1, 2, 3, 4, 5

1 Examine Student Z's work on the following problem. What did the student do correctly? What mistake(s) did the student make? What is a more correct response to the problem, and what would you say to help the student understand how to correctly complete the problem?

Determine the solution to the IVP $y' = -x + y$ with $y(\sqrt{2}) = 0$.

$\int y' = \int -x + \int y$

$y = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

$2y = -x^2 + y^2 + 2C$

Use the initial condition $y(\sqrt{2}) = 0$

$0 = -2 + 0 + 2C \Rightarrow C = 1$

So, $x^2 = y^2 - 2y + 2$

$x^2 - 1 = y^2 - 2y + 1 = (y-1)^2$

$\therefore \quad y - 1 = \pm\sqrt{x^2 - 1}$

Choose the negative sign to get the correct branch of the solution.

$y = 1 - \sqrt{x^2 - 1}$

Check: $y(\sqrt{2}) = 1 - \sqrt{2-1} = 1 - \sqrt{1} = 0$ ✓

2 At midnight, it is 20°F outside and 70°F inside when your furnace breaks. Two hours later, the temperature inside has fallen to 60°F.

> **Newton's Law of Cooling:** The rate of change of the temperature of something is proportional to the temperature difference between that thing and its surroundings.

(a) Assuming that the temperature inside the house obeys Newton's Law of Cooling and that it stays 20°F outside, determine the temperature inside the house as a function of time. When will it reach 40°F inside the house?

(b) Clearly the temperature outside won't stay constant. Suppose that we model the temperature as a sinusoidal function with a minimum value of 20°F at midnight and a maximum value of 40°F at noon. Determine the temperature inside the house and time the house will reach 40°F under these assumptions. (Note: To solve for the time to reach 40°F in this situation, you will need to solve a nonlinear equation. You can use a graph to estimate the solution to this equation, or you can use a computer to approximate the solution for you. Feel free to use `fsolve` in Matlab or `FindRoot` in Mathematica.)

(c) What other assumptions could be relaxed to produce a more accurate prediction? You could also earn some extra credit here if you rework the problem again using your ideas.

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3 Use the Wronskian to determine whether the following functions y_1 and y_2 are linearly independent or linearly dependent on the interval $t \in (0, 1)$.

(a) $y_1(t) = e^{5t}$, $y_2(t) = e^{-3t}$

(b) $y_1(t) = t^2$, $y_2(t) = t^4$

(c) $y_1(t) = \cos t \sin t$, $y_2(t) = \sin 2t$

(d) $y_1(t) = e^{\alpha t} \cos \beta t$, $y_2(t) = e^{\alpha t} \sin \beta t \quad (\alpha, \beta \text{ constants}, \beta \neq 0)$

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4 Consider the equation

$$y'' + p(t)y' + q(t)y = 0, \qquad t \in I,$$

where $p(t)$ and $q(t)$ are continuous on I. Let y_1 and y_2 be solutions to this equation.

(a) Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a *fundamental set* on I.

(b) Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a *fundamental set* on I.

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5 Examine Student R's work on the following problem. What did the student do correctly? What mistake(s) did the student make? (There are at least three.) What is a more correct response to the problem, and what would you say to help the student understand how to correctly complete the problem?

Determine the general solution to the DE $xy' + 2y = \sqrt{x}$.

$$xy' + 2y = \sqrt{x}$$

Find integrating factor $e^{\int 2\,dx} = e^{2x}$.

Mult. by int. factor to make LHS a derivative:

$$e^{2x}(xy' + 2y) = \frac{d}{dx}\left(e^{2x}y\right) = \sqrt{x}$$

Now solve by integrating.

$$e^{2x}y = \frac{2}{3}x^{3/2} + C$$

$$\boxed{y(x) = \frac{2}{3}x^{3/2}e^{-2x} + \tilde{C}}$$

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