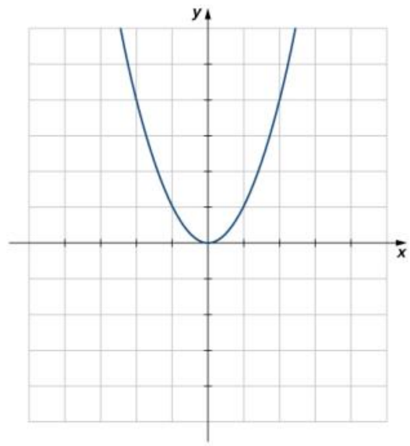
Section 2.1: A Preview of Calculus

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. Recall that for lines, the rate of change is constant, called the slope. This is because as we move from left to right, the function will either increase or decrease at a constant rate.

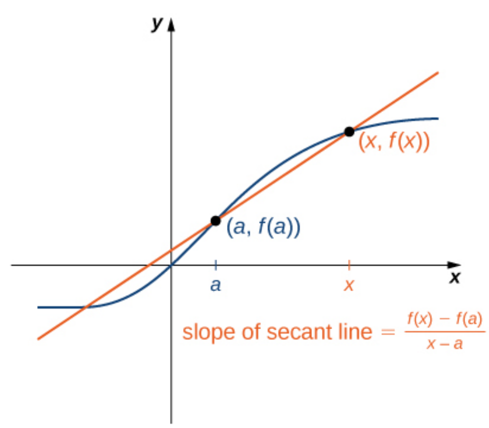
However, for nonlinear functions, the rate of change can vary. For example, in the graph below, if we were to move from left to right, the graph decreases rapidly, decreases more slowly and then levels off, before increasing (slowly at first and then more rapidly). Unlike for a linear function, we cannot represent the rate of change of this function by a single number.



$k(x) = x^2$

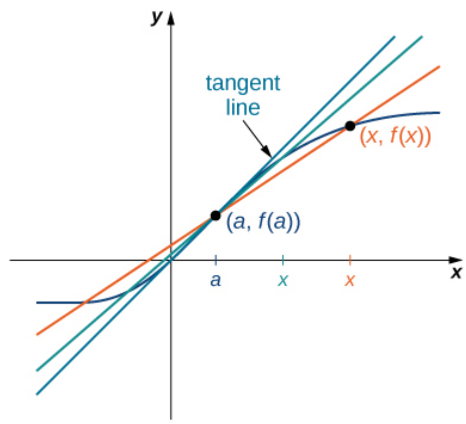
We can, however, approximate the rate of change of a function of $f(x)$ by looking at the slope of the secant line.

> The **secant to the function** $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. It's slope is given by
>
> $$m_{sec} = \frac{f(x)-f(a)}{x-a}$$



However, secant lines are not always the best approximation for the rate of change of the function. It will depend on how close x is to a. As x gets closer to a, the secant lines approach the tangent line.

The secant lines approach a line that is called the **tangent to the function** $f(x)$. The slope of the tangent line is a more accurate measure of the rate of change of the function at a and represents the derivative of the function $f(x)$ at a. The derivative is denoted $f'(a)$.



Media: Watch this video example on estimating the slope of the tangent line.

Examples:

1) Estimate the slope of the tangent line to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1,1)$ and each of the following points on the graph of $f(x) = x^2$.
    a. $(2, 4)$
    b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

2) Points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x) = x^2 + 1$. Complete the following table with the appropriate values: y −coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to six decimal places.

x	y	$Q\ (x,y)$	m_{sec}
1.1			
1.01			
1.001			
1.0001			

Use the values in the right column of the table to guess the value of the slope of the line tangent to f at $x = 1$.

Differential calculus is the field of calculus concerned with the study of derivatives and their applications. For instance, velocity can be thought of as the rate of change of position, $s(t)$, which represents the position of an object along a coordinate axis at any given time t. Using ideas about rate of change (with secant and tangent lines), we can approximate the instantaneous velocity with an average velocity.

