Introduction to Quantum Computing

Exercise 1

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1 Part 1

McMahon Exercises from sections 4,6,7,8 and 9

4.1

Consider the basis in Example 4.1 Show that it is orthonormal.

> **Example 4.1**
> Let H_1 and H_2 be two Hilbert spaces for qubits. Describe the basis of $H = H_1 \otimes H_2$.

Solution

The computational basis for H_1 and H_2 is $\{|0\rangle, |1\rangle\}$. The tensor product basis of $H = H_1 \otimes H_2$ is:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \quad \text{where } |ij\rangle = |i\rangle \otimes |j\rangle.$$

For orthonormality, compute $\langle ij|kl\rangle$:

$$\langle ij|kl\rangle = \langle i|k\rangle\langle j|l\rangle = \delta_{ik}\delta_{jl}.$$

This equals 1 if $i = k$ and $j = l$, and 0 otherwise. Hence, the basis is orthonormal.

4.3

Given $\langle a|b\rangle = \frac{1}{2}$ and $\langle c|d\rangle = \frac{3}{4}$, calculate $\langle\psi|\phi\rangle$ where $|\psi\rangle = |a\rangle \otimes |c\rangle$ and $|\phi\rangle = |b\rangle \otimes |d\rangle$.

Solution

Using the tensor product inner product property:

$$\langle\psi|\phi\rangle = \langle a|b\rangle \otimes \langle c|d\rangle = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \boxed{\frac{3}{8}}.$$

4.4

Calculate the tensor product of

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |\phi\rangle = \frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}.$$

Solution

$$|\psi\rangle \otimes |\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\cdot 1\\1\cdot\sqrt{3}\\1\cdot 1\\1\cdot\sqrt{3}\end{pmatrix} = \boxed{\frac{1}{2\sqrt{2}}\begin{pmatrix}1\\\sqrt{3}\\1\\\sqrt{3}\end{pmatrix}}.$$

4.5

Can $|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$ be written as a product state?

Solution

We know that $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then, by trying to fit it inside $|\psi\rangle$, we find out that:

$$|-\rangle \otimes |-\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) = |\psi\rangle.$$

Thus, $|\psi\rangle$ is a product state. $\boxed{\text{Yes}}$.

4.6

Can $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ be written as a product state?

Solution

Assume $|\psi\rangle = |a\rangle \otimes |b\rangle$. Expanding:

$$a_0 b_0 = \frac{1}{\sqrt{2}}, \quad a_0 b_1 = 0, \quad a_1 b_0 = 0, \quad a_1 b_1 = \frac{1}{\sqrt{2}}.$$

This implies $a_0 = 0$ or $b_1 = 0$, which contradicts $a_1 b_1 = \frac{1}{\sqrt{2}}$. $\boxed{\text{No}}$. Also, we know that $|\psi\rangle$ cannot be written as a product state because it is entangled.

4.7

Find $X \otimes Y|\psi\rangle$, where $|\psi\rangle = \frac{|01\rangle-|10\rangle}{\sqrt{2}}$.

Solution

Pauli matrices: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Applying $X \otimes Y$:

$$X \otimes Y|\psi\rangle = \frac{1}{\sqrt{2}}\left(X|0\rangle \otimes Y|1\rangle - X|1\rangle \otimes Y|0\rangle\right) = \frac{1}{\sqrt{2}}\left(|1\rangle \otimes (i|0\rangle) - |0\rangle \otimes (-i|1\rangle)\right) = \boxed{i\frac{|10\rangle + |01\rangle}{\sqrt{2}}}.$$

4.8

Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Solution

For any operators A, B and states $|\phi\rangle, |\psi\rangle$:

$\langle\phi|(A \otimes B)^\dagger|\psi\rangle = \overline{\langle\psi|A \otimes B|\phi\rangle} = \overline{\langle\psi_1|A|\phi_1\rangle\langle\psi_2|B|\phi_2\rangle} = \langle\phi_1|A^\dagger|\psi_1\rangle\langle\phi_2|B^\dagger|\psi_2\rangle = \langle\phi|A^\dagger \otimes B^\dagger|\psi\rangle.$

Thus, $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

4.9

Find $I \otimes Y|\psi\rangle$, where $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Solution

$$I \otimes Y|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes Y|0\rangle + |1\rangle \otimes Y|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes (-i|1\rangle) + |1\rangle \otimes (i|0\rangle)) = \boxed{i\frac{|10\rangle - |01\rangle}{\sqrt{2}}}.$$

4.10

Calculate the matrix representation of $X \otimes Y$.

Solution

$$X \otimes Y = \begin{pmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}}.$$

6.1

Let P_1 and P_2 be two projection operators. Show that if their commutator $[P_1, P_2] = 0$, then their product P_1P_2 is also a projection operator.

Solution

A projection operator must satisfy $P^2 = P$ (idempotency) and $P^\dagger = P$ (Hermiticity).
1. Hermiticity:

$$(P_1P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2P_1 = P_1P_2 \quad (\text{since } [P_1, P_2] = 0).$$

2. Idempotency:

$$(P_1P_2)^2 = P_1^2P_2^2 = P_1P_2 \quad (\text{since } P_1^2 = P_1,\ P_2^2 = P_2).$$

Thus, P_1P_2 is a projection operator.

6.2

A system is in the state

$$|\psi\rangle = \frac{1}{2}|u_1\rangle - \frac{\sqrt{2}}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle,$$

where the orthonormal basis states $|u_1\rangle, |u_2\rangle, |u_3\rangle$ correspond to measurement results $\hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$. Determine the projection operators, probabilities, and average energy.

Solution

- Projection operators: $P_i = |u_i\rangle\langle u_i|$ for $i = 1, 2, 3$.
- Probabilities:

$$P(|u_1\rangle) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}, \quad P(|u_2\rangle) = \left|\frac{\sqrt{2}}{2}\right|^2 = \frac{1}{2}, \quad P(|u_3\rangle) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

- Average energy:

$$\langle E\rangle = \frac{1}{4}\hbar\omega + \frac{1}{2}(2\hbar\omega) + \frac{1}{4}(3\hbar\omega) = \boxed{2\hbar\omega}.$$

6.3

A qubit in state $|\psi\rangle = |1\rangle$ is measured in the X-basis. Find the projection operators and probabilities for outcomes ± 1.

Solution

- Projection operators:
  $P_+ = |+\rangle\langle +| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $P_- = |-\rangle\langle -| = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
- Probabilities:
  $Pr(+1) = \langle\psi|P_+|\psi\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}$,
  $Pr(-1) = \langle\psi|P_-|\psi\rangle \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}$.

6.4

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

A) Find the probability of measuring $|01\rangle$.
B) Find the probability of the second qubit being $|1\rangle$ and the postmeasurement state.

