Probability 1

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  - For instance, we might want to survey a random subset of American citizens (our data) and estimate the true attitudes of the entire American electorate (the parameter).
  - Alternatively, a game theoretic model may require actors to estimate the location of the median voter given the sequence of prior election outcomes $x = (x_1, x_2, \ldots, x_n)$ and candidate positions $y = (y_1, y_2, \ldots, y_n)$.

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  - Here we *know* the basic features of the data generating process (the parameters) and want to understand what the data is likely to look like.
  - For instance, we might have a fair coin and we want to understand the likelihood of flipping 20 heads before th first tail shows up.
- Obviously, most of the things you are going to be doing in your career will be about inference. Nonetheless, you *really* need to have a grasp of probabilty theory first.

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- For the simplest problems, all you need to know is the number of ways that some set of outcomes X could happen versus the total number of ways things could have turned out.
- So to begin with, you just need to focus on getting a handle on the basic concepts of:
  - How to count events
  - How to think about and handle sets
  - How counting and sets relate to the concept of "probability"
  - Conditional probability, independence, and Bayes' law

Advanced Counting

A preliminary

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- Example: for $x \geq y$,

$$\frac{x!}{(x-y)!} \quad = \quad \frac{x(x-1) \cdot \ldots \cdot (x-y+1) \cdot (x-y) \cdot (x-y-1) \cdot \ldots \cdot 1}{(x-y) \cdot (x-y-1) \cdot \ldots \cdot 1}$$

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&= x \cdot (x-1) \cdot \ldots \cdot (x-y+1)
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Fundamental Theorem of Counting

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- If there are k characteristics, each with n_k alternatives, there are $\prod_{i=1}^{k} n_k$ possible outcomes.
- We often need to count the number of ways to choose a subset from some set of possiblities. The number of outcomes depends on two characteristics of the process: does the *order* matter and is *replacement* allowed?

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  4. Unordered, without replacement: $(n$ choose $k)$: $\frac{n!}{(n-k)!k!} = \left(\begin{array}{c} n \\ k \end{array} \right)$

- Ordered events are sometimes referred to as permutations, while unordered events are combinations.
- You will almost always be working with combinations.

Sets

- **Set**: A set is any well defined collection of elements. If x is an element of S, $x \in S$.

Types of sets

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2. Countably infinite: a set with an infinite number of elements, which can still be mapped onto positive integers.

$$S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$$

3. Uncountably infinite: a set with an infinite number of elements, which cannot be mapped onto positive integers.

$$S = \{x : x \in [0, 1]\}$$

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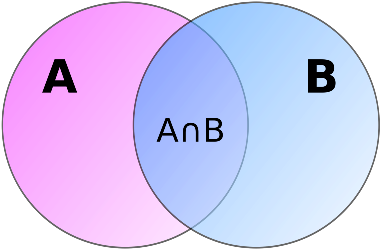
4. Empty: a set with no elements.

$$S = \{\emptyset\}$$

Set operations



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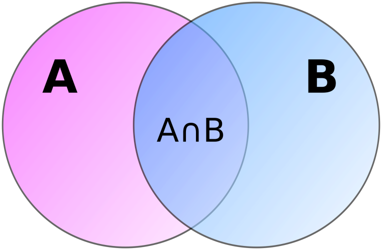

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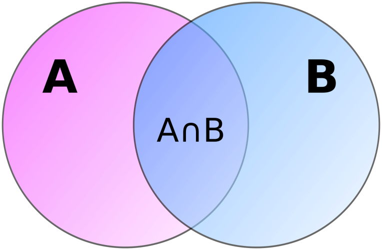

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- **Intersection**: The intersection of sets A and B, $A \cap B$, is the set containing all of the elements in both A and B.
- **Complement**: If set A is a subset of S, then the complement of A, denoted A^C, is the set containing all of the elements in S that are not in A.

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4. de Morgan's laws: $(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C$

Disjointedness and partitions

- Sets are disjoint when they do not intersect, such that $A \cap B = \{\emptyset\}$. A collection of sets is pairwise disjoint if, for all $i \neq j$, $A_i \cap A_j = \{\emptyset\}$.

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- Sets are disjoint when they do not intersect, such that $A \cap B = \{\emptyset\}$. A collection of sets is pairwise disjoint if, for all $i \neq j$, $A_i \cap A_j = \{\emptyset\}$.
- A collection of sets form a partition of set S if they are pairwise disjoint and they cover set S, such that $\bigcup_{i=1}^{k} A_i = S$.

