the following estimate:

$$d(x_0, y_0) \leq d(x_0, x'_0) + d(x'_0, z_0) + d(z_0, y_0) \leq \delta + 2\delta R_0 + \varepsilon. \tag{3}$$

From (3) and (2), we deduce $d(x_0, y_0) \leq 3\varepsilon$. This implies that the Hausdorff distance between $\mathcal{M}_{B_0}$ and $\mathcal{M}_V$ is smaller than 3ε and completes the proof. □

Remark 3.2. In the previous propositions, the real δ depends on R_0 in the following way: it is chosen so that $\delta < \max(\varepsilon, \varepsilon/(2R_0))$. Hence if one considers a family of balls whose radii remain bounded, then one can choose a common $\delta > 0$ for all balls.

3.3. **Continuous variation of medial axis and skeleton: general case.** Let us now turn to the general case. Let K be a compact subset of $\mathbb{R}^n$ and let $\Omega \subset \text{int}(K)$ be an open set such that $\partial\Omega$ is a $\mathcal{C}^2$-manifold. The following result shows that the medial axis of Ω varies continuously under small $\mathcal{C}^2$-perturbations.

Theorem 3.2. *For all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $\delta < \eta$ and all $f \in D^2(K, \delta)$, the following inequality holds*:

$$d(\mathcal{M}_\Omega, \mathcal{M}_{f(\Omega)}) \leq \varepsilon.$$

Proof. Let $\mathcal{C}(\Omega)$ be the core of Ω (Definition 2.2) and let $\varepsilon > 0$ be fixed. Since the hypersurface $\partial\Omega$ is $\mathcal{C}^2$, there exist constants $m, M > 0$ such that the radius R of any ball contained in $\mathcal{C}(\Omega)$ satisfies $m < R < M$. Hence it follows from Remarks 3.1 and 3.2 that one can choose $\delta > 0$ so that propositions and lemmas of previous section remain valid for ε and δ and for all balls in $\mathcal{C}(\Omega)$.

Let $f \in D^2(K, \delta)$, $x_0 \in \mathcal{M}_\Omega$, and B_0 be the maximal ball of radius $R_0 > 0$ centered at x_0. Let $x \in B_0 \cap \partial\Omega$ be an intersection point between B_0 and $\partial\Omega$ and let $y = f(x)$. It follows from Theorem 3.1 that the ball B'

maximal in $f(B_0)$ and passing through y is of radius $R_0 - \varepsilon$ at least. Let B'' be the maximal ball in $f(\Omega)$ containing B'.

Claim. *The radius of B'' is less than $R_0 + 2\varepsilon$.*

Proof. Assume that the radius of B'' is greater than $R_0 + 2\varepsilon$. Therefore, $f^{-1}(B'') \subseteq \Omega$ is a convex set and its boundary contains $x = f^{-1}(y)$. Since the diffeomorphism f is in $D^2(K, \delta)$, the principal radii of curvature at any point of the boundary of $f^{-1}(B'')$ are greater than $R_0 + \varepsilon$. The ball of radius $R_0 + \varepsilon$ passing through x and tangent to $\partial f^{-1}(B'')$ is contained in $f^{-1}(B'') \subseteq \Omega$ and is tangent to B_0. Hence it strictly contains B_0 which is a maximal ball in Ω, a contradiction. □

Now it follows from Proposition 3.1 that

$$d(x_0, \mathcal{M}_{f(B_0)}) \leq \varepsilon.$$

Hence the distance between x_0 and the center y_0 of B' is less than ε. From the previous claim, one deduces that the distance between y_0 and the center z_0 of the ball B'' is less than 3ε. Finally, one has

$$d(x_0, z_0) \leq 4\varepsilon.$$

Therefore, any point of $\mathcal{M}_\Omega$ is at a distance at most 4ε from a point in $\mathcal{M}_{f(\Omega)}$.

In the same way using f^{-1}, one deduces that any point in $\mathcal{M}_{f(\Omega)}$ is at a distance at most 4ε from a point in $\mathcal{M}_\Omega$, which concludes the proof. □

Note that the fact that $\partial\Omega$ has $\mathcal{C}^2$-smooth boundary is only used to prove that the radii of balls in $\mathcal{C}(\Omega)$ are bounded from below by a positive constant.

