Chapter 10

Boundaries in the Plane and in Space

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1 Introduction

Chapter 10 deals with solving wave and diffusion problems the same way we had in previous chapters, but this time in higher dimensions. Specifically, it provides methods for the second and third dimension. In this summary of the chapter, we will structure it a little differently from the text for the sake of simplicity. First, we will go over the first section, which is essentially a reminder of information from previous chapters that will come in handy.

Next, we will discuss the Bessel functions of order n. After this, we will talk about Legendre functions. Last, we will go over angular momentum of a hydrogen atom as a practical quantum mechanics example.

2 Fourier's Method, Revisited

This section is provided as a way to refresh on methods from Chapters 4 and 5, where we learn how to use separation of variables and Fourier series to solve one dimensional wave and diffusion problems. We can denote the Laplacian as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{or} \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

in two or three dimensions respectively. This was first seen in Chapter 1 Section 3. Using this definition, we get the wave equation and the diffusion equation to be

$$u_{tt} = c^2 \Delta u \quad \text{and} \quad u_t = k\Delta u \tag{2}$$

respectively. We then say they have some bounded domain D with some classical homogeneous condition along the boundary of D and with some standard initial condition. For brevity, we will assume we are working with three dimensions and we will also let $\mathbf{x} = (x, y, z)$. Even though we will be assuming three dimensions, it should be noted that this is a general concept which can be applied to two dimensions as well. In three dimensions, however, we get that D is a solid domain and *bdy* D is a surface.

We can then separate the time variable using methods from chapter 4 to get

$$u(x, y, z, t) = T(t)v(x, y, z). \tag{3}$$

Then, plugging this into the wave equation gives us

$$\begin{aligned} T''(t)\mathbf{v} &= c^2 T(t)\Delta\mathbf{v} \\ \frac{T''(t)}{c^2 T(t)} &= \frac{\Delta\mathbf{v}}{\mathbf{v}} = -\lambda, \end{aligned} \tag{4}$$

where λ is a constant. Similarly, plugging this value of u into the diffusion equation gives

$$\begin{aligned} T'(t)\mathbf{v} &= kT(t)\Delta\mathbf{v} \\ \frac{T'(t)}{kT(t)} &= \frac{\Delta\mathbf{v}}{\mathbf{v}} = -\lambda, \end{aligned} \tag{5}$$

where λ is a constant. Both cases give the eigenvalue problem

$$\begin{gathered} -\Delta \mathbf{v} = \lambda \mathbf{v} \text{ in } D \\ v \text{ satisfies } (D),(N),(R), \quad \text{on } bdy\ D, \end{gathered} \tag{6}$$

which gives a solution to the wave equation as

$$u(\mathbf{x}, t) = \Sigma[A_n cos(\sqrt{\lambda_n} ct) + B_n sin(\sqrt{\lambda_n} ct)] v_n(x) \tag{7}$$

and a diffusion equation solution as

$$u(\mathbf{x}, t) = \Sigma A_n e^{-\lambda_n kt} v_n(x). \tag{8}$$

After this example, we define the inner product as

$$(f, g) = \int\int_D\int f(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x} \quad (\text{ where } d\mathbf{x} = dxdydz) \tag{9}$$

for two functions f, g. Then, f and g are said to be orthogonal if $(f, g) = 0$.

The last review we will go over is defining multiplicity. We say that an eigenvalue λ has multiplicity m if it has m linearly independent eigenfunctions. This means the eigenspace for λ is $m-$dimensional.

3 Bessel Functions

We first encounter Bessel functions in section 2. This section has us assuming we have a circular drum $D = \{x^2 + y^2 < a^2\}$ where a is the radius of the drum. If one were to hit the drum, it would create waves in two dimensions. This means we want to solve the problem

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}) & \text{in } D \\ u = 0 & \text{on } bdy\ D \\ u, u_t \text{ are given functions when } t = 0. \end{cases} \tag{10}$$

We are dealing with a circular drum, and so we can do what was done in section 6.3 and convert this to polar coordinates. This gives

$$c^{-2} u_{tt} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. \tag{11}$$

We can then separate the variables like we did in chapter 4 so that $u(r, \theta, t) = T(t)R(r)\Theta(\theta)$. After separating the variables, we get

$$\frac{T''}{c^2 T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}. \tag{12}$$

we then can set $T''/c^2T = -\lambda$ and $\Theta''/\Theta = -\gamma$, which gives three ODEs. These ODEs can be solved with relative ease using methods from section 6.3 again. After doing so, we will let $\rho = \sqrt{\lambda} r$. This then gives us

$$R_r = R_\rho \frac{\mathrm{d}\rho}{\mathrm{d}r} = \sqrt{\lambda} R_\rho, \qquad R_{rr} = \lambda R_{\rho\rho} \tag{13}$$

and

$$R_{\rho\rho} + \frac{1}{\rho} R_\rho + (1 - \frac{n^2}{\rho^2}) R = 0. \tag{14}$$

This is known as Bessel's differential equation of order n. The solution to this ODE is known as the Bessel function of order n and is given by

$$J_n(\rho) = \sum_{j=0}^{\infty} (-1)^j \frac{(\frac{1}{2}\rho)^{n+2j}}{j!(n+j)!} \tag{15}$$

where n is an integer. However, we might be presented with the ODE of a non-integer order. Section 10.5 works through a way of generalizing the Bessel function for this case. We get the Bessel function more generally as

$$J_s(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+s+1)} (\frac{z}{2})^{2j+s}, \tag{16}$$

where Γ represents the gamma function. This is defined as

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds \qquad \text{for } 0 < x < \infty. \tag{17}$$

