An introduction to the theory of spin glasses

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Abstract

We review the main methods used to study spin glasses. In the first part, we focus on methods for fully connected models and systems defined on a tree, such as the replica method, the Thouless-Anderson-Palmer formalism, the cavity method, and the dynamical mean-field theory. In the second part, we deal with the description of low-dimensional systems, mostly in three spatial dimensions, which are mostly studied through numerical simulations. We conclude by mentioning some of the main open problems in the field.

Contents

1 Spin Glasses

Spin glasses are paradigmatic complex systems, for which disorder plays a central role. Although the term was originally coined to describe certain magnetic materials that exhibit an exotic phase behavior, associated models and theory have since found applications in a wide variety of fields, thus making the study of spin glasses an intrinsically interdisciplinary endeavor.

Here, we provide an overview of spin glasses (SGs), from their experimental context to their common theoretical models. We notably present several theoretical methods developed within the context of their study, describe the differences between mean-field and three-dimensional SGs, discuss various interdisciplinary applications, and introduce some open questions in the field.

Experimental SGs As materials, SGs are disordered magnetic alloys containing strongly interacting ions immersed in a weakly interacting substrate [1–4]. They are prepared by rapidly cooling the liquid alloy, thus fixing the strongly interacting particles at random positions within the resulting solid. The pairwise exchange interaction between ions is then positive or negative, depending on the distance vector $\mathbf{r}$ between ions, such as they are for Ruderman-Kittel-Kazuya-Yosida (RKKY) interactions [5–7],

$$J(\mathbf{r}) \sim \frac{1}{|\mathbf{r}|^3} \cos(\mathbf{k} \cdot \mathbf{r}) \,, \tag{1}$$

where the modulus of the frequency $\mathbf{k}$ is of the order of the Fermi vector. Spin glasses also arise in systems with interactions different from RKKY. The general idea is that because ion positions depend on the specific realization of the alloy, distances between them are randomly distributed (and a priori unknown), and therefore values of $J(\mathbf{r})$ are randomly positive and negative.

To make these systems more physically tractable, we can define SG models assuming that the distances between ions are fixed (for example, on a lattice), and that the coupling between two ions – commonly called spins – is a quenched random variable [8]. *Quenched* variables here refer to random quantities that appear in the Hamiltonian as parameters that do not change during the evolution of the system, so as to capture that ions' positions are fixed over the relevant experimental time scales. This formulation leads to the generic SG Hamiltonian

$$\mathcal{H} = -\sum_{i<j}^{N} s_i \cdot J_{ij} s_j \,, \tag{2}$$

where the N dynamical variables are spins s_i coupled through pairwise interactions of quenched magnitude J_{ij}. Spins can be formulated in different ways, but in this section we restrict our consideration to Ising spins, $s_i = \pm 1$, both because of their simplicity, and because experimental systems are typically spatially anisotropic [9–11], thus rendering effective interactions Ising-like [12, 13]. The quenched random variables are extracted from a distribution $P(J_{ij})$, which has support on both positive and negative values. In fully-connected models, all couplings are extracted from $P(J_{ij})$, while in other models some of the couplings are set to zero. For example, the Edwards-Anderson model (EAM) is defined by Hamiltonian (2) on a d-dimensional square lattice, so if s_i and s_j are not nearest neighbors then J_{ij} is suppressed.

Although Eq. (2) is not the only way to model SGs (for example, interactions can be more than pairwise), two of its features are nevertheless generic: quenched disorder and frustration [14]. Frustration here refers to the impossibility to satisfy simultaneously all local constraints, which follows from the couplings being i.i.d. variables that can be both positive and negative. As illustrated in Fig. 1, along a loop local couplings cannot all be minimized.

Self-averaging Because of the quenched disorder, every SG Hamiltonian is different from all others. In other words, every *sample* corresponds to a different set of couplings J_{ij}. Interestingly, even though samples are microscopically different, experimental SG realizations display the same macroscopic behavior. SG descriptions thus work under the assumption, which holds in most relevant cases [15–18], that large SG samples display equivalent average behavior. Therefore, although a *quenched partition function*, Z_J, can be computed for each sample, the quantity of physical interest is the *quenched free energy* averaged $\overline{(\ldots)}$ over the couplings distribution,

$$\mathcal{F} = \overline{F_J} = -T\overline{\log Z_J}, \tag{3}$$

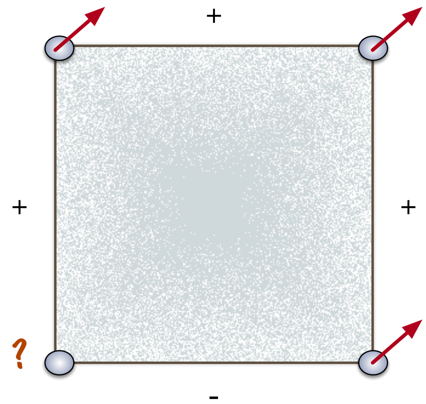

Figure 1: Disordered interactions result in frustration. The signs along the edges indicate whether two neighboring spins prefer to point in the same (+) or opposite (-) directions. It is therefore impossible to satisfy all the couplings simultaneously on the square plaquette.

for Boltzmann constant set to unity $k_{\mathrm{B}} = 1$, temperature T.