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- Modern probability theory is a way of estimating our uncertainty about some future events given specific assumed properties of the world.
- This is a formalization of basic human intuition about how to handle risk.

Sample Space

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  2. **Continuous**: GNP, arms spending, age.

Probability Distribution/Function

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- It is just like any other function.
- We have some event/sample space S we have a probability space (e.g., the probability of event x happening is some number in $[0, 1]$) and we have the function that translates x into the probability space that we denote $p(x)$ or $f(x)$.

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Outcome	X=x
HH	2
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TH	1
TT	0

x	p(x)
TT $\rightarrow$ 0	.25
HT, TH $\rightarrow$ 1	.50
HH $\rightarrow$ 2	.25
Sum	1.00

Exercise: Rolling two fair dice

1. Write out the sample space
2. Write out the empirical probability function

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- Define the number $\Pr(A)$ correponding to each event A in the sample space S such that:
  1. Axiom: For any event A, $\Pr(A) \geq 0$.
  2. Axiom: $\Pr(S) = 1$
  3. Axiom: For any sequence of disjoint events $A_1, A_2, \ldots$ (of which there may be infinitely many),

$$\Pr\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \Pr(A_i)$$

Basic Theorems of Probability

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  3. For any event A, $0 \leq \Pr(A) \leq 1$.

4. If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.
5. For any two events A and B,
   $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
6. For any sequence of n events (which need not be disjoint)

$$A_1, A_2, \ldots, A_n$$

,

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \Pr(A_i)$$

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4. $\Pr\left(\{1, 3, 5\}\right) =$

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4. $\Pr(\{1, 3, 5\}) = 1/6 + 1/6 + 1/6 = 1/2$

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7. Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$. Then $A \cup B = \{1,2,3,4,6\}$, $A \cap B = \{2\}$, and

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4. Calculate $p(A \cup A^C)$. $p(X \cup (Y \cup Z)) = 1$

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  - Conditioning information can be subtlely important

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- Let S be the sample space of some experiment and let the disjoint k events $B_1, \ldots, B_k$ partition S.
- If A is some other event in S, then the events $AB_1, AB_2, \ldots, AB_k$ will form a partition of A and we can write A as

$$A = (AB_1) \cup \cdots \cup (AB_k)$$

- Since the k events are disjoint,

$$\begin{aligned}
\Pr(A) &= \sum_{i=1}^{k} \Pr(A, B_i) \\
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- Sometimes it is easier to calculate the conditional probabilities and sum them than it is to calculate $\Pr(A)$ directly.

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- Since we know $p(B \cap A) = p(A \cap B)$,

$$\begin{aligned} p(A)p(B|A) &= p(B)p(A|B) \\ p(B|A) &= \frac{p(B)p(A|B)}{p(A)} \end{aligned}$$

Then, from the Law of Total Probability,

$$p(B|A) \quad = \quad \frac{p(B)p(A|B)}{p(A)}$$

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These are *Bayes' Law* or *Bayes' Theorem* or *Bayes' Rule*.

Thinking about Bayes' Rule

- Assume that events $B_1, \dots, B_k$ form a partition of the space S.
- Then

$$\Pr(B_j|A) = \frac{\Pr(A, B_j)}{\Pr(A)} = \frac{\Pr(B_j)\Pr(A|B_j)}{\sum_{i=1}^{k} \Pr(B_i)\Pr(A|B_i)}$$

- If there are only two states of B, then this is just

$$\Pr(B_1|A) = \frac{\Pr(B_1)\Pr(A|B_1)}{\Pr(B_1)\Pr(A|B_1) + \Pr(B_2)\Pr(A|B_2)}$$

- If this was a continuouse distribution we could write this as:

$$\Pr(B_j|A) = \frac{\Pr(A, B_j)}{\Pr(A)} = \frac{\Pr(B)\Pr(A|B)}{\int\limits_{\infty}^{\infty} \Pr(A, B)\Pr(B)}$$

- Note that the denominator has an indefinite integral, meaning that there is an unknown integration constant to consider.

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- A test for cancer correctly detects it 90% of the time, but incorrectly identifies a person as having cancer 10% of the time. If 10% of all people have cancer at any given time, what is the probability that a person who tests positive actually has cancer?

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\end{aligned}
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Independence

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- If A and B are independent, then
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  2. $\Pr(B|A) = \Pr(B)$
  3. $\Pr(A \cap B) = \Pr(A)\Pr(B)$

Pairwise independence

- A set of more than two events $A_1, A_2, \ldots, A_k$ is **pairwise independent** if $\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$, $\forall i \neq j$.

