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Any open bounded subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis

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Abstract

Medial axis Transform is sometimes used as an intermediate representation in algorithms for meshing or recognition of shapes from digitized data. This raises the question whether the Medial Axis captures fundamental topological invariants of the object. The (positive) answer has been known already in the case of smooth objects. The main result presented here is *the homotopy equivalence of any bounded open subset of* $\mathbb{R}^n$ *with its medial axis*.

Keywords: Homotopy; Medial axis; Skeleton

1. Introduction

We present here the proof that *any bounded open subset* $\mathcal{O}$ *of* $\mathbb{R}^n$ *has the same homotopy type as its medial axis* $\mathcal{M}(\mathcal{O})$. We call *medial axis* of a bounded open set $\mathcal{O}$ the set of points of $\mathcal{O}$ which have at least two closest points on the boundary $\partial\mathcal{O}$. The definition of the *Homotopy equivalence* is reminded in Section 3.2 below. Homotopy equivalence enforces for example that connected open sets have connected medial axis, or that a bounded open set and its medial axis have the same numbers of cycles. For example, in Fig. 1, the open set and its medial axis, depicted in bold lines, are both connected and have exactly one cycle.

The topology of the medial axis transform without smoothness assumption is addressed in Ref. [12], where the author proves that the closure of the medial axis of a connected open set is connected and considers as 'rather plausible' that its medial axis itself is connected ([12], p. 217). This last fact is a corollary of Theorem 4.19 below. The homotopy equivalence has been previously proved in the particular cases of an open set with a *piecewise* C^2 boundary in the plane or an open set with a C^2 boundary in $\mathbb{R}^n$ [14], using a *deformation retract* technique, described in Section 2. This homotopy equivalence is also known in the case of the complement of a finite number of points in $\mathbb{R}^n$: in this case, the medial axis is the Voronoi diagram. These results are particular cases of Theorem 4.19.

It has been proved that, in the smoother case of a piecewise analytic boundary in the plane, the medial axis is made of a finite number of curves [3]. This finiteness property extends to subanalytic open sets in any dimensions [2]. Because we are considering here general open sets without any smoothness assumption, the medial axis may have a very complex structure (cf. Fig. 4). In this general situation, it is worth to carefully distinguish between several closely related objects, the medial axis, the skeleton and the cut locus [12,14]. The *skeleton* is the set of centers of maximal (maximal for the inclusion order) open balls included in the open set. The *cut locus* [14] of a closed set C is in our setting the closure of the medial axis of the open set C^c, complement of C. Note that, unfortunately, this terminology is not consistent among the literature. For example, some authors call medial axis what is referred here as the closure of the medial axis or consider medial axis or skeleton of closed sets, which should be understood, with our definitions, as the skeleton or medial axis of the (open) complement of the given closed set. The medial axis is a subset of the skeleton, which is itself a subset of the closure of the medial axis.

Section 2 mentions related works, including the deformation retract approach and its limitations. Sections 3 introduces basic tools and gives the general strategy of

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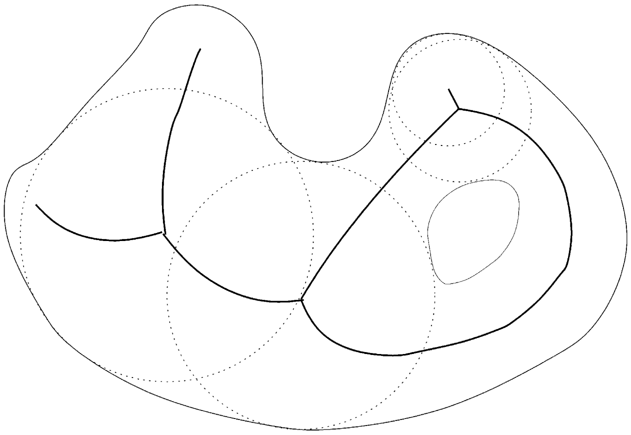

Fig. 1. A set and its medial axis.

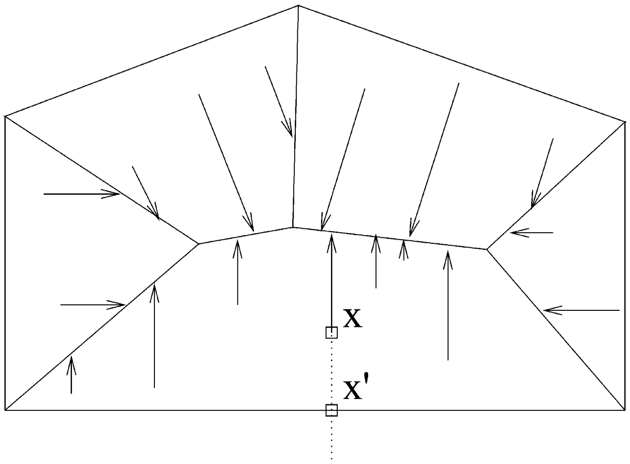

Fig. 2. The map r for some points in a convex polygon.

the proof. Then Section 4 details the lemmas and the complete proof.

2. Related work

Voronoi diagrams are strongly related to medial axis. Given a finite set S of points and a point $p \in S$, the closed set of points closer (or equally close) to p than to any other points of S is called the *Voronoi cell* of p. The boundaries of the Voronoi cells are made of *Voronoi faces* which are intersections of pairs of Voronoi cells. Each Voronoi face is a subset of the bisector plane of the corresponding pair of points. The union of these faces is the medial axis of the complement S^c of S. Our proof of the homotopy equivalence of an open set $\mathcal{O}$ and its medial axis $\mathcal{M}$ consists in constructing a continuous deformation $\mathfrak{C} : \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}$. In fact, in the particular case where $\mathcal{O}$ is the complement of a finite set, that is for Voronoi diagrams, this continuous deformation has been introduced also, independently, in Refs. [7,8] (Flow Diagram) and [4] (Flow relation between Delaunay simplices).

In Ref. [14], the proof of the Homotopy equivalence of the medial axis for smooth objects is based on a deformation retract. The definition of deformation retracts is recalled in Section 3.2. A map $r : \mathcal{O} \to \mathcal{O}$, is defined in the following manner [12,14]:

- If $X \in \mathcal{O} \setminus \mathcal{M}$, there is a unique point $X' \in \partial\mathcal{O}$ that minimizes the distance to X (see Fig. 2). $r(X)$ is the first point where the half line $X'X$ cuts the closure $\bar{\mathcal{M}}$ of $\mathcal{M}$.
- If $X \in \bar{\mathcal{M}}$, it is invariant by the action of $r : r(X) = X$.

In Fig. 2, the arrows represent the images of some points in $\mathcal{O} \setminus \mathcal{M}$ by the action of r. In Ref. [14], the author proves that r is well defined and continuous in the two following cases:

1. In $\mathbb{R}^n$ when the boundary $\partial\mathcal{O}$ is a C^2 $(n-1)$-manifold
2. In $\mathbb{R}^2$ when the boundary $\partial\mathcal{O}$ is a *piecewise* C^2 curve.

Indeed, for some 3D polyhedra with non convex faces, or 2D sets whose boundary is a C^1 only (not C^2) Jordan curve, the map r is not continuous.

Fig. 3 represents a box shaped 'room' with a depression in the floor in shape of an upside down pyramid. The edges of the medial axis of the 'room' are depicted in dotted lines. The vertical edges above vertices A, B, C and D are subsets of the medial axis boundary. Along these edges, r is not continuous. Indeed, the point p belongs to the medial axis and one has $r(p) = p$, while the image $r(q)$ of the point q is on the mid surface between the floor and the ceiling. Because such points p and q can be chosen arbitrarily close to the edge, the map r is not continuous. Therefore, the deformation retract approach with this map r cannot be used in the case of this 3D polyhedron.

In the case of Fig. 4, the lower boundary is the graph of the C^1 function

$$y = \begin{cases} 0 & \text{if } x \le 0 \\ x^3 \sin\dfrac{1}{x} & \text{if } x > 0 \end{cases}$$

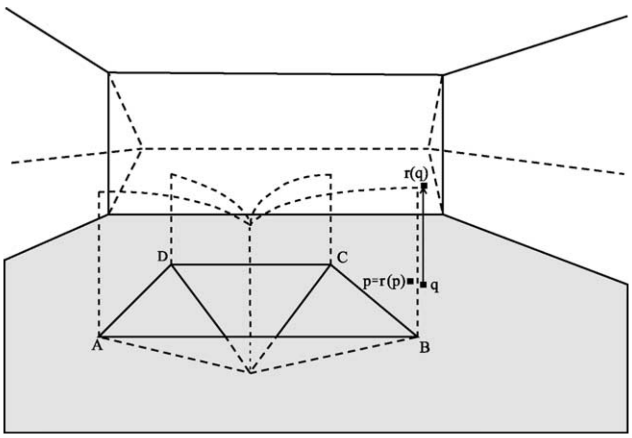

Fig. 3. The map r is not continuous along the edges above the points A, B, C and D.

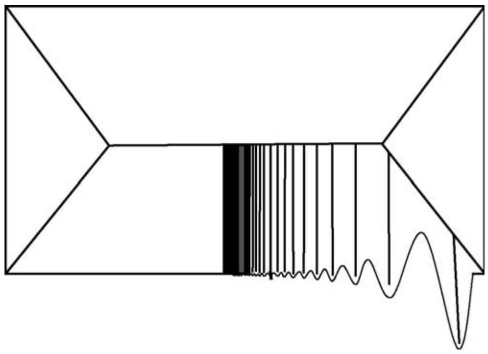

Fig. 4. The map r is not continuous at the limit line.

The map r is not continuous on the limit line, that is the line above the C^2 discontinuity.

3. Overview of the proof

3.1. Preliminary definitions

We use the following definitions and notations. $\mathcal{O}$ and $\mathcal{M}$ denote, respectively, a bounded open subset of $\mathbb{R}^n$ and its medial axis defined below. For any set $X, \bar{X}, X^o, \partial X$ and X^c denote, respectively, the closure, the interior, the boundary and the complement of X. $\mathbb{B}_{x,r}$ and $\mathbb{B}^o_{x,r}$, respectively, denote the closed and the open ball of center x and radius r in $\mathbb{R}^n$. We denote by $\mathbb{S}_{x,r}$ the corresponding sphere, that is $\mathbb{S}_{x,r} = \mathbb{B}_{x,r} \backslash \mathbb{B}^o_{x,r}$.

For any point $x \in \mathcal{O}$, we denote by $\mathbf{\Gamma}(x)$ the set of closest boundary points, that is:

$$\mathbf{\Gamma}(x) = \{y \in \partial\mathcal{O};\ \ d(x,y) = d(x,\partial\mathcal{O})\}$$

Because $\partial\mathcal{O}$ is compact, $\mathbf{\Gamma}(x)$ is a non empty compact set. For a set E, $|E|$ denotes the cardinal of E.

Definition 3.1. (Medial axis) The medial axis $\mathcal{M}$ of the open set $\mathcal{O}$ is the set of points x of $\mathcal{O}$ who have at least 2 closest boundary points:

$$\mathcal{M} = \{x \in \mathcal{O};\ |\mathbf{\Gamma}(x)| \geq 2\}$$

In Fig. 5, the medial axis is depicted in thick line. The strictly positive, real valued function $\mathcal{R}$ defined on $\mathcal{O}$ is the distance to the boundary:

$$\mathcal{R}(x) = d(x, \partial\mathcal{O})$$

It follows from the triangular inequality that $\mathcal{R}$ is 1-Lipschitz. For any $x \in \mathcal{O}$, there exists a unique closed ball of minimal radius enclosing $\mathbf{\Gamma}(x)$. Indeed, the existence follows from the compactness of the set of balls whose radius is bounded by a given value B and containing $\mathbf{\Gamma}(x)$. On another hand, if two distinct balls contain $\mathbf{\Gamma}(x)$ there exists another ball of strictly smaller radius enclosing it. This entails that the minimal ball is unique. The real valued, positive function $\mathcal{F}$ is defined as the radius of this smallest closed ball enclosing $\mathbf{\Gamma}(x)$. (cf. Fig. 5). In other words:

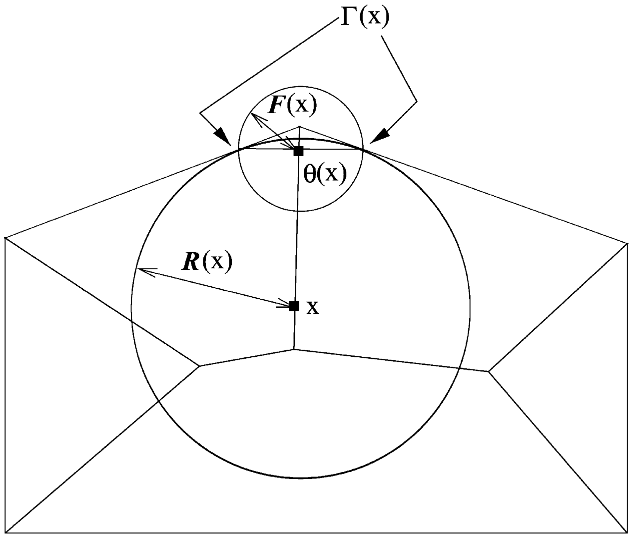

Fig. 5. A 2D example, with two closest points.

$$\mathcal{F}(x) = \inf\{r; \exists y \in \mathbb{R}^n, \mathbb{B}_{y,r} \supseteq \mathbf{\Gamma}(x)\}$$

Similarly, $\Theta(x)$ denotes the center of this smallest enclosing ball. When $x \notin \mathcal{M}$, we have $\mathbf{\Gamma}(x) = \{\Theta(x)\}$ and $\mathcal{F}(x) = 0$. Moreover, one has trivially:

$$x \in \mathcal{M} \Leftrightarrow \mathcal{F}(x) > 0$$

The vector function ∇, that plays a central role in the proof below, is an extension of the gradient of $\mathcal{R}$. The gradient of $\mathcal{R}$ is defined on $\mathcal{O} - \bar{\mathcal{M}}$ only [12]. Instead, $\nabla(x)$ is defined for all $x \in \mathcal{O}$ and coincides with the gradient of $\mathcal{R}$ when $x \in \mathcal{O} - \bar{\mathcal{M}}$. It is defined as follows:

$$\nabla(x) = \frac{x - \Theta(x)}{\mathcal{R}(x)}$$

Even if this is not formally proven at this point, ∇ is in the direction of 'steepest ascent' that is the direction that maximizes the growth of $\mathcal{R}$.

