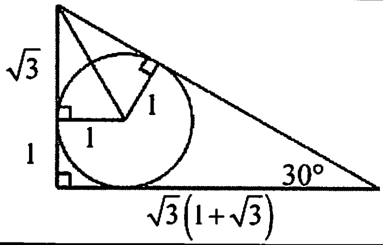
New York State Mathematics Association of Two-Year Colleges

NYSMATYC 1967

Math League Contest ~ Spring 2011 ~ Solutions

1.	Letting $x=1$ gives: $f(1)-2f(2011)=1$, and letting $x=2011$ gives: $f(2011)-2f(1)=2011$. Solving these two linear equations for $f(2011)$, gives $f(2011)=-671$. $\boxed{\text{Answer: } -671}$
2.	$\left(x^2+7x+11\right)^{x^2+3x-10}=1$ will be true only when $x^2+7x+11=1$ or $x^2+3x-10=0$ with $x^2+7x+11\neq 0$. $x^2+7x+11=1 \Rightarrow x^2+7x+10=0 \Rightarrow (x+5)(x+2)=0 \Rightarrow x=-5, x=-2$. $x^2+3x-10=0 \Rightarrow (x+5)(x-2)=0 \Rightarrow x=-5, x=2$. Thus, $x\in\{-5,-2,2\}$. $\boxed{\text{Answer: c}}$
3.	5 will divide every multiple of 5 once, every multiple of $5^2=25$ twice, every multiple of $5^3=125$ three times. $200\div 5=40$, $200\div 25=8$, and 125 divides 200 only once. Thus, there are $40+8+1=49$ fives that will factor out of 200!. $\boxed{\text{Answer: b}}$
4.	Let x represent the number of people in the group, and C be the cost for the rental. Thus, we get the equations: $\frac{C}{x+1}=\frac{C}{x}-5$ and $\frac{C}{x+3}=\frac{C}{x}-12$. Solving gives $x=7$ (and $C=280$). $\boxed{\text{Answer: } 7}$
5.	Noticing the symmetry in the coefficients of the polynomial $ax^4-7x^3+8x^2-7x+a$, divide the equation $ax^4-7x^3+8x^2-7x+a=0$ by x^4 to obtain $a-7x^{-1}+8x^{-2}-7x^{-3}+ax^{-4}=0$, for non-zero x. This new equation can be rewritten as $a-7\left(x^{-1}\right)^1+8\left(x^{-1}\right)^2-7\left(x^{-1}\right)^3+a\left(x^{-1}\right)^4=0$ or $a\left(x^{-1}\right)^4-7\left(x^{-1}\right)^3+8\left(x^{-1}\right)^2-7\left(x^{-1}\right)^1+a=0$. Hence, if x solves the original equation, then so will x^{-1}. Thus, $x=2011^{-1}$ must be the other real root. $\boxed{\text{Answer: } 2011^{-1} \text{ or } \frac{1}{2011}}$
6.	$\sin(t)=\cos\left(\frac{\pi}{2}-t\right)\Rightarrow \cos^{-1}(\sin(t))=\frac{\pi}{2}-t\Rightarrow \cos^{-1}(\sin(t))+t=\frac{\pi}{2}$, now let $x=\sin(t)$, so that $t=\sin^{-1}(x)\Rightarrow \cos^{-1}(x)+\sin^{-1}(x)=\frac{\pi}{2}\Rightarrow f(x)=\frac{1}{\left(\frac{\pi}{2}\right)}=\frac{2}{\pi}$, which is a horizontal line. $\boxed{\text{Answer: d}}$
7.	Let $x=\sin(15^\circ)+\sin(75^\circ)$, but $\sin(15^\circ)=\cos(75^\circ)$. Thus, $x=\cos(75^\circ)+\sin(75^\circ)$. Squaring: $x^2=\cos^2(75^\circ)+2\cos(75^\circ)\sin(75^\circ)+\sin^2(75^\circ)=\underbrace{\cos^2(75^\circ)+\sin^2(75^\circ)}_{1}+\underbrace{2\cos(75^\circ)\sin(75^\circ)}_{\sin(2\cdot 75^\circ)\text{, double angle formula}}$ $x^2=1+\sin(150^\circ)=1+\sin(30^\circ)=1+\frac{1}{2} \Rightarrow x^2=\frac{3}{2} \Rightarrow x=\sqrt{\frac{3}{2}}$, taking the positive square root, since $x>0$. $x=\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{2}{2}}=\frac{\sqrt{6}}{2}$. $\boxed{\text{Answer: b}}$
8.	A regular polygon with n sides has n vertices. Each of the n vertices can form a diagonal with all vertices except itself and the two adjacent ones. Thus, there are $n(n-3)$ diagonals that can be formed. However, that counts each diagonal twice (since it counts the one from vertex A to vertex B and from vertex B to vertex A). Hence, there are $\frac{1}{2}n(n-3)$ diagonals. Solving $\frac{1}{2}n(n-3)=54$, gives $n^2-3n-108=0 \Rightarrow (n+9)(n-12)=0 \Rightarrow n=-9,\ 12$. Taking the positive answer... $\boxed{\text{Answer: } 12}$

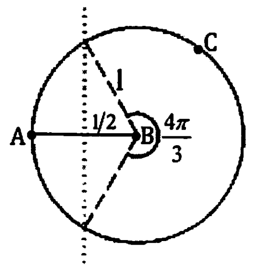
9.



