Lie-Backlund Transformations and Burgers' Equation

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1 Introduction

Lie-Backlund transformations have been a powerful tool when it comes to solving difficult partial differential equations as they allow us to transform them into something more manageable. As is mentioned in [1], Lie-Backlund transformations had their start in differential geometry in the 1880s.

Before the concept of a Lie-Backlund transformation, Sophus Lie had developed the idea of a Lie tangent transformation. This transformation is very heavily rooted in geometry. We start with a curve C with a line τ tangent to it at the point P. Then, performing a Lie tangent transformation gives us a new curve C' with a tangent line τ' at the point P'. That is, the transformation preserves the local tangency (See figure 1).

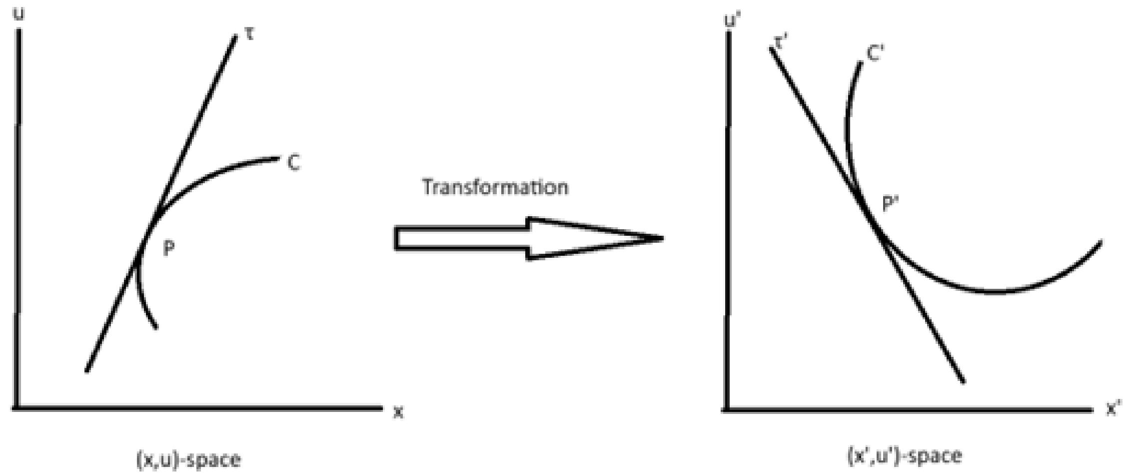

Figure 1: Lie tangent transformation adapted from [1]

The Lie-Backlund transformation gives us a way of generalizing this transformation to apply to partial differential equations. In this sense, the solutions to each PDE are analogous to the tangent point in the figure. Lie-Backlund transformations give us a way to transform the solution of one PDE into the solution of another. This is similar to the Lie tangent transformation above, which transforms a tangent point into another tangent point.

The aim of this paper is to provide an understanding of Lie-Backlund transformations and how they have historically been useful when it comes to solving variations of the Burgers' equation. The first section will be defining what a Lie-Backlund transformation is fundamentally. Throughout the paper, we won't be verifying a choice is indeed a Lie-Backlund transformation, and instead will take this for granted. This is because our focus is understanding how the transformations are useful once we have one. In the second section, we will demonstrate an example of a Lie-Backlund transformation using the Burgers' equation. We will show how we can transform this equation into the heat equation so we can derive a solution.

Following this, we will outline a paper's work on the (2+1)-dimensional Burgers' equation, which extends the equation to take into account two spatial dimensions as opposed to just a single dimension. Last, we will briefly discuss what are known as (2+1)-dimensional strongly coupled Burgers' systems. With these systems, we will see how the Lie-Backlund transformations are derived for (2+1)-dimensional systems. Through these examples, we will see how Lie-Backlund transformations can be incredibly useful when it comes to difficult PDEs while using the Burgers' equation as a reference.