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- Note that this does *not* necessarily imply that $\Pr(\bigcap_{i=1}^{k} A_i) = \prod_{i=1}^{K} \Pr(A_i)$.

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$$
\begin{aligned}
p(A_1 \cap A_2 \cap A_3) &= 0 \\
p(A_1)p(A_2)p(A_3) &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}
\end{aligned}
$$

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    1. $\Pr(A|B \cap C) = \Pr(A|C)$
    2. $\Pr(B|A \cap C) = \Pr(B|C)$
    3. $\Pr(A \cap B|C) = \Pr(A|C)\Pr(B|C)$
- Conditional independence is one of the fundamental assumptions deployed for most statistical estimation techniques. It is a *very* strong assumption.

Random Variables

```
int getRandomNumber()
{
    return 4;  // chosen by fair dice roll.
               // guaranteed to be random.
}
```

Getting oriented

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- In probability theory, random variables are something abstract. A random variable is a yet-to-be observed value.
- What is the probability that a coin will turn up heads? What is the probability the next card will be an ace?

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- These variables have known functional forms, several of which we will discuss today.
- Moreover, these functions have been extensively studied and their properties are well understood.
- The focus of the rest of this lecture is to get you familliar with these “kinds” of variables.

Levels measurement

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- If you do not already understand the difference, please review:
  - Nominal
  - Ordinal
  - Interval
  - Ratio

Types of distribution functions

You will work primarily with three types of distribution functions:

1. Probability mass functions
2. Probability density functions
3. Cumulative distribution functions

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- For joint distn's, $p(x, y) = p(X = x, Y = y)$
- Generally, the joint dist'n is **not** the product of the marginals

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- It assigns a real number to every outcome $s \in S$.
- **Discrete Random Variable**: Y is a discrete random variable if it can assume only a finite or countably infinite number of distinct values.
- Examples: number of wars per year, heads or tails, voting Republican or Democrat, number on a rolled die.

Probability Mass Function

- For a discrete random variable Y, the probability mass function (pmf) $p(x) = \Pr(X = x)$ assigns probabilities to a countable number of distinct x values such that

1. $0 \leq p(x) \leq 1$
2. $\sum_x p(x) = 1$

Example

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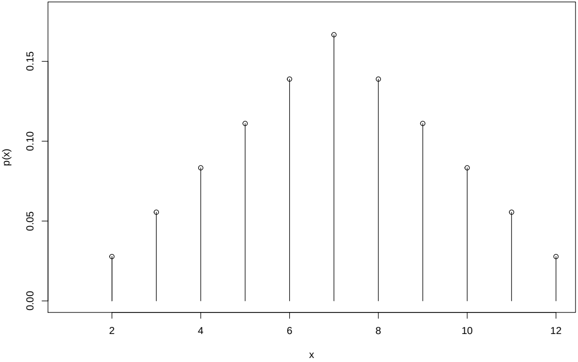
- For one fair six-sided die, there is an equal probability of rolling any number.
- Since there are six sides, the probability mass function is then $p(y) = 1/6$ for $y = 1, \ldots, 6$.
- Each $p(y)$ is between 0 and 1.
- And, the sum of the $p(y)$'s is 1.

- If there are two six-sided dice, the probability mass funciton is shown below.

```
y<-c(1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1)/36; x<-c(2:12)
plot(x, y, xlim=c(1, 12), ylim=c(0, .18),
     xlab="x", ylab="p(x)")
segments(x0=x, y0=rep(0,12), x1=x, y1=y)
```



Cumulative distribution

- The cumulative distribution $F(x)$ or $\Pr(X \leq x)$ is the probability that Y is less than or equal to some value y, or

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. The CDF must satisfy these properties:

1. $F(x)$ is non-decreasing in x.
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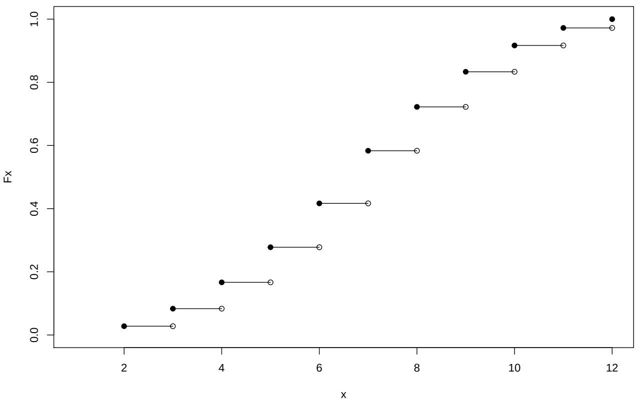
- Example: For a fair die, $\Pr(Y \leq 1) = 1/6$, $\Pr(Y \leq 3) = 1/2$, and $\Pr(Y \leq 6) = 1$.