3.2. Homotopy equivalence

A commonly used technique to prove the homotopy equivalence between a set and one of its subsets is to build a deformation retract [11]. This is the technique used for the smooth case, the map r of Section 2 defining this deformation retract. We use instead a more general characterization of homotopy equivalence.

We first recall the definition of homotopy between maps. Two maps $f_0 : X \to Y$, and $f_1 : X \to Y$, are said *homotopic* if there is a continuous map $H, H : [0,1] \times X \to Y$, such that $\forall x \in X, H(0,x) = f_0(x)$ and $H(1,x) = f_1(x)$. The definition of homotopy equivalence can be found in Refs. [6, p. 171–172] or Ref. [13, p. 108]:

Two spaces X and Y are said to have the same homotopy type if there are continuous maps $f : X \to Y$ and $g : Y \to X$

such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y.

Having the same homotopy type enforces the existence of isomorphisms between respective homology and homotopy groups. Moreover, these isomorphisms are explicitly given by the maps f and g of the definition. The homotopy equivalence between topological sets enforces a one-to-one correspondence between connected components, cycles, holes, tunnels, cavities, or higher dimensional topological features of the two sets, as well as the way these features are related.

We are in the case where $Y \subset X$ and we take as g the canonical inclusion: $\forall y \in Y, g(y) = y$. Our proof use the following characterization.

Proposition 3.2. *If $Y \subset X$ and there exists a continuous map $H, H : [0,1] \times X \to X$ such that*:

(i) $\forall x \in X, H(0,x) = x$
(ii) $\forall x \in X, H(1,x) \in Y$
(iii) $\forall y \in Y, \forall t \in [0,1], H(t,y) \in Y$

then, X and Y *have same homotopy type*.

If, in the previous proposition, one replaces (iii) by the condition $\forall y \in Y, H(1,y) = y$, H defines a *deformation retract* of X towards Y.

In order to get a better intuition of the characterization of Proposition 3.2, the reader may want to check for example that this characterization enforces that X is arcwise connected if and only if Y is arcwise connected. Hints: if X is arcwise connected, considers two points in Y and the image by $x \mapsto H(1,x)$ of the path joining them in X. If Y is arcwise connected, consider a path in Y joining the images by $x \mapsto H(1,x)$ of two points of X.

3.3. Outline of the proof

Let us go back to the reason why the deformation retract technique described in Section 2 does not work in the general case. In the two mentioned examples, the non C^2 example and the polyhedron with non convex faces, the map r, kind of projection on the medial axis, is not continuous. The idea of the proof is to replace this projection by a continuous map that meet the conditions of Proposition 3.2. The intuition of this continuous map is the following. Consider a point x in $\mathcal{O}$, the map that realizes the homotopy $(t,x) \mapsto H(t,x)$ consists in moving from x along a path that locally maximize the increase of the distance to the boundary. The direction that maximise the increase of the distance to the boundary is the gradient of the distance function when x is not in the closure of $\mathcal{M}$. But if x is in the closure of $\mathcal{M}$, even if the distance function is not differentiable, there still exists a direction of steepest ascent, pointed to by the vector field ∇. The idea is to follow this direction. In special cases, such as polyhedron or Voronoi Diagrams, this defines $t \mapsto H(t,x)$ as a piecewise smooth path. The problem consists in finding a formal definition of the path $t \mapsto H(t,x)$ for general open sets. Indeed, general open sets may have very irregular boundaries and this path may be nowhere smooth. The whole proof consists in giving a formal definition of this flow and checking that it satisfies the desired properties for the homotopy equivalence characterisation 3.2. More formally, one builds a continuous map

$$\mathfrak{C} : \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}$$

This map integrates the vector field ∇ (Sections 4.2 and 4.3). In fact $t \mapsto \mathfrak{C}(t,x)$ is not differentiable in general, but it is 1-Lipschitz, right differentiable and its right derivative coincides with $\nabla(\mathfrak{C}(t,x))$. Moreover, one has $\mathfrak{C}(t+s,x) = \mathfrak{C}(s, \mathfrak{C}(t,x))$. In order to build the flow $\mathfrak{C}$, one considers Euler schemes based on the vector field ∇. $\mathfrak{C}$ is defined as the limit, for step size approaching 0, of these Euler schemes. The uniform convergence of these Euler schemes toward a function which is continuous in both variables is based on the fact that the vector field ∇ is semi-Lipschitz (Lemma 4.1). The semi-Lipschitz condition is weaker than the Lipschitz condition, as it holds for the not continuous vector field ∇. In order to be able to use this weaker property, we need to prove a generalization of the classical Cauchy–Lipschitz-Picard theorem [1,10] that states the existence and uniqueness of the solution of a differential equation under the assumption of a Lipschitz condition (Lemmas 4.4 and 4.5). Note that, at this point, the map $\mathfrak{C}$ is defined only locally, that is for small values of t : for any point x, there is a value $t_{\max} > 0$ and a neighborhood $N(x)$ of x such that $\mathfrak{C}$ is defined on $[0, t_{\max}] \times N(x)$.

Then, in Section 4.3, one states the right differentiability of the map $t \mapsto \mathfrak{C}(t,x)$. In the classical case of differential equations defined by *Lipschitz* functions, the fact that the vector field is the derivative of the flow that integrate it, results from the continuity of the vector field. Unfortunately one can only exploit here a kind of semi-continuity (Lemmas 4.6 and 4.8), to prove the right differentiability of $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ and that its right derivative is ∇, first by proving the convergence in direction (Lemma 4.9) and in modulus (Lemma 4.10). This entails the integral equations of Lemma 4.12. One of these equations gives explicitly the rate of growth of the $\mathcal{R}$ function along a path. It enforces in particular that $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ is increasing. It seems obvious that *following the steepest ascent direction* of $\mathcal{R}$ leads to increase $\mathcal{R}$, but the difficulty is hidden in the formal definition of 'following the steepest ascent direction'. Note that $\mathcal{R}$ is not increasing in general on Euler schemes, but only on the limit $\mathfrak{C}$. One consequence of the fact that the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ is increasing is the global existence of $\mathfrak{C}$. Informally, because $\mathcal{R}$ increases, one cannot get out of $\mathcal{O}$ and the flow can be followed 'forever'. This means that we have now a map $\mathfrak{C}$ defined globally:

$$\mathfrak{C} : \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}$$

Note that, in order to apply the characterization 3.2, one need from one hand that, after a finite 'time' T, $\mathfrak{C}(T,x)$ belongs to $\mathcal{M}$ and, for another hand, that, *once in* $\mathcal{M}$, *the path remains in* $\mathcal{M}$, that is, $\forall x \in \mathcal{M}, \forall t \geq 0, \mathfrak{C}(t,x) \in \mathcal{M}$. This last condition is required by the condition (iii) of characterization 3.2. For that, in Section 4.4, we prove that the map $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing (Lemmas 4.14–4.17), which immediately entails the desired property.

Finally, let D be an upper bound of the diameter of $\mathcal{O}$. We prove in Proposition 4.18 in Section 4.5 that, $\forall x \in \mathcal{O}$, $\mathfrak{C}(D,x) \in \mathcal{M}$ and we can define the map H,

$$H : [0,1] \times \mathcal{O} \to \mathcal{O}$$

given by:

$$H(t,x) = \mathfrak{C}(Dt,x)$$

which meets the characterization 3.2. It results that $\mathcal{O}$ and $\mathcal{M}$ have same homotopy type (Theorem 4.19 in Section 4.5).

4. Proofs

4.1. Definitions and first properties

For two sets A and B, we denote by $A \oplus B$ the Minkowski sum of A and B :

$$A \oplus B = \{x; \exists a \in A,\ \exists b \in B;\ x = a + b\}$$

The power of a point with respect to a sphere is the difference of the squared distance to the sphere's center and the square of the radius. The radical plane of two spheres is the locus of the points for which the powers with respect to both spheres are equal. If the two spheres do not intersect, the radical plane separates them. If they do intersect, the radical plane contains their intersection. If $x \neq \Theta(x)$, the sphere $\mathbb{S}_{x,\mathcal{R}(x)}$ is cut by the sphere $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$ along a circle in the tridimensional case and an instance of S^{n-2} in general. As shown in Fig. 6, we denote by $\Pi(x)$ the plane supporting the intersection circle, that is the radical plane of $\mathbb{S}_{x,\mathcal{R}(x)}$ and

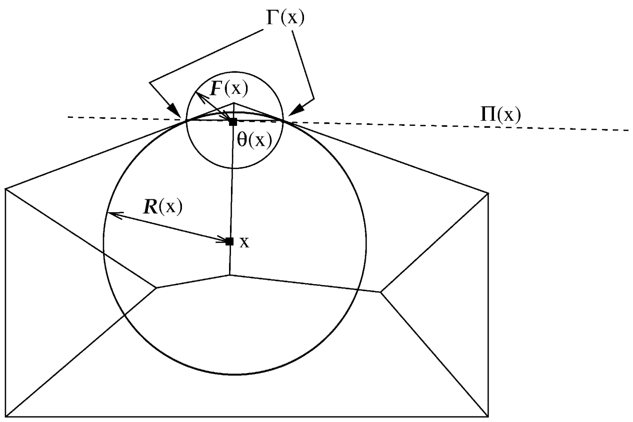

Fig. 6. $\Theta(x) \in \Pi(x)$.

$\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$. $\Pi(x)$ is clearly orthogonal to $\nabla(x)$. When $\mathcal{F}(x) = 0$, $\Pi(x)$ is the plane orthogonal to $\nabla(x)$ and going through $\Theta(x)$. The minimality condition in the definition of $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ entails that $\Pi(x)$ cuts $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$ along a great circle (a maximal $n-2$ sphere), that is:

$$\Theta(x) \in \Pi(x)$$

There is at least one points of $\Gamma(x)$ in $\Pi(x) \cap \mathbb{S}_{\Theta(x),\mathcal{F}(x)}$. If y is such a point, one can apply Pythagoras theorem to the triangle $x, \Theta(x), y$ and we get:

$$\|x - \Theta(x)\|^2 + \mathcal{F}(x)^2 = \mathcal{R}(x)^2 \tag{1}$$

which gives:

$$\|\nabla(x)\|^2 + \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2} = 1 \tag{2}$$

in particular:

$$0 \leq \|\nabla(x)\| \leq 1$$

and:

$$\|\nabla(x)\| < 1 \Leftrightarrow x \in \mathcal{M}$$

We denote by $\mathcal{O}_\rho$ the set of points of $\mathcal{O}$ whose distance to $\partial\mathcal{O}$ is greater or equal to ρ :

$$\mathcal{O}_\rho = \{x \in \mathcal{O};\ \mathcal{R}(x) \geq \rho\}$$

We have, because $\mathcal{O}$ is open, $\mathcal{O} = \bigcup_{\rho>0} \mathcal{O}_\rho$.

We say that a map f, defined on $X \subset \mathbb{R}^n, f : X \mapsto \mathbb{R}$ is *upper semi continuous* if:

$$\forall x \in X,\ \forall \epsilon > 0,\ \exists \alpha > 0,\ \forall y \in X,$$

$$\|y - x\| < \alpha \Rightarrow f(y) < f(x) + \epsilon$$

or, equivalently:

$$\forall \lambda \in \mathbb{R},\ \{x \in X;\ f(x) < \lambda\} \text{ is open}$$

An upper semi continuous function reaches its supremum (its lowest upper bound) on any compact set, for the same reason as the known property for continuous functions. A function f is said *lower semi continuous* if $-f$ is upper semi continuous. The paper makes use of the notions of *right derivative*, *Cauchy sequences*, and *complete spaces*. In particular the fact that *the set of continuous functions equipped with the sup norm is a complete space* is used in the proof of Lemma 4.5. If the reader is not familiar with these notions, he may want to refer to a classic text book in real analysis.

4.2. Local existence of $\mathfrak{C}$

We start with a geometric Lemma.