2 Lie-Backlund Transformations

To introduce the idea of these transformations, we will reference Anderson and Ibragimov [1]. The introduction of this begins with defining a Lie tangent transformation. This allows for us to go from a partial differential equation under one variable and convert it into another partial differential equation in terms of another variable. The text outlines this by considering the space $\mathbb{R}^{2n+1}$ with variables $x = (x_1, ..., x_n), u, \underset{1}{u} = (u_1, ..., u_n)$. The notation of the latter variables is defined as

$$u_i = \frac{\partial u}{\partial x_i} \tag{1}$$

They then define an invertible transformation T as follows:

$$\begin{aligned} x' &= f(x, u, \underset{1}{u}) \\ u' &= \phi(x, u, \underset{1}{u}) \\ \underset{1}{u}' &= \underset{1}{\psi}(x, u, \underset{1}{u}) \end{aligned} \tag{2}$$

where $f = (f_1, ..., f_n)$ and $\underset{1}{\psi} = (\psi_1, ..., \psi_n)$. From this, T can be extended to new variables referred to as "differentials" given as $dx = (dx_1, ..., dx_n), du, d\underset{1}{u} = (du_1, ..., du_n)$ where

$$\begin{aligned} dx_i' &= \frac{\partial f_i}{\partial x_j} dx_j + \frac{\partial f_i}{\partial u} du + \frac{\partial f_i}{\partial u_j} du_j \\ du' &= \frac{\partial \phi}{\partial x_j} dx_j + \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial u_j} du_j \\ du_i' &= \frac{\partial \psi_i}{\partial x_j} dx_j + \frac{\partial \psi_i}{\partial u} du + \frac{\partial \psi_i}{\partial u_j} du_j \end{aligned} \tag{3}$$

Then, the combined action of equations (2) and (3) is a transformation $\tilde{T}$ in the space $(x, u, \underset{1}{u}, dx, du, d\underset{1}{u})-$ space where $\tilde{T}$ is referred to as a prolongation of T. This set up leads to the following definition.

Definition 1 *A transformation T is called a Lie tangent transformation if the first-order tangency condition*

$$du - \sum_{j=1}^{n} u_j dx_j = 0 \tag{4}$$

is invariant with respect to the transformation $\tilde{T}$.

For simplicity, we will adopt the notation Anderson and Ibragimov use where $\sum_{j=1}^{n} u_j dx_j$ is written as just $u_j dx_j$, so we will continue that shortcut for this section. Very simply what this condition is saying is that u must be smooth and have a well-defined plane that is tangent to every point on the surface that u describes. Equation (4) is first-order because it's focused on du, which is the total derivative of u. We can define higher order tangency conditions.

Definition 2 *The transformation T is called a kth-order tangent transformation if the kth-order tangency conditions*

$$\begin{gathered} du - u_j dx_j = 0 \\ du_{i_1 \dots i_s} - u_{i_1 \dots i_s j} dx_j = 0, \;\; s = 1 \dots k-1 \end{gathered} \tag{5}$$

are invariant with respect to the action of $\tilde{T}$

Sophus Lie, whom this transformation is named after, proved that there can be no transformation where the first-order tangency conditions don't hold but any higher order ones do. Keeping this in mind, we can then define a Lie-Backlund Transformation.

Definition 3 *The transformation T is called a Lie-Backlund tangent transformation if the $\infty-$order tangency conditions*

$$\begin{gathered} du - u_j dx_j = 0 \\ du_i - u_{ij} dx_j = 0 \\ \vdots \end{gathered} \tag{6}$$

are invariant with respect to the action of the prolonged transformation $\tilde{T}$.

This means that if T is a Lie-Backlund tangent transformation, then $d\tilde{u} - \tilde{u_j} d\tilde{x_j} = 0$. We can continue this with the transformation of each point in the space $\mathbb{R}^{2n+1}$ and each condition will still hold. For this paper, we will simply refer to these as *Backlund transformations.*

Throughout this paper, we will not make much of a fuss surrounding the above notation, but it's important to see where the idea is coming from geometrically speaking. The main takeaway from this is these transformations give us a way to change the variable and derivatives of a differential equation in such a way that solutions are preserved. This means we can, in theory, transform a difficult partial differential equation into a more familiar and easier one with at least one known solution. We could then take this known solution and transform it back into a solution of the original PDE.

Now that we have a definition for what a Backlund transformation is, we will show how it can be used.