Solution

A) Probability for $|01\rangle$:

$$P(|01\rangle) = \left|\frac{1}{\sqrt{6}}\right|^2 = \boxed{\frac{1}{6}}.$$

B) Second qubit as $|1\rangle$: Let the projection operator for the probability of 2nd qubit being

$|1\rangle$, equal to P_1.Then, the requested probability will be: $\langle\psi|I \otimes P_1|\psi\rangle$. We first compute:

$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle \otimes |1\rangle\langle 1|0\rangle + \frac{1}{\sqrt{6}}|0\rangle \otimes |1\rangle\langle 1|1\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle\langle 1|1\rangle = \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$ So

finally,
$Pr = \langle\psi|I \otimes P_1|\psi\rangle = (\frac{1}{\sqrt{3}}\langle 00| + \frac{1}{\sqrt{6}}\langle 01| + \frac{1}{\sqrt{2}}\langle 11|)(\frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle) = \frac{1}{6} + \frac{1}{2} = \boxed{\frac{4}{6}}$
Postmeasurement state:

$$|\psi'\rangle = \frac{I \otimes P_1|\psi\rangle}{\sqrt{\langle\psi|I \otimes P_1|\psi\rangle}} \frac{\frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle}{\sqrt{\frac{2}{3}}} = \boxed{\frac{|01\rangle + \sqrt{3}|11\rangle}{2}}.$$

6.5

A system in state $|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle$ is measured with observable Y. Find the expectation value.

Solution

Expectaion value is given by the following relationship:

$$\langle Y\rangle = \langle\psi|Y|\psi\rangle = \langle\psi| \left(\frac{1}{\sqrt{6}}Y|0\rangle + \sqrt{\frac{5}{6}}Y|1\rangle\right) = \frac{\sqrt{5}}{6}i - \frac{\sqrt{5}}{6}i = \boxed{0}.$$

6.6

A three-qubit system is in the state

$$|\psi\rangle = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle.$$

A) Probability of measuring $|011\rangle$.
B) What is the probability that a measurement on the second qubit only gives 1? What is the postmeasurement state of the system? Show that the postmeasurement state is normalized.

Solution

A) Probability for $|011\rangle$:

$$P(|011\rangle) = \left|\frac{1}{\sqrt{10}}\right|^2 = \boxed{\frac{1}{10}}.$$

B) Second qubit as $|1\rangle$:

Let the projection operator for the probability of 2nd qubit being $|1\rangle$, equal to P_1.

Then, the requested probability will be: $\langle\psi|I \otimes P_1 \otimes I|\psi\rangle$. We first compute:

$I \otimes P_1 \otimes I|\psi\rangle = \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle$ So finally,

$Pr = \langle\psi|I \otimes P_1 \otimes I|\psi\rangle = \left(\left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)\langle 000| + \frac{1}{\sqrt{2}}\langle 001| + \frac{1}{\sqrt{10}}\langle 011| + \frac{i}{2}\langle 111|\right)\left(\frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle\right) = \frac{1}{10} + \frac{1}{4} = \boxed{\frac{7}{20}}$

Postmeasurement state:

$$|\psi'\rangle = \frac{I \otimes P_1|\psi\rangle}{\sqrt{\langle\psi|I \otimes P_1|\psi\rangle}} \frac{\frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle}{\sqrt{\frac{7}{20}}} = \boxed{\sqrt{\frac{2}{7}}|011\rangle + 5i\sqrt{\frac{1}{35}}|111\rangle}.$$

It is normalized because:

$$\langle\psi|\psi'\rangle = \frac{2}{7} + \frac{25}{35} = \boxed{1}$$

6.7

A two-qubit system in state

$$|\phi\rangle = \frac{1}{\sqrt{6}}|01\rangle + \sqrt{\frac{5}{6}}|10\rangle$$

Is the state normalized? An X gate is applied to the second qubit. After this is done,

what are the possible measurement results if both qubits are measured, and what are the respective probabilities of each measurement result?

Solution

Normalization check:

$$\langle\phi|\phi\rangle = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\sqrt{\frac{5}{6}}\right|^2 = \frac{1}{6} + \frac{5}{6} = 1$$

So $\boxed{Normalized}$. After X-gate on second qubit:

$$|\phi'\rangle = I \otimes X|\phi\rangle = \frac{1}{\sqrt{6}}|0\rangle \otimes X|1\rangle + \sqrt{\frac{5}{6}}|1\rangle \otimes X|0\rangle = \boxed{\frac{1}{\sqrt{6}}|00\rangle + \sqrt{\frac{5}{6}}|11\rangle}.$$

Probabilities:

$$\boxed{P(|00\rangle) = \frac{6}{11},\ P(|11\rangle) = \frac{5}{11}}$$

7.1

Derive the result (7.11).

Solution

$\vec{n} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$, then

$\vec{\sigma}\cdot\vec{n} = \sin(\theta)\cos(\phi)\vec{\sigma_x} + \sin(\theta)\sin(\phi)\vec{\sigma_y} + \cos(\theta)\vec{\sigma_z} = \begin{bmatrix} cos(\theta) & sin(\theta)e^{-i\phi} \\ sin(\theta)e^{i\phi} & -cos(\theta) \end{bmatrix} = A$

$det(A - \lambda I) = 0 \Leftrightarrow -(cos(\theta) - \lambda)(cos(\theta) + \lambda) - sin^2(\theta)e^{i\phi}e^{-i\phi} = 0 \Leftrightarrow \lambda^2 = cos^2(\theta) + sin^2(\theta) \Leftrightarrow \lambda = \pm 1$

For λ= 1:

$(A - \lambda I)\vec{x} = 0 \Leftrightarrow \begin{bmatrix} cos(\theta) - 1 & sin(\theta)e^{-i\phi} \\ sin(\theta)e^{i\phi} & -cos(\theta) - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (cos(\theta) - 1)x_1 & (sin(\theta)e^{-i\phi})x_2 \\ (sin(\theta)e^{i\phi})x_1 & (-cos(\theta) - 1)x_2 \end{bmatrix}$

After, this point we can use trigonometric identities andmake the equations simpler, in order to find the eigenvectors.
So:

$$\begin{cases} (cos(\theta) - 1)x_1 + sin(\theta)e^{-i\phi}x_2 = 0 \ (1) \\ sin(\theta)e^{i\phi}x_1 - (cos(\theta) + 1)x_2 = 0 \quad (2) \end{cases}$$

For (1), using $sin^2(\frac{\theta}{2}) = \frac{1 - cos(\theta)}{2}$ and $sin(\theta) = 2sin(\frac{\theta}{2}cos(\frac{\theta}{2}))$, we get

$(1) \Rightarrow \left(-2\sin^2\left(\frac{\theta}{2}\right)\right)x_1 + \left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)e^{-i\phi}x_2 = 0$ and by dividing with $\sin(\frac{\theta}{2})$

$\left(-2\sin(\frac{\theta}{2})\right)x_1 + \left(2\cos(\frac{\theta}{2})\right)e^{-i\phi}x_2 = 0 \Leftrightarrow x_1 = \frac{\cos(\frac{\theta}{2})}{\sin(\frac{\theta}{2})}e^{-i\phi}x_2$

After those computations, we conclude that: $\vec{x} = c\begin{bmatrix} \cot(\frac{\theta}{2})e^{-i\phi}x_2 \\ x_2 \end{bmatrix}$.