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a,t]$ where $a < t$ (or $[t,a]$ if $t < a$) is

$$v_{\text{average}} = \frac{s(t)-s(a)}{t-a}$$

Finding the average velocity of a position function over a time interval is the same as finding the slope of the secant line to a function.

As t gets closer to a, the average velocity becomes closer to the instantaneous velocity just as when the secant lines approach the tangent line. This process of letting x or t approach a in an expression is called taking a **limit**.

For a position function $s(t)$, the instantaneous velocity at a time $t = a$ is the value that the average velocities approach on intervals $[a,t]$ and $[t,a]$ as the value of t becomes closer to a, provided such a value exists.

Finding the instantaneous velocity of a position function over a time interval is the same as finding the slope of the tangent line to a function.

Media: Watch this video example on estimating instantaneous velocity.

Example

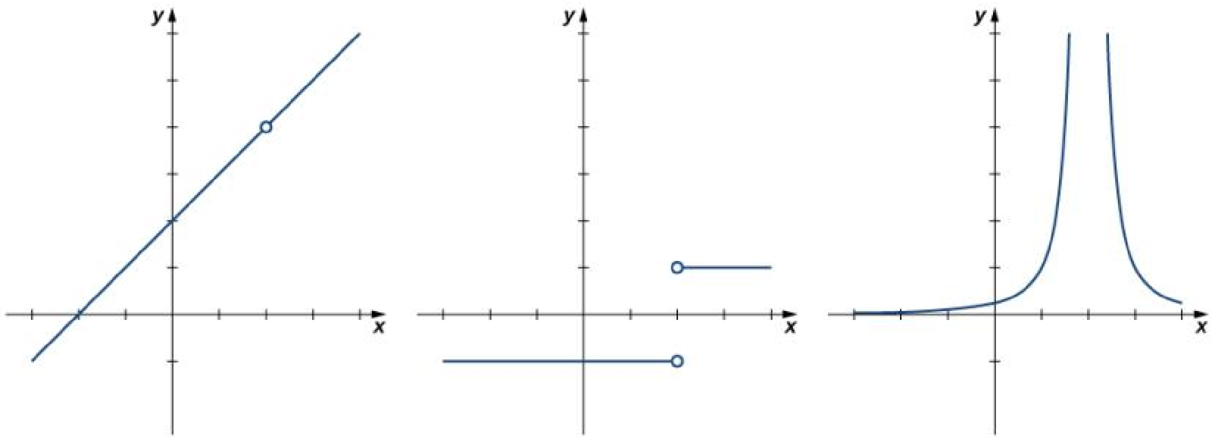
A rock is dropped from a height of 64 feet. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$

b. $[0.5, 0.51]$

Section 2.2: The Limit of a Function

To understand the concept of a limit, first look at graphs of functions. For example, looking at the three functions below, we can see that at $x = 2$, all these functions are undefined.



$f(x) = \frac{x^2 - 4}{x - 2}$ $g(x) = \frac{|x - 2|}{x - 2}$ $h(x) = \frac{1}{(x - 2)^2}$

(a) (b) (c)

Yet, simply stating they are undefined does not give an accurate picture of what is happening around $x = 2$.

Intuitive Definition of a Limit

Looking at the graphs above, we see that the behavior of the function as x approaches 2 can be very different depending on the function.

> Let $f(x)$ be a function defined at all values in an open interval containing a, except maybe at a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of x $(\neq a)$ approach the number a, then the **limit of $f(x)$ as x approaches a is L**. Symbolically,
>
> $$\lim_{x \to a} f(x) = L.$$
>
> In other words, as x gets closer to a, $f(x)$ gets closer to L.

