Newton-Raphson Method for Convex Optimization

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Finding Roots of Functions

Given a differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ we want to find the instances when $f(x) = 0$ (not generally solvable in closed form).

Newton's Method:

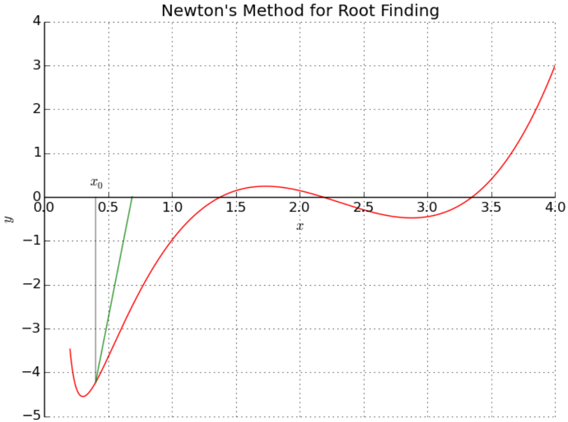
1. Take a starting position x_0
2. Find where the tangent line $y = f(x_n) + (x_{n+1} - x_n)f'(x_n)$ is 0 and iterate:

$$y = f(x_n) + (x_{n+1} - x_n)f'(x_n) = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Finding Roots of Functions

Initial $x_0 = 0.4,\ f(x) = (x-1)(x-3)^2 + e^{\frac{1}{3x}} - \cos\left(\frac{x}{2}\right) - 1.5$



$x_0 = 0.4 \quad \Delta = 0.9805$

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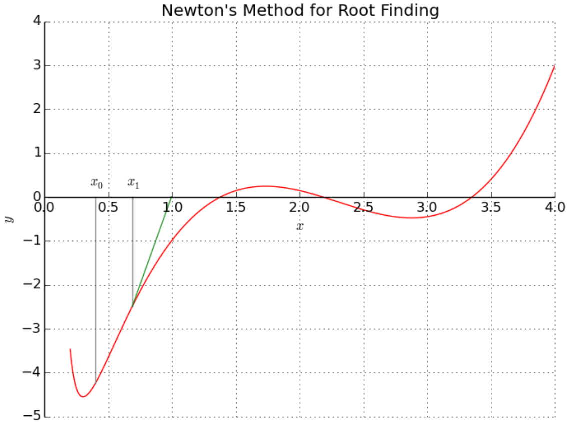

$$x_1 = 0.6866 \quad \Delta = 0.6938$$

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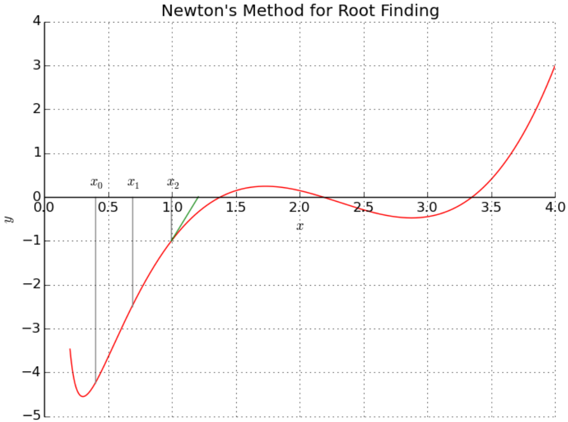

$$x_2 = 0.9938 \quad \Delta = 0.3866$$

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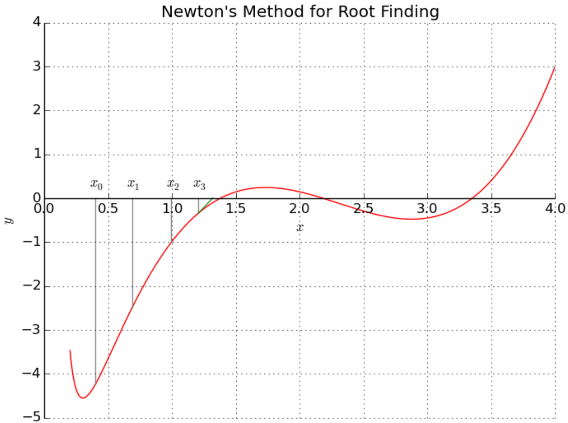

$$x_3 = 1.2050 \quad \Delta = 0.1754$$

Converges in 28 iterations.

Optimizing Function Using Newton's Method

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Instead of:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use derivatives:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

Optimizing $f : \mathbb{R}^n \mapsto \mathbb{R}$

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becomes,

$$x_{n+1} = x_n - H_f^{-1}(x_n)\nabla f(x_n)$$

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Def: The gradient of $f : \mathbb{R}^n \mapsto \mathbb{R}$ is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x_1} f & \frac{\partial}{\partial x_2} f & \dots & \frac{\partial}{\partial x_n} f \end{bmatrix}$$

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$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \ddots & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

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