Technical University of Crete

# Introduction to Quantum Computing

**Exercise 1**

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# 1 Part 1

McMahon Exercises from sections 2 and 3

## 2.1

A quantum system is in the state

$$\frac{(1-i)}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

If a measurement is made, what is the probability the system is in state $|0\rangle$ or in state $|1\rangle$?

### Solution

$P_{|0\rangle} = (\frac{1-i}{\sqrt{3}})^*(\frac{1-i}{\sqrt{3}}) = \frac{1-i^2}{3} = \frac{2}{3}$, so $P_{|1\rangle} = 1 - P_{|0\rangle} = \frac{1}{3}$

## 2.2

Two quantum states are given by

$$|\alpha\rangle = \begin{bmatrix} -4i \\ 2 \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

(A) Find $|\alpha + b\rangle$.

(B) Calculate $3|\alpha\rangle - 2|b\rangle$.

(C) *Normalize* $|\alpha\rangle, |b\rangle$.

### Solution

A) $|\alpha + b\rangle = |\alpha\rangle + |b\rangle = \begin{bmatrix} -4i+1 \\ 1+i \end{bmatrix}$

B) $|c\rangle = \begin{bmatrix} -12i \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2-2i \end{bmatrix} = \begin{bmatrix} -12i-2 \\ 8-2i \end{bmatrix}$

C) If normalized, $\langle\alpha|\alpha\rangle$ and $\langle b|b\rangle$ should be equal to 1. If we compute it thought we see that first is equal to 20 and second to 3. So normalized:

$|\alpha\rangle = \begin{bmatrix} -\frac{i}{5} \\ \frac{1}{10} \end{bmatrix}$ and $|b\rangle = \begin{bmatrix} \frac{1}{3} \\ \frac{-1+i}{3} \end{bmatrix}$

## 2.3

Another basis for $\mathbb{C}^2$ is

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Invert this relation to express $\{|0\rangle, |1\rangle\}$ in terms of $\{|+\rangle, |-\rangle\}$.

**Solution**

I observe that if I add the two basis I get:

$|+\rangle + |-\rangle = \frac{2}{\sqrt{2}}|0\rangle \Leftrightarrow |0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

So $\sqrt{2}|+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle + |1\rangle \Leftrightarrow |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

## 2.4

A quantum system is in the state

$$|\psi\rangle = \frac{3i|0\rangle + 4|1\rangle}{5}$$

(A) Is the state *normalized*?

(B) Express the state in the $|+\rangle, |-\rangle$ basis.

**Solution**

A) If the state is normalized, then the condition $\langle\psi|\psi\rangle = 1$ must be true. Examining our state we observe that:

$\langle\psi|\psi\rangle = \begin{bmatrix} \frac{-3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix} = -i^2 \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

So we can reach to the conclusion that the state is normalized.

B) We know that $|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Also $|\psi\rangle = \frac{1}{5}(3i|0\rangle + 4|1\rangle) = \frac{1}{5}(3i \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix})$.

After that we observe that: $|+\rangle + |-\rangle = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So as we can see, $|\psi\rangle$ can be written as:

$|\psi\rangle = \frac{1}{5\sqrt{2}}[3i(|+\rangle + |-\rangle) + 4\sqrt{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}]$ (1)

Furthermore, we observe that $|+\rangle - |-\rangle = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Finally, after one more replacement in relation (1) and some calculations we find out that:
$|\psi\rangle = \frac{4\sqrt{2}(3i+2)}{5}|+\rangle + \frac{4\sqrt{2}(3i-2)}{5}|-\rangle$

## 2.6

Photon horizontal and vertical polarization states are written as $|h\rangle$ and $|v\rangle$, respectively. Suppose

$$|\psi_1\rangle = \frac{1}{2}|h\rangle + \frac{\sqrt{3}}{2}|v\rangle$$

$$|\psi_2\rangle = \frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|v\rangle$$

$$|\psi_3\rangle = |h\rangle$$

Find

$$|\langle\psi_1|\psi_2\rangle|^2, \quad |\langle\psi_1|\psi_3\rangle|^2, \quad |\langle\psi_3|\psi_2\rangle|^2$$

**Solution**

$\langle\psi_1|\psi_2\rangle = (\frac{1}{2}\langle h| + \frac{\sqrt{3}}{2}\langle u|)(\frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|u\rangle) = \frac{1}{4}\langle h|h\rangle - \frac{\sqrt{3}}{4}\langle h|u\rangle + \frac{\sqrt{3}}{4}\langle u|h\rangle - \frac{3}{4}\langle u|u\rangle = -\frac{1}{2}$
So $|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{4}$