For example, in the first function $f(x) = \frac{x^2-4}{x-2}$, as x approaches 2 from either side, the values of $y = f(x)$ approach 4. So, mathematically,

$$\lim_{x \to 2} f(x) = 4.$$

One way to approximate a limit is to use a table. Choose sets of x —values – one set approaching a from the left (values slightly smaller than a) and another set approaching a from the right (values slightly larger than a).

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\boldsymbol{a-0.1}$	$f(a-0.1)$
$\boldsymbol{a-0.01}$	$f(a-0.01)$
$\boldsymbol{a-0.001}$	$f(a-0.001)$
$\boldsymbol{a-0.0001}$	$f(a-0.0001)$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\boldsymbol{a+0.1}$	$f(a+0.1)$
$\boldsymbol{a+0.01}$	$f(a+0.01)$
$\boldsymbol{a+0.001}$	$f(a+0.001)$
$\boldsymbol{a+0.0001}$	$f(a+0.0001)$

Then, look at each of the $f(x)$ columns, determine whether the outputs seem to be approaching a single value. If both columns approach a common y —value L, then we say

$$\lim_{x\to a} f(x) = L$$

Media: Watch this video example on estimating a limit with a table.

Media: Watch this video example on finding limits from a graph.

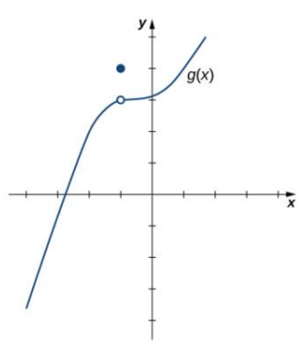
Examples:

1) Evaluate each of the following limits using a table of function values. Then use a graph to confirm your estimate.

   a. $\lim\limits_{x\to 0} \frac{\sin x}{x}$

   b. $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

2) For $g(x)$ shown below, evaluate $\lim\limits_{x\to -1} g(x)$.



The Existence of a Limit

For a limit of a function to exist at a point, the function values must approach a single real-number value at that point. If the function values do not approach a single value, then the limit does not exist.

Media: Watch this video example on finding the limit of the sine function with a table.

Example: Evaluate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ using a table of values.

One-Sided Limits

Indicating that the limit of a function fails to exist at a point does not always provide us with enough information about the behavior of the function at that point. Instead, we look at what happens as we approach the point from the left and right sides.

We define two types of **one-sided limits**.

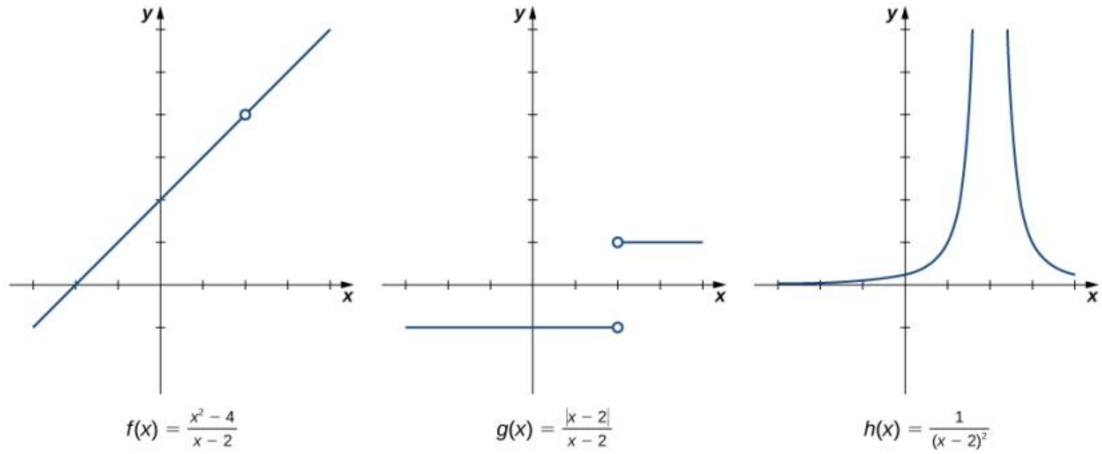
Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

So, looking back at the three graphs from the beginning, we can see that the y −values of the second graph approach different values as we approach $x = 2$ from the left and right.



