Ralated Rates

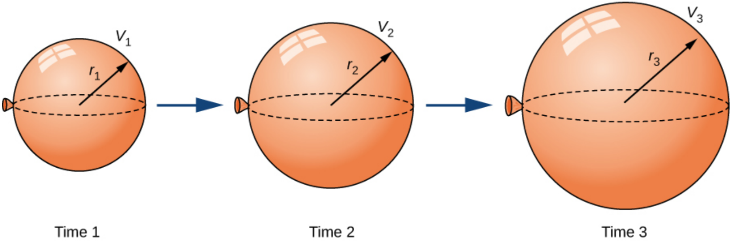
Clotilde Djuikem

Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity based on the rate of change of other quantities.

Application

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r.



Steps

Step 1: Volume of a Sphere

The volume of a sphere of radius r centimeters is:

$$V = \frac{4}{3}\pi r^3 \, \mathrm{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is:

$$V(t) = \frac{4}{3}\pi \left[r(t)\right]^3 \ \mathrm{cm}^3.$$

Step 2: Differentiate and Apply Chain Rule

Differentiating both sides of the equation with respect to time t, and applying the chain rule, we get:

$$\frac{dV}{dt} = 4\pi \left[r(t)\right]^2 \cdot \frac{dr}{dt}.$$

This equation shows that the rate of change of the volume $\frac{dV}{dt}$ is related to the rate of change of the radius $\frac{dr}{dt}$.

Step 3: Known Rate of Volume Change

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so:

$$\frac{dV}{dt} = 2\,\text{cm}^3/\text{sec}.$$

Substituting into the equation, we have:

$$2 = 4\pi \left[r(t)\right]^2 \cdot \frac{dr}{dt} \;\textbf{ Then }\; \frac{dr}{dt} = \frac{1}{2\pi \left[r(t)\right]^2}\,\text{cm}/\text{sec}.$$

Step 4: Substitute r = 3 cm

When the radius $r = 3\,\text{cm}$, substituting this into the equation gives:

$$\frac{dr}{dt} = \frac{1}{18\pi}\,\text{cm/sec.}$$

Therefore, the radius of the balloon is increasing at a rate of $\frac{1}{18\pi}\,\text{cm/sec}$.

Problem-Solving Strategy: Solving a Related-Rates

To solve a related-rates problem, follow these steps:

1. **Assign symbols** to all variables involved in the problem. Draw a figure if applicable.
2. **State the information** given in terms of the variables and identify the rate that needs to be determined.
3. **Find an equation** relating the variables from step 1.
4. **Differentiate** both sides of the equation from step 3 with respect to the independent variable, using the chain rule. This will give an equation relating the derivatives.
5. **Substitute known values** into the equation from step 4 and solve for the unknown rate of change.

Remember

Remember not to substitute values too soon, as this could turn a variable into a constant prematurely.

Problem: Airplane Flying Overhead

An airplane is flying at a constant height of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man at 600 ft/sec.

Question: At what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

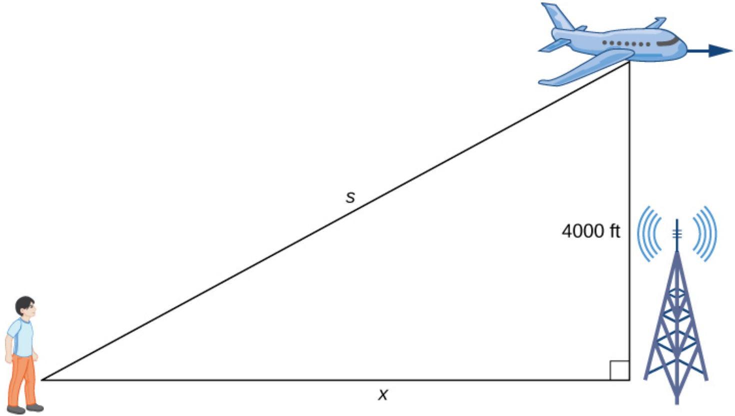
Step 1: Assign Variables

Let:

- $x(t) =$ the horizontal distance between the man and the point on the ground directly below the airplane (changing with time),
- $s(t) =$ the slant distance between the man and the airplane,
- Height of the plane is constant at 4000 ft.

We are tasked to find $\frac{ds}{dt}$ when $x(t) = 3000$ ft.

Graph



Step 2: Write Known Information

The horizontal distance is increasing at a constant rate:

$$\frac{dx}{dt} = 600\,\text{ft/sec}.$$

We need to find the rate of change of the distance between the man and the plane, i.e., $\frac{ds}{dt}$, when $x = 3000\,\text{ft}$.