$\langle\psi_1|\psi_3\rangle = (\frac{1}{2}\langle h| + \frac{\sqrt{3}}{2}\langle u|)|h\rangle = \frac{1}{2}\langle h|h\rangle + \frac{\sqrt{3}}{2}\langle u|h\rangle = \frac{1}{2}$
So $|\langle\psi_1|\psi_3\rangle|^2 = \frac{1}{4}$

$\langle\psi_3|\psi_2\rangle = \langle h|(\frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|u\rangle) = \frac{1}{2}\langle h|h\rangle - \frac{\sqrt{3}}{2}\langle h|u\rangle = \frac{1}{2}$
So $|\langle\psi_3|\psi_2\rangle|^2 = \frac{1}{4}$

## 3.1

Verify that the outer product representations of $X$ and $Y$ are given by

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad Y = -i(|0\rangle\langle 1| + i|1\rangle\langle 0|)$$

by letting them act on the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and comparing with (3.9) and (3.10).

**Solution**

$X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(a|0\rangle + \beta|1\rangle) = |0\rangle\langle 1|(a|0\rangle + \beta|1\rangle) + |1\rangle\langle 0|(a|0\rangle + \beta|1\rangle) = \beta|0\rangle + a|1\rangle$

In that way, we can see that $X \begin{bmatrix} a \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ a \end{bmatrix} \Leftrightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Y|\psi\rangle = -i(|0\rangle\langle 1| + i|1\rangle\langle 0|)(a|0\rangle + \beta|1\rangle) = -i\beta|0\rangle + ia|1\rangle$

Same as before, we see that $Y \begin{bmatrix} a \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ ia \end{bmatrix} \Leftrightarrow Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

## 3.2

Show that the matrix representation of the $X$ operator with respect to the computational basis is

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

### Solution

We know that: $X = |0\rangle\langle 1| + |1\rangle\langle 0|$
By computing its representation on the computational basis, we get:
$X|0\rangle = |0\rangle \langle 1|0\rangle + |1\rangle \langle 0|0\rangle = |1\rangle$ that is equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X|1\rangle = |0\rangle \langle 1|1\rangle + |1\rangle \langle 0|1\rangle = |0\rangle$ that is equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So we can reach to the conclusion that $X = \begin{bmatrix} X|0\rangle & X|1\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

## 3.3

Consider the basis states given by

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Show that the matrix representation of the $X$ operator with respect to this basis is

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

### Solution

$X|+\rangle = \frac{1}{\sqrt{2}}(X|0\rangle + X|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

Also as we can see:

$X|-\rangle = \frac{1}{\sqrt{2}}(X|0\rangle - X|1\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = -|-\rangle$

So we conclude that: $X|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $X|-\rangle = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

## 3.4

Consider the space $\mathbb{C}^3$ with the basis $\{|1\rangle, |2\rangle, |3\rangle\}$. An operator $\hat{A}$ is given by

$$\hat{A} = i|1\rangle\langle 1| + \frac{\sqrt{3}}{2}|1\rangle\langle 2| + 2|2\rangle\langle 1| - |2\rangle\langle 3|$$

Write down the adjoint of this operator $\hat{A}^\dagger$.

**Solution**

It is well known from theory that: $(c|a\rangle\langle\beta|)^\dagger = c^*|\beta\rangle\langle a|$.
So in our case:
$\hat{A}^\dagger = -i|1\rangle\langle 1| + \frac{\sqrt{3}}{2}|2\rangle\langle 1| + 2|1\rangle\langle 2| - |3\rangle\langle 2|$

## 3.5

Find the eigenvalues and eigenvectors of the $X$ operator.

**Solution**

To find the eigenvalues we solve the equation: $det(X - \lambda I) = 0 \Leftrightarrow det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0 \Leftrightarrow \lambda = 1$ or $\lambda = -1$

For $\lambda = 1$: $(X - I)u = 0 \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow u_1 = u_2$. So $u = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -1$: $(X - I)u = 0 \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow v_1 = -v_2$. So $u = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

By normalizing the vectors u and v, we also find out that:
$\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{|1|^2 + |1|^2} = \sqrt{2}$
So, $u = |+\rangle$ and $v = |-\rangle$

## 3.6

Show that the $Y$ operator is traceless.

**Solution**

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. So $Tr(Y) = 0 + 0 = 0 \Rightarrow Y$ traceless.

## 2 Part 2

# 1. Eigenvalues and eigenvectors of the Pauli matrices

Give the eigenvectors and eigenvalues of these four matrices:

$$\sigma_0 \equiv I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 \equiv \sigma_x \equiv X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv \sigma_z \equiv Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Solution**

**1)** $det(I - \lambda I) = 0 \Leftrightarrow (1 - \lambda^2) = 0 \Leftrightarrow \lambda = 1$
For $\lambda = 1$: $(I - I)x = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
So eigenvector can be any two orthonormal vectors.