Finally, $|x\rangle = \cot(\frac{\theta}{2})e^{-i\phi}|0\rangle + |1\rangle$, that requires normalization.

$\langle x|x\rangle = 1 \Leftrightarrow \cot^2(\frac{\theta}{2})x_2^2 + x_2^2 = 1 \Leftrightarrow x_2 = \sin(\frac{\theta}{2})\cdot c$ and $x_1 = \cos(\frac{\theta}{2})e^{-i\phi}\cdot c$

That gives us: $|x\rangle = \cos(\frac{\theta}{2})e^{-i\phi}\cdot c|0\rangle + \sin(\frac{\theta}{2})\cdot c|1\rangle$, considering $c = e^{i\phi} \Leftrightarrow$
$|+_n\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$

By following the same steps for $\lambda = -1$, we get: $|-_n\rangle = \cos(\frac{\theta}{2})|0\rangle - e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$

7.2

The eigenstates of the Y operator are

$$|\pm y\rangle = \frac{|0\rangle \pm i\,|1\rangle}{\sqrt{2}}.$$

Rewrite the singlet state (7.2) in terms of the Y eigenstates. Does it have a similar form?

Solution

The singlet state is given by: $|\psi\rangle = \frac{|01\rangle-|10\rangle}{\sqrt{2}}$
We know that $|0\rangle = \frac{|+i\rangle+|-i\rangle}{\sqrt{2}}$ and $|1\rangle = \frac{|+i\rangle-|-i\rangle}{\sqrt{2}}$

So $|10\rangle$ can be written as: $\frac{|+i\rangle-|-i\rangle}{\sqrt{2}}\frac{|+i\rangle+|-i\rangle}{\sqrt{2}} = \frac{|+i\rangle|+i\rangle+|+i\rangle|-i\rangle-|-i\rangle|+i\rangle-|-i\rangle|-i\rangle}{2}$ and

$|01\rangle$ can be written as: $\frac{|+i\rangle+|-i\rangle}{\sqrt{2}}\frac{|+i\rangle-|-i\rangle}{\sqrt{2}} = \frac{|+i\rangle|+i\rangle-|+i\rangle|-i\rangle+|-i\rangle|+i\rangle-|-i\rangle|-i\rangle}{2}$

So combining them $|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{|+i\rangle|+i\rangle-|+i\rangle|-i\rangle+|-i\rangle|+i\rangle-|-i\rangle|-i\rangle}{2} - \frac{|+i\rangle|+i\rangle+|+i\rangle|-i\rangle-|-i\rangle|+i\rangle-|-i\rangle|-i\rangle}{2}\right) \Leftrightarrow$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{|+i\rangle|+i\rangle-|+i\rangle|-i\rangle+|-i\rangle|+i\rangle-|-i\rangle|-i\rangle-|+i\rangle|+i\rangle-|+i\rangle|-i\rangle+|-i\rangle|+i\rangle+|-i\rangle|-i\rangle}{2}\right) \Leftrightarrow$

$|\psi\rangle = \frac{|-i\rangle|+i\rangle-|+i\rangle|-i\rangle}{\sqrt{2}}$ $\boxed{\text{So we find out that it has a similar form}}$.

7.3

Verify that $Z \otimes Z\,|\beta_{xy}\rangle = (-1)^y\,|\beta_{xy}\rangle$, for

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \text{ and } |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Solution

Recall that $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$
$Z \otimes Z|\beta_{00}\rangle = Z \otimes Z\frac{|00\rangle+|11\rangle}{\sqrt{2}} = \frac{Z|0\rangle\otimes Z|0\rangle+Z|1\rangle\otimes Z|1\rangle}{\sqrt{2}} = \frac{|00\rangle+|11\rangle}{\sqrt{2}} = |\beta_{00}\rangle = (-1)^0|\beta_{00}\rangle$

$Z \otimes Z|\beta_{01}\rangle = Z \otimes Z\frac{|01\rangle+|10\rangle}{\sqrt{2}} = \frac{Z|0\rangle\otimes Z|1\rangle+Z|1\rangle\otimes Z|0\rangle}{\sqrt{2}} = -\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right) = -|\beta_{01}\rangle = (-1)^1|\beta_{01}\rangle$

So we conclude that $Z \otimes Z\,|\beta_{xy}\rangle = (-1)^y\,|\beta_{xy}\rangle$ for $|\beta_{00}\rangle, |\beta_{01}\rangle$

7.4

Show that $X \otimes X\,|\beta_{xy}\rangle \;=\; (-1)^x\,|\beta_{xy}\rangle$.

Solution

Recall that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$, then $X|0y\rangle = |1\bar{y}\rangle$ and $X|1\bar{y}\rangle = |0y\rangle$

So: $X \otimes X|\beta_{xy}\rangle = \frac{X\otimes X|0y\rangle + (X\otimes X)(-1)^x|1\bar{y}\rangle}{\sqrt{2}} = \frac{|1\bar{y}\rangle + (-1)^x|0y\rangle}{\sqrt{2}}$

And because $(-1)^x(-1)^x = 1$, we can write the result as: $\frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}} = (-1)^x|\beta_{xy}\rangle$

7.5

Show that $Y \otimes Y\,|\beta_{xy}\rangle \;=\; (-1)^{x+y+1}\,|\beta_{xy}\rangle$.

Solution

Recall that $Y|0\rangle = i|1\rangle$ and $Y|1\rangle = -i|0\rangle$, then $Y \otimes Y|0y\rangle = \pm i^2|1\bar{y}\rangle$ and $Y \otimes Y|1\bar{y}\rangle = \pm i|0y\rangle$

If y = 0, then $\bar{y} = 1$ and if y = 1, then $\bar{y} = 0$

So, for y = 0: $Y\otimes Y|\beta_{xy}\rangle = \frac{Y\otimes Y|0y\rangle + (Y\otimes Y)(-1)^x|1\bar{y}\rangle}{\sqrt{2}} = \frac{-(|1\bar{y}\rangle + (-1)^x|0y\rangle)}{\sqrt{2}} = -(-1)^y(-1)^x|\beta_{xy}\rangle = (-1)^{x+y+1}|\beta_{xy}\rangle$

For y = 1: $\frac{|1\bar{y}\rangle + (-1)^x|0y\rangle}{\sqrt{2}} = (-1)^{\bar{y}}(-1)^x|\beta_{xy}\rangle = (-1)^{x+y+1}|\beta_{xy}\rangle$

7.6

Show that $X \otimes X$ commutes with $Z \otimes Z$.

