NYS $\int_{1967}^{\infty}$ MATYC

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2010 ~ Solutions

1. $f(i,2010)=\sum_{k=0}^{2010} i^k = i^0+i^1+i^2+i^3+i^4+\ldots+i^{2007}+i^{2008}+i^{2008}+i^{2009}+i^{2010}$

$$=\underbrace{(1+i-1-i)}_{k=0,1,2,3}+\underbrace{(1+i-1-i)}_{k=4,5,6,7}+\ldots+\underbrace{(1+i-1-i)}_{\substack{k=2004,2005,\\2006,2007}}+1+i-1=\underbrace{0+0+\ldots+0}_{502\text{ zeros}}+1+i-1=i$$

Or use the formula for a geometric series, $\sum_{k=0}^{n} r^k=\frac{r^{n+1}-1}{r-1}$ with $r=i$ and $n=2010$. **Answer: i**

2. If we let $y=\sqrt[4]{2010x+1}$, then $\sqrt{2010x+1}=y^2$. Now the equation takes on a simpler form: $y^2-y=2$. Solving for y: $y^2-y-2=0\Rightarrow(y+1)(y-2)=0\Rightarrow y=-1$ or $y=2$. However, y must be positive. Thus, $y=2\Rightarrow\sqrt{2010x+1}=2^2\Rightarrow 2010x+1=4^2\Rightarrow 2010x=15$.

Therefore, $x=\frac{15}{2010}=\frac{1}{134}$. **Answer: $\frac{15}{2010}$ or $\frac{1}{134}$**

3. Let $x=\sqrt{7+\sqrt{48}}+\sqrt{7-\sqrt{48}}$, which is certainly greater than zero. Squaring gives:

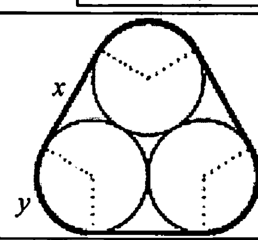
$$x^2=\left(\sqrt{7+\sqrt{48}}+\sqrt{7-\sqrt{48}}\right)^2\Rightarrow x^2=\left(\sqrt{7+\sqrt{48}}\right)^2+2\left(\sqrt{7+\sqrt{48}}\right)\left(\sqrt{7-\sqrt{48}}\right)+\left(\sqrt{7-\sqrt{48}}\right)^2$$

$$=7+\sqrt{48}+2\sqrt{49-48}+7-\sqrt{48}=14+2\sqrt{1}=16$$ Taking the positive root, $x=\sqrt{16}$. **Answer: c**

4. Rich would have 4 pennies, 4 dimes, 1 quarter and 1 half-dollar. **Answer: \$1.19**

5. Let x be the length of the straight segment between two circles, and let y be the length of the arc from one straight segment to the other.
$x=2\cdot\text{radius}=2\cdot1=2$ ft, and $y=(1/3)\cdot\text{circumference}=(1/3)\cdot2\pi=(2/3)\pi$ ft
Thus, the total length $=3x+3y=3\cdot2+3\cdot(2/3)\pi=6+2\pi$ ft.

Answer: $6+2\pi$ ft

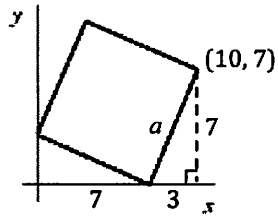


6. $x^4-y^4=x^2-y^2\Rightarrow\left(x^4-y^4\right)-\left(x^2-y^2\right)=0\Rightarrow\left(x^2-y^2\right)\left(x^2+y^2\right)-\left(x^2-y^2\right)=0$

$\Rightarrow\left(x^2-y^2\right)\left[\left(x^2+y^2\right)-1\right]=0\Rightarrow x^2-y^2=0$ or $x^2+y^2-1=0$

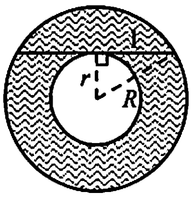
$\Rightarrow y^2=x^2\Rightarrow y=\pm x$ or $x^2+y^2=1$, which gives two lines through the origin with slopes of -1 and 1, and a circle of radius 1 centered at the origin. **Answer: d**

7. $a^2-b^4=345\Rightarrow\left(a-b^2\right)\left(a+b^2\right)=3\cdot5\cdot23$, grouping the three factors of 345 in pairs give $(3\cdot5)\cdot23$, $3\cdot(5\cdot23)$, and $5\cdot(3\cdot23)\Rightarrow15\cdot23$, $3\cdot115$, and $5\cdot69$. Thus, either
❶ $a-b^2=15$ and $a+b^2=23$, ❷ $a-b^2=3$ and $a+b^2=115$, or ❸ $a-b^2=5$ and $a+b^2=69$
Only equations ❶ yield an integer for both a and b, with $a=19$ and $b=2$. **Answer: $a+b=21$**

8. 

Letting a be the length of one side of the square, we get:
$a^2 = 3^2 + 7^2 \Rightarrow a^2 = 9 + 49 = 58 =$ Area of the Square

$\boxed{\text{Answer: 58}}$

9. 

The area of the shaded region is: $\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right)$.
From the diagram, we get the relation: $R^2 = r^2 + 1 \Rightarrow R^2 - r^2 = 1$.
Thus, the area of the shaded region is $\pi \cdot 1 = \pi$.