3 Burgers' Equation

The Burgers' equation has applications in fluid dynamics and traffic flow. This equation was proposed as a way to model turbulent fluid motion. We will start by giving an example of how these transformations are useful. We will do so by finding a solution to the Burgers' equation. This is a nonlinear differential equation, and so it is more difficult than some more familiar equations. The Burgers' equation is given as

$$u_t + uu_x - \nu u_{xx} = 0 \tag{7}$$

where ν is the diffusion coefficient. As is mentioned in [2], the diffusion coefficient is a measure of how much of a substance diffuses through a liquid in one second. For the purposes of simplicity, we will work under the assumption that $\nu = 1$. So, the specific Burgers' equation we will be operating with is

$$u_t + uu_x - u_{xx} = 0. \tag{8}$$

We will note that this equation has only one spatial dimension variable x. This means we are only measuring how the fluid behaves in a singular direction. So, if we were using Burgers' equation to model a ripple in water, we only get information about how the wave reacts in a single arbitrary direction.

The PDE given by the Burgers' equation can be tricky to solve because it has a non-constant coefficient in the term uu_x. However, we can use this idea of a Backlund transformation to transform the Burgers' equation into an easier PDE to solve. We will be outlining Griffiths example (see [3]) for this process.

Griffiths uses a Backlund transformation to go from Burgers' equation to the heat equation. That is, we want the equation

$$v_t = v_{xx}. \tag{9}$$

Then, Griffiths chooses a Backlund transformation given as

$$\begin{aligned} v_x &= -\frac{vu}{2} \\ v_t &= \frac{u^2 v}{4} - \frac{vu_x}{2} \end{aligned} \tag{10}$$

where v_x represents the x part of the Backlund transformation and v_t represents the t part. For now, we won't worry about how this transformation was found and we will just accept that it is in fact a valid Backlund transformation.

We will take advantage of the fact that equation (10) is a Backlund transformation to demonstrate its usefulness. We then can take the derivative of v_x with respect to t and v_t with respect to x. This gives us

$$
\begin{aligned}
v_{xt} &= -\frac{v_t u}{2} - \frac{v u_t}{2} \\
&= -\frac{v_t u + v u_t}{2} \\
v_{tx} &= (\frac{2u u_x v}{4} + \frac{u^2 v_x}{4}) - (\frac{v_x u_x}{2} + \frac{v u_{xx}}{2}) \\
&= \frac{2u u_x v + u^2 v_x}{4} - \frac{v_x u_x + v u_{xx}}{2}.
\end{aligned}
\tag{11}
$$

Then, we can subtract these two equations from one another. For our purposes, we have that $v_{xt} = v_{tx}$. This gives us

$$
0 = -\frac{v_t u + v u_t}{2} - \frac{2u u_x v + u^2 v_x}{4} + \frac{v_x u_x + v u_{xx}}{2}. \tag{12}
$$

Then, we can multiply both sides by 4 to get rid of the fractions. This leaves

$$
0 = -2v_t u - 2v u_t - 2u u_x v - u^2 v_x + 2v_x u_x + 2v u_{xx}. \tag{13}
$$

Then, we can rearrange the equation and simplify as follows

$$
\begin{aligned}
2v u_t + 2u u_x v - 2v u_{xx} &= -2v_t u - u^2 v_x + 2v_x u_x \\
2v(u_t + u u_x - u_{xx}) &= -2v_t u - u^2 v_x + 2v_x u_x \\
u_t + u u_x - v u_{xx} &= -\frac{v_t u}{v} - \frac{u^2 v_x}{2v} + \frac{v_x u_x}{v}.
\end{aligned}
\tag{14}
$$

What we then have at the end of equation (14) is the Burgers' equation on the left. This then means that

$$
-\frac{v_t u}{v} - \frac{u^2 v_x}{2v} + \frac{v_x u_x}{v} = 0. \tag{15}
$$

After this, we will rearrange to solve for v_t. This gives us

$$
\begin{aligned}
-\frac{v_t u}{v} &= \frac{u^2 v_x}{2v} - \frac{v_x u_x}{v} \\
v_t &= -\frac{v}{u}(\frac{u^2 v_x}{2v} - \frac{v_x u_x}{v}) \\
&= -\frac{u v_x}{2} + \frac{v_x u_x}{u}.
\end{aligned}
\tag{16}
$$