The two newly formed (smaller) triangles are both 30°-60°-90° (right) triangles. Which makes the height of the large triangle $1+\sqrt{3}$, thus making its base $\sqrt{3}\left(1+\sqrt{3}\right)$. Hence, the area is $\frac{1}{2}\cdot\sqrt{3}\left(1+\sqrt{3}\right)\cdot\left(1+\sqrt{3}\right)=3+2\sqrt{3}$.

$\boxed{\text{Answer: } 3+2\sqrt{3}}$

10.



Point C can be anywhere on the circumference of the circle of radius 1 centered at Point B. The vertical dotted-line is the perpendicular bisector of segment AB. Thus, if Point C is on any part of the circle to the left of the vertical line, it will be closer to Point A. The angle that subtends the arc closer to Point A is 120° or $\frac{2\pi}{3}$ radians, which has a length of $\frac{2\pi}{3}\cdot 1=\frac{2\pi}{3}$. The circumference of the circle is 2π. Hence, the probability Point C is closer to Point A is $\left(\frac{2\pi}{3}\right)\div\left(2\pi\right)=\frac{1}{3}$.

$\boxed{\text{Answer: } \frac{1}{3}}$

11. Suppose (a,b) is a point on the line with a and b integers. Thus, $b=\pi a-\frac{22}{7}\Rightarrow b+\frac{22}{7}=\pi a$, and $\pi a=\frac{7b+22}{7}\Rightarrow \pi=\frac{7b+22}{7a}$, $a\neq 0$. Which means that π can be expressed as the ratio of two integers, which is impossible, since π is irrational! If $a=0$, then $b=-\frac{22}{7}$, which is not an integer. Hence, there are no points on the line with integer coordinates.

$\boxed{\text{Answer: a}}$

12. Let $\sqrt{9+4\sqrt{5}}=a+\sqrt{b}\Rightarrow\left(\sqrt{9+4\sqrt{5}}\right)^2=\left(a+\sqrt{b}\right)^2\Rightarrow 9+4\sqrt{5}=a^2+b+2a\sqrt{b}$, which gives $a^2+b=9$ and $2a\sqrt{b}=4\sqrt{5}\Rightarrow b=5$ and $a=2\Rightarrow\sqrt{9+4\sqrt{5}}=2+\sqrt{5}$

$\boxed{\text{Answer: b}}$

13.

$$\log_2\left[\log_4(x)\right]=\log_8\left[\log_2(x)\right]\Rightarrow\log_2\left[\log_4(x)\right]=\frac{\log_2\left[\log_2(x)\right]}{\log_2 8}\Rightarrow\log_2\left[\log_4(x)\right]=\frac{\log_2\left[\log_2(x)\right]}{3}$$

$$\Rightarrow 3\log_2\left[\log_4(x)\right]=\log_2\left[\log_2(x)\right]\Rightarrow\log_2\left[\left(\log_4(x)\right)^3\right]=\log_2\left[\log_2(x)\right]\Rightarrow\left(\log_4(x)\right)^3=\log_2(x)$$

$$\Rightarrow\left(\frac{\log_2(x)}{\log_2(4)}\right)^3=\log_2(x)\Rightarrow\left(\frac{\log_2(x)}{2}\right)^3=\log_2(x)\Rightarrow\frac{\left(\log_2(x)\right)^3}{8}=\log_2(x)\Rightarrow\left(\log_2(x)\right)^3=8\log_2(x)$$

$$\Rightarrow\left(\log_2(x)\right)^3-8\log_2(x)=0\Rightarrow\log_2(x)\left[\left(\log_2(x)\right)^2-8\right]=0\Rightarrow\log_2(x)=0\ \text{ or }\ \left(\log_2(x)\right)^2-8=0$$

Since $x>1$, $\left(\log_2(x)\right)^2-8=0\Rightarrow\log_2(x)=\sqrt{8}\Rightarrow x=2^{\sqrt{8}}=2^{2\sqrt{2}}=4^{\sqrt{2}}$

$\boxed{\text{Answer: } 2^{\sqrt{8}} \text{ or } 2^{2\sqrt{2}} \text{ or } 4^{\sqrt{2}}}$

14. Let a = the rate at which water flows from Valve A, b = the rate at which water flows from Valve B, and c = the rate at which water flows from Valve C. Hence, we can write the following equations from the given information (using Work = Rate · Time): ❶ $(a+b+c)\cdot 1=1$, ❷ $(a+c)\cdot\frac{3}{2}=1$, and ❸ $(b+c)\cdot 2=1$, using time in hours and Work = 1 meaning one full tank. No need to solve for a, b, and c, just solve for c in terms of a and b from equations ❷ and ❸, then substitute into ❶. Doing this, we obtain: $c=\frac{7}{12}-\frac{1}{2}a-\frac{1}{2}b$, now substituting this into ❶ gives: $\frac{1}{2}a+\frac{1}{2}b=1-\frac{7}{12}$ $\Rightarrow\frac{1}{2}(a+b)=\frac{5}{12}\Rightarrow\frac{12}{10}(a+b)=1\Rightarrow\frac{6}{5}(a+b)=1$