Example: Two Fair die

```
fx<-c(1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1)/36; x<-c(2:12)
Fx<-sapply(1:11, function(i, fx) sum(fx[1:i]), fx=fx)
plot(x, Fx, xlim=c(1, 12), ylim=c(0, 1), pch=19)
points(3:12, Fx[-11], xlim=c(1, 12), ylim=c(0, 1), pch=1)
segments(x0=2:11, x1=3:12, y0=Fx[-11], y1=Fx[-11])
```



Bernoulli Distribution

- Single trial, binary outcome $0, 1$.

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- "Prob of success in 1 trial?", $x \in \{0, 1\}$
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Political science examples

- Let $X = \begin{cases} 1 & \text{if you turnout} \\ 0 & \text{if you abstain} \end{cases}$.
    - Then, $p(X = 1|p = .4) = .4$ prob of you turning out to vote in next election, given underlying true prob $p = .4$
    - $p(X = 0|p = .4) = .6$ prob of you abstaining in next election.
- What is the probability of a of US-NKorea conflict in 2018?

Binomial distribution

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- $p(X = k|n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$
- $X \sim Bin(1, p) \sim p(X = k|1, p) \sim Bern(p)$

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- $X \sim Bin(n, p)$
- $p(X = k|n, p) = \binom{n}{k} p^k (1-p)^{n-k}$
- $X \sim Bin(1, p) \sim p(X = k|1, p) \sim Bern(p)$
- $X_1 \sim Bin(n_1, p)$, $X_2 \sim Bin(n_2, p)$, $X_1 \perp\!\!\!\perp X_2$, *then* $X_1 + X_2 \sim Bin(n_1 + n_2, p)$

Binomial distribution

- n {*independent*}, {*identically distributed*} *(iid) trials, binary outcome* $0, 1$.
- *"Prob of k successes in n trials?"*, $k \in \{0, \ldots, n\}$
- $X \sim Bin(n, p)$
- $p(X = k | n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$
- $X \sim Bin(1, p) \sim p(X = k | 1, p) \sim Bern(p)$
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- $F(x) = \sum_{i=0}^{\lfloor x \rfloor} \binom{n}{i} p^i (1 - p)^{n-i}$

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Political examples:

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- Prob $\geq$ 3 of 6 opposing Senators support an amendment: $p(X \geq 3|n = 6, p = .3) =$

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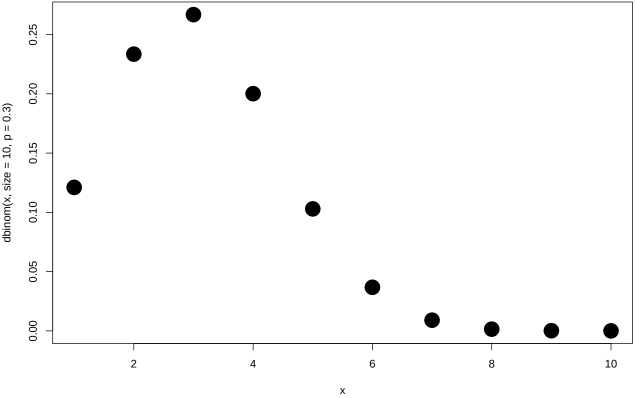
- Prob ≥ 3 of 6 opposing Senators support an amendment: $p(X \geq 3|n = 6, p = .3) =$ `1-pbinom(2,6,prob=.3)` $=$ `pbinom(2,6,prob=.3,lower.tail=FALSE)` $\approx .26$

Binomial PMF: N=10, p=0.3

```
x<-1:10
plot(x, dbinom(x, size=10, p=.3), cex=3, pch=19)
```



Poisson Distribution

- iid Bernoulli trial at every instant

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Political examples

- Political examples:
  - prob of 2 HoR censures this Congress, when usually 4:

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Political examples

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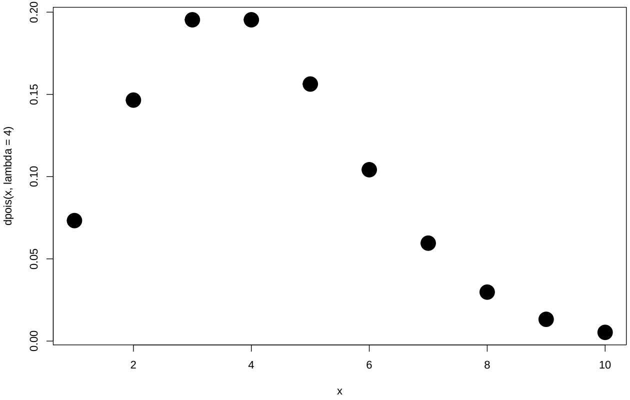
  - prob of 2 HoR censures this Congress, when usually 4: $p(X = 2|\lambda = 4) =$ `dpois(2,4)` $\approx .15$
  - prob of ≥ 2 HoR censures this Congress, when usually 4: $p(X \geq 2|\lambda = 4) =$ `1-ppois(1,4)` $\approx .91$

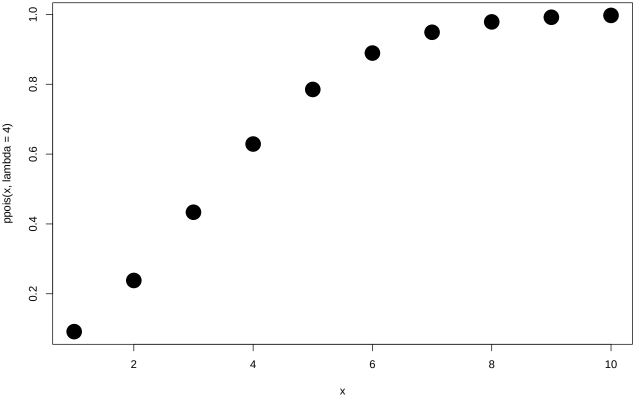
PMF for Poisson with $\lambda = 4$

```r
x<-1:10
plot(x, dpois(x, lambda=4), cex=3, pch=19)
```



CDF for Poisson with $\lambda = 4$

```
x<-1:10
plot(x, ppois(x, lambda=4), cex=3, pch=19)
```



Negative Binomial Distribution

- Waiting: iid binary trials

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- $p(X = k | n, p) = \binom{n+k-1}{k} p^n (1-p)^k$

Political example:

- prob 4 days pass before 3rd roadside bomb, if each day $p = .5$:

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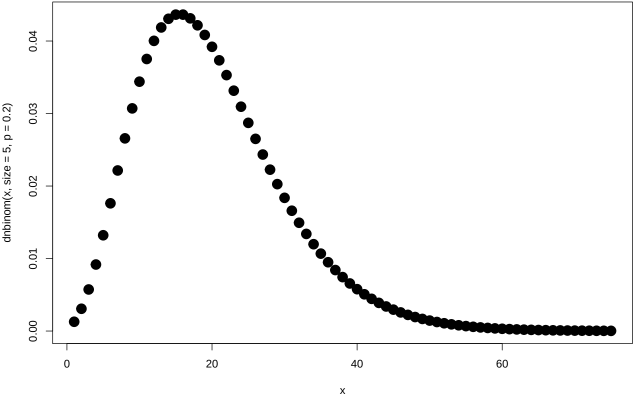
- prob 4 days pass before 3rd roadside bomb, if each day $p = .5$: $p(X = 4|n = 3, p = .5) =$ `dnbinom(4,3,p=.5)` $\approx .12$
- prob ≤ 4 days pass before 3^{rd} roadside bomb, if each day $p = .5$: $p(X \leq 4|n = 3, p = .5) =$ `pnbinom(4,3,p=.5)` $\approx .77$

PMF for Negative Binomial with n=5, p=.2

```
x<-1:75
plot(x, dnbinom(x, size=5, p=.2), cex=2, pch=19)
```



Other distributions you might encounter

- Geometric
- HyperGeometric
- Multinomial (The dice examples)

Continuous Random Variables:

- X is a continuous random variable if there exists a nonnegative function $f(x)$ defined for all real $y \in (-\infty, \infty)$, such that for any interval A,

$$\Pr(x \in A) = \int_A f(x)dx$$

- Examples: income, GNP, temperature

Probability Density function

- The function f above is called the probability density function (pdf) of x and must satisfy
    1. $f(x) \geq 0$
    2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1$