Lemma 4.1. $\forall x_1, x_2 \in \mathcal{O}_\rho$, *we have the inequality*:

$$(\nabla(x_1) - \nabla(x_2)) \cdot (x_1 - x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

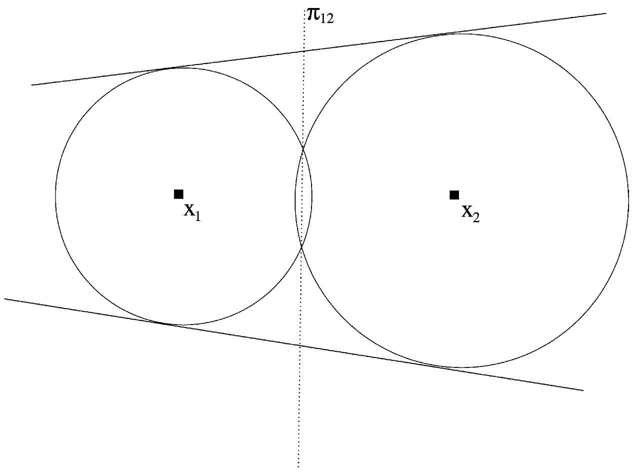

Fig. 7. x_1 and x_2 are on opposite sides of the radical plane.

Proof. We consider x_1 and x_2 in $\mathcal{O}_\rho$. If $x_1 = x_2$, the inequality is trivially satisfied. Let us assume that $x_1 \neq x_2$. We denote by π_{12} the radical plane of the spheres $\mathbb{S}_{x_1,\mathcal{R}(x_1)}$ and $\mathbb{S}_{x_2,\mathcal{R}(x_2)}$, see Figs. 7 and 8. Let p_0 be a point in π_{12} : we denote by π_{12}^+ the closed half-space $\pi_{12}^+ = \{y \in \mathbb{R}^n; (y - p_0)\cdot(x_1 - x_2) \geq 0\}$ and by π_{12}^- the closed half-space $\pi_{12}^- = \{y \in \mathbb{R}^n; (y - p_0)\cdot(x_1 - x_2) \leq 0\}$. Recall that if $\mathbb{B}_{x_1,\mathcal{R}(x_1)}$ and $\mathbb{B}_{x_2,\mathcal{R}(x_2)}$ intersect, π_{12} contains the 'circle' $\mathbb{S}_{x_1,\mathcal{R}(x_1)} \cap \mathbb{S}_{x_2,\mathcal{R}(x_2)}$. We know that:

$$\boldsymbol{\Gamma}(x_1) \subset \mathbb{S}_{x_1,\mathcal{R}(x_1)} \cap [\mathbb{B}^0_{x_2,\mathcal{R}(x_2)}]^c$$

and we can deduce that $\boldsymbol{\Gamma}(x_1) \subset \pi_{12}^+$ which implies:

$$\Theta(x_1) \in \pi_{12}^+$$

In the same manner, one has:

$$\Theta(x_2) \in \pi_{12}^-$$

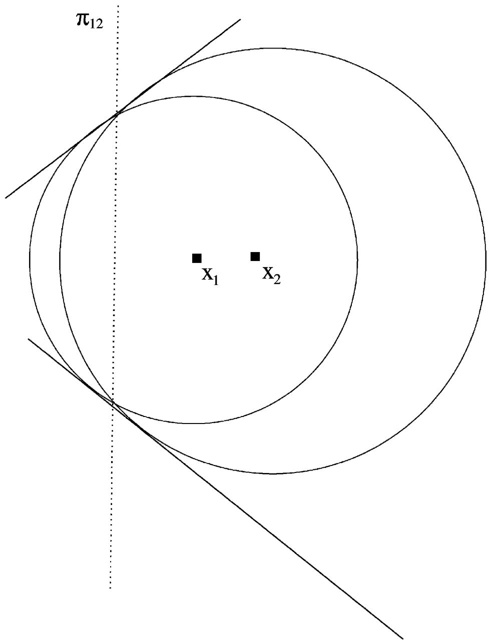

Fig. 8. x_1 and x_2 are on the same side of the radical plane.

which gives us:

$$(\Theta(x_2) - \Theta(x_1))\cdot(x_1 - x_2) \leq 0 \tag{3}$$

We study the map λ, defined by:

$$\lambda(x_1, x_2) = (\nabla(x_1) - \nabla(x_2))\cdot(x_1 - x_2)$$

and we want to prove that:

$$\lambda(x_1, x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

Because λ is symmetric, we can assume, without loss of generality, that $\mathcal{R}(x_1) \leq \mathcal{R}(x_2)$. As depicted in Figs. 7 and 8, one of the following two statement is true: either x_1 and x_2 are on opposite sides of the radical plane, that is $x_1 \in \pi_{12}^+$ and $x_2 \in \pi_{12}^-$, either they both lay on the side of the sphere $\mathbb{B}_{x_2,\mathcal{R}(x_2)}$, that is $x_1 \in \pi_{12}^-$ and $x_2 \in \pi_{12}^-$. From which we get in both cases that:

$$(x_2 - \Theta(x_1))\cdot(x_1 - x_2) \leq 0 \tag{4}$$

We have:

$$\begin{aligned}
\lambda(x_1, x_2) &= (\nabla(x_1) - \nabla(x_2))\cdot(x_1 - x_2) \\
&= \left(\frac{x_1 - \Theta(x_1)}{\mathcal{R}(x_1)} - \frac{x_2 - \Theta(x_2)}{\mathcal{R}(x_2)}\right)\cdot(x_1 - x_2) \\
&= \left[\frac{x_1 - x_2}{\mathcal{R}(x_1)} + \frac{\Theta(x_2) - \Theta(x_1)}{\mathcal{R}(x_2)}\right. \\
&\quad \left. + \left(\frac{1}{\mathcal{R}(x_1)} - \frac{1}{\mathcal{R}(x_2)}\right)(x_2 - \Theta(x_1))\right]\cdot(x_1 - x_2)
\end{aligned}$$

Note that

$$\frac{1}{\mathcal{R}(x_1)} - \frac{1}{\mathcal{R}(x_2)} \geq 0$$

and, from inequalities (3) and (4) we get that:

$$\lambda(x_1, x_2) \leq \left(\frac{x_1 - x_2}{\mathcal{R}(x_1)}\right)\cdot(x_1 - x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

which completes the proof of the Lemma. □

We want to define the flow induced by the vector field ∇. The vector field ∇ is not Lipschitz and not even continuous. Because the classical Lipschitz condition does not apply on ∇ it seems not possible, at first glance, to solve the differential equation:

$$C(0, x) = x, \quad \frac{d}{dt}C(t, x) = \nabla(C(t, x))$$

In order to build a 'kind of solution' of this equation, we have to state Lemmas 4.4 and 4.5 below, that are very similar to the classical theorem [1,10] for differential equations in the Lipschitz case but with a weaker condition. Because these lemmas may have an interest in themselves and to highlight precisely which properties of the vector field ∇ are exploited here, they are expressed independently of the context of the whole proof. For that, we consider a vector field V bounded by a constant B and which is

'K-semi-Lipschitz' (cf. definition below). The integral flow defined by the lemmas is called C. The lemmas are then applied in the context of the proof for the vector field ∇ bounded by the constant 1 and which is $(1/\rho)$−semi-Lipschitz. The resulting flow, central object of the proof, is then called $\mathfrak{C}$. Lemmas 4.4 and 4.5 rely on the two definitions below.

Definition 4.2. (h-sampling) For a real number $T > 0$, we say that a strictly increasing finite sequence of real numbers $(t_i)_{i=0,n} = t_0, t_1, \ldots, t_n$ is an h-sampling of the real interval $[0, T]$ iff:

$$t_0 = 0,\ t_n = T$$

$$\forall i, 1 \le i \le n\ t_i - t_{i-1} < h$$

Definition 4.3. (Euler scheme) Let X be a subset of $\mathbb{R}^n$. Given a vector map $V : X \to \mathbb{R}^n$, a point $x \in X$, a real number $T > 0$ and an h-sampling $(t_i)_{i=0,n}$ of $[0, T]$, Considers the sequence defined by induction:

$$Y_0 = x$$

$$\forall i = 0, \ldots, n-1,\ Y_{i+1} = Y_i + (t_{i+1} - t_i)V(Y_i)$$

We say that the sequence of Y_i is well defined, if the line segments $[Y_{i-1}, Y_i]$ are subsets of $X : \forall i, 1 \le i \le n$; $[Y_{i-1}, Y_i] \subset X$.
If the sequence is well defined, the Euler scheme $E^V_{t_0 \ldots t_n}(x)$ is the continuous, piecewise linear function defined on $[0, T]$ interpolating the points Y_i :

if $t \in [t_i, t_{i+1}]$,

$$E^V_{t_0 \ldots t_n}(x)(t) = Y_i + (t - t_i)V(Y_i)$$

We can now state the two next lemmas allowing the construction of $\mathfrak{C}$ out of ∇. We denote by e the Euler constant, basis of the natural exponential function.

Lemma 4.4. *Let X be a bounded subset of* $\mathbb{R}^n$, $V : X \to \mathbb{R}^n$ *a vector map defined on X that satisfies the conditions*:

- *V is bounded*: $\exists B, \forall x \in X, \|V(x)\| < B$
- *V satisfies the semi-Lipschitz condition for the semi-Lipschitz constant K* :

  $$\exists K > 0,\ \forall x_1, x_2 \in X,$$

  $$(V(x_1) - V(x_2))\cdot(x_1 - x_2) \le K(x_1 - x_2)^2$$

  Then, there is a constant γ *such that, for any real number* $T > 0$, *if* $t_0 \ldots t_n$ *(resp.* $t'_0 \ldots t'_{n'}$*) is a h-sampling of* $[0, T]$ *and* x_1 *(resp.* x_2*) a point of X such that the sequence defining the Euler scheme* $E^V_{t_0 \ldots t_n}(x_1)$ *(resp.* $E^V_{t'_0 \ldots t'_{n'}}(x_2)$*) is well defined, we have*:

  $$\sup_{t \in [0,T]} \|E^V_{t_0 \ldots t_n}(x_1)(t) - E^V_{t'_0 \ldots t'_{n'}}(x_2)(t)\|^2$$

  $$\le \|x_1 - x_2\|^2 e^{2KT} + h\gamma(e^{2KT} - 1)$$

Proof. Under the assumptions of the Lemma, let $u_0 \ldots u_N$ be an H-sampling of $[0, T]$, with $H \le h$, which is a refinement of both $t_0 \ldots t_n$ and $t'_0 \ldots t'_{n'}$. In other words:

$$\{t_0, \ldots, t_n\} \subset \{u_0, \ldots, u_N\}$$

and

$$\{t'_0, \ldots, t'_{n'}\} \subset \{u_0, \ldots, u_N\}$$

Pick a $u_i \in \{u_0, \ldots, u_{N-1}\}$. There is j, $0 \le j \le n-1$ (resp. j', $0 \le j' \le n'-1$) such that $t_j \le u_i < u_{i+1} \le t_{j+1}$ (resp. $t'_{j'} \le u_i < u_{i+1} \le t'_{j'+1}$). We use the following shorter notation in the computation below. E and E' stand, respectively, for $E^V_{t_0 \ldots t_n}(x_1)$ and $E^V_{t'_0 \ldots t'_{n'}}(x_2)$. Moreover one denote by X_1 and X_2 the points $E(t_j)$ and $E'(t'_j)$:

$$X_1 = E^V_{t_0 \ldots t_n}(x_1)(t_j)$$

$$X_2 = E^V_{t'_0 \ldots t'_{n'}}(x_2)(t'_{j'})$$

As shown in Fig. 9, we want to bound the squared distance $[E(u_{i+1}) - E'(u_{i+1})]^2$ at step $i+1$ with respect to the squared distance $[E(u_i) - E'(u_i)]^2$ at step i.
We have:

$$[E(u_{i+1}) - E'(u_{i+1})]^2$$

$$= [(E(u_i) + (u_{i+1} - u_i)V(X_1)) - (E'(u_i) + (u_{i+1} - u_i)V(X_2))]^2$$

$$= [(E(u_i) - E'(u_i)) + (u_{i+1} - u_i)(V(X_1) - V(X_2))]^2$$

$$= [E(u_i) - E'(u_i)]^2 + 2[E(u_i) - E'(u_i)]\cdot(u_{i+1} - u_i)(V(X_1)$$

$$- V(X_2)) + (u_{i+1} - u_i)^2(V(X_1) - V(X_2))^2$$

$$\le [E(u_i) - E'(u_i)]^2 + 2[E(u_i) - E'(u_i)]\cdot(u_{i+1} - u_i)(V(X_1)$$

$$- V(X_2)) + h(u_{i+1} - u_i)(2B)^2$$

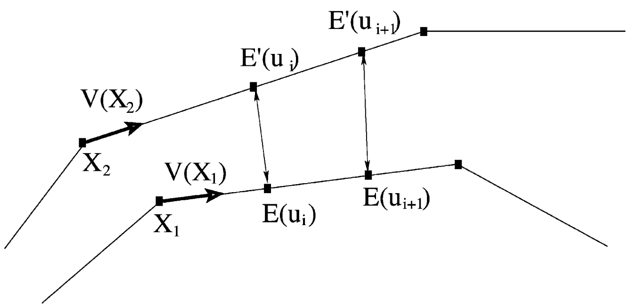

Fig. 9. Euler schemes $E^V_{t_0 \ldots t_n}(x_1)$ and $E^V_{t'_0 \ldots t'_{n'}}(x_2)$.

