Puzzles and Group Theory

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October 2024

Contents

1 History of the Puzzles

In this paper, we will explore three puzzles in particular and how they are related to group theory. These puzzles, as pictured below, are the 15 Puzzle, a Pyraminx, and a 3x3 Rubik's Cube. We will be discussing the notation of each puzzle's legal movements, showing that the combination of moves of each puzzle forms a group, calculating the order of each group, and discussing the Sylow p-subgroups of each. First, we will talk about the history of all three puzzles.

1.1 History of the 15 Puzzle

The 15 Puzzle has an unclear history, however most sources will credit Sam Loyd as the inventor of the puzzle. According to Rezk (2023), Loyd was an American chess player who also developed puzzles in his free time. However, Loyd did not invent the puzzle, and instead it was invented in 1874 by Noyes Palmer Chapman, who was a postmaster in New York.

This misattribution was by no means an accident. Loyd had realized that Chapman's puzzle is not solvable from every possible position. He took advantage of this and set a challenge with a grand prize of $\$1,000$. The challenge was to solve the puzzle with only the 14th and 15th tiles swapped. This is illustrated below. This, as it turns out, is impossible because of the structure of the puzzle.

1.2 History of the Pyraminx

The Pyraminx was developed sometime around 1971 by Uwe Meffert (Pyraminx/Tetraminx 2024). This was actually a few years before the invention of the more well known Rubik's cube. Meffert later got a patent in 1981 after seeing the popularity and success of the Rubik's Cube. Meffert later went on to have his own line of puzzles being designed and manufactured. This puzzle has a similar goal to the Rubik's Cube in that you twist colored sides to make each face of the polyhedra the same color. The only difference is it's in the shape of a pyramid instead of a cube.

1.3 History of 3x3 Rubik's Cube

The 3x3 Rubik's Cube was one of the first kind of puzzle known as "twisty puzzles". These are simply puzzles designed so each side twists and moves around some core to shuffle the pieces of the puzzle around. As Reese (2020) mentions, it's named after its inventor, Erno Rubik, who was an architect. In 1974, he was interested in developing a way to demonstrate movement in three dimensions to his students. He made the first prototype out of wood, paper, gum bands, glue, and paper clips.

After creating the puzzle, it took Rubik an entire month just to solve it. This is a very difficult feat to accomplish given that this prototype was the first puzzle of its kind. Rubik was also unsure if the cube was even solvable. It was hard to say if he could get it solved after scrambling the puzzle, but he kept at it. He did manage to figure it out, though, and it became one of the world's most popular toys to date.

2 Notation

Each puzzle is slightly different in the way it moves, and so it is important to discuss the notation for each puzzle's movements. Below is each notation we will use in our discussions of these groups.

2.1 15 Puzzle

Figure 1: 15 Puzzle

For the 15 puzzle, there are 16 possible places for the tiles to end up, with one of the spaces always being empty. This way, the tiles can move around by essentially swapping places with the empty spot. We will notate this by just saying the position of the tile we will swap with the blank space. The important thing to note is we can only swap tiles adjacent to the blank space as that is the legal way to move pieces in the puzzle.

For example, if we had the string 12-11-7-8-12-16 starting from a solved state, that would give us the following

Figure 2: Scrambled 15 Puzzle

This looks starting with a solved puzzled, then sliding the tile in the 12th position into the open spot, then the tile in the 11th position, then the 8th, and so on.

2.2 Pyraminx

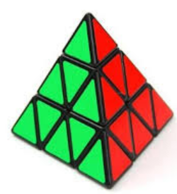


Figure 3: Pyraminx

The Pyraminx is a more complicated puzzle, and so it has more complicated notation. The notation assumes one triangular side is facing the solver, with the flat side on the bottom and the point at the top. Then, we notate U to mean a rotation of the upper layer clockwise if viewed head on. Then, U' is used to denote a rotation of the upper layer counter clockwise. Similar notation is used for the left layer (L and L'), the right layer (R and R'), and the back layer (B and B').

As an example, if we were to do the moves $RUR'U'R'LRL'$, then we end up with

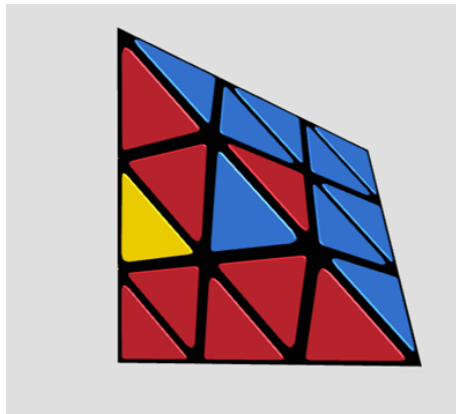


Figure 4: Scrambled Pyraminx

This is a specific algorithm which flips exactly 2 edges in place. The Pyraminx is also capable of another movement. The tips of the puzzle can also rotate in place. This is notated similar to the above, just with lower case letters instead. So, turning the tip on the top of the puzzle is notated with u, u', the left tip is l, l', the right tip is r, r', and the back tip is b, b'.