(a) (b) (c)

In this case, the $\lim_{x \to 2^-} g(x) = -1$ since as x approaches 2 from the left, $f(x)$ (y −values) approach -1.

Similarly, the $\lim_{x \to 2^+} g(x) = 1$ since as since as x approaches 2 from the right, $f(x)$ (y −values) approach 1.

So, if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. If the limit from the left and the limit from the right take on different values, the limit of the function does not exist at that point.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Media: Watch this video example on finding one-sided limits from a table.

Examples

1) Use a table of values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$

2) For the function $f(x) = \begin{cases} x+1 & if\ \ x<2 \\ x^2-4 & if\ x \geq 2 \end{cases}$ evaluate each of the following limits.

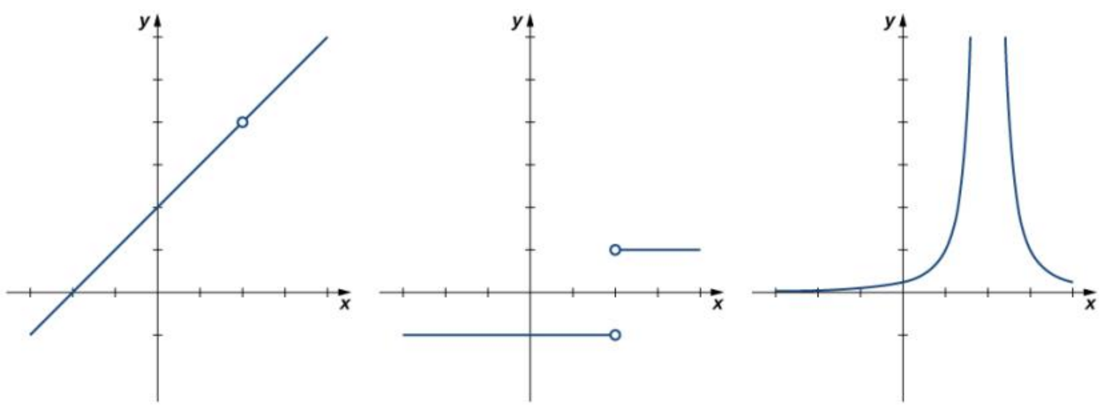
a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

Infinite Limits

Evaluating the limit (or right and left limit) of a function at a point helps us to characterize the behavior of a function around a given value. We can also describe the behavior of functions that do not have finite limits.

Looking back at the original three functions, we see that as x approaches 2, the values of $h(x)$ becomes larger and larger. Hence, we say $\lim\limits_{x \to 2} h(x) = +\infty$. This is called an **infinite limit**.



$f(x) = \frac{x^2-4}{x-2}$

(a)

$g(x) = \frac{|x-2|}{x-2}$

(b)

$h(x) = \frac{1}{(x-2)^2}$

(c)

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty$$

*Note, when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$, we are NOT stating that the limit exists. We are simply describing the behavior of the function.