Order parameter The SG order parameter is also an important matter. In a ferromagnetic system, magnetization, $M = \frac{1}{N}\langle \sum_i s_i \rangle$, distinguishes between the ferromagnetic and the paramagnetic phases. However, for the Hamiltonian in Eq. (2) – if $P(J_{ij})$ is symmetric around zero – the magnetization is zero for all temperatures.

If a low-temperature SG phase exists, there must nevertheless be some configurations that are preferred over others. One way of identifying these preferred configurations is through the *Edwards-Anderson overlap* [19, 20],

$$q_{\mathrm{EA}} = \frac{1}{N} \lim_{t\to\infty} \sum_{i=1}^{N} \langle s_i(0)s_i(t)\rangle_t \ , \tag{4}$$

where $\langle(\ldots)\rangle_t \equiv \frac{1}{t}\int_0^t (\ldots)dt$ marks a time average. If spins s_i have no preferred value, as in the paramagnetic phase, then the products in Eq. (4) vanish on average, and $q_{\mathrm{EA}} = 0$; but if some configurations are preferred, each spin has a preferential value, and $q_{\mathrm{EA}} > 0$.

As further discussed in Sec. 2.1, the overlap can also be expressed without resorting to time averages, through the concept of replicas. Replicas of a system have exactly the same couplings J_{ij}, but evolve independently. For $s_i^{(a)}$ and $s_i^{(b)}$ denoting spins that belong to replicas a and b of the same sample, we can then write the overlap as

$$q_{ab} = \frac{1}{N} \sum_{i=1}^{N} \left\langle s_i^{(a)} \cdot s_i^{(b)} \right\rangle \ , \tag{5}$$

As above, the overlap will be zero if there is no preferred configuration, and positive otherwise.

2 Analytical Methods for Spin Glasses

Some of the most successful methods for studying SG Hamiltonians were developed to describe fully-connected models as well as models defined on tree-like graphs. The first geometry corresponds to the mean-field (MF) approximation, which becomes exact in the limit of $d \to \infty$ spatial dimensions, while the second goes beyond the MF description but neglects contributions from feedback loops.

2.1 The Replica Method

At variance with the partition function, which is typically non-self-averaging, the free energy is key to probing the low-temperature behavior of SGs (as shown in Eq. (3)). However, calculating the

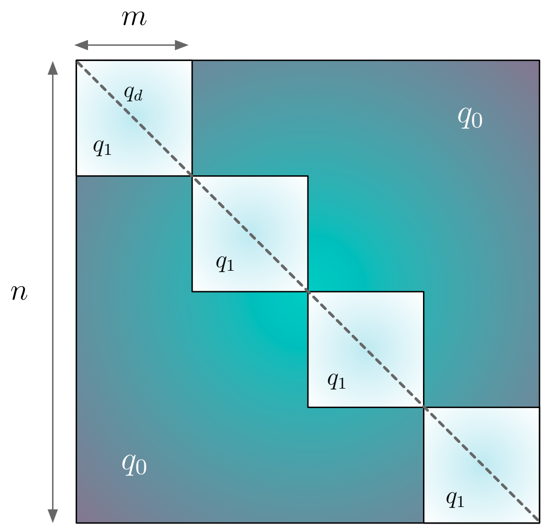

Figure 2: 1-RSB parametrization of the Parisi overlap matrix. q_1 represents the degree of similarity between two replicas inside the innermost block of size $m \times m$, whereas q_0 is the outermost block value.
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average of the logarithm in Eq. (3) can be challenging. The *replica method* overcomes this difficulty by replacing the calculation of $\overline{\ln(Z)}$ with that of $\overline{Z^n}$ through the identity

$$\overline{\ln Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}, \tag{6}$$

and by treating Z^n as the partition function for n replicas of the same sample. Treating the index n as an integer, however, hinges on the assumption that the analytical continuation $n \to 0$ exists. Assuming it does, the replicated partition function can then be rewritten as a function of the overlap matrix Q_{ab} describing the overlap between two replicas a and b:

$$\overline{Z^n} = \int \prod_{(ab)} \frac{dQ_{ab}}{2\pi} e^{N\mathcal{A}[Q_{ab}]} . \tag{7}$$

Taking advantage of the thermodynamic limit $N \to \infty$, the above expression can be evaluated by the Laplace method of saddle-point approximation, which extremizes the action $\mathcal{A}$ with the respect to the reference order parameter. The final result, therefore, depends on the structure of Q_{ab}.

The simplest and most intuitive ansatz for that structure is the replica symmetric (RS) one [21,22]. In this case, the overlap among different replicas is the same for any pair of replicas, with the exception of the overlap between a replica and itself. In the RS scenario, the matrix Q_{ab} therefore admits only two values: a diagonal contribution q_d (for $a = b$) and an off-diagonal one q_0 (for $a \neq b$). As expected, the RS ansatz correctly describes the high temperature and large external magnetic field regimes. But as these control parameters are lowered, some models reach a critical de Almeida-Thouless (dAT) line [21], below which the RS solution becomes unstable. Physically, a *spin glass* phase emerges when the RS solution is no longer stable.