2.3 Rubik's Cube

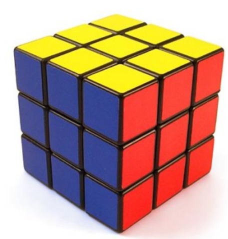


Figure 5: 3x3 Rubik's Cube

The notation for this will be similar to what we had for the Pyraminx, except now we have six sides. Again, we will assume one face of the cube is facing the solver, then choose a rotation of the upper, left, right, and back faces clockwise to be denoted with a U, L, R and B respectively, and counterclockwise to be U', L', R' and B'. Then, we also want to include the front face (F, F') and the bottom face (D, D'). Something we can also do with a 3x3 Rubik's cube is notate double moves. That is, if we have $R2$, this means we should rotate the right face twice. We can see that 2 $90°$ rotations either clockwise or counterclockwise end with the same result. This means $R2$ doesn't need to notate which direction it's being rotated.

An example of this notation is $RUR'URU2R'$. Performing this sequence of moves on a solved cube gives the state

Figure 6: Scrambled Rubik's Cube

3 Are They Groups?

For these to be a group, we will look at Dummit and Foote's (2004) definition of a group. This definition says that a group is an ordered pair $(G, *)$ that satisfies the following axioms:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$
2. There exists an element $e \in G$, such that for all $a \in G, a * e = e * a = a$
3. For each $a \in G$ there is an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

(1)

We will use this definition to see whether the structure of the puzzles forms a group. We will let F be the set of all possible configurations of a 15 puzzle, P to be the set of all possible configurations of a Pyraminx, and R to be the set of all possible configurations. We will the show the ordered pairs $(F, *)$, $(P, *)$, and $(R, *)$ are groups where $*$ represents performing a certain move. It's important to note that all 3 puzzles move in different ways, so $*$ means something slightly different for each puzzle. However, we can generalize this. If we have two elements $a, b \in G$, where G represents the set of possible configurations in one of the three puzzles, then $a * b$ represents starting with a solved puzzle and performing the moves to get to configuration a. Then, our operation tells us to perform the moves to get to configuration b from a solved state.

3.1 Is F a Group?

First, we will show that $(F, *)$ matches this definition of a group. First, it's clear to see that Axiom 2 holds. This is because if we choose some element $a \in F$, then the act of doing nothing, which we will call the identity of F, is the same as $a * e = e * a = a$ where e is the identity. Therefore, the identity element of F is the same as doing nothing.

Next, we can see that Axiom 3 holds. This is because each configuration of the 15 puzzle is achieved by moving one square to the only possible open square next to it. This means we can undo any configuration and get back to e, which is the solve state, by undoing each individual move that got us to that point. For example, let's say $b_1 * b_2 * ... * b_n = a \in F$ is a possible configuration of the puzzle made up of an arbitrary number of $b_i \in F$, which represent single moves only, then $a^{-1} = b_n^{-1} * ... * b_2^{-1} * b_1^{-1}$. This means every element of the puzzle has an inverse.

Because of the design of a 15 puzzle, we can see that every possible set of moves is just a permutation of tiles. This means we can pick permutations $\sigma, \tau, \rho \in F$ and show that these permutations are associative. This means showing that $(\sigma * \tau) * \rho = \sigma * (\tau * \rho)$. We can then define $(\sigma * \tau)(x) = \sigma(\tau(x))$ for any element x being permuted. Then, we can show that $((\sigma * \tau) * \rho)(x) = (\sigma * (\tau * \rho))(x)$ by first evaluating the left hand side, then the right, and showing the two are equal. First, we evaluate the left side to get

$$\begin{aligned} ((\sigma * \tau) * \rho)(x) &= (\sigma * \tau)(\rho(x)) \\ &= \sigma(\tau(\rho(x))) \end{aligned} \tag{2}$$

Next, we can evaluate the right side, which yields

$$\begin{aligned} (\sigma * (\tau * \rho))(x) &= \sigma * ((\tau * \rho)(x)) \\ &= \sigma(\tau(\rho(x))) \end{aligned} \tag{3}$$

This shows the two sides are equal and so $(\sigma * \tau) * \rho = \sigma * (\tau * \rho)$ holds. This tells us that the operation $*$ is associative. This means all three requirements for F to be a group hold, which proves what we wanted to show.