Definition 3.2. Let Ω be a bounded open subset of $\mathbb{R}^n$. The local feature size of Ω is the Hausdorff distance between $\partial\Omega$ and $\mathcal{M}_\Omega$.

Local feature size is a widely used tool for the study of Voronoi diagrams and Delaunay triangulations. Finally, Theorem 3.2 admits the following generalization.

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with positive local feature size. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-$\mathcal{C}^2$-perturbation f,*

$$d(\mathcal{M}_\Omega, \mathcal{M}_{f(\Omega)}) \leq \varepsilon.$$

Remark 3.3. Since we have the following inclusions: $\mathcal{M} \subset \mathcal{S} \subset \overline{\mathcal{M}}$, Theorem 3.2 is also true for the skeleton.

To conclude this section, we give an example showing that the medial axis is unstable under small $\mathcal{C}^1$ perturbations. Let $\varepsilon > 0$ be arbitrarily small and Ω be the open set whose boundary is the square of the plane whose vertices are the points with the coordinates $(-10, 0)$, $(10, 0)$, $(10, -10)$, $(-10, -10)$

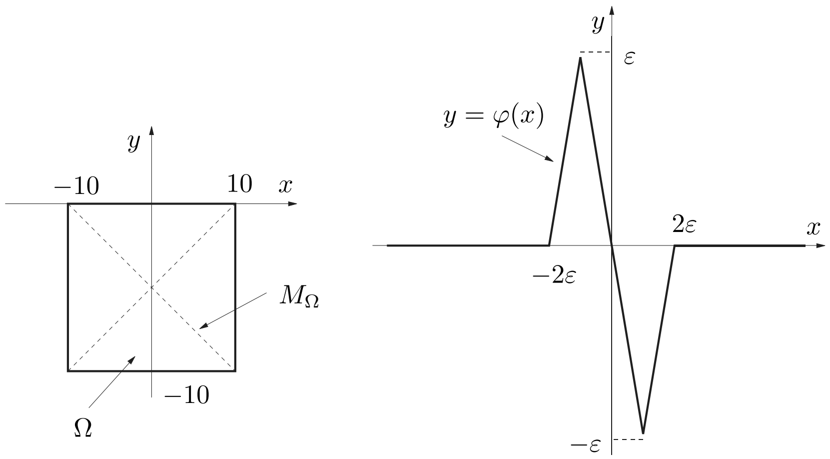

Fig. 7. $\mathcal{C}^1$-perturbation of a square

(the corners of $\partial\Omega$ can be "smoothed" if one prefers to work with $\mathcal{C}^\infty$ objects). Consider the $\mathcal{C}^1$ function $\varphi : \mathbb{R} \to \mathbb{R}$ whose derivative is the piecewise linear function given in Fig. 7. Its second derivative is equal to $1/\varepsilon$ on $[-2\varepsilon^2, 0]$, to $-1/\varepsilon$ on $[0, 2\varepsilon^2]$, and to 0 otherwise. The $\mathcal{C}^1$ diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, y + \varphi(x))$ is an ε perturbation in the $\mathcal{C}^1$ topology. Note that the medial axis of Ω is at a distance greater than 2 from the origin $(0, 0)$. The boundary of $f(\Omega)$ contains the piece of parabola $y = g(x) = \varepsilon^3/2 - x^2/\varepsilon$, $x \in [-\varepsilon^2, \varepsilon^2]$. The curvature of $\partial f(\Omega)$ at the point $m = (0, \varepsilon^3/2)$ is equal to $1/\varepsilon$. Therefore, there is a point of the skeleton (which is contained in the closure of the medial axis) of $f(\Omega)$ at a distance less than ε from m and at a distance less than 2ε from the origin.

4. Appendix: a short introduction to real geometry

The aim of this section is just to give an introduction and to present some basic and classical results (mostly without proof) from real analytic geometry which are necessary to our investigation of medial axis and skeleton. Such results can be considered as generalizations of classical results in real algebraic geometry [5]. For a more detailed introduction, see [6,7,24,26,27].

4.1. **Semianalytic geometry.** In the sequel, V is a real analytic manifold. In most applications, V is the usual n-dimensional Euclidean space $\mathbb{R}^n$.