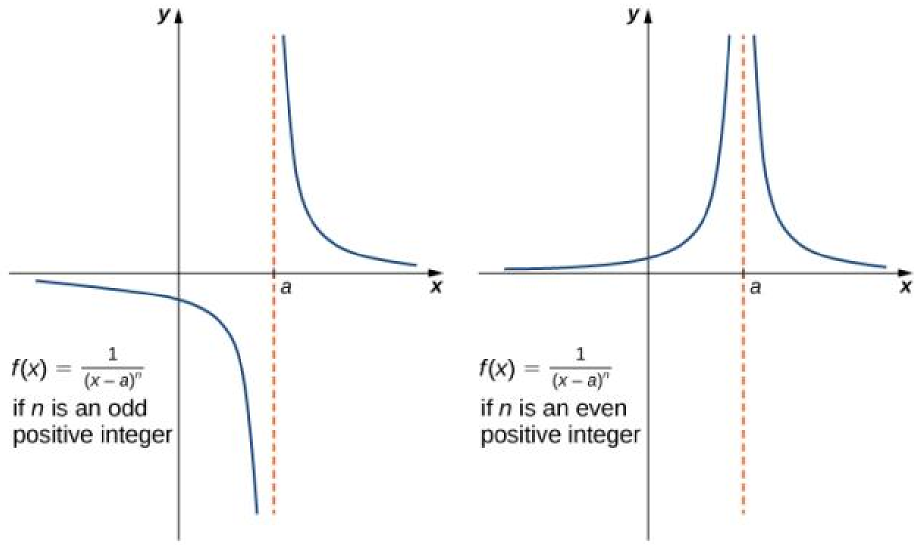
Examples: Evaluate each of the following limits, if possible. Use a table of values and graph $f(x) = \frac{1}{x}$ to confirm your conclusion.

1) $\lim_{x \to 0^-} \frac{1}{x}$

2) $\lim_{x \to 0^+} \frac{1}{x}$

3) $\lim_{x \to 0} \frac{1}{x}$

Functions of the form $f(x) = \frac{1}{(x-a)^n}$ where n is a positive integer, have infinite limits as x approaches a from either the left or the right.



If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

Notice, that for these types of graphs, we also have vertical asymptotes at $x = a$. We can determine whether a function has vertical asymptotes by looking at limits.

Let $f(x)$ be a function. The line $x = a$ is a **vertical asymptote** of $f(x)$, if any of the following conditions hold:

$$\lim_{x \to a^-} f(x) = +\infty \text{ or} -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty \text{ or} -\infty$$

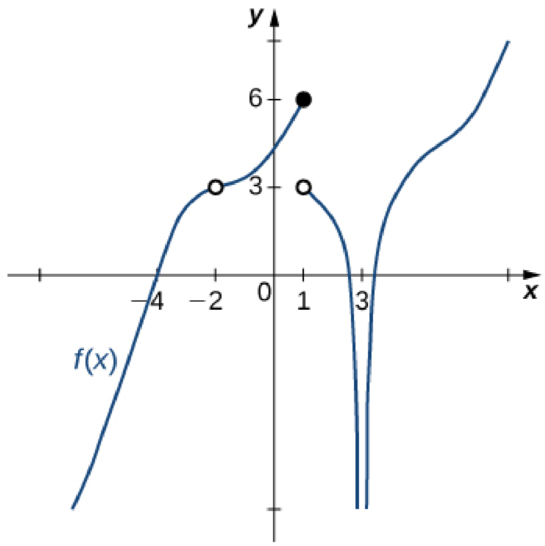
$$\lim_{x \to a} f(x) = +\infty \text{ or} -\infty$$

Media: Watch these [video1]() and [video 2]() examples on finding limits graphically.

Media: Watch this [video]() example on drawing a graph given specific limit properties.

Examples

1) Use the graph of $f(x)$ shown below to determine each of the following:



a. $\lim_{x\to-4^-} f(x)$; $\lim_{x\to-4^+} f(x)$; $\lim_{x\to-4} f(x)$; $f(-4)$

b. $\lim_{x\to-2^-} f(x)$; $\lim_{x\to-2^+} f(x)$; $\lim_{x\to-2} f(x)$; $f(-2)$

c. $\lim_{x\to1^-} f(x)$; $\lim_{x\to1^+} f(x)$; $\lim_{x\to1} f(x)$; $f(1)$

d. $\lim_{x\to3^-} f(x)$; $\lim_{x\to3^+} f(x)$; $\lim_{x\to3} f(x)$; $f(3)$

2) Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3}$

b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3}$

c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$

3) Sketch the graph of a function with the given properties:

- As $x \to -\infty, f(x) \to 2$
- $\lim_{x \to -2} f(x) = -\infty$
- As $x \to \infty, f(x) \to 2$
- $f(0) = 0$

Section 2.3: The Limit Laws

Evaluating Limits with the Limit Laws

In the previous section we estimated limits using a table and graph. In addition to estimating, we look at some properties of limits that will allow us to evaluate limits of many types of algebraic functions.