We bound now the second term:

$$\mathbf{W}=2[E(u_i)-E'(u_i)]\cdot(u_{i+1}-u_i)(V(X_1)-V(X_2))$$

$$=2[(X_1+(u_i-t_j)V(X_1))-(X_2+(u_i-t'_{j'})V(X_2)]\cdot(u_{i+1}-u_i)$$

$$\times[(V(X_1)-V(X_2))$$

$$\leq 2(u_{i+1}-u_i)(V(X_1)-V(X_2))(X_1-X_2)+(2B)^2h)]$$

$$\leq 2(u_{i+1}-u_i)[K(X_1-X_2)^2+(2B)^2h)]$$

We bound now $(X_1-X_2)^2$:

$$(X_1-X_2)^2=[(E(u_i)-E'(u_i))-((u_i-t_j)V(X_1)$$

$$-(u_i-t'_{j'})V(X_2))]^2$$

$$\leq[E(u_i)-E'(u_i)]^2+4hB\|E(u_i)-E'(u_i)\|+4h^2B^2$$

As X is bounded, let D be a constant such that for $y_1,y_2\in X$, $\|y_1-y_2\|\leq D$. In particular, $\|E(u_i)-E'(u_i)\|\leq D$ and $h\leq D$, and we have

$$(X_1-X_2)^2\leq[E(u_i)-E'(u_i)]^2+h(4BD+4DB^2)$$

We can now rewrite the bound on $\mathbf{W}$.

$$\mathbf{W}\leq(u_{i+1}-u_i)[2K[E(u_i)-E'(u_i)]^2+2(K(4BD+4DB^2)$$

$$+(2B)^2)h]$$

We define $\beta=2(K(4BD+4DB^2)+(2B)^2)+(2B)^2$. Notice that $\beta>0$. We can now substitute this upper bound of $\mathbf{W}$ in the initial inequality:

$$[E(u_{i+1})-E'(u_{i+1})]^2$$

$$\leq[E(u_i)-E'(u_i)]^2(1+2K(u_{i+1}-u_i))+\beta h(u_{i+1}-u_i)$$

We define now the sequence $(v_i)_{i=0,\ldots,N}$:

$$v_i=[E(u_i)-E'(u_i)]^2+\frac{\beta}{2K}h$$

The inequality above can be written:

$$v_{i+1}-\frac{\beta}{2K}h\leq v_i(1+2K(u_{i+1}-u_i))-\frac{\beta}{2K}h(1+2K(u_{i+1}-u_i))$$

$$+\beta h(u_{i+1}-u_i)$$

that is:

$$v_{i+1}\leq v_i(1+2K(u_{i+1}-u_i))$$

and:

$$v_i\leq v_0\prod_{k=1,i}(1+2K(u_k-u_{k-1}))$$

And, as both members are strictly positive:

$$\ln(v_i)\leq\ln(v_0)+\sum_{k=1,i}\ln(1+2K(u_k-u_{k-1}))$$

and because $\forall u\geq 0$, $\ln(1+u)\leq u$, we have:

$$\ln(1+2K(u_k-u_{k-1}))\leq 2K(u_k-u_{k-1})$$

We get a telescoping series and we can write, because $u_0=0$:

$$\ln(v_i)\leq\ln(v_0)+2Ku_i$$

that makes:

$$v_i\leq v_0\mathbf{e}^{2Ku_i}$$

We have:

$$v_0=[E(0)-E(x_2)(0)]^2+\frac{\beta}{2K}h=(x_1-x_2)^2+\frac{\beta}{2K}h$$

that gives:

$$v_i\leq\left[(x_1-x_2)^2+\frac{\beta}{2K}h\right]\mathbf{e}^{2Ku_i}$$

and

$$[E(u_i)-E'(u_i)]^2\leq\left[(x_1-x_2)^2+\frac{\beta}{2K}h\right]\mathbf{e}^{2Ku_i}-\frac{\beta}{2K}h$$

$$=(x_1-x_2)^2e^{2Ku_i}+h\frac{\beta}{2K}(e^{2Ku_i}-1)$$

The inequality above holds for any H-sampling $u_0,\ldots,u_N$. For a given $t\in[0,T]$ we can choose an H-sampling $u_0,\ldots,u_N$ such that, for some i, $t=u_i$. We have then:

$$\forall t\in[0,T],[E(t)-E'(t)]^2\leq(x_1-x_2)^2\mathbf{e}^{2Kt}+h\frac{\beta}{2K}(\mathbf{e}^{2Kt}-1)$$

and that ends the proof of the Lemma. □

Remark. The inequality of the Lemma above is very similar to the one used in the Lipschitz case, page 116 in Ref. [1] and page 171 in Ref. [10].

A consequence is the Lemma below that states the local existence and uniqueness of a Lipschitz map, uniform limit of the Euler schemes.

Lemma 4.5. *If X is a bounded subset of $\mathbb{R}^n$, and if $V: X\to\mathbb{R}^n$ is a vector map defined on X that satisfies the conditions*:

- *V is bounded*: $\exists B,\forall x\in X,\|V(x)\|<B$
- *V satisfies the semi-Lipschitz condition for the semi-Lipschitz constant K* :

  $$\exists K>0,\ \forall x_1,x_2\in X,$$

  $$(V(x_1)-V(x_2))\cdot(x_1-x_2)\leq K(x_1-x_2)^2$$

  Let be $x\in X^0$ and $\rho>0$ such that $\mathbb{B}_{x,\rho}\subset X$. Then, there is a unique map

  $$C:\left[0,\frac{\rho}{2B}\right]\times\mathbb{B}_{x,\rho/2}\to\mathbb{B}_{x,\rho}$$

such that:

(i) *For any* $y \in \mathbb{B}_{x,\rho/2}$ *and* $\{h_i\}_{i\in\mathbb{N}}$ *such that* $\lim_{i\to\infty} h_i = 0$, *if* $\{E_i\}_{i\in\mathbb{N}}$ *is a sequence of Euler Schemes starting in* y *such that* E_i *is based on an* h_i-*sampling of* $[0, \rho/2B]$, *then the sequence of* $\{E_i\}_{i\in\mathbb{N}}$ *converges uniformly toward the map* $t \mapsto C(t, y)$.
(ii) *The Map C is Lipschitz with respect to both variables and therefore uniformly continuous.*
(iii) $C(0, y) = y$ *and* $C(t + s, x) = C(t, C(s, x))$

Proof. Let y be in $\mathbb{B}_{x,\rho/2}$. For any h-sampling of $[0, \rho/2B]$ the Euler scheme based on this h-sampling and starting at y is well defined, because at each iteration Y_i is still in the ball $\mathbb{B}_{x,\rho} \subset X$. Let $\{h_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} h_i = 0$. Let $\{E_i(y)\}_{i\in\mathbb{N}}$ be a sequence of Euler Schemes starting in y such that each E_i is based on an h_i sampling. Let $\epsilon > 0$ be a real number. Then there is $N \in \mathbb{N}$ such that $i > N \to h_i < \epsilon$. Then, we get from Lemma 4.4 that, for any $i, i' > N$:

$$\sup_{t\in[0,\rho/2B]} \|E_i(y)(t) - E_{i'}(y)(t)\|^2 \le \epsilon\gamma(\mathbf{e}^{2K(\rho/2B)} - 1) \tag{5}$$

The sequence of $\{E_i(y)\}_{i\in\mathbb{N}}$ is therefore a Cauchy sequence for the *sup* norm on the complete space of continuous map $C[0, \rho/2B] \to \mathbb{B}_{x,\rho}$. The sequence of $\{E_i\}_{i\in\mathbb{N}}$ therefore converges uniformly toward a continuous map. We denote this limit $t \mapsto C(t, y)$.
$t \mapsto C(t, y)$ does not depend on the chosen sequence of $\{E_i(y)\}_{i\in\mathbb{N}}$ as the inequality (5) stands for any sequence of Euler schemes. One checks as well that the bound defined in Lemma 4.4 extends to the limit, that is for $t \in [0, \rho/2B]$ and $y_1, y_2 \in \mathbb{B}_{x,\rho/2}$ we have:

$$\|C(t, y_1) - C(t, y_2)\|^2 \le \|y_1 - y_2\|^2 e^{2K\rho/2B}$$

and that gives:

$$\|C(t, y_1) - C(t, y_2)\| \le \|y_1 - y_2\| e^{K(\rho/2B)}$$

Now, for each Euler Scheme $E_i(y)$ from $\{E_i(y)\}_{i\in\mathbb{N}}$, we have the Lipschitz bound with respect to t induced by the bound B on V. For $t_1, t_2 \in [0, \rho/2B]$:

$$\|E_i(y)(t_1) - E_i(y)(t_2)\| \le B|t_1 - t_2|$$

This extends to the limit:

$$\|C(t_1, y) - C(t_2, y)\| \le B|t_1 - t_2|$$

and we can write:

$$\forall t_1, t_2 \in [0, \rho/2B],\ \forall y_1, y_2 \in \mathbb{B}_{x,\rho/2}\ \|C(t_1, y_1) - C(t_2, y_2)\| \le B|t_1 - t_2| + \|y_1 - y_2\| \mathbf{e}^{K(\rho/2B)}$$

Both the Lipschitz property and the uniqueness are expressed by this inequality. (iii) is trivial. □

From Lemma 4.1, it is clear that the conditions of Lemma 4.5 apply to the vector map ∇ in the set $\mathcal{O}_\rho$ with the semi-Lipschitz constant $K = 1/\rho$ and the bound $B = 1$ on the map ∇. We have then locally, that is on $\cup_{\rho>0}([0, \rho] \times \mathcal{O}_\rho)$, a continuous map that is the limit of the Euler Schemes built on ∇. We denote this map by $\mathfrak{C}$. The 'zigzag trajectory' in Fig. 10, depicts an Euler Scheme applied to the vector field ∇. As h approaches 0, these Euler Schemes converge toward the map $\mathfrak{C}$.

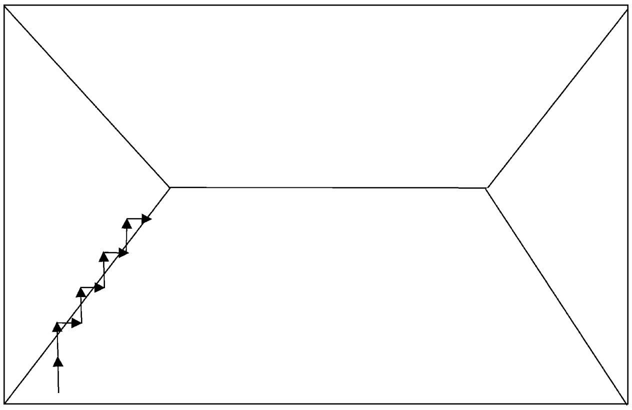

Fig. 10. The 'zigzag trajectory' depicts an Euler Scheme applied to the vector field ∇.

4.3. Global existence of $\mathfrak{C}$

In order to prove a global existence of the map $\mathfrak{C}$, that is that $\mathfrak{C}$ can be extended to a map in $\mathbb{R}^+ \times \mathcal{O}$, we need to state a few propositions.

Lemma 4.6 states a kind of semi-continuity on the map $x \mapsto \mathbf{\Gamma}(x)$. Informally, If y is not too far from x, $\mathbf{\Gamma}(y)$ is not too far from $\mathbf{\Gamma}(x)$. In other words:

Lemma 4.6.

$$\forall x \in \mathcal{O},\ \forall \epsilon > 0,\ \exists \alpha > 0,$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \mathbf{\Gamma}(y) \subset \mathbf{\Gamma}(x) \oplus \mathbb{B}^0_{0,\epsilon}$$

Proof. For any $\epsilon > 0$, the set $(\mathbf{\Gamma}(x) \oplus \mathbb{B}^0_{0,\epsilon})^c$ is closed. Indeed it is the set of points for which the distance to $\mathbf{\Gamma}(x)$ is greater or equal to ϵ. let us define:

$$B_\epsilon = \partial\mathcal{O} \cap (\mathbf{\Gamma}(x) \oplus \mathbb{B}^0_{0,\epsilon})^c$$

B_ϵ, set of points in the boundary of $\mathcal{O}$ for which the distance to $\mathbf{\Gamma}(x)$ is greater or equal to ϵ is then compact.
If B_ϵ is empty, one has $\partial\mathcal{O} \subset \mathbf{\Gamma}(x) \oplus \mathbb{B}^0_{0,\epsilon}$, and the statement of the Lemma is trivially satisfied for any $\alpha > 0$. Let us assume now that B_ϵ is not empty. Fig. 11 depicts B_ϵ on a simple example.
Because B_ϵ is non empty and compact, there is $z \in B_\epsilon$ such that:

$$d(x, z) = d(x, B_\epsilon)$$

Because $z \notin \mathbf{\Gamma}(x)$, we have:

$$d(x, B_\epsilon) = d(x, z) > d(x, \partial\mathcal{O})$$

which entails:

$$d(x, B_\epsilon) - d(x, \partial\mathcal{O}) > 0$$

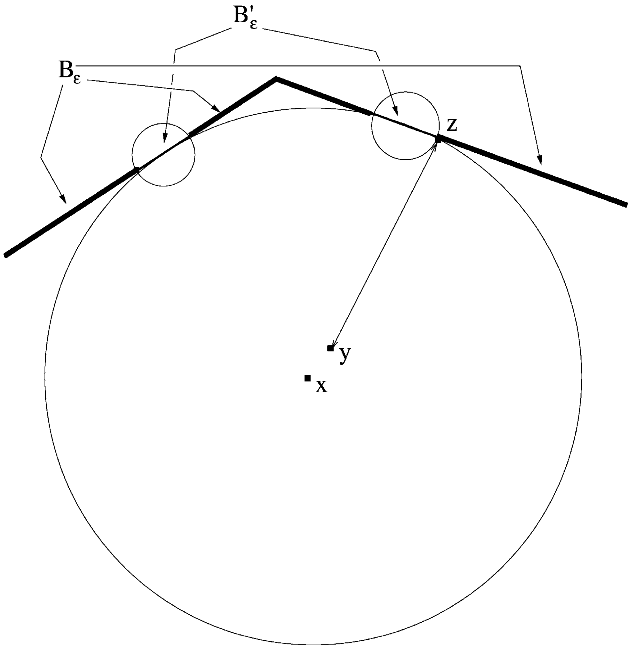

Fig. 11. The sets B_ϵ and B'_ϵ.