3.2 Are R and P Groups?

Then, if we want to show R and P are also groups, we first need to recognize that the identity will be inaction for both puzzles. That is, the identity for all three sets is the action of doing nothing. Similarly, the inverse for any elements of R and P is simply to reverse the actions that led to the given permutation. Last, we can again realize that a 3x3 Rubik's cube and a Pyraminx move each of its pieces by permuting them. This means the operation for each puzzle is once again associative. This tells us R and P must also be groups.

4 Orders of the Groups

When talking about groups, it is very helpful to know how large the group actually is. This section will focus on the calculations necessary to find the order of each puzzle's group. This will in effect be the number of ways the puzzle can be mixed up having only done legal moves from a solved state.

4.1 Order of 15 Puzzle

First, we will find the order of F. We can see that because there are 15 tiles and 1 blank space, there are 16!= 20,922,789,888,000 ways to swap them around. However, not every one of these arrangements is possible. For example, from a solved puzzle, it is impossible to swap the tiles labeled 14 and 15. In fact, the puzzle only allows even permutations. This means the order of F is $\frac{16!}{2} = 10,461,394,944,000$. This tells us that F is isomorphic to the alternating group A_{16}.

4.2 Order of Pyraminxes

Next, we will find the order of P. A Pyraminx has 4 corner pieces, which rotate in place, and 6 edge pieces, which can move around the puzzle. These corners are the triangular piece in between each edge piece and function differently from the tips. We can find the number of ways the corners can be arranged by recognizing that each of the corners only has 3 possible orientations. This means the total number of ways to twist the corners is $3^4 = 81$.

The edge pieces move around the cube, and so there are $6! = 720$ ways to arrange the edges. Each of these edges can be arranged in 2 ways since it has 2 colors on it. This means given a specific arrangement of the edges, there are $2^6 = 64$ ways to flip them. However, not all of these positions are possible. It turns out that we can flip the first 4 edges however we like, but the 5th and 6th ones must be oriented a specific way. This means using legal moves, there are actually $2^4 = 16$ ways to flip the edges.

We can then multiply the 3 results together to find the total number of scrambled Pyraminxes possible. This gives us $|P| = 3^4 * 6! * 2^4 = 933,120$ legal positions.

However, we should also factor in the tips. These pieces move completely independently of any other piece, so they don't add a lot to the puzzle, but each tip increases to total number of permutations. This means we need to multiply by another factor of 3^4, which gives us $|P| = 3^4 * 3^4 * 6! * 2^4 = 75,582,720$.

4.3 Order of 3x3 Rubik's Cube

A 3x3 Rubik's Cube has 8 corner pieces and 12 edge pieces. These are the only pieces that move around the puzzle since each face rotates around a center piece. We can then do what we did for the Pyraminxes and calculate how many ways these two piece types can be arranged and multiply those numbers together.

First, we will calculate how many ways we can arrange the edge pieces. Because there are 12 edges, there are $12! = 479,001,600$ ways to arrange them. Next, we have that each edge has 2 colors, and so there are $2^{12} = 4096$ ways to have an arrangement of edges flipped. However, if we arrange 11 edges, then the last edge's orientation is already decided. This is due to the parity of the puzzle. This means there are $2^{11} = 2048$ ways to flip the edges of a given arrangement. Then, there are $479,001,600 * 2048 = 980,995,276,800$

Next, we will calculate how many ways we can arrange the corner pieces. There are 8 pieces, and so there are $8! = 40,320$ ways to place them. Then, we need to think about rotations. Each corner has 3 colors on it, and so there are $3^8 = 6,561$ ways for a given arrangement of corners to be rotated. However, like what we had above, not all of these positions are possible using legal moves from a solved cube. In fact, we can twist the first 7 corners however we want, but the last one will be required to be rotated a certain way. This means there are $3^7 = 2,187$ ways to twist them for it to still be an element of R. This gives $40,320 * 2,187 = 88,179,840$ ways the corners can be arranged.

The last step is to combine these results together. We multiply the number of ways the edges can be arranged with the number of ways the corners can be arranged. This gives us $980,995,276,800*88,179,840 = 86,504,006,548,979,712,000$. However, one last thing we need to account for is the fact that a Rubik's cube will never have only a set of adjacent corners or edges swapped. If there were adjacent corners swapped, then other pieces must be swapped. The same is true for adjacent edges. This means we need to half the result from earlier, which gives us

$$\begin{aligned} |R| &= \frac{8! * 3^7 * 12! * 2^{11}}{2} \\ &= \frac{1}{2} * 86,504,006,548,979,712,000 \\ &= 43,252,003,274,489,856,000 \end{aligned} \tag{4}$$

This is approximately 43 quintillion ways to scramble the puzzle.