Solution

$[X \otimes X, Z \otimes Z] = 0 \Leftrightarrow (X \otimes X)(Z \otimes Z) - (Z \otimes Z)(X \otimes X) = 0$

Also $(X \otimes X)(Z \otimes Z) = XZ \otimes XZ$ and $(Z \otimes Z)(X \otimes X) = ZX \otimes ZX$ and we know that $XZ = -ZX$

So $(-ZX \otimes -ZX) - (ZX \otimes ZX) = (ZX \otimes ZX) - (ZX \otimes ZX) = 0 \Leftrightarrow 0 = 0$ $\boxed{\text{So it commutes.}}$

7.7

Consider the eigenvectors in Example 7.4. Show that

$$[H, \vec{\sigma}_a \cdot \vec{\sigma}_b] = 0$$

and hence show that the eigenvectors of the Hamiltonian are also eigenvectors of the operator $\bar{\sigma}_a \cdot \bar{\sigma}_b$. In particular, show that

$$\vec{\sigma}_a \cdot \vec{\sigma}_b |\phi_i\rangle = |\phi_i\rangle \quad \text{for } i = 1, 2, 3,$$

and

$$\vec{\sigma}_a \cdot \vec{\sigma}_b |\phi_4\rangle = -3 |\phi_4\rangle.$$

Solution

$[H, \vec{\sigma}_a \cdot \vec{\sigma}_b] = 0 \Leftrightarrow (H \otimes \vec{\sigma}_a \cdot \vec{\sigma}_b) - (\vec{\sigma}_a \cdot \vec{\sigma}_b \otimes H) = 0$, that is true because

$$H = \frac{\mu^2}{r^3} (\vec{\sigma}_A \cdot \vec{\sigma}_B - 3Z_A Z_B),$$

where $\vec{\sigma}_A \cdot \vec{\sigma}_B = X_A X_B + Y_A Y_B + Z_A Z_B$. The commutator is computed as:

$$\begin{aligned} [H, \vec{\sigma}_A \cdot \vec{\sigma}_B] &= \frac{\mu^2}{r^3} [\vec{\sigma}_A \cdot \vec{\sigma}_B - 3Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] \\ &= \frac{\mu^2}{r^3} \left(\underbrace{[\vec{\sigma}_A \cdot \vec{\sigma}_B, \vec{\sigma}_A \cdot \vec{\sigma}_B]}_{0} - 3[Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] \right). \end{aligned}$$

Using the anticommutation relations $\{X, Z\} = 0$ and $\{Y, Z\} = 0$, we find:

$$[Z_A Z_B, X_A X_B] = [Z_A Z_B, Y_A Y_B] = [Z_A Z_B, Z_A Z_B] = 0 \implies [Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0.$$

Thus:

$$\boxed{[H, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0}.$$

2. Eigenvalues of $\vec{\sigma}_A \cdot \vec{\sigma}_B$

The Bell states (eigenvectors of H) are:

$$\begin{aligned} |\phi_1\rangle = |\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\phi_2\rangle = |\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\phi_3\rangle = |\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\phi_4\rangle = |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

Triplet States ($|\phi_{1,2,3}\rangle$):

$$\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_i\rangle = |\phi_i\rangle \quad (i = 1, 2, 3).$$

Example for $|\phi_1\rangle$:

$$\vec{\sigma}_A \cdot \vec{\sigma}_B |\beta_{00}\rangle = (X_A X_B + Y_A Y_B + Z_A Z_B)|\beta_{00}\rangle = |\beta_{00}\rangle.$$

Singlet State ($|\phi_4\rangle$):

$$\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_4\rangle = -3|\phi_4\rangle.$$

Example for $|\phi_4\rangle$:

$$\vec{\sigma}_A \cdot \vec{\sigma}_B |\beta_{11}\rangle = (X_A X_B + Y_A Y_B + Z_A Z_B)|\beta_{11}\rangle = -3|\beta_{11}\rangle.$$

8.3

Find a way to write the Pauli operators X, Y, and Z in terms of the Hubbard operators.

Solution

$$\boxed{X = E^{01} + E^{10}, \quad Y = -iE^{01} + iE^{10}, \quad Z = E^{00} - E^{11}}$$

8.4

Show that the controlled NOT gate is Hermitian and unitary.

Solution

$$(CNOT)^\dagger = ((CNOT)^*)^T = CNOT^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = CNOT$$

Also we know that if $A^\dagger A = I$, A is unitary.

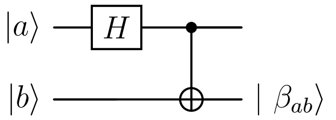
So we can compute: $CNOT^\dagger CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$

$$\boxed{\text{CNOT is Hermitian and unitary.}}$$

8.5

Let $|a\rangle = |1\rangle$, and consider the circuit shown in Figure 8.5. Determine which Bell states are generated as output when $|b\rangle = |0\rangle, |b\rangle = |1\rangle$.

Figure 8.5



Solution

For input $|a\rangle = |1\rangle$: $H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

When $|b\rangle = |0\rangle$: $H|1\rangle \otimes |0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$

When $|b\rangle = |1\rangle$: $H|1\rangle \otimes |1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$

We continue by applying the CNOT gate:
$CNOT\left(\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)\right) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\beta_{10}\rangle$

$CNOT\left(\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\right) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\beta_{11}\rangle$

So we conclude that:

$$\boxed{\begin{aligned} |b\rangle = |0\rangle &\implies |\beta_{10}\rangle, \\ |b\rangle = |1\rangle &\implies |\beta_{11}\rangle. \end{aligned}}$$

8.6

Write down the matrix representation for the controlled Z gate. Then write down its representation using Dirac notation.

Solution

Knowing Z operator whe can find that controlled Z operator, for inputs:

$CZ|00\rangle \rightarrow |00\rangle$, $CZ|01\rangle \rightarrow |01\rangle$, $CZ|10\rangle \rightarrow |10\rangle$, $CZ|11\rangle \rightarrow -|11\rangle$

Then:

$$\text{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \boxed{\text{CZ} = |00\rangle + |01\rangle + |10\rangle - |11\rangle = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z}$$

8.7

Consider the single qubit operators X, Y, Z, S, and T. Find the square of each operator.

Solution

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$Y^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$Z^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$S^2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z,$$

$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S. \text{ Because } e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$$

8.10

By using the tensor product methods developed in chapter 4, show that the controlled-NOT matrix can be generated from $P_0 \otimes I + P_1 \otimes X$.

