NYS MATYC $\int_{1967}^{\infty}$

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2014 ~ Solutions

1. $\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}=\frac{15}{8}\sqrt{\pi}$ **Answer: C**

2. If we remove the "outer" layer of painted $1\times1\times1$ cubes, an unpainted $3\times3\times3$ cube remains. Hence, there will be $3\cdot3\cdot3=27$ paint-free cubes. **Answer: 27**

3. Let's look at each choice. For A, letting $x=\log_2(2014)$ gives $2^x=2014$, and taking the log base 2014 gives $x\log_{2014}(2)=1$ or $\log_{2014}(2)=\frac{1}{x}$. Thus, $\log_2(2014)\cdot\log_{2014}(2)=x\cdot\frac{1}{x}=1$. For B, we know the arcsin function yields results between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (i.e. in Quadrants I or IV), with the "2" in radians (and $2>\frac{\pi}{2}$). Hence, $\arcsin(\sin(2))=\pi-2\approx1.14159$, and choice B is *not* an integer. Choices C and D are seen to be integers as follows. $\left(2+\sqrt{2}\right)^{-2}+\left(2-\sqrt{2}\right)^{-2}=\frac{1}{6+4\sqrt{2}}+\frac{1}{6-4\sqrt{2}}$ $=\frac{6-4\sqrt{2}+6+4\sqrt{2}}{\left(6+4\sqrt{2}\right)\left(6-4\sqrt{2}\right)}=\frac{12}{36-16\cdot2}=\frac{12}{4}=3$ and $4^{\log_2(2014)}=\left(2^2\right)^{\log_2(2014)}=2^{2\log_2(2014)}$ which is $\left(2^{\log_2(2014)}\right)^2=2014^2$, clearly an integer. **Answer: B**

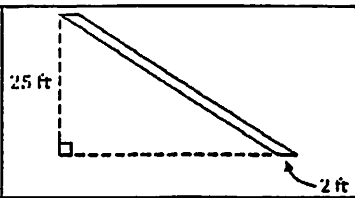
4. We can immediately put a "2" in the upper left corner, to obtain a sum of 15 in the first row. Then, the lower left corner must be $15-(2+x)=13-x$. Similarly, the 3rd row 2nd column entry must be $15-(9+y)=6-y$, the 2nd row 3rd column is $15-(x+y)=15-x-y$, and the lower right corner entry must be $15-(13-x+6-y)=x+y-4$. Summing the entries along the diagonals yields: $2+y+x+y-4=15$ or ① $x+2y=17$, and $13-x+y+4=15$ or ② $x-y=2$. Solving ① and ② for x gives $x=7$ (and $y=5$). **Answer: 7**

2	9	4
x	y	$15-x-y$
$13-x$	$6-y$	$x-y-4$

5. Since $\frac{1}{7}=0.\overline{142857}$, all we need to do is determine where in this 6-digit cycle the 2014th position falls. $2014\div6$ is 335 with a remainder of 4. Thus, the 2014th digit after the decimal point passes 335 full cycles, then lands on the 4th digit in the cycle – which is an 8. **Answer: 8**

6. If 25 is to have four different factors, they must be $-1,\ 1,-5,$ and 5. Since we are only concerned with the sum, the ordering is arbitrary; let $x-a=-1$, $x-b=1$, $x-c=-5$, and $x-d=5$. Now, with $x=-2$ we get $a=-1$, $b=-3$, $c=3$, and $d=-7$. Thus, $a+b+c+d=-8$. **Answer: −8**

7. Only choice B is a perfect square, since $99!\,100!=99!\cdot99!\cdot100=\left(99!\cdot10\right)^2$. **Answer: B**

8. 

The stripe, when unwound, is a parallelogram with a base of 2 ft and height of 25 ft, as shown. Thus, its area is base $\times$ height = 50 square feet.

Answer: A

9. In order for the fly to wind up on a shaded circle, it must first land on an unshaded circle. Thus, the probability is 2 out of 7, or $\frac{2}{7}$.

Answer: A

10. Let x and y be the two numbers. Thus, ① $x+y=3$ and ② $xy=-1$. Squaring equation ① gives $x^2+2xy+y^2=9$, then substituting ② gives $x^2+2(-1)+y^2=9$ or ③ $x^2+y^2=11$. Using equations ① and ③ gives $\left(x^2+y^2\right)(x+y)=11\cdot 3 \Rightarrow$ ④ $x^3+x^2y+xy^2+y^3=33$. Factoring the two middle terms of ④ gives $x^3+xy(x+y)+y^3=33$, but $xy=-1$ and $x+y=3$, which yields $x^3+(-1)(3)+y^3=33$ or $x^3+y^3=36$.