Basic Limits

For any real number a and any constant c,

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} c = c$$

Examples: Evaluate each of the following limits using the Basic Limits.

1) $\lim_{x \to 2} x$

2) $\lim_{x \to 2} 5$

We now look at the limit laws, the individual properties of limits, and practice using them.

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.$$

Let c be a constant. Then, the following statements hold:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} c\, f(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every integer $n > 0$

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$

Media: Watch this video example on determining a limit analytically.

Media: Watch this video example on using the limit laws.

Examples

1) Use limit laws to evaluate each of the following. In each step, indicate the limit law applied.
    a. $\lim_{x \to -3} (4x + 2)$
    b. $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$
    c. $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$

2) For each of the following, assume that $\lim_{x \to 6} f(x) = 4$, $\lim_{x \to 6} g(x) = 9$, and $\lim_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.
    a. $\lim_{x \to 6} \frac{g(x) - 1}{f(x)}$
    b. $\lim_{x \to 6} [(x + 1) \cdot f(x)]$
    c. $\lim_{x \to 6} (f(x) \cdot g(x) - h(x))$

Limits of Polynomial and Rational Functions

In the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

> **Limits of Polynomial and Rational Functions**
>
> Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,
>
> $$\lim_{x \to a} p(x) = p(a) \text{ and}$$
>
> $$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

Examples: Evaluate the following limits.

1) $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$

2) $\lim_{x \to -2} (3x^3 - 2x + 7)$

Additional Limit Evaluation Techniques

We can evaluate the limits of polynomials and limits of some rational functions by direct substitution. However, it is possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined.

> **Calculating a Limit When $\frac{f(x)}{g(x)}$ has the Indeterminate Form $\frac{0}{0}$**
>
> 1) First, make sure that the function has the appropriate form and cannot be evaluated immediately using the limit laws.
>
> 2) Then find a function that is equal to $h(x) = \frac{f(x)}{g(x)}$ for all $x \neq a$ over some interval containing a. To do this, try one or more of the following steps:
>
> a. If $f(x)$ and $g(x)$ are polynomials, factor each function and cancel out any common factors.
>
> b. If the numerator or denominator contains a difference involving square root, try multiplying the numerator and denominator by the conjugate of the expression involving a square root.
>
> c. If $\frac{f(x)}{g(x)}$ is a complex fraction, begin by simplifying it.
>
> 3) Last, apply the limit laws.

Media: Watch these video1 and video2 examples on limits of rational functions by factoring.

Media: Watch these video1 and video2 examples on finding limits by rationalizing.

Examples: Evaluate each of the following limits.

1) $\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}$

2) $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$

3) $\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

4) $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$

Evaluating One-Sided and Two-Sided Limits Using Limit Laws

We can apply the limit laws to one-sided and two-sided limits, just make sure the function is defined over a given interval. For example, to find $\lim_{x\to a^-} h(x)$, we need to make sure $h(x)$ is defined over the interval (b, a). Likewise, to find $\lim_{x\to a^+} h(x)$, make sure $h(x)$ is defined over the interval (a, c).

Media: Watch this video example on finding limits of piecewise functions.

Examples:

1) Evaluate each of the following limits, if possible.
   a. $\lim_{x\to 3^-}\sqrt{x-3}$
   b. $\lim_{x\to 3^+}\sqrt{x-3}$

2) For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

c. $\lim\limits_{x \to 2} f(x)$

3) Evaluate $\lim\limits_{x \to 2^-} \frac{x-3}{x^2-2x}$

The Squeeze Theorem

Although the techniques we have been using work very well for algebraic functions, we also need to evaluate limits of trigonometric functions. The squeeze theorem can help us calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a commonly known limit at a.