5 Sylow P-Subgroup

We have that if a group G has an order of $p^\alpha m$ for some $\alpha \geq 0$ and $p \nmid m$, then there is a subgroup of order p^α called a Sylow p-subgroup of G. We can use the prime factorizations and Sylow's Theorem to analyze the 3 groups we have. We have that the orders can be factored as

$$\begin{aligned}
|F| &= 10,461,394,944,000 = 2^{14} * 3^{6} * 5^{3} * 7^{2} * 11^{1} * 13^{1} \\
|P| &= 75,582,720 = 2^{8} * 3^{10} * 5^{1} \\
|R| &= 43,252,003,274,489,856,000 = 2^{27} * 3^{14} * 5^{3} * 7^{2} * 11^{1}
\end{aligned} \tag{5}$$

We could analyze each individual p-subgroup possible for each of our groups. To analyze each group, we will take advantage of Sylow's Theorems. Specifically, we will use the first theorem, which says that if G has an order of $p^{\alpha}m$ where p is a prime that doesn't divide m, then there is a Sylow p-subgroup of order p^{α}.

When determining if Sylow p-subgroups are normal, one needs to decide if there is a unique subgroup of its kind. Calculating this can be challenging, especially with the size of the groups we're looking at. Because of this, we will use the program GAP to help us better assess the groups normality.

5.1 15 Puzzle's Sylow p-Subgroups

For the 15 Puzzle, we can see that there are Sylow 2-subgroups, 3-subgroups, 5-subgroups, 7-subgroups, 11-subgroups, and 13-subgroups. Moreover, we know the sizes of these subgroups are $2^{14}, 3^{6}, 5^{3}, 7^{2}, 11$, and 13 respectively. Each of these can translate into legal movements of a 15 puzzle's tiles. The order of each Sylow p-subgroup tells us how many combinations of moves will cycle p tiles. Specifically, there are 2^{14} ways to cycle pairs of tiles, 3^{6} ways to cycle tiles in groups of 3, 5^{3} ways to cycle tiles in groups of 5, 7^{2} ways to cycle tiles in groups of 7, 11 ways to cycle tiles in groups of 11, and 13 ways to cycle tiles in groups of 13.

We can then use the fact that a Sylow p-subgroup is normal in G if and only if that subgroup is the only such Sylow p-subgroup in G. We will keep this in mind to see if there may be a unique Sylow 2-subgroup before using GAP.

By the parity of the 15 puzzle, we can't simply swap any 2 tiles. We can, however, swap 2 pairs of tiles. As an example of an element of the subgroup, we will pick the set of moves $n \in$ Sylow 2-subgroup which swaps the tiles in the 14th and 15th positions as well as the tiles in the 10th and 11th positions. This works out with the parity of the puzzle, and thus is possible with legal moves.

Next, we will choose some element of $f \in F$ that cycles the tiles by swapping them with the empty space in a specific order. This element will specifically be in the order of the tile in the positions 12-8-4-3-7-11-15-16. Then, $f^{-1} \in F$ is swapping the tiles in positions 15-11-7-3-4-8-12-16 with the blank tile in that order.

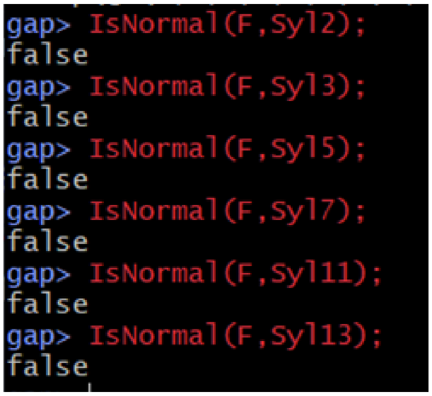
Then, if we check $f * n * f^{-1}$, we end up with the equivalent of swapping the tiles in the 15th position and the 10th position as well as the tiles in the 12th position and the 14th position. This means we are swapping tiles in pairs, and it is then an element of the Sylow 2-subgroup. Then, if we want to see if this is normal, we need to test this with every element $f \in F$ against every $n \in$ Sylow 2-subgroup. However, given that there are over 16,000 elements of the Sylow 2-subgroup to check, this is impractical to do. Despite this, we have some evidence that the Sylow 2-subgroup may be normal within F. If this is the case, then the number of Sylow 2-subgroups in F would necessarily be 1.