Let us choose $\alpha > 0$ such that:

$$\alpha < \frac{1}{2}(d(x, B_\epsilon) - d(x, \partial\mathscr{O}))$$

If $y \in \mathbb{B}_{x,\alpha}$, we have then:

$$d(y, B_\epsilon) \geq d(x, B_\epsilon) - \alpha > d(x, \partial\mathscr{O}) + \alpha$$

On another hand the set of point in the boundary of $\mathscr{O}$ for which the distance to $\boldsymbol{\Gamma}(x)$ is strictly less than ϵ is denoted B'_ϵ :

$$B'_\epsilon = \partial \cap (\boldsymbol{\Gamma}(x) \oplus \mathbb{B}^o_{0,\epsilon})$$

One has:

$$d(y, B'_\epsilon) \leq d(y, \boldsymbol{\Gamma}(x)) \leq d(x, \boldsymbol{\Gamma}(x)) + \alpha = d(x, \partial\mathscr{O}) + \alpha$$

We have then

$$d(y, B'_\epsilon) < d(y, B_\epsilon)$$

which entails

$$\boldsymbol{\Gamma}(y) \subset B'_\epsilon \subset \boldsymbol{\Gamma}(x) \oplus \mathbb{B}^o_{0,\epsilon}$$

□

Corollary 4.7. *The map $\mathscr{F}$ is upper semi continuous.*

Proof. It is a trivial consequence of the Lemma and the definition of $\mathscr{F}$. □

We consider now a given point x such that $\nabla(x) \neq 0$. The next Lemma states some inequalities on $\Theta(y)$ and $\nabla(y)$, when y is sufficiently close to x. When $\nabla(x) \neq 0$, we denote by $\mathbf{e}$ the unit vector collinear to $\nabla(x)$: $\mathbf{e} = (\nabla(x)/\|\nabla(x)\|$.

The significance of Lemma 4.8 below is not immediate. It is a consequence of Lemma 4.6 and is in fact necessary for the proof of Lemmas 4.9–4.11.

Lemma 4.8. *Let x be such that $\nabla(x) \neq 0$. One has the following propositions*:

$$\forall \epsilon > 0,\ \exists \alpha > 0,\ y \in \mathbb{B}_{x,\alpha} \Rightarrow \forall z \in \boldsymbol{\Gamma}(y), (z - \Theta(x))\cdot\mathbf{e} < \epsilon$$

$$\forall \epsilon > 0,\ \exists \alpha > 0,\quad y \in \mathbb{B}_{x,\alpha} \Rightarrow (\Theta(y) - \Theta(x))\cdot\mathbf{e} < \epsilon$$

$$\forall \epsilon > 0,\ \exists \alpha > 0,\quad y \in \mathbb{B}_{x,\alpha} \Rightarrow (\nabla(y) - \nabla(x))\cdot\mathbf{e} > -\epsilon$$

Proof. Recall that, if $x \neq \Theta(x)$, we denote by $\Pi(x)$ the radical plane of $\mathbb{S}_{x,\mathscr{R}(x)}$ and $\mathbb{S}_{\Theta(x),\mathscr{F}(x)}$. $\Pi(x)$ is the plane orthogonal to $\nabla(x)$ and going through $\Theta(x)$. $\Pi(x)$ splits the space into two half-spaces. We denote by $\Pi^+(x)$ the closed half-space that contains x. The closed half-space that does not contains x is denoted $\Pi^-(x)$. Note that, because $\boldsymbol{\Gamma}(x) \subset \mathbb{S}_{x,\mathscr{R}(x)} \cap \mathbb{B}_{\Theta(x),\mathscr{F}(x)}$, we have:

$$\boldsymbol{\Gamma}(x) \subset \Pi^-(x)$$

that is, $\forall z \in \boldsymbol{\Gamma}(x)$, $z\cdot\mathbf{e} \leq \Theta(x)\cdot\mathbf{e}$. If we call $\Pi^{-\epsilon}(x)$ the open half-space of equation

$$\Pi^{-\epsilon}(x) = \{z \in \mathbb{R}^n;\ (z - \Theta(x))\cdot\mathbf{e} < \epsilon\}$$

we get from Lemma 4.6:

$$\forall \epsilon > 0, \exists \alpha > 0,$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \boldsymbol{\Gamma}(y) \subset \boldsymbol{\Gamma}(x) \oplus B^o_{0,\epsilon} \subset \Pi^-(x) \oplus \mathbb{B}^o_{0,\epsilon} = \Pi^{-\epsilon}(x)$$

This proves the first inequality of the Lemma and entails that the center $\Theta(y)$ of the minimal ball enclosing $\boldsymbol{\Gamma}(y)$ is in this half-space: $\Theta(y) \in \Pi^{-\epsilon}(x)$, which proves the second inequality.

Now we know that $\mathscr{R}$ is 1-Lipschitz. Let $\epsilon > 0$ be a positive number. There is α_1 such that:

$$y \in \mathbb{B}_{x,\alpha_1} \Rightarrow \frac{\mathscr{R}(x)}{\mathscr{R}(y)} \leq 2 \tag{6}$$

$$y \in \mathbb{B}_{x,\alpha_1} \Rightarrow \left|1 - \frac{\mathscr{R}(x)}{\mathscr{R}(y)}\right| \|\nabla(x)\| < \frac{\epsilon}{2} \tag{7}$$

Using the second inequality of the Lemma, there is α_2 such that:

$$y \in \mathbb{B}_{x,\alpha_2} \Rightarrow (\Theta(y) - \Theta(x))\cdot\mathbf{e} < \frac{\epsilon\mathscr{R}(x)}{8}$$

Let $\alpha = \min(\alpha_1, \alpha_2, (\epsilon\mathscr{R}(x))/8)$. If $y \in \mathbb{B}_{x,\alpha}$, we have:

$$(y - \Theta(y))\cdot\mathbf{e} - (x - \Theta(x))\cdot\mathbf{e} > -\frac{\epsilon\mathscr{R}(x)}{4}$$

or, using Eq. (6),

$$\frac{(y - \Theta(y))\cdot\mathbf{e}}{\mathscr{R}(y)} - \frac{(x - \Theta(x))\cdot\mathbf{e}}{\mathscr{R}(y)} > -\frac{\mathscr{R}(x)}{\mathscr{R}(y)}\frac{\epsilon}{4} \geq -\frac{\epsilon}{2}$$

that gives:

$$\nabla(y)\cdot\mathbf{e} - \nabla(x)\cdot\mathbf{e} + \left(1 - \frac{\mathscr{R}(x)}{\mathscr{R}(y)}\right)(\nabla(x)\cdot\mathbf{e}) > -\frac{\epsilon}{2}$$

that is, using Eq. (7)

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow (\nabla(y) - \nabla(x))\cdot\mathbf{e} > -\epsilon$$

and that proves the last inequality of the Lemma. □

For a point $x \in \mathcal{O}$, a unit vector v and an angle $\theta \in [0, \pi/2]$, $K_{v,\theta}(x)$ denotes the cone of apex x, axis v and half angle θ:

$$K_{v,\theta}(x) = \{y \in R^n;\ \ v\cdot(y - x) \geq \|y - x\|\cos\theta\}$$

In order to prove that $\nabla(x)$ is the right derivative of $t \mapsto \mathfrak{C}(t,x)$ at $t = 0$, we first show Lemma 4.9, that is, for t sufficiently small, $\mathfrak{C}(t,x)$ remains in a cone of apex x and axis $\mathbf{e}$ with an arbitrary small half angle.

Lemma 4.9. *For any $x \in \mathcal{O}$ such that $\nabla(x) \neq 0$ and any $\theta > 0$ there is $\alpha > 0$ such that, for any $t \in [0,\alpha]$, $\mathfrak{C}(t,x)$ is in $K_{\mathbf{e},\theta}(x)$, in other words, if $\nabla(x) \neq 0$, one has*:

$$\forall\theta > 0,\ \exists\alpha > 0,\ \forall t \in [0,\alpha],\ \mathfrak{C}(t,x) \in K_{\mathbf{e},\theta}(x)$$

Proof. Let us define $\rho = (\mathscr{R}(x))/2$. We have of course $\mathbb{B}_{x,\rho} \subset \mathcal{O}_\rho$. From Lemma 4.8 we know that there is a positive number α_1, $0 < \alpha_1 < \rho$ such that:

$$y \in \mathbb{B}_{x,\alpha_1} \Rightarrow \nabla(y)\cdot\mathbf{e} > \frac{\|\nabla(x)\|}{2} \tag{8}$$

Let us denote by ϕ the angle in $[0,\pi/2]$ defined by: $\cos\phi = \|\nabla(x)\|/2$. Because $\|\nabla(y)\| \leq 1$ and $\nabla(y)\cdot\mathbf{e} > \cos\phi$, any Euler scheme defined on the interval $[0,\alpha_1]$ (or any smaller interval) and starting at x will remain in the cone $K_{\mathbf{e},\phi}(x)$ of apex x, axis $\mathbf{e}$ and half angle ϕ. Notice that $0 < \cos\phi \leq 1/2$ and therefore $\pi/3 \leq \phi < \pi/2$. We denote by ϵ the positive number defined by:

$$\epsilon = \frac{\sin\theta\tan^2\theta\|\nabla(x)\|}{4\tan\phi}$$

From Lemmas 4.1 and 4.8 we know that there is a positive number $\alpha_2 > 0$, with $\alpha_2 < \alpha_1$ and $\alpha_2 < 1$ such that (Lemma 4.1):

$$y \in \mathbb{B}_{x,\alpha_2},\ y \neq x \Rightarrow (\nabla(y) - \nabla(x))\cdot\frac{y - x}{\|y - x\|} \leq \epsilon \tag{9}$$

and (by Lemma 4.8):

$$y \in \mathbb{B}_{x,\alpha_2} \Rightarrow (\nabla(y) - \nabla(x))\cdot\mathbf{e} > -\epsilon \tag{10}$$

Let us consider a point $y^* \in \mathbb{B}_{x,\alpha_2}$ such that $y^* \in K_{\mathbf{e},\phi}(x) - K_{\mathbf{e},\theta}(x)$. Let us define the unit vector $\mathbf{f}$ orthogonal to $\mathbf{e}$ and the angle $\psi \in (\theta,\phi]$ such that:

$$\frac{y^* - x}{\|y^* - x\|} = \cos\psi\mathbf{e} + \sin\psi\mathbf{f}$$

we get from Eq. (9):

$$(\nabla(y^*) - \nabla(x))\cdot(\cos\psi\mathbf{e} + \sin\psi\mathbf{f}) \leq \epsilon$$

and, because $\nabla(x)\cdot\mathbf{f} = 0$:

$$\nabla(y^*)\cdot(\sin\psi\mathbf{f}) \leq \epsilon - (\nabla(y^*) - \nabla(x))\cdot(\cos\psi\mathbf{e})$$

that gives, using Eq. (10):

$$\nabla(y^*)\cdot(\sin\psi\mathbf{f}) < \epsilon(1 + \cos\psi) < 2\epsilon$$

which gives, using the definition of ϵ and the fact that $\psi > \theta$:

$$\nabla(y^*)\cdot f < \frac{\tan^2\theta\|\nabla(x)\|}{2\tan\phi} \tag{11}$$

This inequality is used below.
For any $y \in \mathbb{B}_{x,\alpha_2}$, we use the following notation. $y^\parallel = ((y - x)\cdot\mathbf{e})\mathbf{e}$ and $y^\perp = (y - x) - y^\parallel$. We have from this definition

$$y = x + y^\parallel + y^\perp$$

Similarly, we define: $\nabla^\parallel(y) = (\nabla(y)\cdot\mathbf{e})\mathbf{e}$ and $\nabla^\perp(y) = \nabla(y) - \nabla^\parallel(y)$. We have from this definition

$$\nabla(y) = \nabla(y)^\parallel + \nabla(y)^\perp$$

Let us define the map $g : \mathbb{B}_{x,\alpha_2} \cap K_{\mathbf{e},\phi}(x) \to \mathbb{R}$ by:

$$g(y) = (y^\perp)^2 - \tan^2\theta(y^\parallel)^2$$

For any $y \in \mathbb{B}_{x,\alpha_2} \cap K_{\mathbf{e},\phi}(x)$, we have:

$$g(y) \leq 0 \Leftrightarrow y \in K_{\mathbf{e},\theta}(x)$$

Notice that, because $\alpha_2 \leq 1$, g is 2-Lipschitz.
We consider now, for any $k \in \mathbb{N}$, the Euler Scheme E_k starting at x with the h_k-sampling of $[0,\alpha_2]$ with 2^k constant steps, $t_{i+1} - t_i = h_k$:

$$h_k = 2^{-k}\alpha_2 \text{ and } t_i = i\cdot 2^{-k}\alpha_2$$

We denote by y_i the ith iterate of E_k, that is $y_i = E_k(t_i)$. We have:

$$y_{i+1}^\perp = y_i^\perp + h\nabla^\perp(y_i)$$

and

$$y_{i+1}^\parallel = y_i^\parallel + h\nabla^\parallel(y_i)$$

Let us assume that, for some i, $1 \leq i \leq 2^k$, we have $g(y_i) > 0$. We have then:

$$\begin{aligned} g(y_{i+1}) - g(y_i) &= 2h(y_i^\perp\cdot\nabla^\perp(y_i)) + h^2(\nabla^\perp(y_i))^2 \\ &\quad - 2h(\tan^2\theta y_i^\parallel\cdot\nabla^\parallel(y_i)) - h^2\tan^2\theta(\nabla^\parallel(y_i))^2 \\ &\leq 2h(y_i^\perp\cdot\nabla^\perp(y_i) - \tan^2\theta y_i^\parallel\cdot\nabla^\parallel(y_i)) + h^2 \end{aligned}$$