Step 3: Relating the Variables

From the geometry of the problem (right triangle), we can relate $x(t)$ and $s(t)$ using the Pythagorean theorem:

$$s(t)^2 = x(t)^2 + 4000^2.$$

Differentiating $2s(t)\frac{ds}{dt} = 2x(t)\frac{dx}{dt}$. Simplifying:

$$s(t)\frac{ds}{dt} = x(t)\frac{dx}{dt}.$$

Steps 4: Solve for $\frac{ds}{dt}$

Solving for $\frac{ds}{dt}$, we get:

$$\frac{ds}{dt} = \frac{x(t)\frac{dx}{dt}}{s(t)}.$$

When $x = 3000\,\text{ft}$, we find s using the Pythagorean theorem:

$$s = \sqrt{3000^2 + 4000^2} = 5000\,\text{ft}.$$

Substituting the known values:

$$\frac{ds}{dt} = \frac{3000 \times 600}{5000} = 360\,\text{ft/sec}.$$

Answer: The distance between the man and the airplane is increasing at 360 ft/sec.

Airplane

An airplane is flying at a constant height of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man at 600 ft/sec.

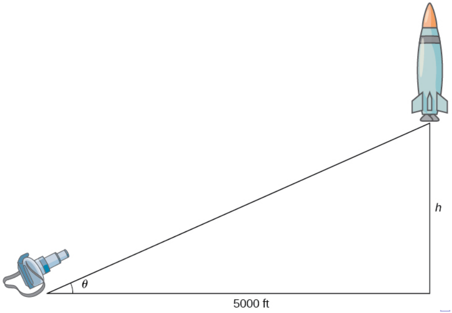
Question: At what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Step 1: Assign Variables

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Objective: We are trying to find $\frac{d\theta}{dt}$ when the rocket is 1000 ft above the ground.
Given: The rocket is moving at a rate of:

$$\frac{dh}{dt} = 600\,\text{ft/sec}.$$



Step 2: Relating Variables with Trigonometry

Relating Variables:

The right triangle formed by the rocket's height, the distance from the camera to the launch pad, and the hypotenuse helps relate h and θ.

From trigonometry, we know:

$$\tan\theta = \frac{h}{5000}.$$

This gives us the equation:

$$h = 5000\tan\theta.$$

Step 3: Differentiate with Respect to Time

Differentiating both sides of the equation $h = 5000 \tan\theta$ with respect to time t:

$$\frac{dh}{dt} = 5000 \sec^2\theta \cdot \frac{d\theta}{dt}.$$

We want to find $\frac{d\theta}{dt}$ when $h = 1000\,\text{ft}$.

So, we need to calculate $\sec^2\theta$ at that point.

Step 4: Determine $\sec^2\theta$ and Hypotenuse

We know the adjacent side is 5000 ft, and the opposite side is $h = 1000$ ft.

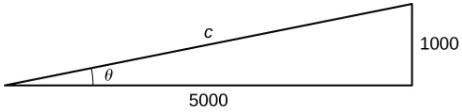
Using the Pythagorean theorem, the hypotenuse c is:

$$c = \sqrt{5000^2 + 1000^2} = 1000\sqrt{26}\,\text{ft}.$$

Therefore:

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Step 4: Triangle Diagram



The triangle shows $h = 1000\,\text{ft}$, adjacent side $5000\,\text{ft}$, and hypotenuse $c = 1000\sqrt{26}\,\text{ft}$.

Step 5: Solve for $\frac{d\theta}{dt}$

From Step 3, we have the equation:

$$\frac{dh}{dt} = 5000 \sec^2 \theta \cdot \frac{d\theta}{dt}.$$

Substituting the known values $\frac{dh}{dt} = 600 \, \text{ft}/\text{sec}$ and $\sec^2 \theta = \frac{26}{25}$:

$$600 = 5000 \cdot \frac{26}{25} \cdot \frac{d\theta}{dt}.$$

Solving for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{3}{26} \, \text{rad}/\text{sec}.$$

Problem: Rate of Change for Camera Angle

Problem:
What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Hint:
Find $\frac{d\theta}{dt}$ when $h = 2000\,\text{ft}$. At that time, we know:

$$\frac{dh}{dt} = 500\,\text{ft/sec}.$$

Solution

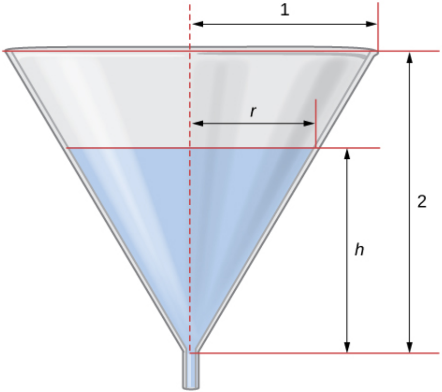
Solution:
The rate of change of the camera's elevation angle is:

$$\frac{1}{10}\,\text{rad/sec.}$$

Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\,\text{ft}^3/\text{sec}$. The height of the funnel is $2\,\text{ft}$ and the radius at the top of the funnel is $1\,\text{ft}$.

At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\,\text{ft}$?