While it seems to be the case that the Sylow 2-subgroup is normal, this isn't rigorous. As mentioned above, this would be an impossibly arduous task. Luckily, we can use *GAP*, which has a function to come up with the Sylow 2-subgroups of some group and identify if it is normal or not. First, we can recall that this group is cycling 16 objects, specifically 15 tiles and 1 blank space. It is also the case that every move must be an even permutation. We need to know this because *GAP* doesn't have a function specifically for the 15 puzzle, but they do have a function for A_{16}. Because the 15 puzzle is isomorphic to A_{16}, the properties of its Sylow p-subgroups will translate nicely.

We can then define our group as below:

```
gap> F := AlternatingGroup(16);
Alt( [ 1 .. 16 ] )
gap> Syl2 := SylowSubgroup(F,2);
<permutation group of size 16384 with 14 generators>
gap> Syl3 := SylowSubgroup(F,3);
Group([ (1,2,3), (4,5,6), (7,8,9), (1,4,7)(2,5,8)(3,6,9), (10,11,12), (13,14,15) ])
gap> Syl5 := SylowSubgroup(F,5);
Group([ (1,2,3,4,5), (6,7,8,9,10), (11,12,13,14,15) ])
gap> Syl7 := SylowSubgroup(F,7);
Group([ (1,2,3,4,5,6,7), (8,9,10,11,12,13,14) ])
gap> Syl11 := SylowSubgroup(F,11);
Group([ (1,2,3,4,5,6,7,8,9,10,11) ])
gap> Syl13 := SylowSubgroup(F,13);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13) ])
```

The first line of code is defining our group F as the group A_{16}. The next few red lines are having the program set up each of the Sylow p-subgroups. This is necessary because we will be using the function $IsNormal$. We can then use this to see if there are any normal Sylow p-subgroups in F. Plugging it into the program gives us



```
gap> IsNormal(F,Syl2);
false
gap> IsNormal(F,Syl3);
false
gap> IsNormal(F,Syl5);
false
gap> IsNormal(F,Syl7);
false
gap> IsNormal(F,Syl11);
false
gap> IsNormal(F,Syl13);
false
```

From this, we can see that there are no normal Sylow p-subgroups within the group F. This means for any $f \in F$ and Sylow p-subgroup N, $fNf^{-1} \nsubseteq N$.

5.2 Pyraminx's Sylow p-Subgroups

Again, we can use Sylow's first theorem to see that the Pyraminx group P has a Sylow 2-subgroup, 3-subgroup, and 5-subgroup. These subgroups then have orders of $2^8, 3^{10}$, and 5 respectively. We can also use this information to tell us that there are 2^8 ways to cycle pairs of pieces, 3^{10} ways to cycle groups of 3 pieces, and 5 ways to cycle groups of 5 pieces.

Next, we can use the fact that a Sylow p-subgroup is normal in G if and only if the subgroup is unique. We will check whether the Sylow 3-subgroup in our group P could be normal.

If a Sylow 3-subgroup is normal in P, then we know that

$$pNp^{-1} \subseteq N \tag{6}$$

where N is the subgroup and p is some element of P. We can see that the moves $R'LRL' \in N$ since they cycle 3 pieces around. Then, if we pick some other element of P, say $p = RUR'U'L'ULU'$, then we can see if the conjugation gives us another element of N. We just need to keep in mind that $p^{-1} = UL'U'LURU'R'$.

Then, we have

$$pnp^{-1} = (RUR'U'L'ULU')(R'LRL')(UL'U'LURU'R') \tag{7}$$

After performing this sequence, we get

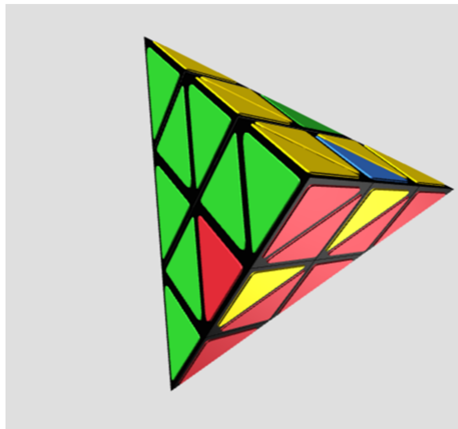
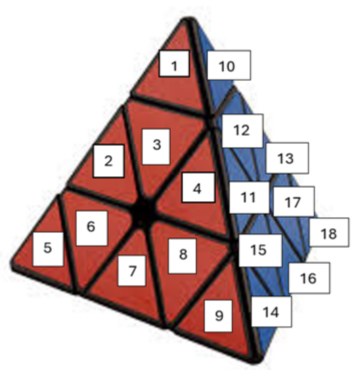
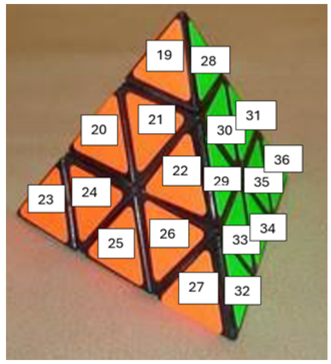