$\boxed{\text{Answer: } \frac{6}{5} \text{ hrs. or 1.2 hrs. or 72 minutes}}$

15. $S(n)=\sum_{k=2}^{n}\frac{1}{\log_k(n!)}=\sum_{k=2}^{n}\frac{1}{\left[\frac{\log_n(n!)}{\log_n(k)}\right]}=\sum_{k=2}^{n}\frac{\log_n(k)}{\log_n(n!)}=\frac{1}{\log_n(n!)}\sum_{k=2}^{n}\log_n(k)$

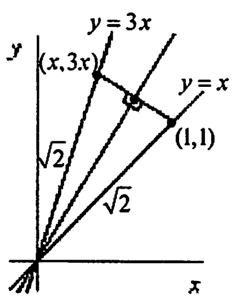
$=\frac{1}{\log_n(n!)}\left[\log_n(2)+\log_n(3)+\log_n(4)+\ldots+\log_n(n-2)+\log_n(n-1)+\log_n(n)\right]$

$=\frac{1}{\log_n(n!)}\left[\log_n(2\cdot3\cdot4\cdot\ldots\cdot(n-2)\cdot(n-1)\cdot n)\right]=\frac{1}{\log_n(n!)}\left[\log_n(n!)\right]=1$ **Answer: b**

16. The check amount is $100x+y$ cents. It was cashed for $100y+x$, with a difference of $(100x+y)-(100y+x)=100(x-y)+(y-x)=100(x-y)-(x-y)=99(x-y)$. Hence, the amount of over payment (in cents) must be a multiple of 99. $198=2\cdot99$, $1089=11\cdot99$, and $495=5\cdot99$, only 972 is not a multiple of 99. **Answer: c**

17. If the teams are equally matched, then the probability of each winning any one game is ½. In order for Team A to win, the team would have to win either the next 2 games, or 2 of the next 3 or 4 games with the final game being a win. We can symbolize this as: AA, ABA, BAA, ABBA, BABA, BBAA, which yields the following probabilities: $\left(\frac{1}{2}\right)^2,\left(\frac{1}{2}\right)^3,\left(\frac{1}{2}\right)^3,\left(\frac{1}{2}\right)^4,\left(\frac{1}{2}\right)^4,\left(\frac{1}{2}\right)^4$. Summing these gives:

$\left(\frac{1}{2}\right)^2\left[1+\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]=\frac{1}{4}\left[2+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right]=\frac{1}{4}\left(\frac{11}{4}\right)=\frac{11}{16}$ **Answer: $\frac{11}{16}$**

18. 

Draw a line segment that is perpendicular to the bisector to create two congruent right triangles, as shown. Taking the point (1,1) on the line $y=x$ yields a length of $\sqrt{2}$ for the hypotenuse of the triangle. Thus, the other hypotenuse must also have a length of $\sqrt{2}$. Now determine the point on the line $y=3x$ labeled $(x,3x)$: $x^2+(3x)^2=\left(\sqrt{2}\right)^2\Rightarrow10x^2=2\Rightarrow x^2=\frac{1}{5}$ $\Rightarrow x=\sqrt{\frac{1}{5}}=\frac{\sqrt{5}}{5}$, taking the positive value for x, which makes the y-value $3\frac{\sqrt{5}}{5}$.

The midpoint of the segment, which is on the bisector, can now be found: $\left(\frac{1+\sqrt{5}/5}{2},\frac{1+3\sqrt{5}/5}{2}\right)$. So the slope of the bisector is: $\frac{\left(\frac{1+3\sqrt{5}/5}{2}-0\right)}{\left(\frac{1+\sqrt{5}/5}{2}-0\right)}=\frac{1+\frac{3\sqrt{5}}{5}}{1+\frac{\sqrt{5}}{5}}=\frac{5+3\sqrt{5}}{5+\sqrt{5}}=\frac{5+3\sqrt{5}}{5+\sqrt{5}}\cdot\frac{5-\sqrt{5}}{5-\sqrt{5}}=\frac{25+10\sqrt{5}-15}{25-5}=\frac{1+\sqrt{5}}{2}$

Answer: b

19. Adding all sums of groups of four: $138+144+151+153+158=744$. This sum includes each age exactly four times, hence $744\div4=186$ is the sum of all five ages. The largest sum (158) of four of the ages must omit the lowest age. Therefore, the youngest is $186-158=28$. **Answer: b**

20. If Statement I were true, then exactly three would be false, making Statement III also true. Hence, Statements I and III cannot be true. Statements II and IV are consistent. **Answer: d**

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