Next, we can see that if we subtract v_{xx} on both sides of equation (16), then we end up with the heat equation on the left. To do so, we first need to calculate v_{xx}. This becomes

$$
v_{xx} = -\frac{v_x u}{2} - \frac{v u_x}{2}. \tag{17}
$$

Subtracting this result from equation (16) gives us

$$
\begin{aligned}
v_t - v_{xx} &= (-\frac{uv_x}{2} + \frac{v_x u_x}{u}) - (-\frac{v_x u}{2} - \frac{vu_x}{2}) \\
&= -\frac{uv_x}{2} + \frac{v_x u_x}{u} + \frac{v_x u}{2} + \frac{vu_x}{2} \\
&= (\frac{v_x u_x}{u} + \frac{vu_x}{2}) + (-\frac{uv_x}{2} + \frac{v_x u}{2}) \\
&= \frac{u_x}{2u}(2v_x + vu) + (-\frac{uv_x}{2} + \frac{uv_x}{2})
\end{aligned} \tag{18}
$$

This gives us

$$
v_t - v_{xx} = \frac{u_x}{2u}(2v_x + vu). \tag{19}
$$

Then, the left side of equation (19) is the heat equation. This tells us the the right side of the equation equals 0. We have a term of u is the denominator of the right side, so we will assume that $u \neq 0$. Furthermore, if u were some constant function in the x direction, then we would be left with a trivial solution. For this reason, we will say $\frac{u_x}{2u} \neq 0$. This means that $2v_x + vu = 0$ must be true. We can then rearrange to get a formula for u in terms of v. This gives us

$$
\begin{aligned}
2v_x + vu &= 0 \\
vu &= -2v_x \\
u &= -\frac{2v_x}{v}.
\end{aligned} \tag{20}
$$

It's important to note that we can rearrange this to solve for v_x as well. This gives us

$$
v_x = -\frac{vu}{2}, \tag{21}
$$

which matches the transformation we chose above. This shows there is consistency in our choice. Specifically, this process is showing that v is in fact a solution of the heat equation while u is simultaneously a solution to the Burgers' equation. What this means is any solution to the Burgers' equation can be ascertained if one knows a solution to the heat equation.

For an example, we will let

$$
v = \frac{1}{\sqrt{4\pi t}} e^{(-\frac{x^2}{4t})} \tag{22}
$$

are equivalent. We can verify this is a solution to the heat equation because

$$
\begin{aligned}
v_t &= \frac{x^2}{8t^2\sqrt{\pi t}} e^{(-\frac{x^2}{4t})} \\
v_{xx} &= ((\frac{-x}{4t\sqrt{\pi t}}) e^{(-\frac{x^2}{4t})})_x = \frac{x^2}{8t^2\sqrt{\pi t}} e^{(-\frac{x^2}{4t})}.
\end{aligned} \tag{23}
$$

Then, we can obtain a solution to the Burgers' equation by

$$
\begin{aligned}
u &= -\frac{2v_x}{v} \\
&= -\frac{2(\frac{-x}{4t\sqrt{\pi t}})e^{(-\frac{x^2}{4t})}}{\frac{1}{\sqrt{4\pi t}}e^{(-\frac{x^2}{4t})}} \\
&= -2(\frac{-x}{4te^{\frac{x^2}{4t}}\sqrt{\pi t}})(e^{\frac{x^2}{4t}}2\sqrt{\pi t}) \\
&= \frac{x}{t}
\end{aligned} \tag{24}
$$

It can then be verified that $u = \frac{x}{t}$ is indeed a solution to $u_t + uu_x - u_{xx} = 0$. We get the following partial derivatives of u :

$$
\begin{aligned}
u_t &= -\frac{x}{t^2} \\
u_x &= \frac{1}{t} \\
u_{xx} &= 0.
\end{aligned} \tag{25}
$$

So, plugging $u = \frac{x}{t}$ into the Burgers' equation gives us

$$
\begin{aligned}
u_t + uu_x - u_{xx} &= 0 \\
-\frac{x}{t^2} + (\frac{x}{t})(\frac{1}{t}) - 0 &= 0 \\
-\frac{x}{t^2} + \frac{x}{t^2} &= 0.
\end{aligned} \tag{26}
$$

On the left side we then clearly get 0, so this is a solution.

