New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Spring 2012 ~ Solutions

1. $a \otimes b = ab - a = a(b-1)$, thus

I. $(a^2+1)\otimes(b^2-1) = (a^2+1)\left[(b^2-1)-1\right] = (a^2+1)(b^2-2)$, which cannot equal zero, since there are no integers that make either $a^2+1=0$ or $b^2-2=0$.

II. $(a^2+b^2)\otimes(a^2-b^2) = (a^2+b^2)(a^2-b^2-1)$ We know $a^2+b^2 \neq 0$ for integers a and b.

Can $a^2-b^2-1=0$? If it does, then $a^2=b^2+1$. However, there is no perfect square that is one more than another positive perfect square.

III. $(a^2-b^2)\otimes(a^2+b^2) = (a^2-b^2)(a^2+b^2-1)$ Which is zero (only) when $a=b$.

Answer: c

2. The product being a multiple of 10 and not zero, requires one of the digits be a 5, another digit be even, and no digit zero. Since the largest digit is the sum of the other three, 5 must be the largest or second largest digit (if not, then the sum of the three smallest digits would be 12 or more and could not equal the largest *digit*). But 5 cannot be the largest digit, since there is no way to get three of the digits 1, 2, 3, and 4 to sum to 5. Thus, 5 must be the second largest digit. Therefore, the two smallest digits may be 1 and 2, or 1 and 3 (any other choice would give a sum of the three smallest digits that yield a two-digit number). Hence, the digits may be: 1, 2, 5, and 8, or 1, 3, 5, and 9. But, there must be an even digit! Consequently, the only four digits satisfying all criteria are 1, 2, 5, and 8, which have 4! = 4·3·2·1=24 arrangements.

Answer: a

3. The average of the 999 numbers is 999,999 ÷ 999 = 1001. Since the integers are consecutive, we also know 1001 is the median. Hence, there are 499 integers above (and 499 below) 1001. The largest being 1001 + 499 = 1500.

Answer: 1500

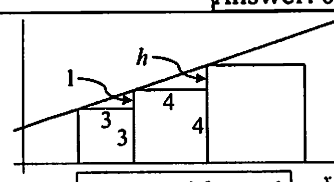
4. Suppose we start with 100 lbs of apricots, then 75 lbs is moisture and 25 lbs is flesh. When dried, 80% of the 75 lbs of moisture is lost, or 20% remains. Thus, when dried we have 15 lbs of moisture, plus the 25 lbs of flesh, for a total of 40 lbs. The moisture content is $15 \div 40 \times 100\%$.

Answer: 37.5%

5. If the lowest point of the hanging rope is 15 feet from the ground, then it is also 25 feet from the top of the poles (measured vertically). The only way to achieve this is to have the rope folded in half. Hence, the two poles must be touching!

Answer: 0

6. Since the slope of the line is constant, the slope obtained from the right-triangle above the 3×3 square, must equal the slope obtained from the right-triangle above the 4×4 square. Thus, $\frac{1}{3} = \frac{h}{4} \Rightarrow h = \frac{4}{3}$



Therefore, the larger square has sides of length $4 + \frac{4}{3} = \frac{16}{3}$.

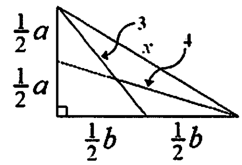
Answer: $\frac{16}{3}$ or $5\frac{1}{3}$

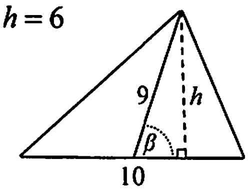
7. For x to be minimized, its denominator must be maximized. Hence, $a = 2$ and the fraction with b, c, and d must be maximized as well. Thus, its denominator must be minimized, giving $b=1$, $c=2$, and $d=1$.