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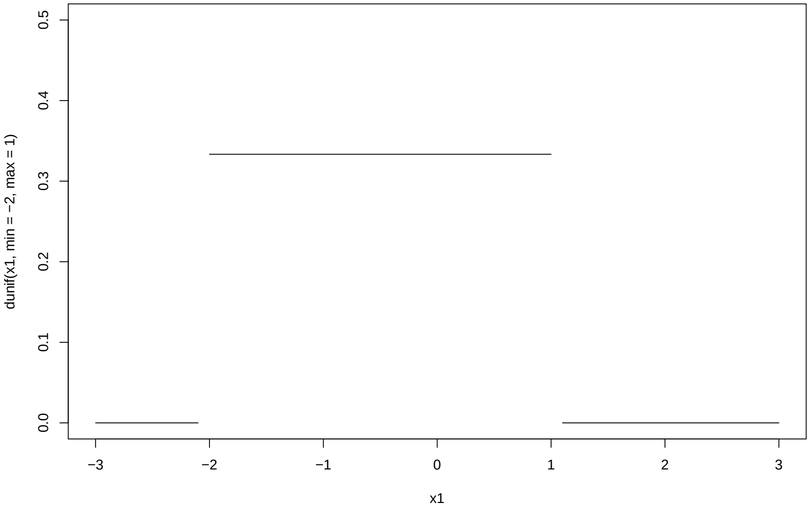
  1. $f(x) \geq 0$
  2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1$
- Note also that $\Pr(X = x) = 0$ — i.e., the probability of any point x is zero.

Example: Uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

PDF for Uniform(-2, 1)



Cumulative Distribution

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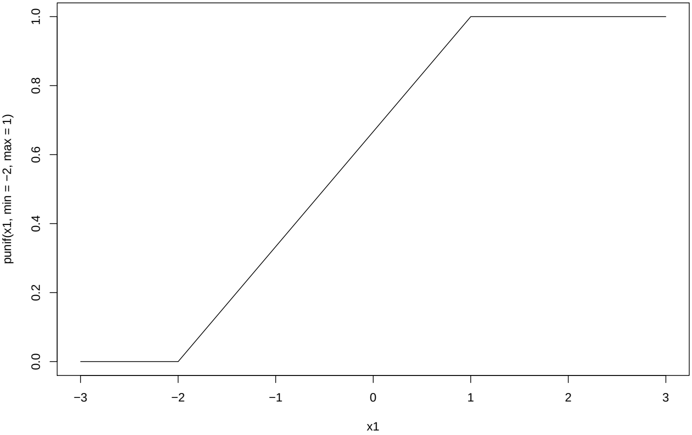
- Note that $F(x)$ has similar properties with continuous distributions as it does with discrete
  - non-decreasing, continuous (not just right-continuous),
  - and $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

CDF for Uniform(-2, 1)

```r
x1<-seq(-3, 3, by=.1)
plot(x1, punif(x1, min=-2, max=1), cex=2, pch=19,
     type="l", xlim=c(-3,3), ylim=c(0, 1))
```



- Similarly, we can also make probability statements about X falling in an interval $a \leq x \leq b$.

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$$F(y) = \int\limits_0^y f(s)ds = \int_0^y 1ds = s|_0^y = y$$

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$$F(y) = \int\limits_0^y f(s)ds = \int_0^y 1ds = s|_0^y = y$$

$$\Pr(.5 < y < .75) = \int_{.5}^{.75} 1ds = s|_{.5}^{.75} = .25$$

- Finally, note that:

$$F'(y) = \frac{dF(y)}{dy} = f(y)$$

Uniform Distribution

- $X \sim Unif(a, b)$

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Political examples:

- “Suppose voter’s probability of turnout is draw from uniform”
- Suppose the ideology of an agent is drawn from the uniform distribution

Exponential Distribution

- $X \sim Expo(\beta)$

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χ^2 Distribution

- $X \sim \chi_n^2$

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χ^2 Distribution

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- $x > 0$
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- n sometimes called "degrees of freedom"
- $X_1 \sim \chi^2_{n_1}$, $X_2 \sim \chi^2_{n_2}$, $X_1 \perp\!\!\!\perp X_2$, then $X_1 + X_2 \sim \chi^2_{n_1+n_2}$

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- $\chi^2_n \sim \mathit{Gamma}\left(\frac{n}{2}, 2\right)$

