Math 45 - Section —
Homework 3
Friday, April 8, 2016

D.{1,2,3,4,5,6,7,8}

D1 For each of the following differential equations find its general solution:

(a) $y'' - y' - 12y = 0$

(b) $y'' + 3y' + y = 0$

(c) $y'' + 3y' + 3y = 0$

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D2 For each of the following differential equations find its general solution:

(d) $y'' + 4y' + 4y = 0$

(e) $y'' + 2y' + 2y = 0$

(f) $4y'' + 12y' + 9y = 0$

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D3 Find the solution that satisfies $y(0) = 1$, $y'(0) = 0$ in equations (a), (c) and (f) of Problems D1 and D2.

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D4 Consider the initial value problem

$$\begin{cases} 2y'' - y' - 3y = 0 \\ y(0) = A \\ y'(0) = B. \end{cases}$$

(a) Find the solution to this problem in terms of A and B.

(b) For what values of A and B does the solution tend to 0 as $t \to \infty$?

(c) For what values of A and B does the solution tend to ∞ as $t \to \infty$?

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D5 For each of the following initial value problems, determine the largest interval on which a unique twice differentiable solution, $y(t)$, is certain to exist.

(a) $ty'' + 3y = t^3, \quad y(1) = 1, \quad y'(1) = 2$

(b) $(t-3)y'' + ty' + \ln|t|y = 0, \quad y(1) = 0, \quad y'(1) = 1$

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D6 Consider the initial value problem

$$\begin{cases} y'' = -y \\ y(0) = \cos(a) \\ y'(0) = -\sin(a). \end{cases} \tag{1}$$

(a) Find the solution to (1).

(b) Prove $\cos(t+a)$ also solves the above initial value problem.

(c) Use the uniqueness theorem to prove that $\cos(t+a) = \cos(t)\cos(a) - \sin(t)\sin(a)$ for any $t, a \in \mathbb{R}$.

(d) Now show that another form of the general solution to (1) is $b\cos(t+a)$.

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D7 Prove that if a, b, c are three different real numbers then e^{ax}, e^{bx}, e^{cx} are linearly independent functions.

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D8 (Conservation of Energy) Let $V(x)$ be a twice continuously differentiable function. Using Newtons second law of motion we can model a marble of mass 1 rolling along the graph of $V(x)$ starting at position x_0 and with initial velocity v_0 (where positive velocity indicates moving to the right) with the initial value problem

$$\begin{cases} x'' = -V'(x) \\ x(0) = x_0 \\ x'(0) = v_0. \end{cases} \tag{2}$$

This model assumes that the marble does not fly off the surface of the graph and that there is no friction. The function $V(x)$ in this context is referred to as a potential function. *Note: A particularly interesting potential is $V(x) = \frac{1}{2}kx^2$ with $k > 0$, the potential energy for a spring. In this case the equation becomes $x'' = kx$, whose fundamental solutions are periodic. This implies that a spring system is equivalent to a marble rolling up and down a parabola.*

(a) Show that any solution, $x(t)$, to (2) conserves the energy, i.e. E is constant,

$$E(t) = \frac{1}{2}\left(x'(t)\right)^2 + V(x(t)).$$

(b) Show that if the right hand side in (2) is replaced by $-V'(x) - cx'$, $c > 0$ the energy is dissipated. *Note: This DOES NOT model friction, as friction is not proportional to the velocity of the marble. Since the force of friction is discontinuous (static vs kinetic friction) it causes issues for existence of solutions.*

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