More technically, below the dAT line, the correct solution requires a matrix Q_{ab} with an iterative block structure, which breaks the symmetry between different pairs of replicas [23–27]. For the first iteration of replica symmetry breaking (noted 1-RSB), the $n \times n$ matrix is parametrized by a diagonal value q_d and two off-diagonal values that can be either q_1, if the two replicas belong to the same block of size $m \times m$, or q_0, if the replicas fall outside the innermost block (see Fig. 2). If the low-temperature phase remains unstable after 1-RSB, this procedure can be iterated within each of these blocks, leading to a k-step RSB (or k-RSB). Depending on the specifics of the Hamiltonian, SG phases can have different levels of RSB. The limit $k \to \infty$ corresponds to the full-RSB scenario, according to which the overlap matrix is parametrized by a continuous function $q(x)$, with $x \in [0, 1]$ [25]. The overlap density distribution, $P(q)$, being non-zero over a continuous range of q, is a signature of the presence of an infinite number of symmetry-breaking points [28].

In the replica formalism, all the mutual information about pairs of equilibrium configurations is encoded in the overlap, which is a measure of their mutual distance [29, 30]. Given the hierarchical way in which the full-RSB construction is obtained, in systems with a full-RSB phase, the states have an *ultrametric* structure, meaning that their mutual distance can be described through a taxonomic or genealogical tree [31–33].

The replica structure of two common MF spin glass models. For the sake of concreteness, let's consider two paradigmatic MF spin glass models. Both have a low-temperature SG phase, but with different levels of RSB, and therefore present different free energy minima structures.

- The **spherical $\boldsymbol{p}$-spin** model has Hamiltonian

$$\mathcal{H} = -\sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,i_2,\ldots,i_p} s_{i_1} \ldots s_{i_p}, \tag{8}$$

  where p indicates p-wise interactions to which each spin participates. Spins are continuous variables subject to the global constraint $\sum_{i=1}^{N} s_i^2 = N$, and the random couplings are extracted from a Gaussian distribution $P(J) = \exp\left(-J^2 \frac{N^{p-1}}{p!}\right)$, where the factor N^{p-1} guarantees the extensivity of the free energy in the thermodynamic limit. For $p \geq 3$, this class of mean-field systems is characterized by a 1-RSB low-temperature phase, with an emergent number of minima that grows exponentially with N, and those are separated by extensive barriers.

  Small variations to this model result in significantly different physical behaviors. In particular, setting $p = 2$ results in only one single free-energy minimum (so there is no SG phase) [34], whereas replacing the spherical with Ising spins [35] results in the 1-RSB phase becoming unstable toward a full-RSB phase at low temperatures [36–38].

- The **Sherrington-Kirkpatrick** model (SK) [39, 40] has a Hamiltonian with an external uniform field h

$$\mathcal{H} = -\sum_{(ij)} J_{ij} s_i s_j - h \sum_i s_i \tag{9}$$

  where the sum runs over distinct pairs. The spins are Ising variables, $s_i = \pm 1$, and the random couplings are extracted from a Gaussian distribution $\mathcal{P} \simeq e^{-NJ_{ij}^2/(2J^2)}$ with zero mean and variance J^2/N. The low-temperature phase of this model is characterized by a hierarchical organization of energy minima, as given by the Parisi solution. The emergent number of minima is sub-exponential in system size, and those are separated by sub-extensive barriers [41]. The transition from single equilibrium (RS) to multiple equilibria (full-RSB) is second-order-like, with diverging correlation lengths and power-law singularities.

The solution of the SK model obtained by the replica method was rigorously proven thirty years later [15, 42–44]. Even though there is no rigorous demonstration that the replica method is generally correct, it has since been proven to provide the correct result in several specific cases [45–47], and no counter-example is known.

2.2 The Thouless-Anderson-Palmer (TAP) Approach

The TAP approach aims to probe complex energy and free-energy landscape properties through a perturbative high-temperature expansion (also known as *Plefka* or *Georges-Yedidia expansion* [48, 49]) by defining a Legendre transform $\mathcal{F}[\mathbf{m}]$ of the free energy as a function of the average magnetization $\mathbf{m}$. Such an expansion can detect the metastable states of the system, which correspond to the local minima of an appropriately-defined non-convex functional.