If x is an integer, then $\Gamma(x+1) = x!$, so we can see how equations (15) and (16) are related. We then get that equation (16) is a solution to Bessel's differential equation, given by

$$u_{zz} + \frac{1}{z} u_z + (1 - \frac{s^2}{z^2}) u = 0 \tag{18}$$

for any $s \in \mathbb{R}$. We also are able to find that as z approaches ∞, the Bessel function is bounded such that

$$[J_s(z) - \sqrt{\frac{2}{\pi z}} cos(z - \frac{s\pi}{2} - \frac{\pi}{4})] z^{\frac{3}{2}} < \infty. \tag{19}$$

We also see the Bessel function show up in Section 3, which discusses vibrations in a ball. We're given the example of the wave equation with Dirichlet BCs end up separating variables in spherical coordinates. At the end of solving this PDE, we end up with the solution being any constant multiple of

$$w(r) = J_{\sqrt{\gamma + \frac{1}{4}}}(\sqrt{\lambda} r), \tag{20}$$

which is a Bessel function of order $n = \sqrt{\gamma + \frac{1}{4}}$.

4 Legendre Functions

We first encounter the idea of Legendre functions in Section 3. This section does not go into depth on these functions outside of briefly mentioning them, so we will go ahead to Section 6. This is where we get details on these types of functions.

When talking on vibrations in spheres in Section 3, we see the differential equation

$$[(1-z^2u']' + \gamma u = 0, \tag{21}$$

where $\gamma = l(l+1)$ for some nonnegative integer l. This is referred to as Legendre's differential equation. A solution to this ODE is given by the Legendre polynomials, which are given by

$$P_l(z) = \frac{1}{2^l}\sum_{j=0}^{m}\frac{(-1)^j}{j!}\frac{(2l-2j)!}{(l-2j)!(l-j)!}z^{l-2j} \tag{22}$$

where $m = \frac{l}{2}$ if l is even and $m = \frac{l-1}{2}$ if l is odd. Then, we can choose the value of l, which will then fix γ to a certain value. Then, $P_l(z)$ is a solution to equation (21). We also have that the Legendre polynomials satisfy the orthogonality relation. This means if $l \neq l'$, then

$$\int_{-1}^{1} P_l(z)P_{l'}(z)dz = 0. \tag{23}$$

With each of these Legendre polynomials, it can be difficult to compute each of these sums. Instead, we can use Rodrigues' Formula, which says

$$P_l(z) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}z^l}\ (z^2-1)^l. \tag{24}$$

Then, we can see if $l = 0$, we get the solution

$$\begin{aligned} P_0(z) &= \frac{1}{2^0 0!}\frac{\mathrm{d}^0}{\mathrm{d}z^0}\ (z^2-1)^0 \\ &= \frac{1}{1} * 1 = 1. \end{aligned} \tag{25}$$

Then, if we let $u = 1$, we can see this is indeed a solution to equation (21).

We then are given the associated Legendre equation as

$$[(1-z^2)u']' + (\gamma - \frac{m^2}{1-z^2})u = 0, \tag{26}$$

where $\gamma = l(l+1)$, $m \leq l$, and $m, l \in \mathbb{Z}$. Then, we define the associated Legendre functions as

$$P_l^m(z) = (1-z^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\ P_l(z). \tag{27}$$

We can then see that if $m = 0$, we just end up with equation (24). For these functions, we have the normalizing constants as

$$\int_{-1}^{1}[P_l^m(z)]^2 dz = \frac{2(l+m)!}{(2l+1)(l-m)!}. \tag{28}$$

5 Angular Momentum

In the discussion of angular momentum, the text calls back to Section 3, where the Y equation, given as

$$\frac{1}{sin^2\theta}Y_{\theta\theta} + \frac{1}{sin\theta}(sin\theta\ Y_\theta)_\theta + \gamma Y = 0 \tag{29}$$

with γ being the separation constant. This comes up in separating the time variable from Schrodinger's equation, followed by the radial variable. This gives $v = R(r)Y(\theta, \phi)$ where

$$Y(\theta, \phi) = Y_l^m(\theta, \phi) = P_l^m(cos\theta)e^{im\phi}. \tag{30}$$

So, the Y equation involved is just an associated Legendre function.

If we were to observe the case of a hydrogen atom where $V(r) = \frac{1}{r}$, then we can separate the variables again to get

$$R_{rr} + \frac{2}{r}R_r + [\lambda + \frac{2}{r} - \frac{l(l+1)}{r^2}]R = 0. \tag{31}$$

The case of $l = 0$ is discussed in Chapter 9 Section 5. Then, choosing the boundary conditions such that $R(0)$ is finite and $R(\infty) = 0$, we can follow the method used when l was zero. Doing so, we get the eigenfunctions that take on the form

$$v_{nlm}(r, \theta, \phi) = e^{-\frac{r}{n}}L_n^l(r) \cdot Y_l^m(\theta, \phi), \tag{32}$$

where $l \in \mathbb{Z}, 0 \leq l < n$. These are then the wave functions of the hydrogen atom. As mentioned above, we separated Schrodinger's equation into the radial variable and the Y equation. Here, we are solving just for $R(r)$, so the separated solutions for Schrodinger's equation are

$$e^{\frac{it}{2n^2}} \cdot e^{-\frac{r}{n}} \cdot L_n^l(r) \cdot Y_l^m(\theta, \phi). \tag{33}$$