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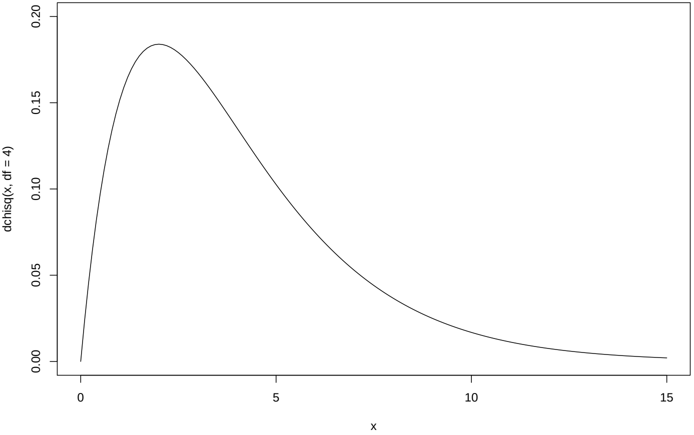
- $\chi^2_n \sim Gamma\left(\frac{n}{2}, 2\right)$

PDF of χ^2 distribution with $n = 4$

```
x<-seq(0, 15, by=.1)
plot(x, dchisq(x, df=4), cex=2, pch=19,
     type="l", xlim=c(0,15), ylim=c(0, .2))
```



Political examples:

- Model relationships between table rows/columns
  $X^2 = \sum_{i=1}^{r} \sum_{i=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2_{(r-1)(c-1)}$
- regression statistics

Normal (Gaussian) Distribution

- $X \sim N(\mu, \sigma^2)$

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- $x \in \mathbb{R}$

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- $x \in \mathbb{R}$
- $\mu \in \mathbb{R}$

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- $\mu \in \mathbb{R}$
- $\sigma > 0$

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- $\mu \in \mathbb{R}$
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- $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

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- $\Phi(x) =$ standard normal CDF – that no one knows

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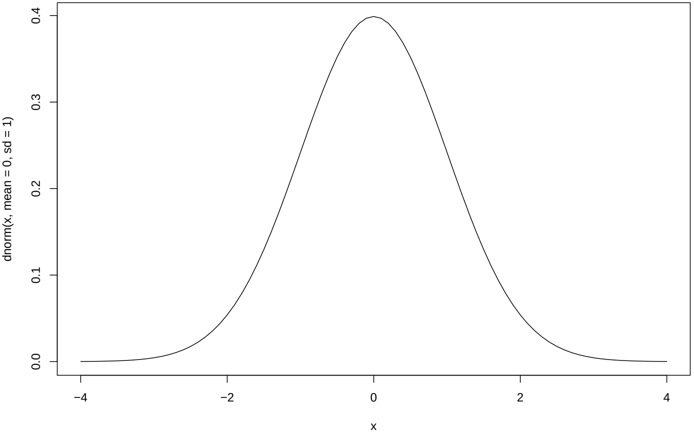
- $\Phi(x) =$ standard normal CDF – that no one knows

Political examples:

- population quantities, asymptotic/known variance sampling distributions
- $\Phi(x) = p(X = 1)$ is the basic *probit model*

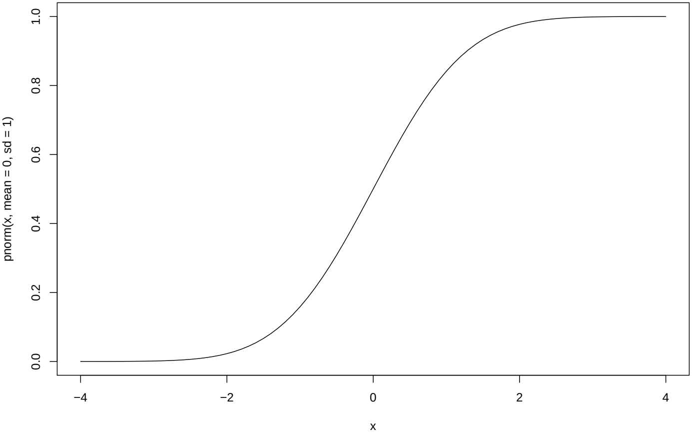
PDF for standard normal distribution

```
x<-seq(-4, 4, by=.1)
plot(x, dnorm(x, mean=0, sd=1),  type="l")
```



CDF for standard normal distribution

```
x<-seq(-4, 4, by=.1)
plot(x, pnorm(x, mean=0, sd=1),  type="l")
```