Solution

We know that $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, so we can write the controlled-NOT operator as:

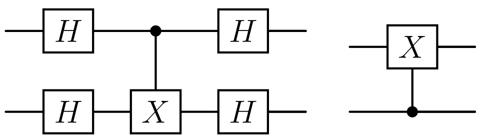
$$CNOT = P_0 \otimes I + P_1 \otimes X = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$CNOT$

8.11

Show that the following circuits are equivalent:



Solution

As we see the 2nd circuit is $CNOT_{2\to1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, because the control qubit is the second one and the target qubit is the first one.
The first circuit is:

$H|CNOT_{1\to2}|H = H|0\rangle\langle 0|H + H|1\rangle\langle 1|H \otimes HXH \Leftrightarrow$
$H|CNOT_{1\to2}|H = \frac{1}{2}\left[((I+X)\otimes I) + ((I-X)\otimes Z)\right] \Leftrightarrow$
$H|CNOT_{1\to2}|H = \frac{1}{2}(I\otimes I + X\otimes I + I\otimes Z - X\otimes Z)$

We proceed by making the calculations for the tensor products:

$I_{2,2}\otimes I_{2,2} = I_{4,4}$

$X\otimes I = \begin{bmatrix} 0\cdot I & 1\cdot I \\ 1\cdot I & 0\cdot I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ by adding them we get $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$I\otimes Z = \begin{bmatrix} 1\cdot Z & 0\cdot Z \\ 0\cdot Z & 1\cdot Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$-X\otimes Z = -\begin{bmatrix} 0\cdot Z & 1\cdot Z \\ 1\cdot Z & 0\cdot Z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ by adding them we get $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Finally, we can add the 2 matrices together:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

So we conclude that:

$$\mathrm{H|CNOT_{1\to2}|H} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = CNOT_{2\to1}$$

9.1

Using the matrix representation of the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Write down the matrix $H \otimes H$ and find $(H \otimes H)(|0\rangle \otimes |1\rangle)$. Show that this is equivalent to

$$|\psi\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \left(\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \right)$$

Solution

$H \otimes H = \frac{1}{2} \begin{bmatrix} 1 \cdot H & 1 \cdot H \\ 1 \cdot H & -1 \cdot H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$H \otimes H = H|0\rangle \otimes H|1\rangle = \frac{(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)}{2} = |\psi\rangle$

9.4

Derive (9.17) through (9.19).

$$U_f|01\rangle = |0, 1 \oplus f(0)\rangle = f(0)|00\rangle + (1 - f(0))|01\rangle \tag{9.17}$$

$$U_f|11\rangle = |1, 1 \oplus f(1)\rangle = f(1)|10\rangle + (1 - f(1))|11\rangle \tag{9.19}$$

Solution

As we know from theory $U_f|xy\rangle = |x, y \oplus f(x)\rangle$, so $U_f|11\rangle = |1, 1 \oplus f(1)\rangle$

Case f(1) $=$ 0: $U_f|11\rangle = |1, 1 \oplus 0\rangle = |1, 1\rangle$
Case f(1) $=$ 1: $U_f|11\rangle = |1, 1 \oplus 1\rangle = |1, 0\rangle$

So we can write it as: $U_f|11\rangle = f(1)|10\rangle + (1 - f(1))|11\rangle$

Then $U_f|01\rangle = |0, 1 \oplus f(0)\rangle$

Case f(0) $=$ 0: $U_f|01\rangle = |0, 1 \oplus 0\rangle = |0, 1\rangle$
Case f(0) $=$ 1: $U_f|01\rangle = |0, 1 \oplus 1\rangle = |0, 0\rangle$

So we can write it as: $U_f|01\rangle = f(0)|00\rangle + (1 - f(0))|01\rangle$

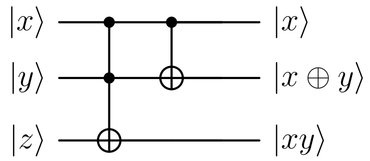
9.5

Quantum gates are universal in the sense that quantum gates can be designed that do anything a classical gate can do. Design a quantum adder, a gate that takes three inputs $|x\rangle$, $|y\rangle$, $|z\rangle$ and that has three output qubits $|x\rangle$, $|x \otimes y\rangle$, $|xy\rangle$, where $|x \otimes y\rangle$, is the sum and $|xy\rangle$ is the carry.

Solution

In order to create $|x \oplus y\rangle$ we need a CNOT gate and in order to create $|xy\rangle$ we need a Toffoli gate, because it gives us $|z\rangle \oplus |xy\rangle$, and $|z\rangle = |0\rangle$.

So we can create a circuit like this:



2 Part 2

Exercise 1

1.Compute the eigenvectors $|u_j\rangle$ and the eigenvalues λ_j for the gate $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and verify in Qiskit that the relation holds:

$$U|u_j\rangle = \lambda_j|u_j\rangle \Rightarrow X|u_j\rangle = \lambda_j|u_j\rangle, j = 0, 1 \tag{1}$$

Solution

We start by computing the eigenvalues of the matrix and for each one its eigenvector$X$:
$det(X - \lambda I) = 0 \Leftrightarrow \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda_0 = 1, \lambda_1 = -1$

For $\lambda_0 = 1$:

$$(X - I)\vec{u_0} = 0 \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow u_1 = u_2 \Rightarrow \vec{u_0} = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_1 = -1$:

$$(X + I)\vec{u_1} = 0 \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow u_3 = -u_4 \Rightarrow \vec{u_1} = c \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

By applying normalization to the eigenvectors we get:

$$\vec{u_0} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = |+\rangle, \quad \vec{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = -|-\rangle$$

So we conclude that:

$$X|+\rangle|-\rangle = X|+\rangle X|-\rangle = |+\rangle - |-\rangle = 1 \cdot |+\rangle - 1 \cdot |-\rangle$$

<u>Qiskit code:</u>

```
# ========== |+ Verification ==========
qc_plus = QuantumCircuit(1)
qc_plus.h(0)  # Create |+> state
qc_plus.x(0)  # Apply X gate

# Simulate and format output
result_plus = simulator.run(qc_plus).result()
state_plus = np.round(result_plus.get_statevector().data, 3)
print("\n=== X|+> Verification ===")
print(f"Expected: [1/sqrt(2), 1/sqrt(2)] ~ [0.707, 0.707]")
print(f"Actual:    {state_plus} (global phase)")
print("Conclusion: X|+> = +|+>\n")

# ========== |-> Verification ==========
qc_minus = QuantumCircuit(1)
qc_minus.x(0)  # Prepare |1s
qc_minus.h(0)  # Create |-> state
qc_minus.x(0)  # Apply X gate

# Simulate and format output
result_minus = simulator.run(qc_minus).result()
state_minus = np.round(result_minus.get_statevector().data, 3)
print("=== X|-> Verification ===")
print(f"Expected: [-1/sqrt(2), 1/sqrt(2)] ~ [-0.707, 0.707]")
print(f"Actual:    {state_minus}")
print("Conclusion: X|-> = -|->\n")
```

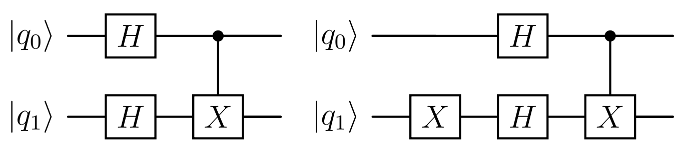
Qiskit output:

```
=== X|+⟩ Verification ===
Expected: [1/√2, 1/√2] ≈ [0.707, 0.707]
Actual:   [0.707+0.j 0.707+0.j] (global phase)
Conclusion: X|+⟩ = +1·|+⟩

=== X|−⟩ Verification ===
Expected: [-1/√2, 1/√2] ≈ [-0.707, 0.707]
Actual:   [-0.707+0.j  0.707-0.j]
Conclusion: X|−⟩ = -1·|−⟩
```

Figure 1: Eigenvalues and eigenvectors of the X gate

2. Create and then execute the following circuits:



Can you observe the phase kickback effect? Where does it occur and why?