The underlying idea is to enforce the system to have single-site magnetization m_i by means of Lagrange multipliers $\lambda_i^{(\beta)}$ that depend on the inverse temperature β. Then, from the Legendre transform, one obtains

$$\mathcal{F}^{(\beta)}[\boldsymbol{m}] = \log \sum_{\boldsymbol{s}} e^{-\beta\mathcal{H}[\boldsymbol{s}] + \sum_i \lambda_i^{(\beta)}(s_i - m_i)} \tag{10}$$

given the stationarity condition $\lambda_i^{(\beta)} = -\frac{\partial \mathcal{F}^{(\beta)}[\boldsymbol{m}]}{\partial m_i}$. In the $\beta \to 0$ limit, spins are uncorrelated, making the computation of the first derivatives [1] of the free-energy functional with respect to β very straightforward:

$$\left.\frac{d\mathcal{F}^{(\beta)}[\boldsymbol{m}]}{d\beta}\right|_{\beta=0} = -\langle \mathcal{H} \rangle \,, \tag{11}$$

$$\left.\frac{d^2\mathcal{F}^{(\beta)}[\boldsymbol{m}]}{d\beta^2}\right|_{\beta=0} = \left\langle \left[\mathcal{H}[\boldsymbol{s}] - \langle \mathcal{H} \rangle - \sum_i \frac{\partial \lambda_i^{(\beta)}}{\partial \beta}(s_i - m_i) \right]^2 \right\rangle . \tag{12}$$

The second derivative involves both the connected part of the Hamiltonian and the partial derivatives of the Lagrange multipliers. The TAP free energy can then be written as a second-order expansion around the $\beta = 0$ limit. For the SK model, it reads

$$\mathcal{F}_{\mathrm{TAP}}^{\mathrm{SK}}[\boldsymbol{m}] = -\frac{1}{\beta}\sum_i s_0(m_i) - \frac{1}{2}\sum_{i \neq j} J_{ij} m_i m_j - \frac{N\beta}{4}(1-q)^2 \,, \tag{13}$$

where the first term on the right-hand side accounts for entropic effects, the second term is average energy contribution, and the last piece, the *Onsager reaction term*, represents a first correction to the MF approximation. It is also worth noting that since the couplings in the SK model are of order $1/\sqrt{N}$, $O(\beta^2)$ terms do matter in the final TAP expression. This contrasts with the purely ferromagnetic case (the fully-connected Ising model), for which the couplings are of order $1/N$.

In the case of the spherical p-spin model, the TAP free energy reads

$$\mathcal{F}_{\mathrm{TAP}}^{\mathrm{pspin}}[\mathbf{m}] = -\frac{1}{\beta}\sum_i s_0(m_i) - \frac{1}{p!}\sum_{i_1,\ldots,i_p} J_{i_1,\ldots,i_p} m_1 \ldots m_p - \frac{\beta N}{4}\left[1 - pq^{p-1} + q^p(p-1)\right] \,, \tag{14}$$

where, again, the last term represents the Onsager reaction term. By setting $m_i = 0$, one can immediately check that the TAP free energy is equal to $-\beta/4$, which is the free energy of the paramagnetic state.

In principle, the equilibrium probability distribution – hence the partition function – can always be decomposed into a combination of *pure states* α, with a free-energy density f_α. These pure states are entirely determined by a set of local observables, such as local magnetizations. The partition function can therefore be expressed as a sum over such states

$$Z = e^{-\beta NF} \simeq \sum_\alpha e^{-\beta N f_\alpha} = \int df \, \sum_\alpha \delta(f - f_\alpha) e^{-\beta N f} = \int df \; e^{N[\Sigma(f,\beta) - \beta f]} \simeq e^{N[\Sigma(f^*,\beta) - \beta f^*]} \,, \tag{15}$$

where in the last step we applied the saddle-point method [50, 51]. The configurational entropy Σ represents the logarithm of the total number of states. Evaluated at the states with f^*, which contribute maximally to Z, it leads to a simple temperature dependence of the TAP free energy with the identification $\partial\Sigma(f,\beta)/\partial f|_{f^*(\beta)} = \beta$.

In the p-spin model, this decomposition in pure states discloses a rich behavior in terms of the number of pure states that, at a given temperature, participate in the thermodynamic behavior of the system. At high temperatures, the free-energy density is ruled by a single state, the paramagnetic (Boltzmann-Gibbs) state. Upon lowering the temperature, one reaches a *dynamical transition* at temperature T_d, at which the emergence of an exponentially large number of clusters of pure states is accompanied by a dramatic slowing down of the dynamics. Surprisingly, the transition is not associated with any thermodynamic transition, because the free energy preserves its analyticity. The thermodynamic transition occurs at a lower temperature, T_s, known as *static* or *condensation transition*, where all different clusters of states collapse to the same state as a clear signature of vanishing configurational entropy [52].

Although the TAP approach was conceived as a high-temperature expansion, it can also be developed in terms of other perturbative quantities, such as a fictitious coupling associated with an effective energy cost. In this case, the Plefka-like expansion offers a playground for the definition of suitable effective high-dimensional potentials in situations where the energy is either ill-defined or trivially zero, such as in non-convex constraint satisfiability problems [53, 54].

[1] In the thermodynamic limit all higher-order terms but the first two can be neglected in a fully-connected system.

2.3 The Cavity Method

The cavity method [2] was initially devised as a way to solve the SK model without needing to resort to the replica formalism, but can also be used to go beyond the MF approximation. It can notably consider correlations between spins, thanks to the fact that it is exact in systems with a loopless interaction graph – or with divergingly wide loops – such as in trees and some random graphs. In the following, we will focus on finite-connectivity graphs, $\mathcal{G} = (V, E)$, defined by V nodes and E edges. The Hamiltonian in Eq. (2) can be then rewritten as $E(\mathbf{s}) = -\sum_{(i,j)\in E} J_{ij} s_i s_j$, where i, j belong to the same edge E. The graph $\mathcal{G}$ together with the set of random couplings J_{ij} then define a sample.

