1, 2, 3, 4, 5, 6, 7

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1 Which of these differential equations is separable? For those equations that are separable, separate them. You dont have to solve any of these equations.

(a) $\dfrac{dy}{dx} = \dfrac{e^{x+y}}{xy}$

(b) $C'(q) = q^2 + 2qC(q)$

(c) $\dot{z} = \omega, \omega$ is constant.

(d) $y' = \ln\left(\dfrac{x}{y}\right)$

(e) $x' + 2 = tx - 2x + t$

■

2

(a) Find the general solution to the equation

$$\frac{dy}{dt} = \frac{4\sin(2t)}{y}$$

Your answer should be an explicit function $y(t)$ with two branches.

(b) Now impose the initial condition $y(0) = 1$. What is the solution to the DE that satisfies this initial condition?

(c) What is the largest t-interval on which the solution is defined?
Note: You may be interested to look at the last two slides of Lecture 1, posted on Sakai, which give a bit more information on how to find the interval (domain) of validity for the solution to a DE.

■

3 Find all solutions to the ODE

$$yy' = (1 - y^2)\sin x.$$

Your answer will be an implicit function of y and x. Make sure you don't "lose" any solutions along the way.

4 Using the integrating factor method find the general solution to the following differential equations.

(a) $2y' + 3y = e^{4t}$

(b) $y' + ty = 5t$

5 Examine Student X's work on the following problem. What did the student do correctly? What mistake(s) did the student make? What is a more correct response to the problem, and what would you say to help the student understand how to correctly complete the problem?

Determine the solution to the IVP $y' = 5 - ty$ with $y(0) = 1$.

$$y' = 5 - ty$$

Integrate both sides with respect to t:

$$\int y' \, dt = \int (5 - ty) \, dt$$

$$y = 5t - \tfrac{1}{2}t^2 y + C$$

Use the initial condition $y(0) = 1$

$$1 = 5 \cdot 0 - 0 + C \implies C = 1$$

So $y = 5t - \tfrac{1}{2}t^2 y + 1$

$$y + \tfrac{1}{2}t^2 y = 5t + 1$$

$$\therefore y(t) = \frac{5t + 1}{1 + \tfrac{1}{2}t^2}$$

■

6 You can solve $y'' = x\,(y')^2$ with the initial conditions $y'(0) = -2$ and $y(0) = 0$, using a change of variables combined with separation of variables. This simple example illustrates a technique that can be applied to harder problems. Here's the idea:

- First define a new variable, $w = y'$.

- Rewrite the original DE in terms of w to get a separable ODE.

- Solve this equation to find $w(x)$ (Note: You may want to use one of the initial condition at this point).

- Substitute the answer you get for $w(x)$ into $w = y'$ and solve for y.

- Show that your solution satisfies the original DE and initial conditions.

■

7 Satellite dishes, radar antennae and microwave relays are designed so that they collect and focus an incoming signal into a single point (see Figure 1). In this problem you will use differential equations to determine the dish shape that imparts this focusing property. For simplicity assume that we are working with a cross-section of the dish, represented by a curve $y(x)$. Incoming waves traveling parallel to the y-axis reflect off the dish according to the *law of reflection*: the angle of incidence is equal to the angle of reflection.
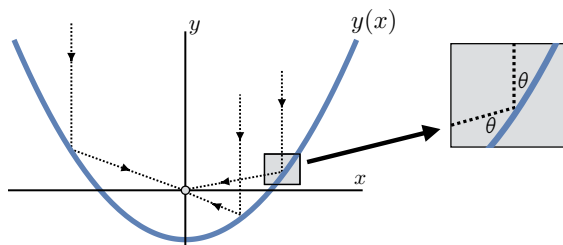

Figure 1: A cross-section of a satellite dish, with law of reflection illustrated.

(a) For convenience, let's place the focus point at the origin. Consider the triangle formed by the tangent line to the dish at (x, y), the reflected ray, and the y-axis, as shown on the left in Figure 2. Explain why this triangle must be isosceles.
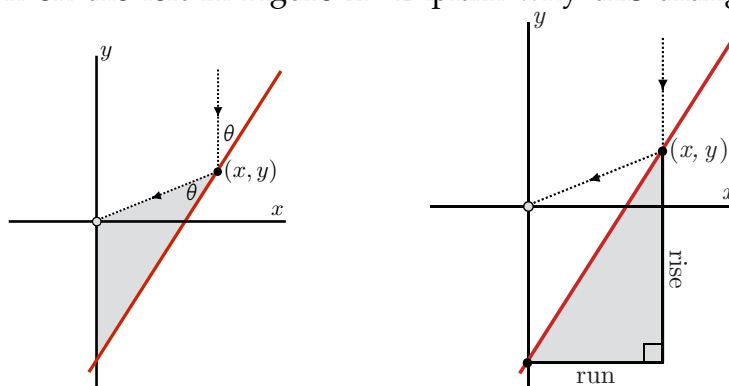

Figure 2: Two triangles to help you derive the differential equation.

(b) Based on your work on part (a), show that if the dish focuses incoming rays into a single point, then the curve of the dish, $y(x)$, must satisfy the differential equation

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

Hint: Think of dy/dx above as a slope, which is calculated as the "rise over run" in the right triangle on the right in Figure 2.

(c) Using the substitution $y(x) = xz(x)$, show that the differential equation found in part (b) can be transformed into the equation

$$z' = \frac{\sqrt{1 + z^2}}{x}.$$

You may assume that $x > 0$ throughout your computation.

(d) Solve the ODE from part (c) and determine the shape of the dish, $y(x)$.

■