Because $y_i \notin K_{\mathbf{e},\theta}(x)$, we can use the inequality Eq. (11) above with $y^* = y_i$. Note that $y_i^\perp = \|y_i^\perp\|f$ and $\nabla(y_i)\cdot f = \nabla^\perp(y_i)\cdot f$. That gives:

$$y_i^\perp\cdot\nabla^\perp(y_i) < \|y_i^\perp\|\frac{\tan^2\theta\|\nabla(x)\|}{2\tan\phi}$$

Because $y_i \in K_{\mathbf{e},\phi}(x)$, we have $\|y_i^{\perp}\| \le \tan\phi \|y_i^{\|}\|$ and we have:

$$y_i^{\perp} \cdot \nabla^{\perp}(y_i) < \|y_i^{\|}\| \frac{\tan^2\theta \|\nabla(x)\|}{2} \tag{12}$$

On the other hand, we have $y_i \in \mathbb{B}_{x,\alpha_1}$ and we get from Eq. (8):

$$\nabla(y_i)\cdot\mathbf{e} > \frac{\|\nabla(x)\|}{2}$$

and that gives:

$$y_i^{\|} \cdot \nabla^{\|}(y_i) > \|y_i^{\|}\| \frac{\|\nabla(x)\|}{2}$$

We can then write, using Eq. (12):

$$y_i^{\perp} \cdot \nabla^{\perp}(y_i) - \tan^2\theta y_i^{\|} \cdot \nabla^{\|}(y_i) < 0$$

which finally leads to:

$$g(y_{i+1}) - g(y_i) \le h^2$$

We know that $g(y_0) = g(x) = 0$ and $g(y_1) = g(y_0 + h\nabla(x)) < 0$. Because g is 2-Lipschitz and E_k is 1-Lipschitz, if i is the first index for which $g(y_i) > 0, g(y_i) \le 2h$. Then for any $i' > i$, we have:

$$g(y_{i'}) \le 2h + (i' - i)h^2 \le 2h + 2^k h^2 \le (2\alpha_2 + \alpha_2^2)\cdot 2^{-k}$$

Because the sequence of Euler schemes E_i converges uniformly toward $t \mapsto \mathfrak{C}(t,x)$:

$$\forall \eta > 0,\ \exists k,\ \forall t \in [0, \alpha_2],$$

$$\|\mathfrak{C}(t,x) - E_k\| < \eta \text{ and } (2\alpha_2 + \alpha_2^2)\cdot 2^{-k} < \eta$$

and, because g is 2-Lipschitz we have:

$$\forall t \in [0, \alpha_2], g(\mathfrak{C}(t,x)) < 2\eta + \eta$$

We get then, because η can be arbitrarily small:

$$\forall t \in [0, \alpha_2], g(\mathfrak{C}(t,x)) \le 0$$

and that completes the proof of the Lemma. □

Lemma 4.10. *The map $t \mapsto \nabla(\mathfrak{C}(t,x))$ is the right derivative of $t \mapsto \mathfrak{C}(t,x)$. That is*:

$$\forall x \in \mathcal{O},\ \forall \epsilon > 0,\ \exists \alpha > 0,\ \forall t \in [0, \alpha],$$

$$\|\mathfrak{C}(t,x) - (x + t\nabla(x))\| < t\epsilon$$

Proof. If $\nabla(x) = 0$ any Euler scheme starting at x is stationary: $\forall t \ge 0,\ \mathfrak{C}(t,x) = x$ and the proposition of the Lemma is trivially satisfied. Let assume now that $\nabla(x) \ne 0$. Let $\epsilon > 0$ be the positive number given in the Lemma. We can assume, without loss of generality that $\epsilon < \|\nabla(x)\|/2$. Let $\theta > 0$ be an angle such that $\tan\theta < \epsilon/4$ and $\cos\theta > 1/2$. From Lemma 4.9 one can choose α_1 such that

$$\forall t \in [0, \alpha_1],\ \ \mathfrak{C}(t,x) \in K_{\mathbf{e},\theta/2}(x)$$

and from Lemma 4.8 one can choose α_2 such that

$$\forall y \in \mathbb{B}_{x,\alpha_2},\ \ \nabla(y)\cdot\mathbf{e} > \|\nabla(x)\| - \frac{\epsilon}{2} > \frac{\|\nabla(x)\|}{2} \tag{13}$$

Let E_k be a sequence of Euler schemes, each E_k being based on an h_k sampling of an interval $[0, \alpha]$, with $\alpha \le \min(\alpha_1, \alpha_2)$ and such that:

$$\lim_{k\to\infty} h_k = 0$$

Then, we get from Lemmas 4.4 and 4.5 that the sequence E_k converges uniformly toward $t \mapsto \mathfrak{C}(t,x)$ and that there is a constant γ such that:

$$\|E_k(x)(t) - \mathfrak{C}(t,x)\| \le \sqrt{h_k\gamma(\mathbf{e}^{2Kt} - 1)} \tag{14}$$

On another side, from $\forall y \in \mathbb{B}_{x,\alpha_2},\ \nabla(y)\cdot\mathbf{e} > \|\nabla(x)\|/2$ we get that:

$$E_k(x)(t)\cdot\mathbf{e} > t\frac{\|\nabla(x)\|}{2} \tag{15}$$

and, because $\forall t \in [0, \alpha_1],\ \mathfrak{C}(t,x) \in K_{\mathbf{e},\theta/2}(x)$ we get from the inequalities (14) and (15) above that for any $\eta > 0$, there is $K(\eta)$ such that

$$\forall k \ge K(\eta),\ \forall t \in [\eta, \alpha_1],\ E_k(t) \in K_{\mathbf{e},\theta}(x) \tag{16}$$

From Lemma 4.1 we can choose α_3 such that:

$$\forall y \in \mathbb{B}_{x,\alpha_3}, (\nabla(y) - \nabla(x))\cdot\frac{y - x}{\|y - x\|} \le \frac{\epsilon}{8} \tag{17}$$

Let $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$. Let be $t \in [0, \alpha]$ and a sequence of Euler scheme E_k as above and such that, for some $i(k)$, $t_{i(k)} = t$. Let y_j be an iterate of the Euler scheme E_k such that $y_j \in \mathbb{B}_{x,\alpha} \cap K_{\mathbf{e},\theta}(x)$. We use, as in the proof of Lemma 4.9 the decomposition of $y_j - x$ on the orthogonal unit vectors $\mathbf{e}$ and $\mathbf{f}$:

$$\frac{y_j - x}{\|y_j - x\|} = \cos\psi\mathbf{e} + \sin\psi\mathbf{f}$$

with $0 \le \psi \le \theta$ which gives, from Eq. (17):

$$(\nabla(y_j) - \nabla(x))\cdot(\cos\psi\mathbf{e} + \sin\psi\mathbf{f}) \le \frac{\epsilon}{8}$$

and that gives:

$$\cos\psi\nabla(y_j)\cdot\mathbf{e} \le \cos\psi\nabla(x)\cdot\mathbf{e} - \sin\psi\nabla(y_j)\cdot f + \frac{\epsilon}{8}$$

that is:

$$\nabla(y_j)\cdot e \le \nabla(x)\cdot\mathbf{e} - \mathrm{tans}\,\psi\nabla(y_j)\cdot f + \frac{\epsilon}{8\cos\psi}$$

Because $0 \le \psi \le \theta$, we have $\tan\psi < \epsilon/4$ and $\cos\psi > 1/2$ and:

$$\nabla(y_j)\cdot e \le \nabla(x)\cdot\mathbf{e} + \frac{\epsilon}{2}$$

We have then, using Eq. (13):

$$\|\nabla(x)\| - \frac{\epsilon}{2} \leq \nabla(y_j)\cdot\mathbf{e} \leq \|\nabla(x)\| + \frac{\epsilon}{2} \tag{18}$$

Because (hereafter i stand for $i(k)$):

$$y_i - x = \sum_{j=0,i-1} (t_{j+1} - t_j)\nabla(y_j)$$

we have

$$(y_i - x)\cdot\mathbf{e} = \sum_{j=0,i-1} (t_{j+1} - t_j)[\nabla(y_j)\cdot e]$$

Let us choose η such that $0 < \eta < t = t_i$ and $\eta < t(\epsilon/8)$. According to Eq. (16), for any $k > K(\eta)$, and any $t_j > \eta$, if y_j is the iterate of the Euler scheme E_k corresponding to t_j, then we have $y_j \in \mathbb{B}_{x,\alpha} \cap K_{\mathbf{e},\theta}(x)$. We can write:

$$(y_i - x)\cdot\mathbf{e} = \sum_{0\leq j\leq i-1, t_j\leq\eta} (t_{j+1} - t_j)[\nabla(y_j)\cdot\mathbf{e}] + \sum_{0\leq j\leq i-1, t_j>\eta} (t_{j+1} - t_j)[\nabla(y_j)\cdot\mathbf{e}]$$

and, from Eq. (18), we have:

$$\left(\|\nabla(x)\| - \frac{3\epsilon}{4}\right)t_i \leq (y_i - x)\cdot\mathbf{e} \leq \left(\|\nabla(x)\| + \frac{3\epsilon}{4}\right)t_i$$

and that entails

$$|(y_i - [x + t_i\nabla(x)])\cdot\mathbf{e}| \leq t_i\frac{3\epsilon}{4}$$

Because the sequence of E_k converges toward $t \mapsto \mathfrak{C}(t,x)$ the inequality remains true for the limit:

$$\forall t \in [0,\alpha], |(\mathfrak{C}(t,x) - [x + t\nabla(x)])\cdot\mathbf{e}| \leq t\frac{3\epsilon}{4}$$

and the Lemma is a consequence of this inequality and Lemma 4.9. □

The previous Lemma 4.10 that the map $t \mapsto \nabla(\mathfrak{C}(t,x))$ is the right derivative of the map $t \mapsto \mathfrak{C}(t,x)$. Similarly, Lemma 4.11 claims that the map $t \mapsto \nabla(\mathfrak{C}(t,x))^2$ is the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$.