4 (2+1)-Dimensional Burgers' Equation

As is used by Liu, Li, and Zhang in their paper, (see [4]), there is a general form for the Burgers' equation given as

$$
u_t = au_x^2 + bu_{xx} \tag{27}
$$

where u is the unknown real function and $a, b \in \mathbb{R}$ such that $ab \neq 0$. In this paper, the authors perform a Lie symmetry analysis to find the general solution. The methods of how to perform a Lie symmetry analysis are beyond the scope of this paper, so we will simply note the solution they got was

$$
u(x,t) = \beta - \frac{b}{2a}log(t) - \frac{x^2}{4at} + \sum_{n=0}^{\infty}(-1)^n \frac{a^n}{(n+1)b^n}\eta^{n+1}(xt^{-1})^{n+1}. \tag{28}
$$

where β, η are arbitrary constants. However, as discussed above, the Burgers' equation is only focused on one spatial dimension. This is referred to as the (1+1)-dimensional Burgers' equation where the initial 1 is denoting how many spatial dimensions there are and the second 1 is how many temporal dimensions.

In everyday physics, it's very beneficial to understand how a fluid is behaving in more than one direction. This led to the authors of [5] finding the exact solution for the general (2+1)-dimensional Burgers' equation, which is given by

$$u_t + \alpha u_{xy} + \beta u_x u_y = 0 \tag{29}$$

where $\alpha, \beta \in \mathbb{R}$ such that $\alpha\beta \neq 0$. Again, the equation being "(2+1)-dimensional" means there are two spatial dimensions and one temporal one. We will use this section to outline their work. We can see in this instance we have x, y as our spatial variables.

Remark: It's also helpful to note that in the case that $x \equiv y$, we get

$$u_t + \alpha u_{xx} + \beta u_x^2 = 0. \tag{30}$$

We have that α, β are arbitrary, so this is equivalent to equation (27). This means we will want $x \not\equiv y$ to be true. Next, the authors mention symmetry of a differential equation. We get the following definition of symmetry from Olver, (see [6]):

Definition 4 *A point transformation g is called a symmetry of a system of partial differential equations if $u = f(x)$ is a solution and the transformed function $\overline{f} = g \cdot f$ is well-defined, then $\overline{u} = \overline{f}(\overline{x})$ is a solution as well.*

They use this definition by saying that $\sigma = \sigma(x, y, t, u)$ is a symmetry of equation (29) if and only if

$$\sigma_t + \alpha\sigma_{xy} + \beta u_y \sigma_x + \beta u_x \sigma_y = 0. \tag{31}$$

So, we know that every symmetry σ must satisfy equation (31). Following this, the authors get the first theorem of the paper.

Theorem 1 *Suppose that u is a solution to equation (29). Then, for a symmetry σ of this equation,*

$$\overline{u} = u + \frac{\alpha}{\beta} log(\sigma) \tag{32}$$

is also a solution to equation (29).

This solution $\overline{u}$ is then a Backlund transformation of equation (29).

Remark: We can then see that a Backlund transformation for the (1+1)-dimensional Burgers' equation can be obtained from equation (30). That is, if we let $\alpha = b$ and $\beta = a$, then the Backlund transformation for equation (27) is given by

$$\overline{u} = u + \frac{b}{a} log(\sigma). \tag{33}$$

The next step in solving this (2+1)-dimensional Burgers' equation is to use the vector fields of equation (29). They find in their paper that for $aV_1 + V_2$, we have

$$u = f(\xi, \tau), \tag{34}$$

where $\xi = y - (\frac{1}{a})x, \tau = t$. Then, if we plug this solution in to equation (29), we get

$$f_\tau - \frac{\alpha}{a} f_{\xi\xi} \frac{\beta}{a} f_{xi}^2 = 0. \tag{35}$$

From this, we get the explicit solution of equation (29), which is

$$u(x,y,t) = -\frac{\epsilon(ay-x)^2}{a+4a\beta\epsilon t} - \frac{\alpha}{2\beta} log(1+4\beta\epsilon t) \tag{36}$$

where $a \neq 0$ and $\epsilon \in \mathbb{R}$ is an arbitrary constant. Then, if we have a symmetry σ, we can obtain new solutions to the equation. For example, we will let

$$\sigma = u_t = \frac{1}{\beta(x-y+t)}. \tag{37}$$

Then, the (2+1)-dimensional Burgers' equation has the solution

$$\overline{u}(x,y,t) = \frac{1}{\beta}x + \frac{1}{\beta} log(x-y+t) - \frac{\alpha}{\beta} log(\beta(x-y+t)). \tag{38}$$