**2)** $det(X - \lambda I) = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda = 1$ or $\lambda = -1$
For $\lambda = 1$: $(X - I)x = 0 \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = x_2$
So eigenvector is $+x = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \forall c \in \mathbb{R}$.
For $\lambda = -1$: $(X + I)x = 0 \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = -x_2$
So eigenvector is $-x = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \forall c \in \mathbb{R}$.
Then by normalizing the eigenvectors we find that $+x = |+\rangle$ and $-x = |-\rangle$

**3)** $det(Y - \lambda I) = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda = 1$ or $\lambda = -1$
For $\lambda = 1$: $(Y - I)x = 0 \Leftrightarrow \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = -ix_2$
So eigenvector is $+x = c \begin{bmatrix} 1 \\ -i \end{bmatrix}, \forall c \in \mathbb{R}$.
For $\lambda = -1$: $(Y + I)x = 0 \Leftrightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = ix_2$
So eigenvector is $-x = c \begin{bmatrix} 1 \\ i \end{bmatrix}, \forall c \in \mathbb{R}$.
$\|-\mathbf{x}\| = \|+\mathbf{x}\| = \sqrt{|1|^2 + |\pm i|^2} = \sqrt{2}$
Then by normalizing the eigenvectors we find that $+x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $-x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

**4)** $det(Z - \lambda I) = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda = 1$ or $\lambda = -1$

For $\lambda = 1$: $(Z - I)x = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = 0$

So eigenvector is $+x = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \forall c \in \mathbb{R}$.

For $\lambda = -1$: $(Z + I)x = 0 \Leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = 0$

So eigenvector is $-x = c \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \forall c \in \mathbb{R}$.

$\|-\mathbf{x}\| = \|+\mathbf{x}\| = \sqrt{|1|^2 + |0|^2} = 1$

So the eigenvectors are already normalized.

## 2. Eigenvalues and eigenvectors of a $4 \times 4$ matrix

Give the eigenvalues and eigenvectors of this matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Solution**

$$det(A - \lambda I) = 0 \Leftrightarrow det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$det(A - \lambda I) = (1-\lambda) det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} =$

$(1-\lambda)(1-\lambda)(\lambda^2 - 1) = 0 \Leftrightarrow \lambda = 1$ or $\lambda = -1$

For $\lambda = 1$: $(A - I)x = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = x_3$

So eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -1$: $(A + I)x = 0 \Leftrightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$

$x_1 = 0, x_2 = -x_3, x_4 = 0$

So eigenvector is $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

In case eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, we need to normalize it. Finding that $\|\mathbf{x}\| = \sqrt{2}$, we can easily say that they become $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$.

# 3. Inner products

For matrix $M$, let $M^\dagger = (M^T)^*$, where $M^T$ is the transpose of $M$, and $*$ denotes the complex conjugate of $M$. We call $M^\dagger$ the adjoint of $M$.

Let

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) What is $v^\dagger v$?

(b) What is $v^\dagger w$?

(c) What is $vw^\dagger$?

(d) What is $v^\dagger X w$?

### Solution

In general we observe that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, equal to $v$ and w respectively.

**a)** $v^\dagger v = \langle 0|0\rangle = 1$
**b)** $v^\dagger w = \langle 0|1\rangle = 0$
**c)** $vw^\dagger = |0\rangle\langle 0| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
**d)** $v^\dagger X w = \langle 0|\hat{\sigma_X}|1\rangle = \langle 0|0\rangle = 1$

# 4. Hermitian matrices

A matrix $M$ is Hermitian if $M^\dagger = M$. Let $M$ be Hermitian.

(a) Prove that all its eigenvalues are real.

(b) Prove that $v^\dagger M v$ is real, for all vectors $v$. When $v^\dagger M v > 0$, we say that $M > 0$.

**Solution**

**a)** Supposing an eigenvalue $\lambda$ of M that has an eigenvector v then $Mv = \lambda v$, by multiplying with $v^\dagger$ we get $Mvv^\dagger = \lambda v^\dagger v$.

Then, by taking the complex conjugate:
$(Mvv^\dagger)^* = (\lambda v^\dagger v)^* \Leftrightarrow Mvv^\dagger = \lambda^* v^\dagger v = \lambda v^\dagger v$.
So, we see that $\lambda^* = \lambda$ that shows that the eigenvalues are real.

**b)** $(v^\dagger M v)^* = v^\dagger M^* v = v^\dagger M v$

# 5. Unitary matrices

Let $M$ be Hermitian, and define

$$U = e^{iM} = \sum_k \frac{(iM)^k}{k!}$$

Prove that $U^\dagger U = I$, where $I$ is the identity matrix.

**Solution**

By finding the dagger of $U$, we find $U^\dagger = (e^{iM})^\dagger = e^{-iM^*} = e^{-iM}$
So $UU^\dagger = e^{iM} e^{-iM} = e^0 = I$