The Squeeze Theorem

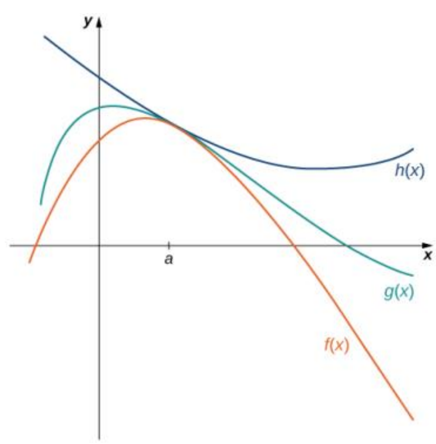
Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim\limits_{x \to a} g(x) = L$.



Media: Watch these video1 and video2 examples on the squeeze theorem.

Examples

1) Evaluate the following limits by applying the squeeze theorem.
   a. $\lim_{x \to 0} x \cos x$
   b. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$

2) True or False? If $2x - 1 \leq g(x) \leq x^2 - 2x + 3$, then $\lim_{x \to 2} g(x) = 0$.

Section 2.4: Continuity

Continuity at a Point

A function is considered continuous if we can trace the graph with a pencil without lifting the pencil from the page. Points where you must lift the pencil are considered **points of discontinuity**.

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

- $f(a)$ is defined (no holes)
- $\lim_{x \to a} f(x)$ exists (no breaks)
- $\lim_{x \to a} f(x) = f(a)$ (no jumps)

A function is **discontinuous at a point** a if it fails to be continuous at a.

Media: Watch this video example on continuity of piecewise functions.

Examples: Using the definition of continuity, determine whether the given function is continuous at the given point. Justify the conclusion.

1) $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$

3) $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $x = 0$

2) $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ at $x = 3$

4) $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ at $x = 1$

Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Media: Watch this video example on continuity of polynomial and rational functions.

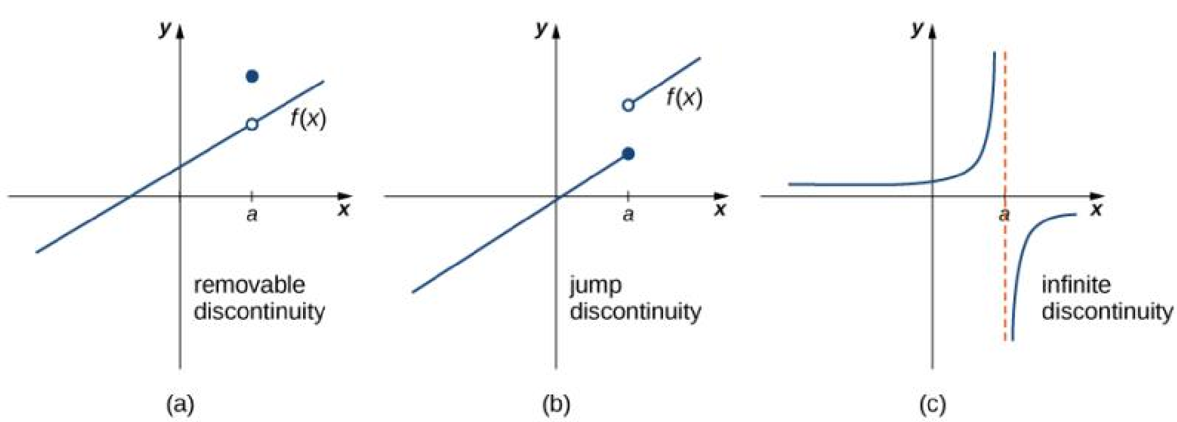
Examples

1) For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

2) For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Types of Discontinuities

Discontinuities can take on several different appearances. A **removable discontinuity** is a discontinuity for which there is a hole in the graph. A **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up. An **infinite discontinuity** is a discontinuity located at a vertical asymptote.