Then, we can revisit equation (27) and find a new solution. This means $x = y, \beta = a$, and $\alpha = b$. So, we get

$$\overline{u}(x,t) = \frac{1}{a}x + \frac{1}{a} log(t) - \frac{b}{a} log(at). \tag{39}$$

Plugging this in to the differential equation gives us

$$\begin{aligned}
u_t &= a u_x^2 + b u_{xx} \\
\frac{1}{a} \cdot \frac{1}{t} - \frac{b}{a} \cdot \frac{1}{t} &= a(\frac{1}{a})^2 + b(0) \\
\frac{1-b}{at} &= \frac{1}{a} \\
\frac{1-b}{t} &= 1 \\
t &= 1-b
\end{aligned} \tag{40}$$

where $t \neq 0$. What this shows is that the solution is given for an instant in time and is not general for all t. This is helpful because if one were interested in how some sort of flow behaved at a specific time t, we can do just that by picking our b accordingly. We can also see that the value of a is independent of this. For instance, if we had a wave modeled by the Burgers' equation and we wanted a solution at, say, $t = 3.4$ seconds, then we can let

$$\begin{aligned}
b &= 1 - 3.4 \\
&= -2.4.
\end{aligned} \tag{41}$$

So, a solution at this moment would be

$$u(x,t) = \frac{1}{a}x + \frac{1}{a} log(t) + \frac{2.4}{a} log(at) \tag{42}$$

where a is some arbitrary constant.

5 Strongly Coupled Burgers' System

The authors of [7] used a nonlocal residual symmetry analysis on the (2+1)-dimensional Burgers' equation. In this section, we will look at a process of deriving a set a Backlund transformations.

We will write the form this took using the notation of [8] to allow for consistency with the original authors. For a solution to the Burgers' equation p, we have

$$\begin{aligned} p_t &= pp_y + arp_x + bp_{yy} + abp_{xx} \\ p_x &= r_y \end{aligned} \tag{43}$$

where $a, b \in \mathbb{R}$ are arbitrary. The authors of [8] then define a new (2+1)-dimensional strongly coupled Burgers' system. They define it by taking values in a commutative subalgebra $\mathbb{Z}_2 = \mathbb{C}[\Gamma]\backslash(\Gamma^2)$. Then, we can replace p and r in equation (43) with the commutative matrices

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix}, \qquad \begin{pmatrix} r & s \\ s & r \end{pmatrix} \tag{44}$$

respectively. Plugging these matrices into equation (43) gives

$$\begin{aligned} p_t &= pp_y + qq_y + arp_x + asq_x + bp_{yy} + abp_{xx} \\ q_t &= pq_y + qp_y + arq_x + asp_x + bq_{yy} + abq_{xx} \\ p_x &= r_y \\ q_x &= s_y. \end{aligned} \tag{45}$$

Then, equation (45) is called (2+1)-dimensional strongly coupled Burgers' system. This is essentially a (2+1)-dimensional Burgers' equation where the two solutions p and q influence one another and are not independent. The authors then perform a residual symmetry analysis on this system. A residual symmetry is simply a non-local symmetry that meets certain requirements based on what's known as the truncated Painleve expansion. This is beyond the scope of the paper, so we will stay focused on the Backlund transformations and only note the following result from this analysis:

$$\begin{aligned} a\psi_x r + a\phi_x s + ab\psi_{xx} + p\psi_y + q\phi_y + b\psi_{yy} - \psi_t = 0 \\ a\phi_x + a\psi_x s + ab\phi_{xx} + q\psi_y + p\phi_y + b\phi_{yy} - \phi_t = 0, \end{aligned} \tag{46}$$

where ϕ, ψ are arbitrary functions that satisfy the equation. We also have the original residual symmetry of the (2+1)-dimensional strongly coupled Burgers' system is given by

