1 Prompt

Later this year, aliens will visit Earth and announce that they intend to blow up the planet. (Lovely.)

However, they present you with a challenge. If you successfully complete the challenge, they will blow up another planet instead (probably Neptune, because why not).

The aliens have telepathically assigned each of the $N = 8$ billion human beings on Earth a unique random number, uniformly distributed between 0 and 1. Each human being knows their own number, but no one else?s. Your challenge is to identify the person with the highest number.

The aliens will allow you to ask a single yes-or-no question to all 8 billion people. This question must be the same for everyone and will be answered simultaneously by everyone. The aliens have courteously agreed to aggregate the data for you as to who answers your question yes or no.

What question would you ask, and what are your chances of saving the world?

2 Strategy: Choosing a Threshold

The natural question to ask each person is: "Is your number greater than x?", and our goal is to determine the optimal value of x to maximize our change of survival.

2.1 Computing the Expected Win Probability

If $n > 0$ people answer yes to this question, then we have a probability $1/n$ of choosing the correct person with the maximum. If $n = 0$, then we're pretty well out of luck as we have a $1/N \approx 10^{-10}$ chance of guessing correctly.

The probability of exactly n people answering yes to the question is $P(n, x) = \binom{N}{n} x^{N-n}(1-x)^n$, and for a given threshold x the combined probability we guess the person correctly is

$$E(x) = \frac{x^N}{N} + \sum_{n=1}^{N} \frac{P(N, x)}{n} \tag{1}$$

Using the fact that

$$\frac{\partial P(n, x)}{\partial x} = \left(\frac{N-n}{p} - \frac{n}{1-p} \right) P(n, x) \tag{2}$$

We find a first order differential equation

$$\frac{dE}{dx} = \frac{NE}{x} - \frac{1-x^N}{x(1-x)} \ ; \ E(0) = E(1) = \frac{1}{N} \tag{3}$$

We see immediately that at the optimal value of $x = x_0$ (to be determined), the derivate vanishes and

$$E(x_0) = \frac{1}{N} \frac{1-x_0^N}{1-x_0} \tag{4}$$

As a curiosity, by solving the ODE, we can derive an alternative series expression for the expected value (however it won't be amenable to our approach for finding the optimal value[1]).

$$E(x) = x^N \left[\sum_{n=1}^{N} \frac{1}{nx^n} - H_{N-1} \right] \tag{5}$$

Here H_n is the nth harmonic number.

[1] One challenge is that $x_0 \approx 1$ and the denominator in the sum decays slowly as $1/n$

2.2 Finding x_0

It is well know that for N IID uniform random variables U_i

$$E[\max(U_i)] = \frac{N}{N+1} = 1 - 1/N + O(1/N^2). \tag{6}$$

This suggests using a heuristic $x_0 = 1 - \lambda/N$, where will we now attempt to determine λ. We know $N \gg 1$ and will assume $\lambda \ll N$. This allows us to derive parameters that match a Poisson distribution

$$\begin{aligned} x_0^N &= e^{-\lambda} && (7) \\ \left(\frac{1}{x_0} - 1\right)^n &= \frac{\lambda^n}{N^n} && (8) \\ & && (9) \end{aligned}$$

And reduce the expected value expression (Eq 1) as

$$\begin{aligned} E(x_0) &= \frac{e^{-\lambda}}{N} + \sum_{n=1}^{N} \frac{1}{n} \binom{N}{n} e^{-\lambda} \frac{\lambda^n}{N^n} && (10) \\ &\approx e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\lambda^n}{n!} && (11) \end{aligned}$$

Here we dropped the first $1/N$ term as negligible and approximated[2] $\binom{N}{n} \approx N^n/n!$. Minimizing wrt λ we need to solve

$$\frac{\partial E}{\partial \lambda} = e^{-\lambda} \left[\frac{(e^{\lambda} - 1)}{\lambda} - \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\lambda^n}{n} \right] = 0 \tag{12}$$

We can numerically solve this equation[3] yielding

$$\begin{aligned} \lambda &= 1.50286 && (13) \\ e^{-\lambda} &= 0.222493 && (14) \\ x_0 &= 1 - \frac{\lambda}{N} = 0.9999999998121425 && (15) \\ E(x_0) &= \frac{1 - e^{-\lambda}}{\lambda} = 0.517352 && (16) \end{aligned}$$

The expression for $E(x_0)$, a reduction from (Eq 4.), is fairly intuitive. λ tells us the expected number of people to say yes, and $1 - e^{-\lambda}$ is the probability at least one person says yes.

2.3 The Answer

By asking every person the question, $\boxed{\text{“Is your number greater than 0.9999999998121425”}}$, and guessing evenly among those who say yes[4], we ensure a $\boxed{51.7\%}$ chance of correctly guessing the person with the highest number. Further, we find the distribution of responses following the typical Poisson distribution with values

$$\begin{aligned} P(n=0) &= 22.2\% && (17) \\ P(n=1) &= 33.4\% && (18) \\ P(n=2) &= 25.1\% && (19) \\ P(n=3) &= 12.6\% && (20) \\ P(n=4) &= 4.7\% && (21) \\ P(n=5) &= 1.4\% && (22) \\ P(n=6) &= 0.4\% && (23) \\ & && (24) \end{aligned}$$

[2] We see that factorial suppression ensures us that only the first few values of n will contribute, and that this approximation holds well

[3] e.g. Taylor expanding λ to order 10

[4] Or if everyone says no, choosing any random person and accepting our fate