The main goal of the cavity method is to calculate the probability distribution of each spin, $P(s_j)$. The idea behind the approach is that $P(s_j)$ is determined by the local fields induced by all the neighbors of s_j, which are in turn determined by their own neighbors, and so on (Fig. 3). Because of

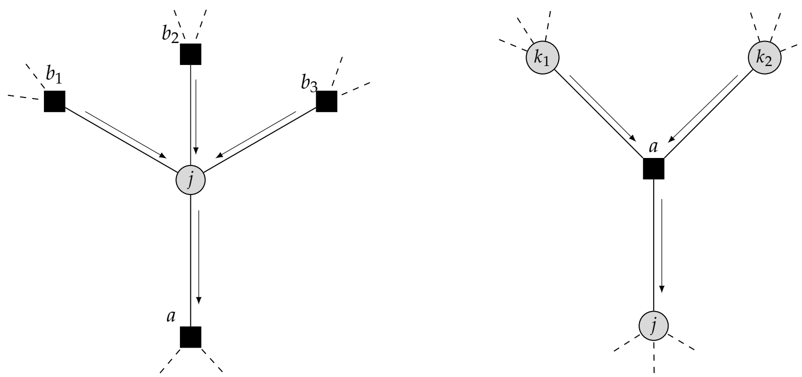

Figure 3: Left: Graph representation for the message $\hat{\nu}_{b\to j}(s_j)$, from the function node (square) to the variable node (circle). Right: corresponding part of the graph for the computation of the message $\hat{\nu}_{a\to j}(s_j)$, function of $\nu_{k\to a}(\sigma_k)$ from the variable node to the function node.

the tree-like structure of the graph, the neighbors of s_j are mutually independent, and hence $P(s_j)$ can be factorized and determined through the incoming local fields, $\hat{\nu}_{a\to j}(s_j)$ (where a indicates a neighboring interaction): $P(s_j) \simeq \prod_{a\in\partial j} \hat{\nu}_{a\to j}(s_j)$. At the same time, the interaction a influencing s_j is defined by the marginal distribution of all the spins k in the neighborhood of a, once s_j is removed, $\nu_{k\to a}(s_j)$. Therefore, the cavity formalism results in a set of coupled equations for the marginal probability laws

$$\begin{cases} \nu_{j\to a}(s_i) \propto \prod_{b\in\partial j\setminus a} \hat{\nu}_{b\to j}(s_j) \ , \\ \hat{\nu}_{a\to j}(s_j) \propto \sum_{\mathbf{s}\in\partial a\setminus j} \psi_a(\mathbf{s}_{\partial a}) \prod_{k\in\partial a\setminus j} \nu_{k\to a}(s_k) \, , \end{cases} \tag{16}$$

where ψ_a is called *compatibility function*, which is analogous to the Boltzmann weight, and $\partial a\setminus j$ indicates the spins near a excluding j.

The resulting free energy as a function of fixed-point messages turns out to be a combination of three contributions coming from: all function nodes, all variable nodes, and the edges connecting i to any possible function node (see Fig. 3). When the size of the loops is larger than the correlation length, the signals entering j are independent. In this case, if there is a unique solution for the state in j, the cavity method can recover it exactly. For instance, at the RS level, where there is only one pure state, the cavity approach is formally equivalent to the replica method [28]. It the system is not RS, or if the loops are short, the approach needs to be refined. For example, 1-RSB cavity protocols have been developed to extract properties of the pure state decomposition and satisfiability conditions in generic optimization problems [56, 57]. Such 1-RSB message passing equations go by the name of *survey propagation* and have been generalized as *population dynamics algorithms*.

Beyond Mean-Field: multi-layer construction and loop corrections The cavity method stops being exact in a non-tree-like topology and is indeed hindered by the presence of loops. Furthermore,

[2] In computer science and artificial intelligence, the cavity method is also known as *Belief Propagation*, whereas in statistical physics is referred to as *Bethe-Peierls* [55].

because of its non-perturbative nature, any small parameter expansion used to compute corrections to mean-field theory [58, 59] appears to be unfeasible. One can nevertheless build a M-layer model, where M copies of the original lattice are stacked on top of each other assuming the same distribution of random couplings. The idea – originally proposed in computer science by Vontobel [60] – has been reworked recently in disordered systems based on a rewiring procedure. By inducing random permutations of the links, inter-layer connections, and hence spatial loops, are automatically generated by a generalized transfer matrix formalism on uncorrelated one-dimensional chains. The perturbative computation of finite-size corrections in powers of $1/M$ notably offers a reliable formal method to study critical phenomena in finite-dimensional systems [61]. The advantage of this tree-based approach is that it can recover phase transitions that deviate from the behavior of fully-connected models and are instead more similar to the finite-dimension phenomenology. This situation arises, for instance, in the Random Field Ising Model [62, 63], which is strongly affected by non-perturbative effects in low dimensions, and in Anderson localization [64] in the quantum realm.

