Newton's Method for Convex Optimization

Conner DiPaolo, Jeffrey Rutledge, Colin Adams

Harvey Mudd College

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Finding Roots of Functions

Given a differentiable function $f : \mathbf{R} \mapsto \mathbf{R}$ we want to find the instances when $f(x) = 0$ (not generally solvable in closed form).

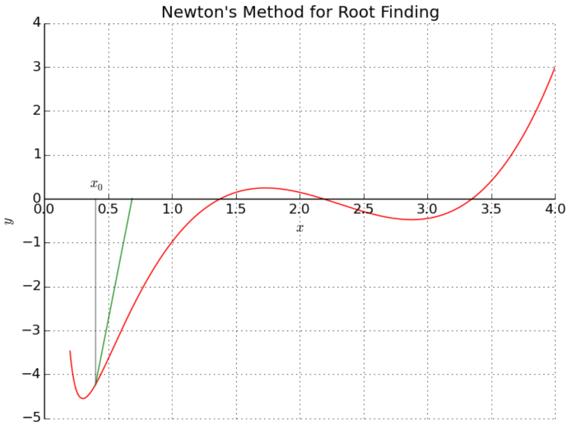
Newton's Method:

1. Take a starting position x_0
2. Find where the tangent line $y = f(x_n) + (x_{n+1} - x_n)f'(x_n)$ is 0 and iterate:

$$y = f(x_n) + (x_{n+1} - x_n)f'(x_n) = 0$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Finding Roots of Functions

Initial $x_0 = 0.4,\ f(x) = (x-1)(x-3)^2 + e^{\frac{1}{3}/x} - \cos\left(\frac{x}{2}\right) - 1.5$



$$x_0 = 0.4 \quad \Delta = 0.9805$$

Finding Roots of Functions

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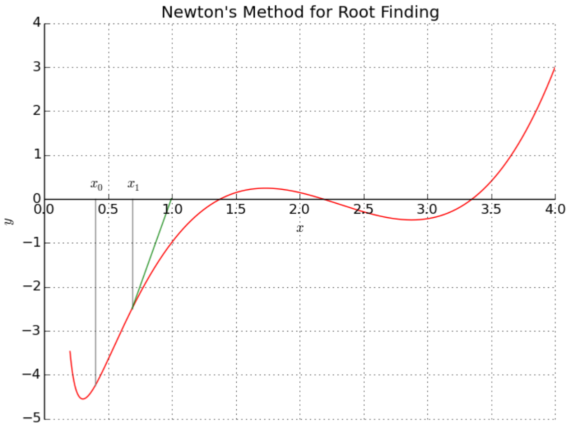
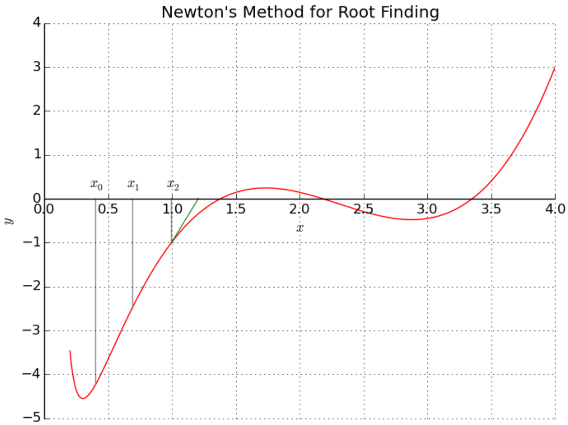


$x_1 = 0.6866 \quad \Delta = 0.6938$

Finding Roots of Functions

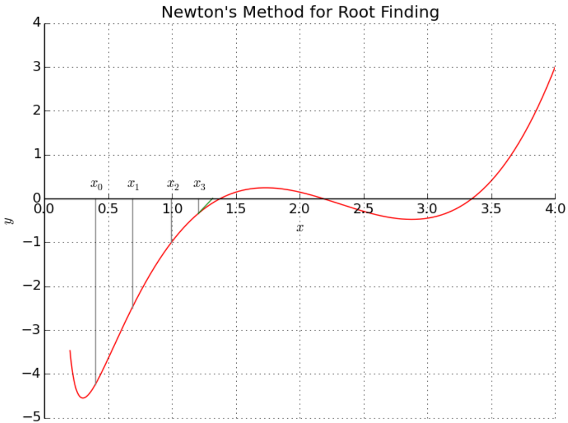
Initial $x_0 = 0.4,\ f(x) = (x-1)(x-3)^2 + e^{\frac{1}{3}/x} - \cos\left(\frac{x}{2}\right) - 1.5$



$$x_2 = 0.9938 \quad \Delta = 0.3866$$

Finding Roots of Functions

Initial $x_0 = 0.4$, $f(x) = (x-1)(x-3)^2 + e^{\frac{1}{3}/x} - \cos\left(\frac{x}{2}\right) - 1.5$
Converges in 28 iterations.



$x_3 = 1.2050 \quad \Delta = 0.1754$

Optimizing Function Using Newton's Method

Instead of finding wheren $f(x) = 0$, how can we find where $f'(x) = 0$? Same method:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

Def: The gradient of $f : \mathbf{R}^n \mapsto \mathbf{R}$ is

$$\nabla f = \left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_n} f\right].$$

Def: The Hessian of $f : \mathbf{R}^n \mapsto \mathbf{R}$ is

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \ddots & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Optimizing $f : \mathbf{R}^n \mapsto \mathbf{R}$

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