Definition 4.1. A set $X \subseteq V$ is a *semianalytic* subset of V if for any $a \in V$, there exists a neighborhood U of a such that

$$X \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} X_{i,j},$$

where $X_{i,j} = \{f_{i,j} = 0\}$ or $X_{i,j} = \{f_{i,j} > 0\}$ and $f_{i,j}$ is a real analytic function on U.

In other words, a semianalytic set is *locally* defined by analytic equations and inequalities. It is very important to note that X must be defined by analytic equations and inequalities in a neighborhood of any point of V and not only of X. For example, the curve C given by the equation

$$y = \sin\frac{1}{x} \cdot \exp\left(-\frac{1}{x^2}\right), \qquad x > 0,$$

is defined by an analytic equation in a neighborhood of each of its points but is not a semianalytic subset of $\mathbb{R}^2$. It is not possible to define C by an analytic equation in any neighborhood of the origin $(0,0)$.

Example 4.1. Let X be the plane curve defined as follows: X is the union of the pieces of curves

$$y = \begin{cases} 1 & \text{if } \ -1 \le x \le 0, \\ \exp(x) & \text{if } \quad 0 \le x \le 1, \\ 1 + 1/2(e-1)(x+1) & \text{if } \ -1 \le x \le 1. \end{cases}$$

It is an easy exercise to show that X is semianalytic.

More generally, every piecewise analytic curve, as defined in [13], is a semianalytic curve.

Example 4.2. The surface $X \subset \mathbb{R}^3$ defined by the equation

$$z = \frac{1}{x^2 + y^2} \sin(x^2 + y^2)$$

is analytic. Its intersection with the horizontal plane $\{z = 0\}$ is clearly a semianalytic subset of $\mathbb{R}^3$. It is an infinite union of disjoint circles. It has an infinite number of connected components. Therefore, general semianalytic sets do not have the same finiteness properties as semialgebraic sets. Nevertheless, for relatively compact semianalytic sets, these finiteness properties still hold.

In general, the linear projection of a semianalytic set is no longer a semianalytic set. The following set (see [6, pp. 10–11]), known as Osgood's example, is not semianalytic although it is obtained by projection of a semianalytic set:

$$X = \{(x,y,z) \in \mathbb{R}^3 \mid \exists (u,v) \in \mathbb{R}^2,\ x = u,\ y = uv,\ z = uve^v\}.$$

This is the reason why in the next section we consider a larger class of subsets for which the Tarski–Seidenberg theorem (i.e., stability under linear projection) holds.

We sum up some useful properties of semianalytic sets in the following theorem (for the definition of the *stratification*, see [5, p. 68]).

Theorem 4.1. *Let V be a real analytic manifold and X be a relatively compact semianalytic subset of V. We have*:

1. *X is locally connected, its closure $\overline{X}$, its interior* $\mathrm{int}(X)$*, its boundary ∂X, and its complement $V \setminus X$ are semianalytic subsets of V*;
2. *X admits a finite semianalytic stratification* (*each stratum is a semianalytic subset of V and an analytic submanifold of V*);
3. *the number of connected components of X is finite*;
4. *the volume of X is finite*;
5. *the boundary ∂X has dimension strictly less than* $\dim(X)$.

4.2. **Projection of semianalytic sets: subanalytic sets.** The smallest class of subsets of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, which contains relatively compact semianalytic sets and which is stable under linear projection is the class of *subanalytic sets*. They are defined as follows.

Definition 4.2. A subset X of a real analytic manifold V is *subanalytic* if for each point $a \in V$, there exists a neighborhood U of a such that $X \cap U$ is a linear projection of a relatively compact semianalytic set. More precisely, there exists a real analytic manifold W and a relatively compact semianalytic subset Y of $V \times W$ such that $X \cap U = \pi(Y)$, where $\pi : V \times W \to V$ is the canonical projection.

In other words, a subanalytic set is *locally* the projection of a semianalytic set. Note that if X and Y are two subanalytic sets, then $X \times Y$ is also a subanalytic set.

Now, we have a new larger class of sets wich satisfy Tarski–Seidenberg theorem. But we have a new problem: do the finiteness properties of semianalytic sets remain valid for subanalytic sets? Fortunately, the answer is yes! But the proofs are not easy. The most important difficulty is to show that the complement of a subanalytic set is still a subanalytic set. This result belongs to Gabrielov [20].