2.4 Dynamical Mean-Field Theory (DMFT)

Understanding the dynamics of SGs is one of the oldest and most challenging problems in the theory of matter. One characteristic feature of the SG dynamics is that its relaxation depends on the history of the system itself, *i.e.,* it ages. In other words, the autocorrelation between a time t and $t' > t$ does not only depend on $(t' - t)$, but also on the *age* of the system, t. An aging system relaxes extremely slowly without ever reaching an equilibrium state. During its exploration, the system wanders across states that are not the relevant ones from a thermodynamic viewpoint and whose static properties cannot be defined rigorously.

A MF solution for the dynamics in glassy systems was first suggested by Sompolinsky and Zippelius for equilibrium properties [65]. Intriguing off-equilibrium features came emerged from a deeper investigation of specific models, for which a closed set of integro-differential equations could be solved [66–68].

For the sake of simplicity, in the following, we shall consider the over-damped Langevin dynamics of a system in contact with a thermal bath,

$$\frac{ds_i}{dt} = -\frac{\partial \mathcal{H}}{\partial s_i} + \eta_i(t) \tag{17}$$

whose behavior is captured by Gaussian white noise with zero mean and variance $\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$. The index $i = 1, ..., N$ runs over the total number of spins in the system. The Hamiltonian $\mathcal{H}$ typically incorporates a single-spin potential $V(s)$ plus a disordered part, which explicitly depends on the quenched disordered couplings (see *e.g.* Eq. (9)), which need to be averaged out. The dynamical MF procedure allows to average over the couplings and coarsen the time-dependence of the system as that of a single average spin $s(t)$. For the p-spin model, the resulting DMFT equation reads [65, 69]:

$$\dot{s}(t) = -\frac{\partial V(s(t))}{\partial s} + \frac{p(p-1)}{2}\int_0^t dt'\; R(t,t')C^{p-2}(t,t')s(t') + \xi(t) \tag{18}$$

where the noise $\xi(t)$ accounts for the interaction with the rest of the system and has an extra colored term, *i.e.* $\langle \xi(t)\xi(t')\rangle = 2T\delta(t-t') + C(t,t')$.

The dynamical equation above is expressed in terms of the two-time correlation function

$$C(t,t') = \frac{1}{N}\sum_i s_i(t)s_i(t') \;, \tag{19}$$

and of the response function to an external pulse perturbation

$$R(t,t') = \frac{1}{N}\sum_i \left.\frac{\delta s_i(t)}{\delta h_i(t')}\right|_{h_i=0} \;. \tag{20}$$

In the thermodynamic limit, the above equations converge to a unique non-fluctuating solution, which is the only one allowed by causality. At equilibrium, $C(t-t')$ and $R(t-t')$ are related by the fluctuation-dissipation theorem, $R(t-t') = -\frac{1}{T}\Theta(t-t')\frac{d}{dt}C(t-t')$. While, for finite N, the correlation and response functions are expected to decay exponentially in time, this property is no longer generically true in the thermodynamic limit, as the relaxation time might diverge, thus signaling a dynamical transition (Sec. 2.2).

Based on the long-time limit analysis first performed in the spherical p-spin model [66], one can separately analyze the fast regime, in which relevant degrees of freedom rapidly equilibrate preserving time translation invariant (TTI) properties, and the slow regime, in which violations of fluctuation-dissipation relations and non-equilibrium phenomena emerge. In the large-time limit, for $t, t' \to \infty$, the correlation function can be split as $C(t,t') = C_{\mathrm{TTI}}(t-t') + C_{\mathrm{Aging}}(t,t')$. Depending on the appearance either of a single diverging timescale or a multiplicity of progressively slower timescales, the slow part of the correlator, $C_{\mathrm{Aging}}(t,t')$, can be captured by a combination of involved functions each associated with a slow timescale [70].

Notably, by assuming the MF picture of aging as a starting point (such as Eq. (18)), it is possible to investigate the properties of asymptotic regimes without explicitly solving the integro-differential equations for the correlation and response functions (*e.g.* Eq. (19), (20)). The outcome is different depending on whether the low-T phase is 1-RSB or full-RSB [71]. For the latter, a dynamical effective stochastic process for the slow-evolving effective part of Eq. (18) has been worked out [71], which exactly maps into the ultrametricity condition associated with the Parisi solution in the replica formalism [28]. Hence, DMFT both in the presence of a single slow timescale and in the ultrametric scenario explicitly links the aging dynamics with the replica-based description.

3 Spin Glasses in Three Dimensions

3.1 Numerical Methods

Although MF treatments provide elegant exact solutions for SGs defined on a fully-connected or tree-like graph, extending those findings to three dimensions is quite challenging. For instance, even the existence of a SG phase in the $3d$ EAM has yet to be proven analytically. Most progress, therefore, arise from numerical simulations. In particular, there is convincing numerical evidence that, as their experimental counterparts, SG models in three dimensions exhibit a continuous phase transition at a temperature T_c [72–88].