Solutuion

The first circuit is:

Qiskit code:

```
# Create circuit 1
qc1 = QuantumCircuit(2)
qc1.barrier([0, 1])
qc1.h(0)
qc1.h(1)
qc1.barrier([0, 1])
qc1.cx(0, 1)

qc1.measure_all()
qc1.draw('mpl')
qc1.remove_final_measurements()

sim = Aer.get_backend('statevector_simulator')
result = sim.run(qc1).result()
state = result.get_statevector()
plot_bloch_multivector(state)

counts = simulator.run(qc1, shots = 1024).result().get_counts()
plot_histogram(counts)
```

Qiskit output:

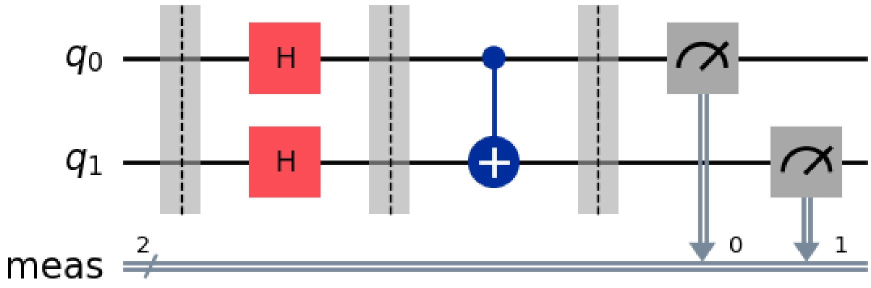


Figure 2: Quantum Circuit 1 Visualization

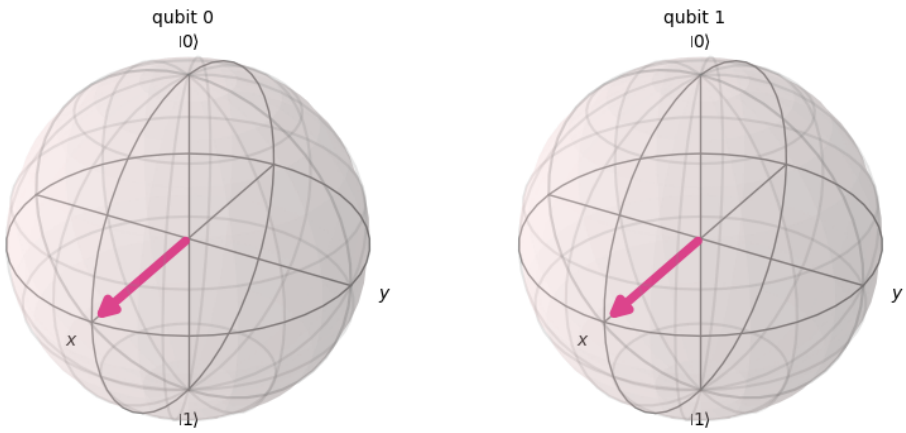


Figure 3: Quantum Circuit 1 Bloch sphere output

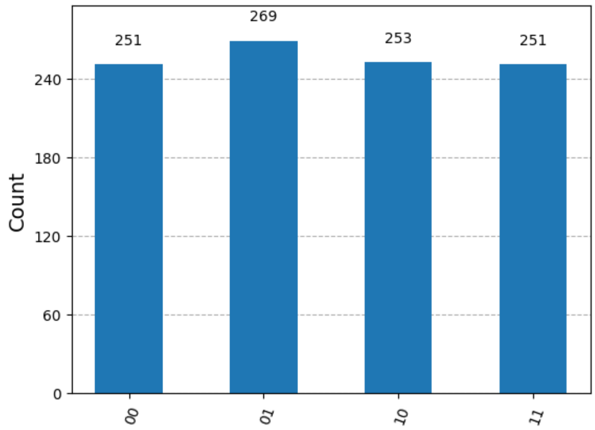


Figure 4: Quantum Circuit 1 results

As we can see here there is **no phase kickback effect**, because **the output is the same as the input**.

Now we create the second circuit:

Qiskit code:

```
# Create circuit 2
qc2 = QuantumCircuit(2)
qc2.barrier([0, 1])
qc2.x(1)

qc2.barrier([0, 1])
qc2.h(0)
qc2.h(1)

qc2.barrier([0, 1])
qc2.cx(0, 1)
qc2.measure_all()
qc2.draw('mpl')
qc2.remove_final_measurements()

sim = Aer.get_backend('statevector_simulator')
result = sim.run(qc2).result()
state = result.get_statevector()
plot_bloch_multivector(state)

counts = simulator.run(qc2, shots = 1024).result().get_counts()
plot_histogram(counts)
```

Qiskit output:

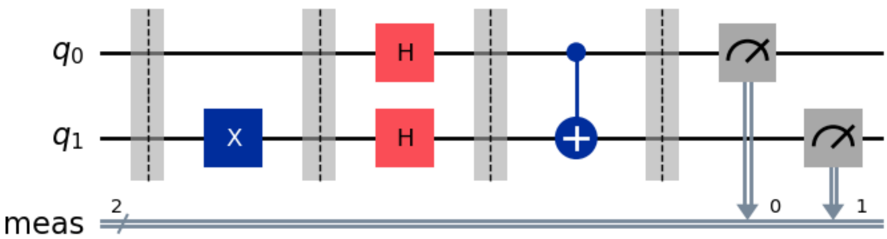


Figure 5: Quantum Circuit 1 Visualization

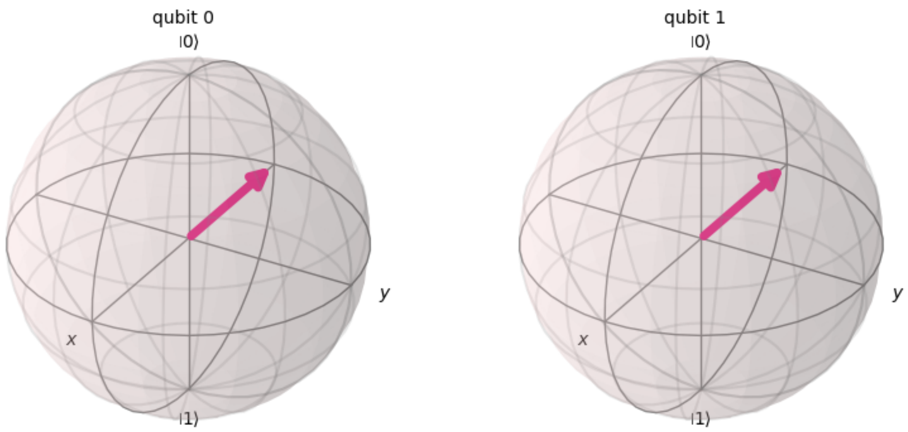


Figure 6: Quantum Circuit 1 Bloch sphere output

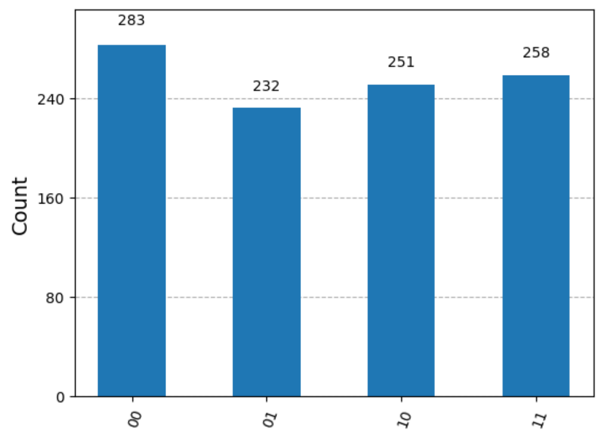
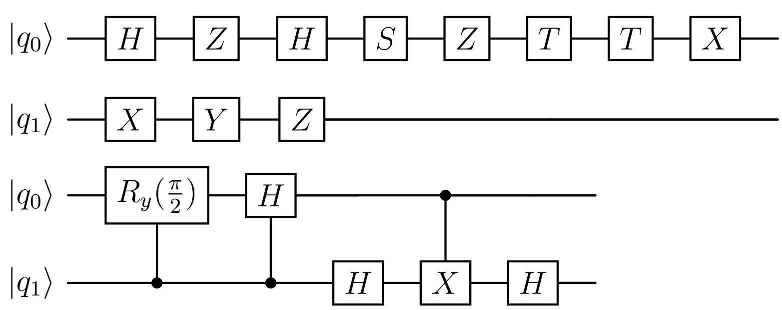


Figure 7: Quantum Circuit 1 results

As we can see here **there is phase kickback effect**, because **the control qubit's phase is modified**.

Exercise 2

Prove analytically that the following circuits implement the unitary transformation I and verify the results in Qiskit.



Solution

For the first circuit, $|q_0\rangle$:

We know that $HZH = X$. After that we make the needed computations.
$SX = \begin{bmatrix}1 & 0\\0 & i\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\i & 0\end{bmatrix}$, $ZSX = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\i & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\-i & 0\end{bmatrix}$,
$TZSX = \begin{bmatrix}1 & 0\\0 & e^{i\frac{\pi}{4}}\end{bmatrix}\begin{bmatrix}0 & 1\\-i & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\-ie^{i\frac{\pi}{4}} & 0\end{bmatrix}$,
$TTZSX = \begin{bmatrix}1 & 0\\0 & e^{i\frac{\pi}{4}}\end{bmatrix}\begin{bmatrix}0 & 1\\-ie^{i\frac{\pi}{4}} & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = X, \quad X \cdot X = I$

For $|q_1\rangle$:

$YX = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}-i & 0\\0 & i\end{bmatrix}$, $ZYX = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-i & 0\\0 & i\end{bmatrix} = \begin{bmatrix}-i & 0\\0 & -i\end{bmatrix} = -i \cdot I$

So 1st circuit is equivalent to I

Qiskit code:

```
qc2_1 = QuantumCircuit(2)
qc2_1.h(0)
qc2_1.x(1)  # Apply the Hadamard gate to the target qubit
qc2_1.z(0)  # Apply the X gate to the second qubit
qc2_1.y(1)  # Apply the Hadamard gate to the target qubit
qc2_1.h(0)
qc2_1.z(1)
qc2_1.s(0)
qc2_1.z(0)
qc2_1.t(0)
qc2_1.t(0)
qc2_1.x(0)

qc2_1.draw('mpl')

# unitary matrix simulation
sim = Aer.get_backend('unitary_simulator')
result = sim.run(qc2_1).result()
unitary = result.get_unitary()

# Print the unitary matrix
print("Unitary matrix:")
array_to_latex(unitary, prefix="\\text{Unitary} = ")
```

Qiskit output:

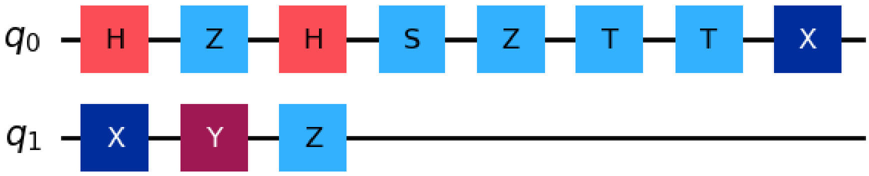


Figure 8: Quantum Circuit 1

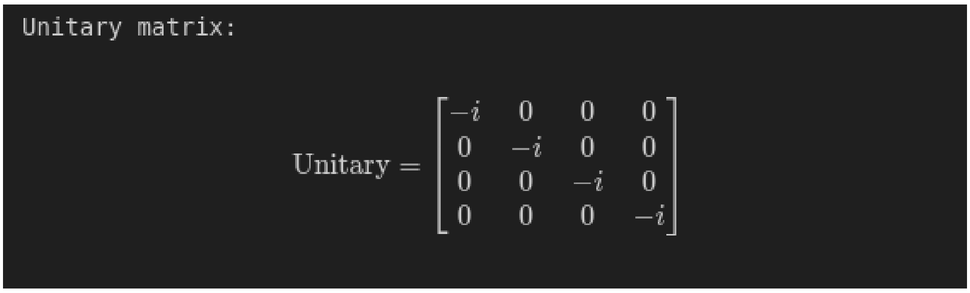


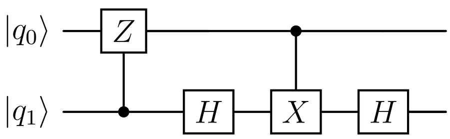
Figure 9: Quantum Circuit 1 results

So we verify with the use of Qiskit that the first circuit is equivalent to I

For the second circuit, $|q_0\rangle$:

We know that $H \cdot R_y(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = Z$

So we can tell that the circuit is equivalent to:



Let $U = CZ \cdot H \cdot CX \cdot H$. U is unitary if $U^\dagger U = I$.
And $U^\dagger U = H^\dagger \cdot CX^\dagger \cdot H^\dagger \cdot CZ^\dagger \cdot CZ \cdot H \cdot CX \cdot H \cdot = I$