Student's t Distribution

- $X \sim t_n(\mu, \sigma^2)$

Student's t Distribution

- $X \sim t_n(\mu, \sigma^2)$
- $x \in \mathbb{R}$
- $n \in \mathbb{Z}_+$, often called degrees of freedom
- $p(x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sigma\sqrt{n\pi}} \left(1 + \frac{1}{n}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}}$
- Common: $t_n \sim \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$

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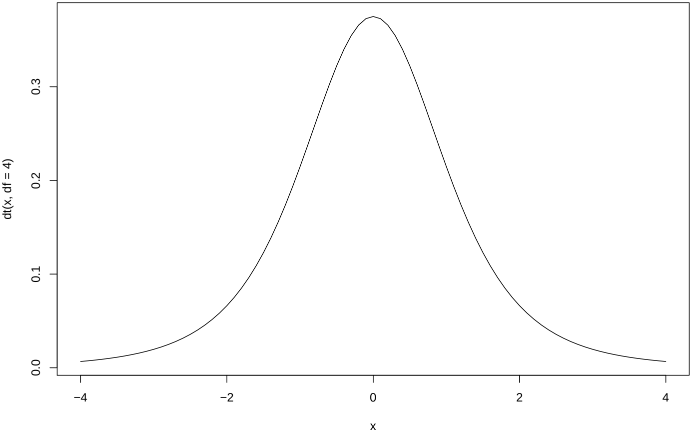
Political examples

- Finite sample/unknown variance distributions
- robust estimation

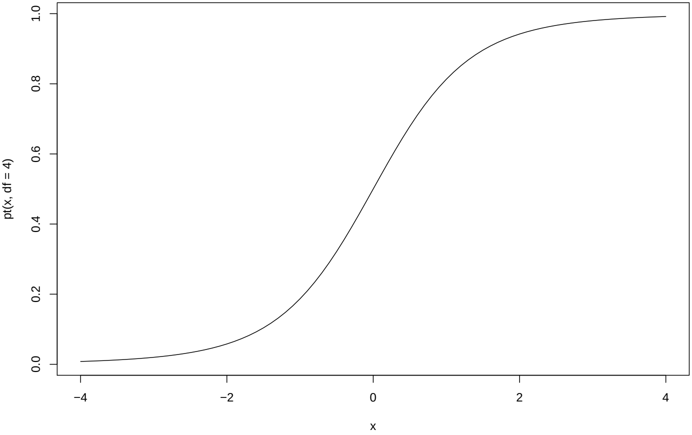
PDF for Student-t with df=4

```
x<-seq(-4, 4, by=.1)
plot(x, dt(x, df=4),  type="l")
```



CDF for Student-t with df=4

```
x<-seq(-4, 4, by=.1)
plot(x, pt(x, df=4),  type="l")
```



Other distributions you may encounter

- Logistic distribution
- F(isher's) distribution
- Gamma distribution
- Laplace distribution
- Weibull disribution
- Log-normal distribution
- Pareto distribution
- Dirichlet distribution

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- Joint distributions can be made up of any combination of discrete and continuous random variables.

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- We can define two random variables X and Y such that $X = 1$ if heads and $X = 0$ if tails, while Y equals the number on the die.
- We can then make statements about the joint distribution of X and Y.

Joint discret random variables

- If both X and Y are discrete, their joint probability mass function assigns probabilities to each pair of outcomes

$$p(x, y) = \Pr(X = x, Y = y)$$

- Again, $p(x, y) \in [0, 1]$ and $\sum \sum p(x, y) = 1$.

Marginal pmf

- If we are interested in the marginal probability of one of the two variables (ignoring information about the other variable), we can obtain the marginal pmf by summing across the variable that we don't care about:

$$p_X(x) = \sum_i p(x, y_i)$$

Conditional pmf

- We can also calculate the conditional pmf for one variable, holding the other variable fixed.
- Recalling from the previous lecture that $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$, we can write the conditional pmf as

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}, \quad p_X(x) > 0$$

Joint continuous random variables

- If both X and Y are continuous, their joint probability density function defines their distribution:

$$\Pr((X, Y) \in A) = \iint_A f(x, y) dx dy$$

- Likewise, $f(x, y) \geq 0$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

Marginal pdf

- Instead of summing, we obtain the marginal probability density function by integrating out one of the variables:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

Conditional pdf

- Finally, we can write the conditional pdf as

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \quad f_X(x) > 0$$

Class business

- Problem set 1 will be given out on Thursday. You will have one week.
- It should not be hard, it will be long.
- PROOFS
- Next class will cover Wasserman Chpts 3 and 5 (but carefully read the very short Chapter 4.)