The low-temperature dynamics of SGs is, however, very slow, due to the competition between short and long-range interactions, and the presence of temperature chaos. In the SG phase, the free energy profile indeed changes drastically even for infinitesimal changes in temperature [89]. The numerical study of SGs at low T is therefore highly challenging [90]. Numerical strategies advanced for studying SGs notably include: the construction and use of special-purpose computers [91–99], GPUs [13, 100–103], the formulation parallelization techniques, such as multi-spin coding [102–108], which, by encoding every spin on a single bit and restricting to bit-to-bit operations, allows one to simulate 64 replicas (the number of bits in a long integer) in the time of one; and the development of algorithms, such as *parallel tempering*, which consists of simultaneously simulating several replicas at different temperatures, and proposing Monte-Carlo updates, which exchange the temperatures among replicas [109–111].

3.2 Spatial Correlations

A key feature of low-dimensional systems is the characteristic extent of spatial correlations, ξ. Its typical experimental determination is through the magnetic response to an external magnetic field h [112–115], which provides the coherence length ξ_{Zeeman}. In numerical simulations, the size of the correlated domains is instead calculated through correlation functions of the form [116, 117]

$$C(\mathbf{r}) = \frac{1}{N} \sum_{\mathbf{x}}^{N} q_{\mathbf{x}} q_{\mathbf{x}+\mathbf{r}} \,, \tag{21}$$

where $\mathbf{x}$ indicates a position in the lattice, $\mathbf{r}$ is a displacement vector, and $q_{\mathbf{x}}$ is usually the overlap at site $\mathbf{x}$, $q_{\mathbf{x}} = q_{\mathbf{x}}^{(ab)} = s_{\mathbf{x}}^{(a)} s_{\mathbf{x}}^{(b)}$, or the link-overlap, $q_{\mathbf{x}} = q_{\mathrm{link},\mathbf{x}}^{(ab)} = q_{\mathbf{x}}^{(ab)} q_{\mathbf{x}+\hat{e}}^{(ab)}$ ($\hat{e}$ is a unit vector) [118, 119], since these quantities are equivalent in the limit $d \to \infty$. There are several ways to extract a correlation length ξ_{micro} from $C(\mathbf{r})$ [117, 120–122]. For example, $\xi_{\mathrm{micro}} = \sqrt{\frac{\int r^2 C(r) dr}{\int C(r)}}$. While, out of equilibrium, ξ_{Zeeman} and ξ_{micro} have different behaviors [123], at equilibrium they match [124]. These length scales can be used to identify phase transitions through finite-size scaling [125–127], by comparing simulations performed in systems of different linear sizes L. Because quantities such as $\frac{\langle \xi(T) \rangle}{L}$ are scale-invariant at T_c, one can identify critical points by investigating where the curves $\frac{\langle \xi(T) \rangle}{L}$ cross for different L. For the study of SGs, it has also proven useful to consider other quantities, such as ratios of susceptibilities [128], possibly conditioned to given values of the overlap [129].

3.3 The Spin-Glass Phase in Low Dimensions

Pictures for the nature of the low-temperature phase in three dimensions. Just like there is no rigorous proof of the existence of T_c in three dimensions, there is no first-principles theory on the nature of the spin-glass phase in this case either[3]. Two main phenomenological scenarios have been proposed: the Droplet and the RSB pictures. The former [130–133] is based on a renormalization group approach on the EAM [134, 135], which is exact in one dimension, and depicts the SG phase as having only two pure states, with $q = \pm q_{\mathrm{EA}}$, reminiscently of the behavior of the ferromagnetic Ising model [136]. One consequence of this proposal is that the size of the surface separating different magnetic domains scales as L^{d_s}, with $d_s < d$ (for the Ising ferromagnet $d_s = d - 1$), and the energy of the smallest excitation grows with L. Another consequence is that the low-temperature phase disappears as soon as an external magnetic field is applied to the system.

The RSB picture [137–139] is based on the opposite limit, $d = \infty$. It assumes that the SG phase of the EAM is qualitatively similar to that of the SK model, with an overlap distribution $P(q)$ which is non-zero over a wide interval of q. This picture prescribes that the domain surfaces are space-filling ($d_s = d$) and the smallest excitation remains $\mathcal{O}(1)$ when $L \to \infty$. In addition, the SG phase survives when a finite magnetic field is applied, with a dAT line $h_c(T)$ separating the paramagnetic from the SG phase (as described in Sec. 2.1). Whether either of the two theories holds remains, however, a matter of debate [139–149].

Evidence on the nature of the spin glass phase in $3d$. Evidence to falsify or support the Droplet and RSB scenarios has principally been sought through numerical simulations. For example, equilibrium runs of the $3d$ EAM with linear size $L \leq 32$ deep in the SG phase show that $P(q)$ has wide support, and $P(0)$ is stable as L increases [116, 150]. This observation is in contradiction with the Droplet prediction, $P(q) \propto [\delta(q - q_{\mathrm{EA}}) + \delta(q + q_{\mathrm{EA}})]$, which assumes $P(0) = 0$.