$\boxed{\text{Answer: a}}$

10. Three tickets can be selected in 3! ways, or 6 ways. Only 1 of the 6 permutations would have the tickets in increasing order.

$\boxed{\text{Answer: } \tfrac{1}{6}}$

11. Since the maximum value of $\sin(x)$ is 1, the line will not intersect the sine curve for $x > 100\pi$. Thus, from $x = 0$ to $x = 100\pi$, there will be 2 points of intersection per period. The period of $y = \sin(x)$ is 2π, which means there are 50 periods from $x = 0$ to $x = 100\pi$. Hence, there will be 50·2=100 points of intersection from $x = 0$ to $x = 100\pi$. Similarly from $x = -100\pi$ to $x = 0$, but that would count the origin again. Therefore, there are $100 + 100 - 1$ points of intersection.

$\boxed{\text{Answer: 199}}$

12. If there is only one such number between 2 and 2^{100}, then it must be the smallest. Hence, it must be 2^n, where n is the least common multiple of 2, 3, 4, 5 and 6 (so the 2nd, 3rd, 4th, 5th and 6th roots are whole numbers). Thus, $n = 3 \cdot 4 \cdot 5 = 60$ and the number is 2^{60}.
<u>Note</u>: The next such number is $2^{2\cdot 60} = 2^{120} > 2^{100}$.

$\boxed{\text{Answer: } 2^{60}}$

13. $$\log_{2011}(2010) - \frac{1}{\log_{2010}(2011)} = \frac{\log(2010)}{\log(2011)} - \frac{1}{\left(\frac{\log(2011)}{\log(2010)}\right)} = \frac{\log(2010)}{\log(2011)} - \frac{\log(2010)}{\log(2011)} = 0$$

$\boxed{\text{Answer: b}}$

14. Ray works at the rate of $\frac{1}{9}$ of a job/hr. Let r be the rate at which Tim works. Together, they work at $\left(\frac{1}{9} + r\right)$ job/hr, and since Work = Rate · Time, after 4 hours they complete $4\left(\frac{1}{9} + r\right)$ of the job. However, it takes Ray 2 more hours to finish. Thus, there was only $2\left(\frac{1}{9}\right) = \frac{2}{9}$ of the job remaining. Therefore, together they must have completed $\frac{7}{9}$ of the job. From this we get: $4\left(\frac{1}{9} + r\right) = \frac{7}{9}$. Solving for r, gives $r = \frac{1}{12}$ job/hr. Hence, working alone, Tim would need 12 hours.

$\boxed{\text{Answer: 12 hours}}$

15. Let $p =$ the probability of obtaining *heads up* on one toss of the coin.
Then $1 - p =$ the probability of obtaining *tails up* on one toss of the coin.
The probability of getting two heads on two tosses is p^2, which is equal to $1 - p$.

Thus, $p^2 = 1 - p \Rightarrow p^2 + p - 1 = 0 \Rightarrow p = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2}$.

However, only $\dfrac{-1 + \sqrt{5}}{2}$ is positive (and between 0 and 1).

$\boxed{\text{Answer: } \dfrac{\sqrt{5} - 1}{2}}$

16.	I must pick up 7 shirts to hold me over until the following Monday. Hence, I must drop off 7 shirts each Monday. Counting the shirt I wear on Monday, the required total is 7+7+1=15. <u>Note</u>: I cannot get by with only 14 shirts, as I would not have a clean shirt to wear the following Monday. $\boxed{\text{Answer: c}}$
17.	When Sophia completed half the race, Ida was 3.5 km ahead. Thus, by the time Sophia completed the entire race, Ida would have been 7 km ahead (if she continued at the same rate). Hence, Sophia completed the race at 11:52 AM plus the time it would have taken Ida to go an additional 7 km. At 42 km/hr , it would have taken Ida $(7 \div 42)$ hour = ⅙ hour = 10 minutes. Therefore, Sophia completed the race at 12:02 PM. $\boxed{\text{Answer: 12:02 PM}}$
18.	Since each match eliminates one competitor, and 2009 competitors must be eliminated so that only one person (the winner) remains, 2009 matches must be played. $\boxed{\text{Answer: b}}$
19.	Let w represent the hourly wage. Then the tax rate is $(2w)$% and the tax on w will be $\frac{2w}{100} \cdot w = \frac{w^2}{50}$. Thus, the after taxes income (per hour) is $w - \frac{w^2}{50}$. This quadratic is maximized along the axis of symmetry, i.e. for $w = \frac{-b}{2a} = \frac{(-1)}{2\left(-\frac{1}{50}\right)} = 25$. $\boxed{\text{Answer: \$25 per hour}}$
20.	Look at each clue, knowing *exactly* one of each person's statements is true. Artie's: If *it was Barbara* is true, then we know the other statement is false, therefore it was Edward. This is a contradiction. Hence we now know it wasn't Barbara, nor Edward (as *it wasn't Edward* must be the true statement). Looking at Carmine's statements, we can similarly determine that it wasn't Artie. Since we know it wasn't Barbara, Darci's statements tell us *it was Carmine* is true. This also checks against the other clues. $\boxed{\text{Answer: Carmine}}$

NYSMATYC 1967