Theorem 4.1 is thus true when replacing the word "semianalytic" by the word "subanalytic" and adding the Tarski–Seidenberg property of stability under linear projections. Moreover, we have the following result due to Lojasievicz [26]. An elementary proof is also given in [24].

Theorem 4.2. *Each subanalytic set of dimension not greater than* 1 *and each subanalytic subset of an analytic manifold of dimension not greater than* 2 *is semianalytic.*

Let us recall the notion of *subanalytic function*.

Definition 4.3. A mapping $f : X \to Y$ between two subanalytic (respectively, semianalytic) sets X and Y is said to be subanalytic (respectively, semianalytic) if its graph $G = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is a subanalytic (respectively, semianalytic) subset of $X \times Y$.

Recall that a mapping is a *proper mapping* if the inverse image of any compact set is compact. We know yet that images of relatively compact subanalytic sets under linear projection are subanalytic. In fact, we have a more general statement: *The image of any subanalytic set under a proper subanalytic mapping is subanalytic.*

The stability of subanalytic sets under linear projections and complements is equivalent to the so-called existence of *quantifier elimination* for this class of subsets. This notion (coming from logic) can be illustrated as follows: let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a subanalytic function. Then we know that the set $A = \{(x, y) \in \mathbb{R}^{n+m} \mid F(x, y) \,?\, 0\}$ is subanalytic (where $? \in \{=, >, \geq\}$). Let us define:

$$B = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ F(x, y) \,?\, 0\}$$

and

$$C = \{x \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^m,\ F(x, y) \,?\, 0\}.$$

The sets B and C are said to be obtained from A by quantifying a variable (the variable y in this example). Moreover, we have:

$$B = \pi(A), \quad C = \pi(A^c)^c,$$

where π denotes the canonical projection from $\mathbb{R}^n \times \mathbb{R}^m$ on $\mathbb{R}^n$. Hence B and C remain subanalytic.

This property of quantifier elimination is also a useful tool to show that a set is subanalytic. As a good illustration, the reader can easily prove that if A is subanalytic, then its closure $\overline{A}$, its interior $\operatorname{int}(A)$, its boundary, and its frontier remain subanalytic.

We complete this section by an easy application of previous results which is useful in this paper.

Lemma 4.1. *If X is a compact subanalytic subset of $\mathbb{R}^n$ and if $d_X : \mathbb{R}^n \to \mathbb{R}$ is the distance function defined by $d_X(p) = \inf\{\|p - q\| \mid q \in X\}$, then d_X is a continuous subanalytic function.*

Proof. The continuity of d_X is an immediate consequence of the compactness of X. The graph of the mapping d_X is defined by

$$\begin{aligned} G = \{(p, r) \in \mathbb{R}^n \times \mathbb{R} \mid \forall q \in X,\ \|p - q\|^2 - r^2 \geq 0 \\ \text{and} \quad \exists q' \in X,\ \|p - q'\|^2 = r^2\}. \end{aligned}$$

The following subsets of $\mathbb{R}^n \times X \times \mathbb{R}$,

$$U_1 = \{(p, q, r) \in \mathbb{R}^n \times X \times \mathbb{R} : \|p - q\|^2 = r^2\},$$
$$U_2 = \{(p, q, r) \in \mathbb{R}^n \times X \times \mathbb{R} : \|p - q\|^2 < r^2\}$$

are clearly subanalytic. Let $\pi : \mathbb{R}^n \times X \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ be the canonical linear projection. The graph of d_X is defined by $G = \pi(U_1) \cap \pi(U_2)^c$, where $\pi(U_2)^c$ is the complement of $\pi(U_2)$ in $\mathbb{R}^n \times \mathbb{R}$. Therefore, G is a subanalytic set and d_X is a subanalytic mapping. □

4.3. **Subanalytic geometry and computational geometry.** The interest of real analytic geometry for computational geometry is the following: most general curves and surfaces encountered in applications are "piecewise analytic" (or even algebraic) and the geometrical objects associated with them (medial axis, skeleton, cut locus) belong in a very natural way to the subanalytic settings. The notion of piecewise analytic surface—or, more generally, piecewise analytic manifold—is usually used in computer scientists community without precise definition. The framework of subanalytic geometry allows to give a precise definition.

Definition 4.4. A (topological) submanifold X of an analytic manifold V is said to be piecewise analytic if it is a subanalytic set of V.

Such a definition is motivated by the fact that X admits an analytic stratification.

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