Figure 7: Pyraminx

This is also simply a solved Pyraminx with 3 pieces cycled. This means that for our choice of p and n, we have that $pnp^{-1} \in N$. Once again, there are far too many positions to check by exhaustion, but this provides us some evidence that the Sylow 3-subgroup may be normal in P. This then means the subgroup may be the only one of its kind.

We will now use *GAP* to build a representation of the Pyraminx group and determine if there are any normal Sylow p-subgroups on the Pyraminx. When we're dealing with this group on *GAP* we need to keep in mind that it doesn't have a designated function for the puzzle. Instead, we can build up the basis of the function by seeing how each sticker cycles around the puzzle. To be able to do this, we will label the puzzle as follows:





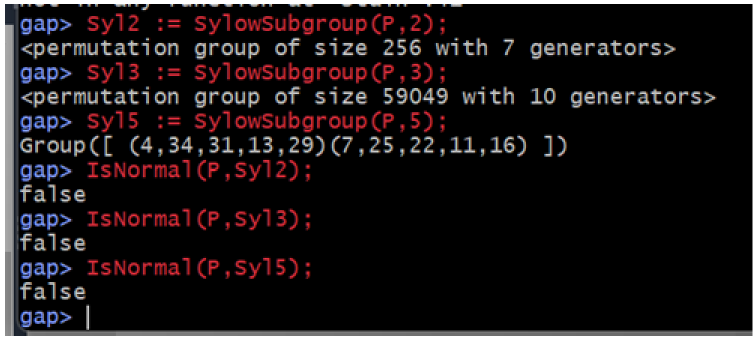
It's important to not that the labeling is arbitrary and it's just a way to tell the program where there is a unique and distinct sticker. Next, we need to work out all of the generating moves. These are essentially the generators of our group P. We can see that these moves are simply twisting each corner as well as each of the tips. From these few moves, we get all of the possible combinations of a Pyraminx.

Then, when we take these into account, we can tell GAP how the pieces are able to cycle around. This looks like

```
gap> gen1 := (1,10,23);
(1,10,23)
gap> gen2 := (5,19,28);
(5,19,28)
gap> gen3 := (9,14,36);
(9,14,36)
gap> gen4 := (18,32,27);
(18,32,27)
gap> gen5 := (2,11,25)(3,12,24)(4,13,20);
(2,11,25)(3,12,24)(4,13,20)
gap> gen6 := (21,30,6)(20,29,7)(2,22,31);
(2,22,31)(6,21,30)(7,20,29)
gap> gen7 := (8,35,15)(4,31,16)(11,7,34);
(4,31,16)(7,34,11)(8,35,15)
gap> gen8 := (33,26,17)(34,22,13)(29,25,16);
(13,34,22)(16,29,25)(17,33,26)
gap> P := Group(gen1,gen2,gen3,gen4,gen5,gen6,gen7,gen8);
<permutation group with 8 generators>
gap> Size(P);
75582720
```

In this, we are defining variables labeled $gen1$ to $gen8$. These are the generators of our group. The elements $gen1$ to $gen4$ represent twisting the tips of the puzzle. Then, $gen5$ to $gen8$ represent twisting each corner excluding the tips. The second to last line is then defining P as a group built up by these elements, which makes them generators of the group. Then, the last line just checks to make sure it has the same order to ensure we have the same group represented. As we can see, this group also has an order of $75,582,720$.

Then, we can determine the normality of each Sylow p-subgroup. Using $GAP's$ functions, this looks like



```
gap> Syl2 := SylowSubgroup(P,2);
<permutation group of size 256 with 7 generators>
gap> Syl3 := SylowSubgroup(P,3);
<permutation group of size 59049 with 10 generators>
gap> Syl5 := SylowSubgroup(P,5);
Group([ (4,34,31,13,29)(7,25,22,11,16) ])
gap> IsNormal(P,Syl2);
false
gap> IsNormal(P,Syl3);
false
gap> IsNormal(P,Syl5);
false
gap> |
```

The first 3 lines define for the computer what each of the Sylow subgroups look like. It identifies each of the Sylow p-subgroups in P. Then, the last 3 lines use the function $IsNormal$ to tell if the Sylow p-subgroups are normal in P, and thus if they're normal.