$$\begin{aligned} \sigma^p = 2b\psi_y,\ \sigma^q = 2b\phi_y \\ \sigma^r = 2b\psi_x,\ \sigma^s = 2b\phi_x, \end{aligned} \tag{47}$$

which is related to equation (46). However, by the linear property of symmetry equations, multiple residual symmetries are expressed as

$$\begin{aligned}
&\sigma_n^p = \sum_{i=1}^{n} c_i \psi_{i,y}, \quad \sigma_n^q = \sum_{i=1}^{n} c_i \phi_{i,y}, \\
&\sigma_n^r = \sum_{i=1}^{n} c_i \psi_{i,x}, \quad \sigma_n^s = \sum_{i=1}^{n} c_i \phi_{i,x}, \quad (n = 1, 2, 3, ...)
\end{aligned} \tag{48}$$

where ψ_i, ϕ_i are different solutions of equation (46).

We then will say $f = \psi_x, g = \phi_x, h = \psi_y, k = \phi_y$ for simplicity of notation. Then, we get the next theorem, which helps us use these residual symmetries to perform a Backlund transformation.

Theorem 2 *If $\{p, q, r, s, f_i, g_i, h_i, k_i, \psi_i, \phi_i\}, (i = 1, 2, ..., n)$ is a solution of the enlarged system*

$$\begin{aligned}
p_t - pp_y - qq_y - arp_x - asq_x - bp_{yy} - abp_{xx} = 0 \\
q_t - pq_y - qp_y - asp_x - arq_x - bq_{yy} - abq_{xx} = 0 \\
p_x = r_y,\ q_x = s_y
\end{aligned} \tag{49}$$

$$\begin{aligned}
a\psi_{i,x}r + a\phi_{i,x}s + ab\psi_{i,xx} + p\psi_{i,y} + q\phi_{i,y} + b\psi_{i,yy} - \psi_{i,t} = 0 \\
a\phi_{i,x}r + a\psi_{i,x}s + ab\phi_{i,xx} + q\psi_{i,y} + p\phi_{i,y} + b\phi_{i,yy} - \phi_{i,t} = 0 \\
f_i = \psi_{i,x},\ g_i = \phi_{i,x} \\
h_i = \psi_{i,y},\ k_i = \phi_{i,y},
\end{aligned} \tag{50}$$

$(i = 1, 2, ..., n)$, then the symmetry of equation (48) is localized to the Lie point symmetry.

We can note that equation (49) is equivalent to the definition given in equation (45). What this all means is if we have a set of functions $\{p, q, r, s, f_i, g_i, h_i, k_i, \psi_i, \phi_i\}$ that satisfy all of the above equations, then the original non-local symmetry becomes a Lie point symmetry, which, as mentioned in [6], means we can change each variable by a small amount and the equation still holds true.

Then, the author's use Lie's first theorem. This theorem allows us to construct full transformations if we know how the system changes infinitesimally. We found above that the symmetry becomes localized, which means this theorem applies. Then, we get the N-th Backlund transformation by solving the following IVP:

$$
\begin{aligned}
\frac{\mathrm{d}P(\epsilon)}{\mathrm{d}\epsilon} &= \sum_{j=1}^{n} c_j \Psi_{j,y}(\epsilon), \quad \frac{\mathrm{d}Q(\epsilon)}{\mathrm{d}\epsilon} = \sum_{j=1}^{n} c_j \Phi_{j,y}(\epsilon), \\
\frac{\mathrm{d}R(\epsilon)}{\mathrm{d}\epsilon} &= \sum_{j=1}^{n} c_j \Psi_{j,x}(\epsilon), \quad \frac{\mathrm{d}S(\epsilon)}{\mathrm{d}\epsilon} = \sum_{j=1}^{n} c_j \Phi_{j,x}(\epsilon), \\
\frac{\mathrm{d}\Psi_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{2b}(\Psi_i^2(\epsilon) + \Phi_i^2(\epsilon)) - \sum_{j\neq i}^{n} \frac{c_j}{2b}(\Psi_j(\epsilon)\Psi_i(\epsilon) + \Phi_j(\epsilon)\Phi_i(\epsilon)), \\
\frac{\mathrm{d}\Phi_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{b}\Psi_i(\epsilon)\Phi_i(\epsilon) - \sum_{j\neq i}^{n} \frac{c_j}{2b}(\Psi_j(\epsilon)\Phi_i(\epsilon) + \Phi_j(\epsilon)\Psi_i(\epsilon)), \\
\frac{\mathrm{d}F_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{b}(F_i(\epsilon)\Psi_i(\epsilon) + G_i\Phi_i(\epsilon)) \\
&\quad - \sum_{j\neq i}^{n} \frac{c_j}{2b}(F_i(\epsilon)\Psi_j(\epsilon) + G_i(\epsilon)\Phi_j(\epsilon) + F_j(\epsilon\Psi_i(\epsilon) + G_j(\epsilon)\Phi_i(\epsilon)) \\
\frac{\mathrm{d}G_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{b}(F_i(\epsilon)\Phi_i(\epsilon) + G_i(\epsilon)\Psi_i(\epsilon) \\
&\quad - \sum_{j\neq i}^{n} \frac{c_j}{2b}(F_i(\epsilon)\Phi_j(\epsilon) + G_i(\epsilon)\Psi_j(\epsilon) + F_j(\epsilon)\Phi_i(\epsilon) + G_j(\epsilon)\Psi_i(\epsilon) \\
\frac{\mathrm{d}H_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{b}(H_i(\epsilon)\Psi_i(\epsilon) + K_i(\epsilon)\Phi_i(\epsilon)) \\
&\quad - \sum_{j\neq i}^{n} \frac{c_j}{2b}(H_i(\epsilon)\Psi_j(\epsilon) + K_i(\epsilon)\Phi_j(\epsilon) + H_j(\epsilon)\Psi_i(\epsilon) + K_j(\epsilon)\Phi_i(\epsilon) \\
\frac{\mathrm{d}K_i(\epsilon)}{\mathrm{d}\epsilon} &= -\frac{c_i}{b}(H_i(\epsilon)\Phi_i(\epsilon) + K_i(\epsilon)\Psi_i(\epsilon)) \\
&\quad - \sum_{j\neq i}^{n} \frac{c_j}{2b}(H_i(\epsilon)\Phi_j(\epsilon) + K_i(\epsilon)\Psi_j(\epsilon) + H_j(\epsilon)\Phi_i(\epsilon) + K_j(\epsilon)\Psi_i(\epsilon)),
\end{aligned} \tag{51}
$$

with the initial conditions below

$$
\begin{aligned}
&P(0) = p, \quad Q(0) = q, \quad R(0) = r, \quad S(0) = s, \quad \Psi_i(0) = \psi_i, \\
&\Phi_i(0) = \phi_i, \quad F_i(0) = f_i, \quad G_i(0) = g_i, \\
&H_i(0) = h_i, \quad K_i(0) = k_i, \quad i = 1, 2, ..., n.
\end{aligned} \tag{52}
$$

6 Conclusion

We started by defining what a Backlund transformation is. It has its roots in differential geometry wherein we are preserving a tangency criterion. This translates to differential equations as allowing us to transform a difficult PDE into a simpler one while preserving integral properties. We can use this transformation, then, to convert a solution of an easy PDE into a solution of a harder one. We demonstrated this by using the known solution to the heat equation $v = \frac{1}{\sqrt{4\pi t}} e^{(-\frac{x^2}{4t})}$ and converted it to a solution of the Burgers' equation $u = \frac{x}{t}$.

We then extended the idea of the Burgers' equation to two spatial dimensions. We then performed a Backlund transformation that allowed for us to find a new solution, given we already have one solution $u(x, y, t)$. We then showed how this could be applied to the (1+1)-dimensional Burgers' equation by letting $x = y$.

Lastly, we looked at the (2+1)-dimensional strongly coupled Burgers' system and saw a process for how to derive a set of Backlund transformations. To do so, we must solve the specified IVP. We can then see how much complexity there is in finding these Backlund transformations for more sophisticated PDEs.

In this paper, we have shown how Backlund transformations are beneficial in solving the Burgers' equation. This equation can be used to model different types of flow, which means this is very helpful in applications of fluid dynamics, including monitoring traffic.

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