When an external magnetic field is applied to the system, equilibrium simulations appear perturbed by finite-size effects [129], and there is disagreement on the existence of a dAT line. Arguments stemming from numerical simulations on the $3d$ EAM state that a phase transition could be present at temperatures that are 40% lower than those accessible [129, 151]. Those are opposed by arguments from simulations on SG chains with long-ranged interactions, which mimic $3d$ systems, claiming that no transition exists [152]. We note, however, that the upper critical dimension is $d_{\mathrm{up}} \geq 6$ [28, 153], and that there is strong evidence for a dAT transition in $d = 4$ [128]. Perturbative renormalization group analysis in $d = 6 - \epsilon$ is consistent with the presence of a dAT line [139, 146, 154, 155], with a non-trivial fixed point appearing at second order in ϵ [156, 157].

Outlook on the nature of the spin glass phase The current numerical evidence does not confirm the Droplet picture, but it has been argued that the observations might change if larger system sizes were simulated [147, 152]. It is to be noted, however, that times and system sizes of numerical simulations nowadays closely approach those of experiments [116, 158–160]. Therefore, if even there is a limit of very large L in which the Droplet theory applies, this might not be experimentally relevant [149]. Evidence in favor of the RSB picture is, however, not decisive, and relevant beyond-MF mechanisms might arise in low dimensions. Alternative or intermediate scenarios could also be valid [146], such as the trivial-non-trivial picture [161, 162], according to which the domain surfaces are not space-filling, as in the Droplet picture, but there are large-scale excitations whose energy does not increase with size [163, 164], as prescribed by the RSB scenario.

[3]The results in this section refer to the $3d$ EAM, which is the most studied model.

4 Conclusions

Universality classes While our discussion thus far focused on few specific SG models, changing details of the Hamiltonian can change general features of the energy landscape, such that new physics emerges. As discussed above, dimensionality plays a crucial role, with the $d = 1, \infty$ limits having dissimilar behaviors that might both differ from $d = 3$. Altering the distribution of the couplings [165, 166], $P(J_{ij})$, does not seem to influence the universality class of the models as long as they are not fat-tailed [85, 166, 167]. Changing the interaction *range* is also a relevant perturbation, in that a system with longer-ranged interactions is more MF like [168]. This effect has notably led to the search for a correspondence between short-ranged high-dimensional models and long-ranged SGs in $d = 1$ [152, 169–172].

Another important is the nature of the spins. As described in Sec. 2.1, one can soften spins, passing from Ising to spherical spins. For pairwise interactions in $d = \infty$, this completely changes the landscape, which passes from complex to simple. (The dynamics nevertheless remains slow due to a large number of flat directions [66, 173, 174].) Alternatively, one can use spins that are normalized vectors with m components [175]. For high m, the landscape becomes trivial, both in high- [176] and low-dimensional systems [177], which points toward using the limit $m \to \infty$ for studies of the $1/m$ expansions [178, 179]. Because the $m \to \infty$ behavior seems radically different from any finite m [180], however, this approach has not been extensively pursued.

One can also consider Hamiltonians that mix interactions with different numbers of bodies, such as *mixed p-spin models* [181, 182]. These models display several crossover temperatures at which the dynamical behavior changes qualitatively, apparently without any thermodynamic transitions [183–185]. This behavior is reminiscent of that of supercooled liquids [186].

An important aspect, which goes beyond the description of the specific model, is that over the years SGs have become a theoretical paradigm for modeling many different complex systems across disciplines, with an extremely broad spectrum of equilibrium and out-of-equilibrium properties. In fact, theories based on disordered systems often reveal to be beneficial for the treatment of problems outside the domain of SGs. Heterogeneous systems, either with quenched or self-generated disorder, are much more general than it might be believed at first glance, ranging from neural networks and optimization problems [53, 54, 57, 187–193], signal reconstruction [194, 195], random lasers [196, 197], financial markets [198–200], supercooled liquids and jammed packings [201–205] and theoretical ecology [206–214], and thermodynamic and dynamic formalisms rooted in spin-glass theory can still be used to solve them.

Open problems SGs can be defined simply, but many questions remain open. We note: the nature of the SG phase *[does replica symmetry breaking occur in finite-dimensional systems [147, 149]?]*; their out-of-equilibrium behavior *[how and which length scales describe their evolution [123, 149, 159, 215]? how are equilibrium states related to the closest local minima [185]?]*; or activated dynamics *[how does the dynamics take place at times $t \gg N$ [216–228]?]*; temperature chaos *[how and why does the landscape change drastically even with small temperature changes and how can we harness it to obtain low-temperature configurations [229–233]?]*; the identification of metastates *[can we explicitly measure the pure states of the SG phase [234–237]?]*; connection to other kinds of systems *[can we map the SG behavior onto other systems such as supercooled liquids or deep neural networks? [184, 201, 238–247]?]*.

These questions are not mere academic curiosities about a disordered magnetic alloy, because – as it has happened several times in the past already – spin-glass theory can significantly impact other fields. We take theoretical ecology as a final example. It is well known that the kind of sparsity of the interaction matrix of couplings between species has a crucial influence on the behavior of the ecosystem [248] and that these couplings are typically sparse in nature. Therefore, if we could harness low-dimensional SGs, which *e.g.* have spatial fluctuations, our understanding of ecological and biological communities would significantly advance.

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