Answer: 36

11. Recalling the difference of two squares factors as: $x^2-y^2=(x+y)(x-y)$, then letting $x=444{,}444{,}444{,}445$ and $y=444{,}444{,}444{,}444$, gives x^2-y^2
$=(444{,}444{,}444{,}445+444{,}444{,}444{,}444)(444{,}444{,}444{,}445-444{,}444{,}444{,}444)$
$=(888{,}888{,}888{,}889)(1)$

Answer: B

12. A triangle must have the sum of the lengths of the two smaller sides (i.e. the legs) greater than the length of its longest side (i.e. the hypotenuse). Thus, the sum of the two legs must be greater than $15\div 2$ and the hypotenuse must be less than $15\div 2$. Listing all groups of three positive integers that satisfy the requirements gives: {1, 7, 7}, {2, 6, 7}, {3, 5, 7}, {3, 6, 6}, {4, 4, 7}, {4, 5, 6}, and {5, 5, 5}.

Answer: 7

13. Squaring both sides of $\cos(x)+\sin(x)=\cos(x)\sin(x)$, gives $\cos^2(x)+2\cos(x)\sin(x)+\sin^2(x)=\cos^2(x)\sin^2(x)$, or $\cos^2(x)+\sin^2(x)+2\cos(x)\sin(x)=\cos^2(x)\sin^2(x)$. Now use $\cos^2(x)+\sin^2(x)=1$ to give $1+2\cos(x)\sin(x)=\cos^2(x)\sin^2(x)$. Letting $y=\cos(x)\sin(x)$ simplifies the last result to $1+2y=y^2$ or $y^2-2y-1=0$. The quadratic formula gives $y=1\pm\sqrt{2}$, but the product of $\cos(x)$ and $\sin(x)$ cannot exceed 1, thus $y=1-\sqrt{2}$.

Answer: B

14. Since the graph of the cosine curve is simply the graph of the sine curve that has been horizontally shifted by $\frac{\pi}{2}$ to the left (i.e. $\cos(y)=\sin(y+\frac{\pi}{2})$) and $\sin(x)=\sin(x+2\pi k)$, we get $\sin(y+\frac{\pi}{2})=\sin(x+2\pi k)$, where k is any integer. This gives $y+\frac{\pi}{2}=x+2\pi k$ or $y=x+2\pi k-\frac{\pi}{2}$, i.e. a family of parallel lines with slopes of 1. We also know that $\cos(y)=\sin(\frac{\pi}{2}-y)$, and again using $\sin(x)=\sin(x+2\pi k)$, gives $\sin(\frac{\pi}{2}-y)=\sin(x+2\pi k)$. Thus, $\frac{\pi}{2}-y=x+2\pi k$ or $y=-x-2\pi k+\frac{\pi}{2}$, which is a family of parallel lines with slopes of -1.

Answer: A

15. The final value for each choice is:
A) $\$1000(1.03)^6(1.02)^4$, B) $\$1000(1.02)^2(1.03)^6(1.02)^2=\$1000(1.03)^6(1.02)^4$, C) $\$1000(1.02)^4(1.03)^6$.
Thus, the results are the same in any case.

Answer: D

16. Let the number be $x=10a+b$, thus $\hat{x}=10b+a$, where a and b are digits (i.e. from 0 through 9) with $a\neq 0$ and $a\geq b$ (so that $x-\hat{x}\geq 0$). This gives ① $x-\hat{x}=9(a-b)$ and ② $x+\hat{x}=11(a+b)$. In order for ① to be a perfect square, $a-b$ must be either 0, 1, 4, or 9. In order for ② to be a perfect square, $a+b$ must be 11. Thus, the only digits whose sum is 11 and difference is either 0, 1, 4, or 9 are 5 and 6 (whose difference is 1). Thus, $a=6$ and $b=5$. $\boxed{\text{Answer: 65}}$