Lemma 4.11. *The map $t \mapsto \nabla(\mathfrak{C}(t,x))^2$ is the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$. In other words, we have*:

$$\forall x \in \mathcal{O},\ \forall\epsilon > 0,\ \exists\alpha > 0,\ \forall t \in [0,\alpha]$$

$$|\mathcal{R}(\mathfrak{C}(t,x)) - [\mathcal{R}(x) + t\nabla(\mathfrak{C}(t,x))^2]| < t\epsilon$$

Proof. As for Lemma 4.10, it is enough to prove the proposition for $t = 0$. Again, the proposition is trivial if $\nabla(x) = 0$. We can then assume that $\nabla(x) \neq 0$ and we define the angle ψ defined by $\cos\psi = -\|\nabla(x)\| = -\nabla(x)\cdot\mathbf{e}$, with $\pi/2 < \psi \leq \pi$.
From Lemma 4.6, we can choose $\alpha_1 > 0$ such that

$$\forall t \in [0,\alpha_1],\ \mathbf{\Gamma}(\mathfrak{C}(t,x)) \subset \mathbf{\Gamma}(x)\oplus\mathbb{B}^o_{0,|\cos\psi|(\epsilon\mathcal{R}(x)/8)}$$

Let us pick $t \in [0,\alpha_1]$ and $z' \in \mathbf{\Gamma}(\mathfrak{C}(t,x))$. We have:

$$d(x,z') \leq \mathcal{R}(x)\left(1 + \frac{\epsilon}{8}\right) \tag{19}$$

and, as in Lemma 4.8

$$(z' - x)\cdot\mathbf{e} \leq \mathcal{R}(x)\cos\psi\left(1 - \frac{\epsilon}{8}\right) < 0 \tag{20}$$

If we call ψ' the angle between the vectors $\mathbf{e}$ and $z' - x$ we have:

$$\cos\psi' = \frac{(z' - x)\cdot\mathbf{e}}{d(x,z')}$$

and from Eqs. (19) and (20) (remember that $\cos\psi < 0$):

$$\cos\psi' \leq \cos\psi\frac{1 - \frac{\epsilon}{8}}{1 + \frac{\epsilon}{8}} \leq \cos\psi\left(1 - \frac{\epsilon}{4}\right) \tag{21}$$

The minimality condition on the definition of $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ and the compactness of $\mathbf{\Gamma}(x)$ imply that there exists at least one $z \in \mathbf{\Gamma}(x)$ such that the angle between the vector $\mathbf{e}$ and $z - x$ is ψ. We have:

$$\mathcal{R}(x) = d(x,z)$$

$$\mathcal{R}(\mathfrak{C}(t,x)) \leq d(\mathfrak{C}(t,x),z)$$

and:

$$\mathcal{R}(\mathfrak{C}(t,x)) = d(\mathfrak{C}(t,x),z')$$

$$\mathcal{R}(x) \leq d(x,z')$$

which gives:

$$d(\mathfrak{C}(t,x),z')^2 - d(x,z')^2 \leq \mathcal{R}(\mathfrak{C}(t,x))^2 - \mathcal{R}(x)^2 \leq d(\mathfrak{C}(t,x),z)^2 - d(x,z)^2 \tag{22}$$

Using Lemma 4.10, we can write:

$$\mathfrak{C}(t,x) = x + t\nabla(x) + o(t)$$

which gives:

$$\begin{aligned}[d(\mathfrak{C}(t,x),z)]^2 - [d(x,z)]^2 &= [z - (x + t\nabla(x) + o(t))]^2 - [z - x]^2 \\ &= -2t(z - x)\nabla(x) + o(t) = -2t\mathcal{R}(x)\cos\psi\|\nabla(x)\| + o(t) \\ &= 2t\mathcal{R}(x)\|\nabla(x)\|^2 + o(t)\end{aligned}$$

Similarly, we get:

$$[d(\mathfrak{C}(t,x),z')]^2 - [d(x,z')]^2 = -2t\mathcal{R}(x)\cos\psi'\|\nabla(x)\| + o(t)$$

and, using Eq. (21):

$$[d(\mathfrak{C}(t,x),z')]^2 - [d(x,z')]^2 \geq 2t\mathcal{R}(x)\|\nabla(x)\|^2\left(1 - \frac{\epsilon}{4}\right) + o(t)$$

That is, from Eq. (22):

$$2t\mathscr{R}(x)\|\nabla(x)\|^2\left(1-\frac{\epsilon}{4}\right)+o(t)\leq\mathscr{R}(\mathfrak{C}(t,x))^2-\mathscr{R}(x)^2$$

$$\leq 2t\mathscr{R}(x)\|\nabla(x)\|^2+o(t)$$

By choosing $\alpha\leq\alpha_1$ small enough, the $o(t)$ in the above equations can be made smaller than $(\epsilon t)/2$ for any $t\in[0,\alpha]$ and we have:

$$|\mathscr{R}(\mathfrak{C}(t,x))^2-\mathscr{R}(x)^2-2t\mathscr{R}(x)\nabla(x)^2|<\epsilon t$$

which means that the right derivative of the map $t\mapsto\mathscr{R}(\mathfrak{C}(t,x))^2$ at $t=0$ is $2\mathscr{R}(x)[\nabla(x)]^2$. We can conclude that the right derivative of the map $t\mapsto\mathscr{R}(\mathfrak{C}(t,x))$ at $t=0$ is $\nabla(x)^2$ and that completes he proof of the Lemma. □

Lemma 4.12. *We have the following equalities*:

$\forall x\in\mathscr{O},\ \forall t\in[0,\mathscr{R}(x)/4]$

$$\mathfrak{C}(t,x)=x+\int_0^t\nabla(\mathfrak{C}(\tau,x))d\tau$$

$$\mathscr{R}(\mathfrak{C}(t,x))=\mathscr{R}(x)+\int_0^t\nabla(\mathfrak{C}(\tau,x))^2d\tau$$

The curve image of $t\mapsto\mathfrak{C}(t,x)$ is rectifiable and its arc length is an increasing function of t given by:

$$s(t)=\int_0^t\|\nabla(\mathfrak{C}(\tau,x))\|d\tau$$

Notice that the Lemma asserts as well that these expressions are well defined, that is the functions under the symbol $\int$ are Lebesgues integrable.

Proof. The maps $t\mapsto\mathfrak{C}(t,x)$ and $t\mapsto\mathscr{R}(\mathfrak{C}(t,x))$ are Lipschitz. Therefore they are absolutely continuous which entails ([5, p. 49–54]), that they are differentiable almost everywhere and that they are the integral of their derivatives. On another hand, we know that these maps are also right differentiable (everywhere). Almost everywhere, their derivatives exist and have to be equal to the right derivative. This proves the two first equations. Notice that because $t\mapsto\mathfrak{C}(t,x)$ is Lipschitz, it defines a rectifiable arc, whose arc length is the right hand side of the third equation. □

It results from the previous Lemma that *the map $t\mapsto\mathscr{R}(\mathfrak{C}(t,x))$ is increasing* and this allows to show the Lemma below, that is the global existence of $\mathfrak{C}$.

Lemma 4.13. *The local map C extends to a global map on $\mathbb{R}^+\times\mathscr{O}$:*

$\mathfrak{C}:\mathbb{R}^+\times\mathscr{O}\to\mathscr{O}$

$\mathfrak{C}$ is continuous, it is 1-Lipschitz with respect to the real variable t and, for any ρ, its restriction to $\mathbb{R}^+\times\mathscr{O}_\rho$ is Lipschitz with respect to both variables. To be more precise, If $x_1,x_2\in\mathscr{O}_\rho$, one has

$$\|\mathfrak{C}(t_2,x_2)-\mathfrak{C}(t_1,x_1)\|\leq|t_2-t_1|+\|x_2-x_1\|\mathbf{e}^{(1/\rho)\min(t_1,t_2)}$$

Proof. For any $x\in\mathscr{O}$, $\mathfrak{C}$ is defined on $[0,\mathscr{R}(x)/2]\times\mathbb{B}_{x,\mathscr{R}(x)/4}$. But, because $t\mapsto\mathscr{R}(\mathfrak{C}(t,x))$ is increasing, we know that $\mathscr{R}(\mathfrak{C}(t,x))\geq\mathscr{R}(x)$ and it is therefore possible to extend $t\mapsto\mathfrak{C}(t,x)$ at least on $[\mathscr{R}(x)/4,\mathscr{R}(x)/4+\mathscr{R}(x)/4]$. This can be repeated an arbitrary number of times, each time extending the right end of the interval of definition of $t\mapsto\mathfrak{C}(t,x)$ of at least $\mathscr{R}(x)/4$. This means that $\mathfrak{C}$ is defined on $\mathbb{R}^+\times\mathscr{O}$. The Lipschitz property is inherited from the Euler Scheme (Lemmas 4.4 and 4.5). □

4.4. The map $t\mapsto\mathscr{F}(\mathfrak{C}(t,x))$ is increasing

In order to get the homotopy equivalence of $\mathscr{O}$ and $\mathscr{M}$ using the map $\mathfrak{C}$, we need to prove that the map $t\mapsto\mathscr{F}(\mathfrak{C}(t,x))$ is increasing. We start with a few lemmas.

The technical Lemma 4.14 below is worth some intuitive restatement. One can check that the map $\mathfrak{R}_0$ of the Lemma satisfies the integral equation

$$\forall t\in[0,T],\ \mathfrak{R}_0(t)=R_0+\int_0^t\sqrt{1-\frac{f(0)^2}{\mathfrak{R}_0(u)^2}}du$$

Roughly speaking, the Lemma states that, if the map f is decreasing, then, any map $\mathfrak{R}$ satisfying the integral equation:

$$\forall t\in[0,T],\ \mathfrak{R}(t)=R_0+\int_0^t\sqrt{1-\frac{f(u)^2}{\mathfrak{R}(u)^2}}du$$

should be greater than $\mathfrak{R}_0$, for some t ($\exists t\in[0,T],\mathfrak{R}(t)>\mathfrak{R}_0(t)$).

Lemma 4.14. *Let $T>0$ and $R_0>0$ be positive numbers and $f:[0,T]\to\mathbb{R}$ an upper semi continuous function such that*:

$\forall t\in[0,T],\ f(t)\leq f(0)$ *and* $\exists t\in[0,T],\ f(t)<f(0)$

Let $\mathfrak{R}:[0,T]\to\mathbb{R}$ be an increasing continuous map such that:

$\forall t\in[0,T],\ f(0)<\mathfrak{R}(t)$

and satisfying the integral equation:

$$\forall t\in[0,T],\ \mathfrak{R}(t)=R_0+\int_0^t\sqrt{1-\frac{f(u)^2}{\mathfrak{R}(u)^2}}du$$

Let $\mathfrak{R}_0:[0,T]\to\mathbb{R}$ be the increasing continuous map defined by:

$$\mathfrak{R}_0(t)=\sqrt{R_0^2+2t\sqrt{R_0^2-f(0)^2}+t^2}$$

Then, we have:

$\exists t \in [0,T],\ \Re(t) > \Re_0(t)$

Proof. First, note that

$\forall t \in [0,T], f(0) < \Re_0(t)$

and a simple computation shows that:

$$\forall t \in [0,T],\ \Re_0(t) = R_0 + \int_0^t \sqrt{1 - \frac{f(0)^2}{\Re_0(u)^2}}\, du$$

The equation on $\Re$ can be expressed as a fix point of an operator ω_f. Given a continuous increasing map g defined on $[0,T]$ such that $\forall t \in [0,T], f(t) < g(t)\omega_f(g)$ is defined by:

$$\forall t \in [0,T],\quad \omega_f(g)(t) = g(0) + \int_0^t \sqrt{1 - \frac{f(u)^2}{g(u)^2}}\, du$$

and we have:

$\Re = \omega_f(\Re)$

Note that ω_f is monotonic in the following sense. If two continuous maps g_1 and g_2 are such that $\forall t \in [0,T], f(t) < g_1(t)$ and $f(t) < g_2(t)$ we check easily that if:

$\forall t \in [0,T],\quad g_1(t) \le g_2(t)$

We have:

$\forall t \in [0,T],\quad \omega_f(g_1)(t) \le \omega_f(g_2)(t)$

We start with the function $g_0 = \Re_0$ and we build the sequence of functions defined by:

$g_{i+1} = \omega_f(g_i)$

Recall that:

$\forall t \in [0,T],\ \ f(t) \le f(0)$ and $\exists t \in [0,T],\ f(t) < f(0)$

and, because f is upper semi continuous, there is $\epsilon > 0$ and $\alpha > 0$ such that

$\exists t_0 \in [0,T]\ \ \forall t \in [t_0 - \alpha, t_0 + \alpha],\ \ f(t) \le f(0) - \epsilon$

and, if we compare the expressions giving

$$g_0 = \Re_0 \text{ and } g_1 = \omega_f(g_0) = g(0) + \int_0^t \sqrt{1 - \frac{f(u)^2}{g_0(u)^2}}\, du,$$

this entails:

$$\forall t \in [0,T],\ \ \Re_0(t) = g_0(t) \le g_1(t) \text{ and } \exists t \in [0,T],$$
$$\Re_0(t) = g_0(t) < g_1(t)$$

We have then, from the monotony of ω_f and by induction on i :

$\forall i \in \mathbb{N},\ \ \forall t \in [0,T],\ \ g_i(t) \le g_{i+1}(t)$ and
$\exists t \in [0,T],\ \Re_0(t) < g_i(t)$

On the other hand, we have:

$$\forall i \in \mathbb{N}, \forall t \in [0,T],\ \ g_i(t) \le R_0 + t$$

Then, for a fixed t, the real number sequence $i \mapsto g_i(t)$ is increasing and bounded. Therefore, g_i converges pointwise toward a map g. From the Beppo Levi theorem of monotone convergence ([5, p. 36], [9, p. 18]) we get that:

$$\forall t \in [0,T], \lim_{i \to \infty} \omega_f(g_i)(t) = \omega_f(g(t))$$

which gives

$g = \omega_f(g)$

and of course, $\forall i,\ \forall t,\ g_i(t) \le g(t)$ and:

$$\forall t \in [0,T],\ \ \Re_0(t) \le g(t) \text{ and } \exists t \in [0,T],\ \Re_0(t) < g(t)$$

In order to complete the proof of the Lemma, we still have to prove that $\Re = g$, that is the uniqueness of the solution of the fix point equation. Recall that $\forall t \in [0,T]$:

$$0 \le f(t) \le f(0),\ \ g(t) \ge g(0) = R_0 \ge 0 \text{ and } \Re(t) \ge \Re(0)$$
$$= R_0 \ge 0$$

Because g and $\Re$ are continuous and f upper semi continuous, $t \mapsto \frac{f(t)}{g(t)}$ and $t \mapsto \frac{f(t)}{\Re(t)}$ are upper semi continuous. Recall that an upper semi continuous function defined on a compact set reaches its least upper bound. Therefore, there is $\epsilon > 0$ such that:

$$\forall t \in [0,T],\ \ \frac{f(t)}{g(t)} < 1 - \epsilon$$

and

$$\forall t \in [0,T],\ \ \frac{f(t)}{\Re(t)} < 1 - \epsilon$$

On another hand, there is a uniform bound K to the derivative of the map:

$$G \mapsto \sqrt{1 - \frac{f(t)^2}{G^2}} \text{ for } G \in \left[\frac{1}{1-\epsilon} f(t), \infty\right) \text{ and } t \in [0,T]$$