If $f(x)$ is discontinuous at a, then f has

1) a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists.
2) a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
3) an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

Media: Watch these video1 and video2 examples on classifying discontinuities.

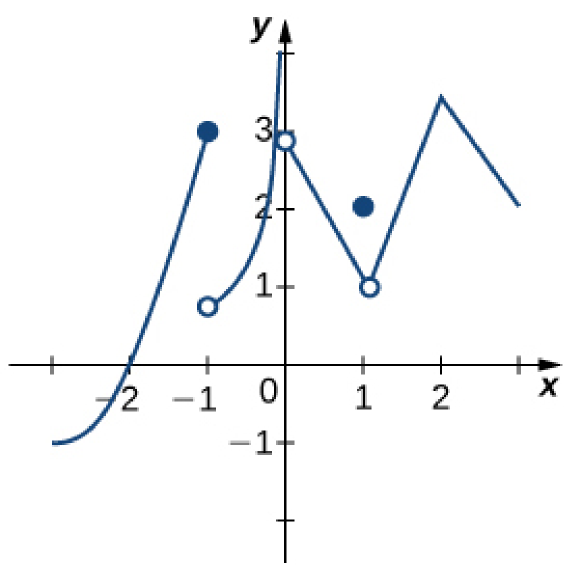
Media: Watch this video example on determining a limit analytically.

Examples

1) Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

2) For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If the function is discontinuous at 1, classify the discontinuity as removable, jump, or infinite.

3) Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.

b. For each value in part a, state why the formal definition of continuity does not apply.

c. Classify each discontinuity as either jump, removable, or infinite.

4) Suppose $y = f(x)$. Sketch a graph with the indicated properties:

- Discontinuous at $x = 1$
- $\lim_{x \to -1} f(x) = -1$
- $\lim_{x \to 2} f(x) = 4$

Continuity over an Interval

A function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. Before looking at what it means to be continuous over an interval, we need to understand what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

A function $f(x)$ is said to be **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Continuity over an Interval

A function is continuous over an open interval if it is continuous at every point in the interval.

A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b.

A function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b.

Continuity over other types of intervals are defined in a similar fashion.

Media: Watch this video example on finding intervals of continuity.

Media: Watch this video example on sketching graphs with given conditions.

Examples: State the interval(s) over which the given function is continuous.

1) $f(x) = \frac{x-1}{x^2+2x}$

2) $f(x) = \sqrt{4 - x^2}$

3) Sketch the graph of the function $y = f(x)$ with the following properties:

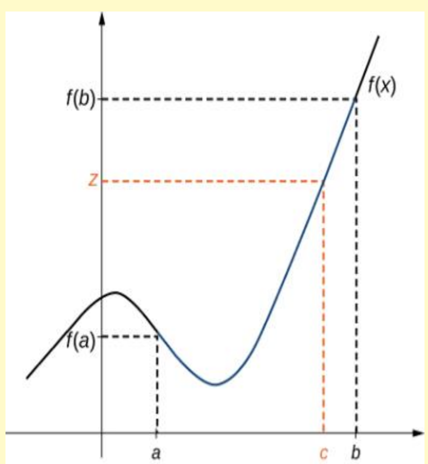
- The domain of f is $(-\infty, \infty)$.
- f has an infinite discontinuity at $x = -6$.
- $f(-6) = 3$
- $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$
- $f(-3) = 3$
- f is left continuous but not right continuous at $x = 3$
- $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

The Intermediate Value Theorem

Functions that are continuous over intervals for the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. The Intermediate Value Theorem helps us determine whether solutions exist before going through the process to find them.

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfyling $f(c) = z$.



Media: Watch this video example on the Intermediate Value Theorem.

Examples

1) Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0,1]$.

2) Show that $f(x) = x - \cos x$ has at least one zero.