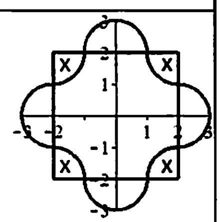
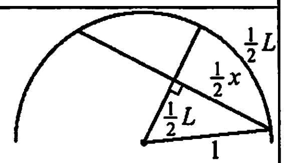
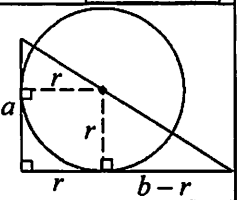
New York State Mathematics Association of Two-Year Colleges

NYSMATYC 1967

Math League Contest ~ Fall 2011 ~ Solutions

1.	$A(1,2)=A(0,A(1,1))=A(0,A(0,A(1,0)))=A(0,A(0,A(0,1)))=A(0,A(0,2))=A(0,3)=4$ $\boxed{\text{Answer: 4}}$
2.	If the parabola formed by $f(x)=3x^2+kx+1$ is tangent to the x-axis, then it has only one (repeated) real root. Hence, the discriminant (i.e. b^2-4ac) must be zero. Thus, $k^2-4\cdot 3\cdot 1=0 \Rightarrow k^2=12$ $\Rightarrow k=\pm\sqrt{12}$, with $x=\dfrac{-k}{2\cdot 3}$. Since the problem requires the point of tangency to be on the *positive* x-axis, k must be negative. Thus, $k=-\sqrt{12}=-2\sqrt{3}$. $\boxed{\text{Answer: } -\sqrt{12} \text{ or } -2\sqrt{3}}$
3.	Completing the square for both equations: $x^2+y^2-2x-2y+1=0 \Rightarrow (x-1)^2+(y-1)^2=1 \Rightarrow$ circle of radius 1 centered at (1,1) $x^2+y^2-8x-10y+25=0 \Rightarrow (x-4)^2+(y-5)^2=16 \Rightarrow$ circle of radius 4 centered at (4,5) The distance between centers is $\sqrt{(4-1)^2+(5-1)^2}=\sqrt{25}=5$, which is the sum of both radii. Hence, the circles are tangent to each other and have exactly one point of intersection. $\boxed{\text{Answer: b}}$
4.	$1!+2!+3!+\ldots+2011!=1+2+6+24+120+\cdots$(all remaining terms have a zero in the units position) Hence, the sum ends with a 3 in the units position. Therefore, the sum cannot have an even number as a factor, nor can 55 be a factor (since it does not end with a 0 or 5). Only 33 can be a factor. $\boxed{\text{Answer: b}}$
5.	Let V be the volume (in teaspoons) of the containers. After pouring a teaspoon of tea in the coffee, the concentration of tea in the coffee container is $\frac{1}{V+1}$. So, the amount of tea remaining in the coffee container (after taking a teaspoon of the mixture out) is $V\left(\frac{1}{V+1}\right)=\frac{V}{V+1}$. The amount of coffee in the teaspoon of mixture (which is poured into the tea container) is $1-\frac{1}{V+1}=\frac{V}{V+1}$. Thus, the amounts are equal. $\boxed{\text{Answer: c}}$
6.	Drawing x and y-axes and a square over the figure, as shown, helps illustrate the areas involved. The four regions that are x'ed are quarter-circles and are not included in the area, while the four semi-circles outside the square are included. Thus, the area of the region is the area of the 4×4 square (16), minus the four quarter-circle areas $(4\cdot\frac{\pi}{4}=\pi)$, plus the four semi-circle areas $(4\cdot\frac{\pi}{2}=2\pi)$. Hence, the area is $16-\pi+2\pi=16+\pi$. $\boxed{\text{Answer: } 16+\pi}$ 
7.	$\ln(\tan 1^\circ)+\ln(\tan 2^\circ)+\ln(\tan 3^\circ)+\ldots+\ln(\tan 89^\circ)=\ln\left(\frac{\sin 1^\circ}{\cos 1^\circ}\right)+\ln\left(\frac{\sin 2^\circ}{\cos 2^\circ}\right)+\ln\left(\frac{\sin 3^\circ}{\cos 3^\circ}\right)+\ldots+\ln\left(\frac{\sin 89^\circ}{\cos 89^\circ}\right)$ $=\ln(\sin 1^\circ)-\ln(\cos 1^\circ)+\ln(\sin 2^\circ)-\ln(\cos 2^\circ)+\ln(\sin 3^\circ)-\ln(\cos 3^\circ)+\ldots+\ln(\sin 89^\circ)-\ln(\cos 89^\circ)$ $=\left[\ln(\sin 1^\circ)+\ln(\sin 2^\circ)+\ln(\sin 3^\circ)+\ldots+\ln(\sin 89^\circ)\right]-\left[\ln(\cos 1^\circ)+\ln(\cos 2^\circ)+\ln(\cos 3^\circ)+\ldots+\ln(\cos 89^\circ)\right]$ $=\left[\ln(\sin 1^\circ)+\ln(\sin 2^\circ)+\ln(\sin 3^\circ)+\ldots+\ln(\sin 89^\circ)\right]-\left[\ln(\sin 89^\circ)+\ln(\sin 88^\circ)+\ln(\sin 87^\circ)+\ldots+\ln(\sin 1^\circ)\right]^*$ $=0$ *Since $\cos(x^\circ)=\sin(90^\circ-x^\circ)$ $\boxed{\text{Answer: a}}$