Then, for any real number $\alpha, 0 < \alpha < 1$, and $t, 0 \le t < T$, if $t + \alpha/K \le T$ we have:

$$\sup_{h \in [0,\alpha/K]} \left| \sqrt{1 - \frac{f(t+h)^2}{\Re(t+h)^2}} - \sqrt{1 - \frac{f(t+h)^2}{g(t+h)^2}} \right|$$
$$\le K \sup_{h \in [0,\alpha/K]} |\Re(t+h) - g(t+h)|$$

and therefore:

$$\sup_{h\in[0,\alpha/K]}\left|\int_t^{t+h}\sqrt{1-\frac{f(\tau)^2}{\Re(\tau)^2}}d\tau-\int_t^{t+h}\sqrt{1-\frac{f(\tau)^2}{g(\tau)^2}}d\tau\right|$$

$$\le \alpha \sup_{h\in[0,\alpha/K]} |\Re(t+h)-g(t+h)|$$

Because $g=\omega_f(g)$ and $\Re=\omega_f(\Re)$, this inequality entails that:

$$\Re(t)=g(t)\Rightarrow \sup_{h\in[0,\alpha/K]}|\Re(t+h)-g(t+h)|$$

$$\le \alpha \sup_{h\in[0,\alpha/K]}|\Re(t+h)-g(t+h)|$$

which gives, because $\alpha<1$:

$$\Re(t)=g(t)\Rightarrow \sup_{h\in[0,\alpha/K]}|\Re(t+h)-g(t+h)|=0$$

that is:

$$\Re(t)=g(t)\Rightarrow \forall h\in\left[0,\frac{\alpha}{K}\right],\ \ \Re(t+h)=g(t+h)$$

We have of course $g(0)=\Re(0)=R_0$ and that entails $g=\Re$ which ends the proof. □

Lemma 4.15. *We have, for any $x,y\in\mathcal{O}$, such that $\nabla(x)\neq 0$:*

$$\mathcal{R}(y)\le\sqrt{\mathcal{R}(x)^2+2d(x,y)\sqrt{\mathcal{R}(x)^2-\mathcal{F}(x)^2}+d(x,y)^2}$$

Proof. Let be $z\in\mathbf{\Gamma}(x)$. For $y\neq x$, we define $d=d(x,y)$. Using the unit vector $u=(y-x)/d$ we can write:

$$\|z-y\|^2=\|z-(x+du)\|^2=((z-x)-du)^2$$

$$=(z-x)^2-2d(z-x)\cdot u+d^2$$

which gives

$$\mathcal{R}(y)^2\le\inf_{z\in\mathbf{\Gamma}(x)}\|z-y\|^2=\mathcal{R}(x)^2+d^2-2d\sup_{z\in\mathbf{\Gamma}(x)}(z-x)\cdot u$$

For a unit vector u, $\|u\|=1$, we define $r(u)=\sup_{z\in\mathbf{\Gamma}(x)}(z-x)\cdot u$.
Because $\nabla(x)\neq 0$, for some unit vector u, $r(u)<0$ and there is a ball of radius $\sqrt{\mathcal{R}(x)^2-r(u)^2}$ enclosing $\mathbf{\Gamma}(x)$. Among these balls, the one with the smallest radius is $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ and it corresponds to the unit vector $\mathbf{e}=\nabla(x)/\|\nabla(x)\|$ which then minimize $u\mapsto r(u)$:

$$\forall u,\ \ \|u\|=1,\ \ r(\mathbf{e})\le r(u)$$

We have then:

$$\mathcal{R}(y)^2\le\mathcal{R}(x)^2+d^2-2dr(u)\le\mathcal{R}(x)^2+d^2-2dr(\mathbf{e})$$

and, because $r(\mathbf{e})=-\sqrt{\mathcal{R}(x)^2-\mathcal{F}(x)^2}$, we have:

$$\mathcal{R}(y)^2\le\mathcal{R}(x)^2+2d(x,y)\sqrt{\mathcal{R}(x)^2-\mathcal{F}(x)^2}+d(x,y)^2$$

and we obtain the inequality of the Lemma. □

In fact, Lemma 4.15 remains true when $\nabla(x)=0$: in this special case, $\mathcal{R}(x)=\mathcal{F}(x)$ and the Lemma reduces to the triangular inequality.

Lemma 4.16. *If $\nabla(\mathfrak{C}(T,x))\neq 0$, the arc length map $t:[0,T]\to[0,s(T)]$, $t\mapsto s(t)$ given in Lemma 4.12 is strictly increasing and we can define the inverse function $t,[0,s(T)]\to[0,T]:s\mapsto t(s)$.*
In other words, one can parameterize the curve by the arc length. Using this parameterization, one has:

$$\mathcal{R}(\mathfrak{C}(t(s),x))=\mathcal{R}(x)+\int_0^s\|\nabla(\mathfrak{C}(t(\sigma),x))\|d\sigma$$

Proof. For a given $x\in\mathcal{O}$, let be T such that $\|\nabla(\mathfrak{C}(T,x))\|\neq 0$. If, for some $t_0<T$, we had $\|\nabla(\mathfrak{C}(t_0,x))\|=0$, the map $t\mapsto\mathfrak{C}(t,x)$ would be stationary for $t\ge t_0$ which contradicts $\|\nabla(\mathfrak{C}(T,x))\|\neq 0$. This means that as far as $t<T$, we have $\|\nabla(\mathfrak{C}(t,x))\|>0$. From Lemma 4.12 we get that, on $[0,T]$, the arc length map s is strictly increasing.
If we denote by S the arc length of the image of $[0,T]$ by $t\to\mathbb{C}(t,x)$, one can define the map $t:[0,S]\to[0,T]$ inverse of the arc length map s.
From the right derivatives of the maps $t\to s(t)$ and $t\to\mathcal{R}(\mathfrak{C}(t,x))$ as given in Lemmas 4.11 and 4.12, one can check the right differentiability and compute the right derivative of the map $s\mapsto\mathcal{R}(\mathfrak{C}(t(s),x))$. We get:

$$\mathcal{R}(\mathfrak{C}(t(s+h),x))=\mathcal{R}(\mathfrak{C}(t(s),x))+h\|\nabla(\mathfrak{C}(t,x))\|+o(h)$$

As in the proof of Lemma 4.12, because this map is Lipschitz, it is differentiable almost everywhere. It is then the indefinite integral of its right derivative, which completes the proof. □

Lemma 4.17. *For fixed x, $t\mapsto\mathcal{F}(\mathfrak{C}(t,x))$ is an increasing, right continuous function.*

Proof. Let be $x\in\mathcal{O}$ and $T>0$ such that $\nabla(\mathfrak{C}(T,x))\neq 0$ and let us define:

$$f_{\max}=\sup_{t\in[0,T]}\mathcal{F}(\mathfrak{C}(t,x))$$

From the upper semi-continuity of $\mathcal{F}$ we know that

$$\exists t^*\in[0,T]\ \text{s.t.}\mathcal{F}(\mathfrak{C}(t,x))=f_{\max}$$

Let us denote by $t,t:s\mapsto t(s)$, the inverse of the arc length function defined in Lemma 4.16 of the arc defined by the map $t\mapsto\mathfrak{C}(t,x)$ starting at $\mathfrak{C}(t^*,x)$. If S denote the length of the arc image of $[t^*,T]$ by $t\mapsto\mathfrak{C}(t,x)$, we have:

$$t(0)=t^*,\ \ t(S)=T,$$

and $\forall s \in [0, S]$:

$$\mathscr{R}(\mathfrak{C}(t(s), x)) = \mathscr{R}(\mathfrak{C}(t^*, x)) + \int_0^s \|\nabla(\mathfrak{C}(t(\sigma), x))\| d\sigma$$

$$= \mathscr{R}(\mathfrak{C}(t^*, x)) + \int_0^s \sqrt{1 - \frac{\mathscr{F}(\mathfrak{C}(t(\sigma), x))^2}{\mathscr{R}(\mathfrak{C}(t(\sigma), x))^2}} d\sigma$$

and, because $s \geq d(\mathfrak{C}(t^*, x), \mathfrak{C}(t(s), x))$ we get from Lemma 4.15:

$$\mathscr{R}(\mathfrak{C}(t(s), x))$$

$$\leq \sqrt{\mathscr{R}(\mathfrak{C}(t^*, x))^2 + 2s\sqrt{\mathscr{R}(\mathfrak{C}(t^*, x))^2 - \mathscr{F}(\mathfrak{C}(t^*, x))^2} + s^2} \tag{23}$$

If we assume now that there is t_0, $t^* < t_0 \leq T$ such that $\mathscr{F}(\mathfrak{C}(t_0, x)) < f_{\max}$ the map $\mathfrak{R}$ defined by

$$\mathfrak{R}(s) = \mathscr{R}(\mathfrak{C}(t(s), x))$$

meets the conditions of Lemma 4.14 with:

$$f(s) = \mathscr{F}(\mathfrak{C}(t(s), x))$$

and

$$\mathfrak{R}_0(s) = \sqrt{\mathscr{R}(\mathfrak{C}(t^*, x))^2 + 2s\sqrt{\mathscr{R}(\mathfrak{C}(t^*, x))^2 - \mathscr{F}(\mathfrak{C}(t^*, x))^2} + s^2}$$

and we get a contradiction between the statement of Lemma 4.14 and inequality Eq. (23) above. We can conclude that $s \mapsto f(s) = \mathscr{F}(\mathfrak{C}(t(s), x))$ cannot be decreasing on $[t^*, T]$.
This means that, for any $T > 0$ such that $\nabla(\mathfrak{C}(T, x)) \neq 0, t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ reaches its maximum at $t = T$. $t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ is then increasing on $[0, T]$.
If there is some $T_{\max}$ which is the supremum (the least upper bound) of the T for which $\nabla(\mathfrak{C}(T, x)) \neq 0$, we know that, from the lower semi-continuity of $\|\nabla(\mathfrak{C}(T, x))\|$, we must have $\nabla(\mathfrak{C}(T_{\max}, x)) = 0$. $t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ is then increasing on $[0, T_{\max})$ and stationary on $[T_{\max}, \infty)$. From the upper semi-continuity of $\mathscr{F}$ we get that:

$$\mathscr{F}(\mathfrak{C}(T_{\max}, x)) \geq \lim_{t \to T_{\max}, t < T_{\max}} \mathscr{F}(\mathfrak{C}(t, x))$$

and this finally proves that $t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ is increasing on $\mathbb{R}^+$.
The right continuity of $t \mapsto \mathscr{F}(\mathbb{C}(t, x))$ is an immediate consequence of the fact that it is upper semi continuous and increasing. □

4.5. Main theorem

It is enough, to prove the homotopy equivalence of $\mathscr{M}$, to apply the deformation $\mathfrak{C}$ on a compact interval $[0, D]$ defined by the next proposition.

Proposition 4.18. *If D is an upper bound of the diameter of $\mathscr{O}$, then*: $\forall x \in \mathscr{O}$, $\mathfrak{C}(D, x) \in \mathscr{M}$.

Proof. Let us assume that for some $x \in \mathscr{O}$, $\mathfrak{C}(D, x) \notin \mathscr{M}$, in other words, $\mathscr{F}(\mathfrak{C}(D, x)) = 0$. Because the map $t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ is increasing, this implies that $\forall t \in [0, D]$, $\mathscr{F}(\mathfrak{C}(t, x)) = 0$. Then, from Eq. (2) of Section 4.1, this implies that $\forall t \in [0, D]$, $\|\nabla((\mathfrak{C}(t, x)))\| = 1$ and we have then, using Lemma 4.12

$$\mathscr{R}(\mathfrak{C}(D, x)) = \mathscr{R}(x) + D$$

and this contradicts the fact that D is an upper bound of the diameter of $\mathscr{O}$. □

We can now state the main theorem.

Theorem 4.19. *If $\mathscr{O}$ is a bounded open subset of $\mathbb{R}^n$ and $\mathscr{M}$ its medial axis, then $\mathscr{O}$* and *$\mathscr{M}$ have same homotopy type.*

Proof. Let us consider the map H,

$$H : [0, 1] \times \mathscr{O} \to \mathscr{O}$$

given by:

$$H(t, x) = \mathfrak{C}(Dt, x)$$

where D is the upper bound of the diameter of $\mathscr{O}$ introduced in Proposition 4.18. From this proposition, we know that $\forall x \in \mathscr{O}, H(1, x) \in \mathscr{M}$.
Moreover, $t \mapsto \mathscr{F}(\mathfrak{C}(t, x))$ is increasing (Lemma 4.17) which implies that, if $x \in \mathscr{M}$ then, $\forall t, \mathfrak{C}(t, x) \in \mathscr{M}$. It follows that H meets the condition of the characterization of Proposition 3.2. □

5. Conclusion

This work establishes for the first time the Homotopy Equivalence between any bounded open subset of $\mathbb{R}^n$ and its Medial Axis $\mathscr{M}$. This gives a theoretically sound justification to the significance of the medial axis in shape classification and reconstruction.

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