Therefore, the minimum value is

$$x = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}}} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3}}} = \cfrac{1}{2+\cfrac{1}{\left(\frac{4}{3}\right)}} = \cfrac{1}{2+\frac{3}{4}} = \cfrac{1}{\left(\frac{11}{4}\right)} = \frac{4}{11}$$

Answer: a

8.	The flight from NYC to LA takes 6 hours and the flight from LA to NYC takes 5 hours. Using EST time as our time frame of reference, we get that the first plane lands in LA at 4 PM (EST). Thus, the second plane left LA for NYC at 2 PM (EST), takes 5 hours and lands in NYC at 7 PM. $\boxed{\text{Answer: d}}$
9.	Let a = the length of the base of the triangle, and b = the length of the height. Using the two smaller right triangles (one whose hypotenuse is 3, the is 4) with the Pythagorean theorem: $a^2+\left(\frac{1}{2}b\right)^2=3^2$ and $\left(\frac{1}{2}a\right)^2+b^2=4^2$.  $\Rightarrow\ a^2+\frac{1}{4}b^2=9$ and $\frac{1}{4}a^2+b^2=16\ \Rightarrow\ 4a^2+b^2=36$ and $a^2+4b^2=64$ Summing the last two equations gives: $5a^2+5b^2=100\ \Rightarrow\ a^2+b^2=20$, but $x^2=a^2+b^2$. Thus, $x^2=20$, so $x=\sqrt{20}$. $\boxed{\text{Answer: } \sqrt{20}=2\sqrt{5}}$
10.	We must use the fact that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. Thus, we get $P(A\cup B)=\frac{3}{4}+\frac{2}{3}-P(A\cap B)\ \Rightarrow\ P(A\cup B)=\frac{17}{12}-P(A\cap B)$ Since no probability can be greater than 1, we require $\frac{17}{12}-P(A\cap B)\le 1$. Giving the minimum value of $P(A\cap B)=\frac{5}{12}$. The maximum value for $P(A\cap B)$ is the minimum of $P(A)$ and $P(B)$, i.e. $A\cap B$ cannot occur more frequently than either A or B. Giving the maximum $\frac{2}{3}$. Hence, $\frac{5}{12}\le P(A\cap B)\le\frac{2}{3}$. $\boxed{\text{Answer: b}}$
11.	$\frac{x^2-2011y^2}{x^2+2012y^2}=\frac{1}{2}$, dividing the numerator and denominator of the left-hand-side by y^2 gives $\frac{\left(\frac{x}{y}\right)^2-2011}{\left(\frac{x}{y}\right)^2+2012}=\frac{1}{2}$, letting $z=\frac{x}{y}$: $\frac{z^2-2011}{z^2+2012}=\frac{1}{2}\ \Rightarrow\ 2z^2-4022=z^2+2012$ $\Rightarrow\ z^2=6034\ \Rightarrow\ z=\pm\sqrt{6034}$ Since $xy<0,\ z<0$. $\boxed{\text{Answer: } -\sqrt{6034}}$
12.	The addition of each new L-shaped region adds 4 (i.e. 10 – 3 – 3) to the new perimeter. Hence, $P_n=10+4(n-1)$. Therefore, $P_{2012}=10+4(2012-1)=10+8044=8054$. $\boxed{\text{Answer: b}}$
13.	$[\cos(x)+\sin(x)]^2=\left(\frac{1}{2}\right)^2\ \Rightarrow\ \cos^2(x)+2\cos(x)\sin(x)+\sin^2(x)=\frac{1}{4}$ $\Rightarrow\ 1+2\cos(x)\sin(x)=\frac{1}{4}\ \Rightarrow\ \cos(x)\sin(x)=-\frac{3}{8}$ $[\cos(x)+\sin(x)]^3=\left(\frac{1}{2}\right)^3\ \Rightarrow\ \cos^3(x)+3\cos^2(x)\sin(x)+3\cos(x)\sin^2(x)+\sin^3(x)=\frac{1}{8}$ $\Rightarrow\ \cos^3(x)+3\underbrace{\cos(x)\sin(x)}_{-\frac{3}{8}}\underbrace{[\cos(x)+\sin(x)]}_{\frac{1}{2}}+\sin^3(x)=\frac{1}{8}$ $\Rightarrow\ \cos^3(x)-\frac{9}{8}\cdot\frac{1}{2}+\sin^3(x)=\frac{1}{8}\ \Rightarrow\ \cos^3(x)+\sin^3(x)=\frac{1}{8}+\frac{9}{16}=\frac{11}{16}$ $\boxed{\text{Answer: d}}$
14.	$\frac{2011n+1}{2012}=\frac{2012n-n+1}{2012}=\frac{2012n}{2012}-\frac{n-1}{2012}=n-\frac{n-1}{2012}$ Thus, $\frac{n-1}{2012}$ must be an integer. The smallest integer, greater than 1, is for $n-1=2012$. Hence, $n=2013$. $\boxed{\text{Answer: 2013}}$