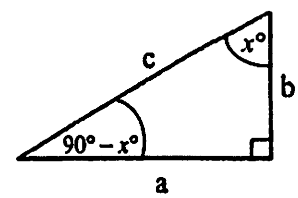
17. $\tan(1°)\tan(2°)\tan(3°)\cdots\tan(87°)\tan(88°)\tan(89°)=\frac{\sin(1°)}{\cos(1°)}\frac{\sin(2°)}{\cos(2°)}\frac{\sin(3°)}{\cos(3°)}\cdots\frac{\sin(87°)}{\cos(87°)}\frac{\sin(88°)}{\cos(88°)}\frac{\sin(89°)}{\cos(89°)}$

$=\frac{\sin(1°)}{\cos(1°)}\frac{\sin(2°)}{\cos(2°)}\frac{\sin(3°)}{\cos(3°)}\cdots\frac{\sin(44°)}{\cos(44°)}\frac{\sin(45°)}{\cos(45°)}\frac{\sin(46°)}{\cos(46°)}\cdots\frac{\sin(87°)}{\cos(87°)}\frac{\sin(88°)}{\cos(88°)}\frac{\sin(89°)}{\cos(89°)}$,

with $\cos(x°)=\sin(90°-x°)$ the product becomes

$\frac{\sin(1°)}{\sin(89°)}\frac{\sin(2°)}{\sin(88°)}\frac{\sin(3°)}{\sin(87°)}\cdots\frac{\sin(44°)}{\sin(46°)}\frac{\sin(45°)}{\sin(45°)}\frac{\sin(46°)}{\sin(44°)}\cdots\frac{\sin(87°)}{\sin(3°)}\frac{\sin(88°)}{\sin(2°)}\frac{\sin(89°)}{\sin(1°)}=1.$ $\boxed{\text{Answer: B}}$

<u>Alternate Solution</u>: Referring to the right triangle shown, we know that $\tan(x°)=\frac{a}{b}$ and $\tan(90°-x°)=\frac{b}{a}$, which gives $\tan(x°)\cdot\tan(90°-x°)=1$. Hence,



$\tan(1°)\tan(2°)\cdots\tan(45°)\cdots\tan(88°)\tan(89°)$

$=\tan(1°)\tan(89°)\tan(2°)\tan(88°)\cdots\tan(44°)\tan(46°)\tan(45°)$

$=1\cdot 1\cdot\ldots\cdot 1\cdot\tan(45°)=\tan(45°)=1.$

18. A parabola of the form $y=ax^2+bx+c$ with vertex at (h,k) can be written as $y=a(x-h)^2+k=ax^2-2ahx+ah^2+k$. Thus, $b=-2ah$ and $c=ah^2+k$. The reflected parabola, $y=dx^2+ex+f$, can be written as $y=-a(x-h)^2+k=-ax^2+2ahx-ah^2+k$. Thus, $d=-a$, $e=2ah$, and $f=-ah^2+k$, giving $a+b+c+d+e+f=a+(-2ah)+(ah^2+k)+(-a)+(2ah)+(-ah^2+k)=2k$. $\boxed{\text{Answer: D}}$

19. Let $n=$ the total number of socks in the drawer, and $b=$ the number of blue socks. Thus, the probability of selecting two blue socks is: $P=\frac{b}{n}\cdot\frac{b-1}{n-1}$, which is $\frac{2}{5}$. Hence, $\frac{b^2-b}{n^2-n}=\frac{2}{5}$ $\Rightarrow 5b^2-5b=2n^2-2n \Rightarrow 2n^2-2n+5b-5b^2=0$. Using the quadratic formula to solve for n gives: $n=\frac{2\pm\sqrt{4-4\cdot 2\cdot(5b-5b^2)}}{4}=\frac{2\pm 2\sqrt{1-(10b-10b^2)}}{4}=\frac{1\pm\sqrt{10b^2-10b+1}}{2}$. Now we need to determine the smallest value for $b\geq 2$ that yields a positive integer for n. The first such value we encounter is $b=4$, which gives $n=\frac{1\pm\sqrt{121}}{2}$, taking the positive result we get $n=6$. $\boxed{\text{Answer: 6}}$

(The next smallest value is $b=16$, giving $n=25$, and $b=133$ with $n=210$ after that.)

20. There are only two problems being referenced: *this one* (#20) and the one *before this one* (#19). The entire question can be rephrased like this:
If the problem you solved before this one (#19) was harder than this one (#20), was the problem you solved before this one (#19) harder than this one (#20)? The answer is obviously "yes." $\boxed{\text{Answer: B}}$