5.3 3x3 Rubik's Cube p-Subgroups

Lastly, we can find that the Rubik's Cube group R has a Sylow 2-subgroup, 3-subgroup, 5-subgroup, 7-subgroup, and 11-subgroup each of order $2^{27}, 3^{14}, 5^3, 7^2$ and 11 respectively. This tells us that there are 2^{27} ways to cycle pairs of pieces at a time, 3^{14} ways to cycle groups of 3 pieces, 5^3 ways to cycle groups of 5 pieces, 7^2 ways to cycle groups of 7 pieces, and 11 ways to cycle groups of 11 pieces.

We will do what we did above and show that there is a possibility that the Sylow 2-subgroup is normal in R, and is thus unique. Again, the number of positions that would need to be tested is too exhaustive to check each, so we will do one example.

For this one, we will choose $n \in N$ where $n = RU'R'U'RUR'F'RUR'U'R'FR$. We will then pick an $r \in R$ where $r = R2UL'$ and $r' = LU'R2$. Then we have

$$rnr^{-1} = (R2UL')(RU'R'U'RUR'F'RUR'U'R'FR)(LU'R2) \tag{8}$$

This results in

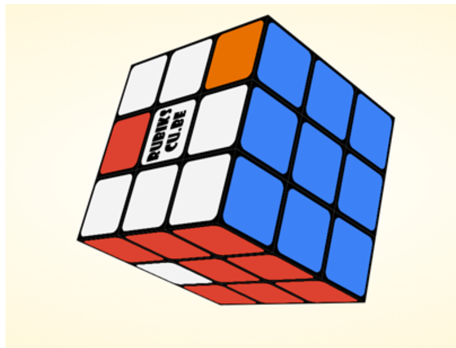
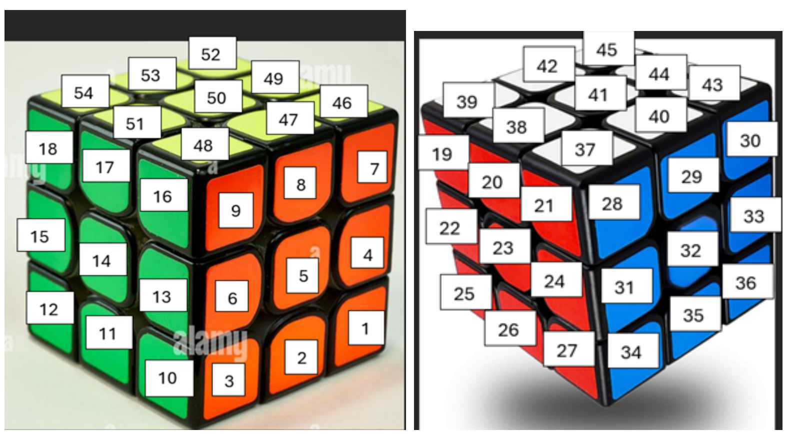


Figure 8: Ruibk's Cube

where 2 edges and 2 corners are swapped. This tells us that $rnr^{-1} \in N$. This suggests that N could be normal, and so the Sylow 2-subgroup could be unique.

However, we can once again define the group R in GAP so that it can do these calculations for us. As we did with the Pyraminx, we will just arbitrarily assign numbers to each sticker. An example of this is the following setup:



This numbering is completely arbitrary. The important thing is each sticker has a unique number. Then, we can represent this in GAP as a group generated by cycles. This group is easier to generate since the possible moves are slightly more limited. That is, we only can move the faces of the cubes. Plugging this in looks like

```
gap> gen1 := (1,10,19,28)(2,11,20,29)(3,12,21,30)(45,39,37,43)(44,42,38,40);
(1,10,19,28)(2,11,20,29)(3,12,21,30)(37,43,45,39)(38,40,44,42)
gap> gen2 := (7,43,21,52)(4,40,24,49)(1,37,27,46)(36,30,28,34)(35,33,29,31);
(1,37,27,46)(4,40,24,49)(7,43,21,52)(28,34,36,30)(29,31,35,33)
gap> gen3 := (9,36,27,18)(8,35,26,17)(7,34,25,16)(52,54,48,46)(53,51,47,49);
(7,34,25,16)(8,35,26,17)(9,36,27,18)(46,52,54,48)(47,49,53,51)
gap> gen4 := (3,48,25,39)(6,51,22,42)(9,54,19,45)(18,12,10,16)(17,15,11,13);
(3,48,25,39)(6,51,22,42)(9,54,19,45)(10,16,18,12)(11,13,17,15)
gap> gen5 := (46,16,45,30)(47,13,44,33)(48,10,43,36)(9,3,1,7)(8,6,2,4);
(1,7,9,3)(2,4,8,6)(10,43,36,48)(13,44,33,47)(16,45,30,46)
gap> gen6 := (54,34,37,12)(53,31,38,15)(52,28,39,18)(25,27,21,19)(26,24,20,22);
(12,54,34,37)(15,53,31,38)(18,52,28,39)(19,25,27,21)(20,22,26,24)
gap> P := Group(gen1,gen2,gen3,gen4,gen5,gen6);
<permutation group with 6 generators>
gap> Size(P);
43252003274489856000
```