15.	The only lockers that will be left open are those that get open/closed an odd number of times. Each locker gets open/closed when counting by its factors. For example, locker #20 will be switched when every 1st, 2nd, 4th, 5th, 10th, and 20th locker gets switched. The lockers that get switched an odd number of times are only those whose numbers are perfect squares, since only the perfect squares have an odd number of factors. Thus, only locker numbers $1^2, 2^2, 3^2, \ldots, 11^2$, and 12^2. Answer: 12
16.	Let $x = 2012^{\ln(2)} \Rightarrow \ln(x) = \ln(2)\ln(2012) = \ln(2012)\ln(2) = \ln\left(2^{\ln(2012)}\right) \Rightarrow x = 2^{\ln(2012)}$ Answer: a
17.	Clearly any score between -20 and -1 can be obtained by answering questions incorrectly, with none correct and none blank, and a score of 0 by submitting a blank answer sheet (just one of five ways to get zero!). Now looking at the high end. A perfect score of 80 is obtained by correctly answering all 20 questions...19 correct with 1 blank gives a score of 76...19 correct with 1 incorrect gives a 75. There is no way to obtain scores 77, 78, and 79...18 correct with 2 blank gives a score of 72. There is no way to achieve scores of 73 or 74...18 correct with 1 wrong and 1 blank gives a 71...18 correct with 2 wrong gives a 70...17 correct with 3 blank gives a 68...There is no way to obtain a 69. All other scores, down to 1, are obtainable since for each correct answer (+4 points) one can get 1, 2, or 3 incorrect (-1, -2, or -3 points). Thus, any multiple of 4, from 4 through 68 can be achieved, and then lose 1, 2, or 3 points. Hence, the only unobtainable scores are the six scores: 69, 73, 74, 77, 78, and 79. Answer: 6
18.	Since the area of the triangle is 30, $\frac{1}{2} \cdot 10 \cdot h = 30 \Rightarrow h = 6$ Thus, $\sin\beta = \frac{6}{9} = \frac{2}{3}$.  Answer: $\frac{2}{3}$
19.	$\log(d+1) - \log(d) = \log\left(\frac{d+1}{d}\right)$, so the probability of selecting digit d is $P(d) = \log\left(\frac{d+1}{d}\right)$. $P(2) = \log\left(\frac{3}{2}\right)$, for choice (a) $P(3 \text{ or } 4) = \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) = \log\left(\frac{4}{3} \cdot \frac{5}{4}\right) = \log\left(\frac{5}{3}\right)$ For choice (b) $P(4 \text{ or } 5 \text{ or } 6 \text{ or } 7 \text{ or } 8) = \log\left(\frac{5}{4}\right) + \log\left(\frac{6}{5}\right) + \log\left(\frac{7}{6}\right) + \log\left(\frac{8}{7}\right) + \log\left(\frac{9}{8}\right) = \log\left(\frac{9}{4}\right)$ Thus, $P(4 \text{ or } 5 \text{ or } 6 \text{ or } 7 \text{ or } 8) = \log\left[\left(\frac{3}{2}\right)^2\right] = 2\log\left(\frac{3}{2}\right) \Rightarrow P(2) = \frac{1}{2}P(4 \text{ or } 5 \text{ or } 6 \text{ or } 7 \text{ or } 8)$ Answer: b
20.	Not taking leap years into account, each year shifts a date by one day forward, since 365 days is 1 day more than 52 weeks. That means a date will land on the same day of the week every 7 years. Einstein was born 133 years ago, which is a multiple of 7. Hence, without considering leap years, Einstein would have been born also on a Wednesday. Now count leap years. During the 133 years, there have been 33 leap years (remembering that 1900 was *not* a leap year). Now "shift" the Wednesday by 33 days backward. 33 ÷ 7 is 4 with a remainder of 5. Therefore, we only need to "shift" 5 days back from Wednesday, giving Friday. Answer: Friday

NYS MATYC 1967