8.	Multiplying by the LCD $xy(x+y)$, provided $x \neq 0,\ y \neq 0$, and $x+y \neq 0$, gives: $y(x+y)-x(x+y)=xy \Rightarrow yx+y^2-x^2-xy=xy \Rightarrow y^2-xy-x^2=0$ Solving for y, using the quadratic formula gives: $y=\left(\frac{1\pm\sqrt{5}}{2}\right)x$, i.e. two lines through the origin (open at the origin). $\boxed{\text{Answer: a}}$
9.	Multiplying by the conjugate $\sec(x)+\tan(x)$ gives: $[\sec(x)-\tan(x)][\sec(x)+\tan(x)]=2[\sec(x)+\tan(x)]$ $\Rightarrow \sec^2(x)-\tan^2(x)=2[\sec(x)+\tan(x)]$, the identity $1+\tan^2(x)=\sec^2(x) \Rightarrow 1=\sec^2-\tan^2(x)$ $\Rightarrow 1=2[\sec(x)+\tan(x)] \Rightarrow \sec(x)+\tan(x)=\frac{1}{2}$ $\boxed{\text{Answer: c}}$
10.	Let m = the number of men, and w = the number of women in the town $\Rightarrow \frac{2}{3}m=\frac{3}{5}w \Rightarrow w=\frac{10}{9}m$, thus m must be a multiple of 9...the smallest integer greater than or equal to 100 that is a multiple of 9 is 108 $\Rightarrow w=\frac{10}{9}(108)=120$ $\boxed{\text{Answer: 120}}$
11.	For a matching color, the second die must match the first. The first die can show any color, red say. Then the probability the second die matches is $\frac{2}{6}=\frac{1}{3}$. $\boxed{\text{Answer: } \frac{1}{3}}$
12.	$\frac{1}{\log_2 2011!}+\frac{1}{\log_3 2011!}+\ldots+\frac{1}{\log_{2011} 2012!}=\frac{1}{\left(\frac{\log 2011!}{\log 2}\right)}+\frac{1}{\left(\frac{\log 2011!}{\log 3}\right)}+\ldots+\frac{1}{\left(\frac{\log 2011!}{\log 2011}\right)}$ $=\frac{\log 2}{\log 2011!}+\frac{\log 3}{\log 2011!}+\ldots+\frac{\log 2011}{\log 2011!}=\frac{\log 2+\log 3+\ldots+\log 2011}{\log 2011!}=\frac{\log(2\cdot 3\cdot\ldots\cdot 2011)}{\log 2011!}$ $=\frac{\log 2011!}{\log 2011!}=1$ $\boxed{\text{Answer: c}}$
13.	Let x be Larry's rowing speed, y the speed of the stream, and t the time (in minutes) he takes to row back to reach the hat. Thus, we get ❶ $(x+y)t=(x-y)\cdot 5+1$, the distance traveled back down stream to retrieve the hat equals the distance traveled from the turn-around point to the bridge plus 1 mile, and ❷ $y(5+t)=1$, the distance the hat traveled downstream. Expanding ❶ gives $xt+yt=5x-5y+1$, now substituting $1=5y+yt$ from ❷ $\Rightarrow xt+yt=5x-5y+5y+yt \Rightarrow xt=5x \Rightarrow t=5$ Substituting back into ❷ gives $10y=1 \Rightarrow y=\frac{1}{10}$ mile/minute, which is 6 miles/hour. $\boxed{\text{Answer: 6}}$
14.	Drawing a radius as a perpendicular bisector of the chord and the arc, forms a right triangle with a hypotenuse of 1 and a base of $\frac{1}{2}x$. The central angle, in radians, equals the subtended arc (since the radius of the circle is 1). Hence,  $\sin\left(\frac{L}{2}\right)=\frac{\left(\frac{x}{2}\right)}{1} \Rightarrow \sin\left(\frac{L}{2}\right)=\frac{x}{2} \Rightarrow \frac{L}{2}=\sin^{-1}\left(\frac{x}{2}\right) \Rightarrow 2\sin^{-1}\left(\frac{x}{2}\right)$ $\boxed{\text{Answer: d}}$
15.	Let m be the first numbered page that is missing, and n be the last numbered page that is missing. The sum of all the missing pages is then $(1+2+3+\cdots+(m-1)+m+\cdots+n)-(1+2+3+\cdots+(m-1))$ $=\frac{1}{2}n(n+1)-\frac{1}{2}(m-1)(m-1+1)=\frac{1}{2}\left[(n^2+n)-(m^2-m)\right]=\frac{1}{2}\left[(n^2-m^2)+(n+m)\right]$ $=\frac{1}{2}[(n+m)(n-m)+(n+m)]=\frac{1}{2}(n+m)[(n-m)+1]=\frac{1}{2}(n+m)(n-m+1)$, which must equal 355 $\Rightarrow (n+m)(n-m+1)=710$ The factorization of 710 is $1\cdot 2\cdot 5\cdot 71$. We want the maximum number of pages, so we seek to maximize $n-m$. Also, since $m\geq 1$, $n+m>n-m+1$. Thus, $n+m=71$ and $n-m+1=1\cdot 2\cdot 5$. Solving gives $m=31$ and $n=40$, which is 10 missing pages. $\boxed{\text{Answer: 10}}$

