New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2019

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. The *floor function*, denoted $\lfloor x \rfloor$, is defined as *the greatest integer less than or equal to x*; while the *ceiling function*, denoted $\lceil x \rceil$, is defined as *the smallest integer greater than or equal to x*. For example, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.8 \rfloor = -3$, and $\lfloor -2 \rfloor = -2$; with $\lceil 2.1 \rceil = 3$, $\lceil -2.1 \rceil = -2$, and $\lceil -2 \rceil = -2$. What real number, if any, satisfies the equation: $x\lfloor x \rfloor \lceil x \rceil = 99$?

2. In eight years, I will be twice as old as my daughter. Eight years ago, I was three times as old as my daughter. What is the sum of our ages today?

3. For $\cos(x) \neq 0$ and $\sin(x) \neq 0$, the expression $\frac{\sqrt{1-\sin^2(x)}}{\cos(x)} + \frac{\sqrt{1-\cos^2(x)}}{\sin(x)}$ could equal which of the following values?
   A) -2 only B) 0 only C) 2 only D) -2 or 2 only E) -2, 0, or 2

4. For positive integers *m* and *n*, $\frac{1}{m} + \frac{1}{n} = \frac{2}{2019}$. What is the smallest possible value for *m*?

5. Some integer, *n*, satisfies the equation $3!\,5!\,7! = n!$. What is *n*?

6. Two sides of a triangle have length 4 and 5. What should be the length of the third side in order to maximize the area of the triangle?

7. I have 10 socks in a drawer, some black and some white. If 2 socks are selected at random, the probability that they are both black is $\frac{1}{3}$. What is the probability the 2 socks selected are *not* a matching pair? Assume each sock has the same probability of being selected.

A) $\frac{2}{9}$ B) $\frac{4}{15}$ C) $\frac{1}{3}$ D) $\frac{4}{9}$ E) $\frac{8}{15}$

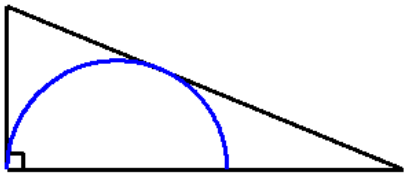
8. Which of the following is/are *not* a whole number? Note: $n! \equiv n(n-1)(n-2)\cdots 2\cdot 1$, $googol \equiv 10^{100}$ and $googolplex \equiv 10^{googol}$.

I. $\frac{100!}{2^{98}}$ II. $\frac{\log_{10}\left(googol^{\,googol}\right)}{\log_{100}(googolplex)}$ III. $\frac{\sqrt{1000^{\,googol}}}{googolplex}$

A) I only B) II only C) III only D) I and II only E) II and III only

9. Place the following numbers in numerical order, from smallest to largest (left to right): $w = \frac{1}{2}\cdot googol$, $x = \sqrt[100]{googolplex}$, $y = \ln(googolplex)$, $z = googol \cdot \cos\left(googol^{\circ}\right)$. Thus, if you believe $w < x < y < z$, for example, then your answer should be: *w*, *x*, *y*, *z*. See the previous problem for the definitions of a *googol* and a *googolplex*.

10. A semicircle is inscribed in a right triangle, with legs of length 2 and 5, so it is tangent to both the hypotenuse and the vertical leg of the triangle, as shown. What is the radius of the semicircle?



11. If $f(x)$ is a linear function, then which of the following *must* be true?

I. $f^{-1}(x)$ is also a linear function.

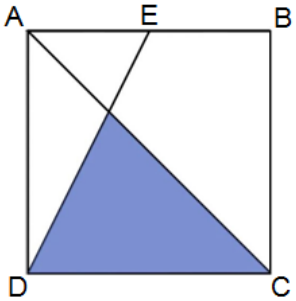
II. $f(ax+b)$ is also a linear function for all real numbers *a* and *b*.

III. $f(\cos(x))$ *may* also be a linear function.

A) I only B) II only C) I and II only D) II and III only E) I, II, and III

12. The figure shows a square with an area of 1. Segment AC is a diagonal, and DE extends from a corner of the square to the midpoint of AB. What is the area of the shaded region?

A) $\frac{3}{10}$ B) $\frac{5}{16}$ C) $\frac{1}{3}$ D) $\frac{3}{8}$ E) $\frac{2}{5}$



13. Three sides of a triangle have lengths $\ln(3)$, $\ln(4)$, and $\ln(x)$. How many different integer values are possible for *x*?

14. What is the product of all solutions of the equation $\log_x(4) = \log_4(4x)$?

A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) 1

15. If a triangle has interior angles A, B, and C such that $\sin(A):\sin(B):\sin(C) = 4:5:6$, then $\cos(A):\cos(B):\cos(C) = ?$

A) $4:5:6$ B) $6:5:4$ C) $12:9:2$ D) $15:12:10$ E) $\sqrt{84}:\sqrt{75}:8$

16. If $\arccos(x) = \arctan(x)$, then x is

A) $\frac{\pi}{5}$ B) $\sqrt{\sqrt{5}-\sqrt{3}}$ C) $\sqrt{3}-1$ D) $\sqrt{\frac{\sqrt{5}-1}{2}}$ E) $\frac{1}{2}\sqrt{1+\sqrt{3}}$

17. The graph of $13x^2 + 4xy + y^2 - 6x + 1 = 0$ consists of a single point in the xy-plane. What are the coordinates of that point?

18. The equation $(x-1)(x-3)(x-5)(x-7) = 20$ has two distinct real solutions (and two complex solutions). What is the larger of the two real solutions?

19. What positive values of k make the equation $\sin(k\pi x) = \frac{1}{2}$ have exactly four solutions on $x \in [0,4]$? Express your answer in interval form.

20. A certain family has several children, some girls and some boys. Each girl has just as many sisters as brothers. Each boy has twice as many sisters as brothers. How many children are in this family?

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