Each of these generators just represents each of the faces rotating once. Then, we are defining the group to be generated by these moves, and we can double check the order of the group given. The last line of the code confirms that it has the same order we calculated above, and so we have accurately represented the Rubik's cube in *GAP*.

Next, we need to use the functions for Sylow p-subgroups in *GAP*. This looks like

```
gap> Syl2 := SylowSubgroup(P,2);
<permutation group of size 134217728 with 27 generators>
gap> Syl3 := SylowSubgroup(P,3);
<permutation group of size 4782969 with 14 generators>
gap> Syl5 := SylowSubgroup(P,5);
<permutation group of size 125 with 3 generators>
gap> Syl7 := SylowSubgroup(P,7);
<permutation group of size 49 with 2 generators>
gap> Syl11 := SylowSubgroup(P,11);
Group([ (4,51,24,11,38,49,6,22,29,8,53)(13,15,40,47,26,33,17,31,42,20,35) ])
gap> IsNormal(P,Syl2);
false
gap> IsNormal(P,Syl3);
false
gap> IsNormal(P,Syl5);
false
gap> IsNormal(P,Syl7);
false
gap> IsNormal(P,Syl11);
false
gap>
```

So, what we end up with is that the Rubik's cube group R also has no normal Sylow p-subgroups.

6 Conclusion

We have made a connection between these puzzles and group theory. In showing that these puzzles form a group under their legal moves, we have shown a certain symmetry underlying each of their structures. We have also evaluated what each of the orders of these groups would be, which also helps in understanding the internal structure of the groups.

Not only that, but using code, we have determined that there are no normal Sylow p-subgroups for any of the three puzzles. This was entirely counterintuitive to what it might have seemed. This tells us that each of the Sylow p-subgroups are not invariant under conjugation.

If there were normal Sylow p-subgroups, then we would be able to consider some quotient group, which retains the structure of the puzzle, but reduces the order to make it more manageable. This would allow us to simplify our group analysis, but we are unable to do so. This implies that there is some level of complexity to these puzzles and their corresponding groups.

7 Sources

Rezk, M. (2023, July 18). The Fifteen Puzzle. The Computation Side of Things. https://mohamedrezk122.github.io/fifteen-puzzle

Reese, H. (2020, September 25). A Brief History of the Rubik's Cube. Smithsonian Magazine; Smithsonian Magazine. https://www.smithsonianmag.com/innovation/brief-history-rubiks-cube-180975911/

Pyraminx / Tetraminx. (2024). Jaapsch.net. https://www.jaapsch.net/puzzles/pyraminx.htm

Dummit, D., & Foote, R. (2004) *Abstract Algebra.* John Wiley & Sons, Inc.

Gallian, J. A., & Cengage Learning. (2017). Contemporary abstract algebra. Cengage Learning.

Animesh, C. (2018, December 13). Calculating the Number of Permutations of the Rubik's Cube. Medium. https://medium.com/@chaitanyaanimesh/calculating-the-number-of-permutations-of-the-rubiks-cube-121066f5f054

The Fifteen puzzle. (n.d.). Personal.math.ubc.ca. https://personal.math.ubc.ca/ cass/courses/m308-02b/projects/grant/fifteen.html

Stillman, Z., & Shan, B. (2016). Group Theory and the Pyraminx. https://pi.math.cornell.edu/ apatotski/II project.pdf

O'Neil, C. (2012, July 18). HCSSiM Workshop, day 14. Mathbabe. https://mathbabe.org/2012/07/18/hcssi workshop-day-14/

Learn how to solve a Pyraminx - Beginners Notations Guide — KewbzUK. (2024). KewbzUK. https://ukspeedcubes.co.uk/blogs/notations/beginners-pyraminx-notation-guide

Rubik's Cube Notation - What the rotation letters mean: F R' U2. (n.d.). Ruwix.com. https://ruwix.com/the-rubiks-cube/notation/