16.	Let m = the number of black socks (an even number), and n = the number of blue ones. The probability of getting two blue socks is $\frac{n}{n+m}\cdot\frac{n-1}{n+m-1}=\frac{1}{2}$. Expanding, cross-multiplying, and combining like terms (for n) gives: $n^2-(1+2m)n+m-m^2=0$. If $m=2$, then $n^2-5n-2=0$, which has no positive integer solution. If $m=4$, then $n^2-9n-12=0$, which also has no positive integer solution. If $m=6$, then $n^2-13n-30=0$, which factors as: $(n-15)(n+2)=0$. Thus, $n=15$. We know this is the smallest solution, since as m increases so does n. Answer: 15
17.	Draw two radii that are perpendicular to the legs of the triangle, as shown, then use similar triangles to obtain the desired relationship. Hence, $\frac{a}{b}=\frac{r}{b-r}\Rightarrow br=ab-ar$ $\Rightarrow ar+br=ab \Rightarrow r=\dfrac{ab}{a+b}$. Answer: $\dfrac{ab}{a+b}$ 
18.	If 100 pounds are 99% water, then 99 pounds are water with only 1 pound of flesh. After dehydration, some water evaporates but we still have 1 pound of flesh — which now represents 2% of what remains (since 98% is water). Letting x represent the number of pounds remaining, we get $\dfrac{1}{x}=0.02 \Rightarrow \dfrac{1}{x}=\dfrac{2}{100}=\dfrac{1}{50} \Rightarrow x=50$ Answer: 50
19.	In the first race Julie runs $50-5=45$ meters further than in the second race, and requires an additional $5+5=10$ seconds to do it. Hence, her speed is $\frac{45 \text{ meters}}{10 \text{ seconds}}=4.5$ meters/sec. Answer: a
20.	Tim's wife and mother are not blood relatives. So from III, if the mathematician is a female, the engineer is a male. But from I, if the engineer is a male, then the mathematician is a male. Thus, there is a contradiction, if the mathematician is a female. Hence, either Tim or his son is the mathematician. Tim's son is the youngest of all four and is a blood relative of each of them. So from II, Tim's son is not the mathematician. Hence, Tim is the mathematician. Answer: a <u>Note</u>: From II, Tim's mother cannot be the mathematician. So the engineer is either his wife or his son. It is not possible to determine anything further.

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