As we can also see with the help of Qiskit that our result is correct.
Qiskit Code

```
qc2_2 = QuantumCircuit(2)
#qc2_2.cry(pi/2, 1, 0)  # Specify the control qubit (0) and target qubit
    (1)
qc2_2.cz(1, 0)  # Apply the controlled-Hadamard gate
qc2_2.h(1)  # Apply the Hadamard gate to the target qubit
qc2_2.cx(0, 1)  # Apply the X gate to the second qubit
qc2_2.h(1)  # Apply the Hadamard gate to the target qubit
qc2_2.draw('mpl')

```

```
# unitary matrix simulation
sim = Aer.get_backend('unitary_simulator')
result = sim.run(qc2_2).result()
unitary = result.get_unitary()

# Print the unitary matrix
print("Unitary matrix:")
array_to_latex(unitary, prefix="\\text{Unitary} = ")
```

Qiskit output:

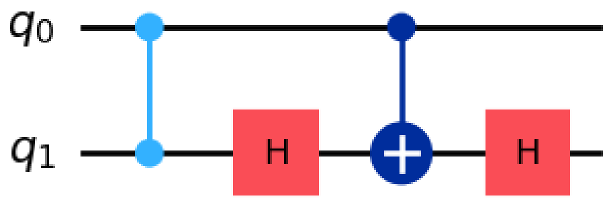


Figure 10: Quantum Circuit 2

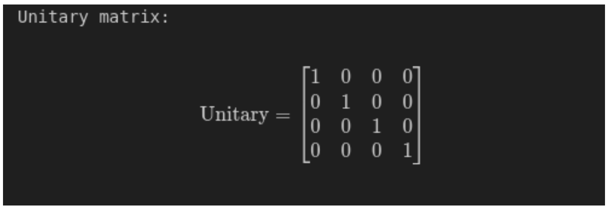


Figure 11: Quantum Circuit 2 results

Exercise 3

Write in Qiskit code to implement Deutsch algorithm Create 2 Oracles. One Oracle that implements a constant function and one Oracle that implements a balanced function. Verify the correct operation of the algorithm.

Solution

Qiskit code:

```
def deutsch_algorithm(oracle):
    qc = QuantumCircuit(2, 1)
    # Initialize qubits
    qc.x(1)            # Set ancillary qubit (q1) to |1>
    qc.h([0, 1])       # Apply Hadamard to both qubits
    qc.barrier()
    # Append the oracle
```

```python
    qc.compose(oracle, inplace=True)
    qc.barrier()
    # Measure the first qubit
    qc.h(0)
    qc.measure(0, 0)
    qc.draw('mpl')
    return qc

oracle_constant = QuantumCircuit(2)
oracle_constant.x(1)  # Flip the ancillary qubit (q1) unconditionally

oracle_balanced = QuantumCircuit(2)
oracle_balanced.cx(0, 1)  # CNOT: q0 controls q1

# Create circuits
qc_constant = deutsch_algorithm(oracle_constant)
qc_balanced = deutsch_algorithm(oracle_balanced)

# Simulate using qasm_simulator
simulator = Aer.get_backend('qasm_simulator')

# Execute circuits
job_constant = simulator.run(qc_constant, shots=1024)
job_balanced = simulator.run(qc_balanced, shots=1024)

# Get results
counts_constant = job_constant.result().get_counts()
counts_balanced = job_balanced.result().get_counts()

# Plot results
plot_histogram([counts_constant, counts_balanced], legend=['Constant', 'Balanced'])
plt.show()
```

Qiskit output:

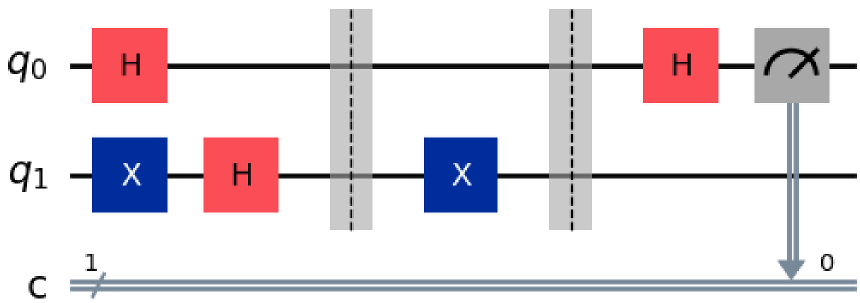


Figure 12: Circuit for constant oracle

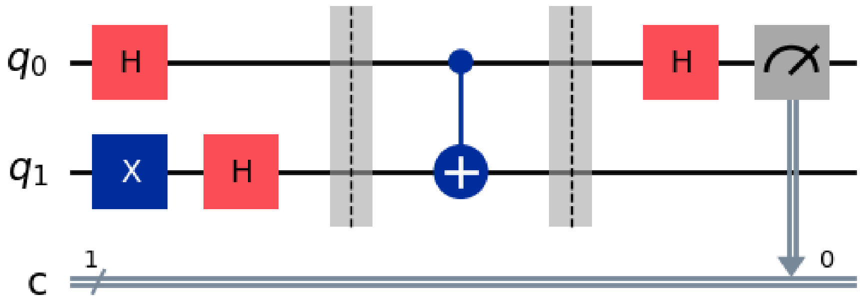


Figure 13: Circuit for balanced oracle

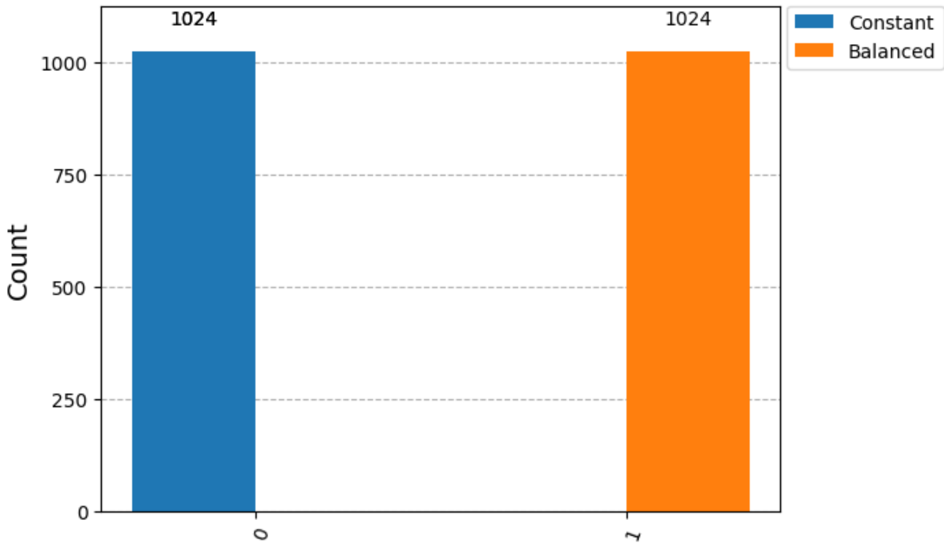


Figure 14: Deutsch Algorithm results