Step 2: Determine $\frac{dh}{dt}$

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that:

$$\frac{dV}{dt} = -0.03\,\mathrm{ft}^3/\mathrm{sec}.$$

Step 3: Volume of Water in the Cone

The volume of water in the cone is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we know that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same:

$$\frac{r}{h} = \frac{1}{2} \quad \text{or} \quad r = \frac{h}{2}.$$

Using this, the equation for volume becomes:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

Step 4: Apply Chain Rule

Applying the chain rule, we differentiate both sides of the equation with respect to time t:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Step 5: Solve for $\frac{dh}{dt}$

We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know:

$$\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}.$$

Therefore:

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt},$$

which simplifies to:

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

Final Solution

Solving for $\frac{dh}{dt}$, we get:

$$\frac{dh}{dt} = \frac{-0.48}{\pi} \approx -0.153\,\text{ft}/\text{sec}.$$

Problem: Water Level Rate of Change

Problem:

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Hint:

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Step 1: Volume Formula

The volume of water in the cone is:

$$V = \frac{1}{3}\pi r^2 h.$$

Since $\frac{r}{h} = \frac{1}{2}$, we have $r = \frac{h}{2}$. Substituting this into the volume equation:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 2: Differentiate the Volume

Differentiating both sides of the volume equation with respect to time:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Step 3: Solve for $\frac{dh}{dt}$

We know that water is draining at a rate of $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Thus:

$$-0.03 = \frac{\pi}{4} \left(\frac{1}{4}\right)^2 \frac{dh}{dt}.$$

Simplifying the equation:

$$-0.03 = \frac{\pi}{4} \times \frac{1}{16} \frac{dh}{dt} = \frac{\pi}{64} \frac{dh}{dt}.$$

Therefore, solving for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-0.03 \times 64}{\pi} = \frac{-1.92}{\pi} \approx -0.611\,\text{ft/sec}.$$

Step 4

The height of the water is decreasing at a rate of approximately:

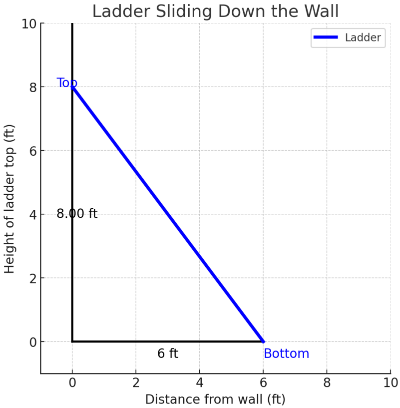
$$\frac{dh}{dt} \approx -0.611\,\mathrm{ft}/\mathrm{sec}$$

when the height of the water is $\frac{1}{4}$ ft.

Problem: Ladder Sliding Down a Wall

A ladder 10 feet long is leaning against a vertical wall. The bottom of the ladder is pulled away from the wall at a rate of 2 feet per second.

Question: How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet away from the wall?



Problem: Ladder Sliding Down a Wall

- **Step 1: Assign Variables**
- **Step 2: Use the Pythagorean Theorem**
  The relationship between x, y, and L is given by:
- **Step 3: Differentiate with Respect to Time** Differentiate both sides of the equation with respect to time t:
- **Step 4: Solve for $\frac{dy}{dt}$**
- **Step 5: Find y When $x = 6$**
  Use the Pythagorean theorem to find y when $x = 6\,\text{ft}$:
- **Step 6: Final Substitution**
  Substitute $x = 6$, y, and $\frac{dx}{dt} = 2\,\text{ft/sec}$ into the equation to find $\frac{dy}{dt}$.

Key Concepts

To solve a related rates problem:

1. First, **draw a picture** that illustrates the relationship between the two or more related quantities that are changing with respect to time.
2. In terms of the quantities, **state the information** given and the rate to be found.
3. **Find an equation** relating the quantities.
4. **Use differentiation**, applying the chain rule as necessary, to find an equation that relates the rates.
5. Be sure **not to substitute** a variable quantity for one of the variables until after finding an equation relating the rates.

Maxima and Minima

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Learning Objectives

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- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical numbers of a function over a closed interval.
- Describe how to use critical numbers to locate absolute extrema over a closed interval.

Practical Significance of Extrema

Important Note

We are often interested in determining the largest and smallest values of a function. This information is important for:

- Creating accurate graphs.
- Solving optimization problems such as maximizing profit or minimizing material usage.

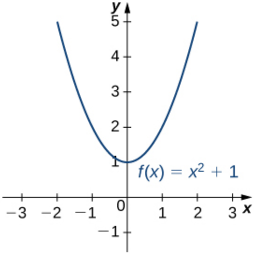
Absolute Extrema Example

Example

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$:

- As $x \to \pm\infty$, $f(x) \to \infty$.
- Therefore, the function does not have a largest value.
- However, $f(x) \geq 1$ for all x, and $f(0) = 1$.

Conclusion: The function has an **absolute minimum** of 1 at $x = 0$, but no absolute maximum.



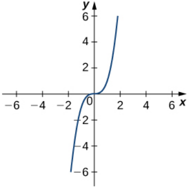
Definition of Absolute Extrema

Definition

Let f be a function defined over an interval I, and let $c \in I$.

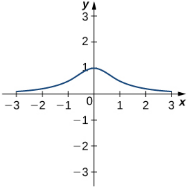
- f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$.
- f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$.
- If f has an absolute maximum or minimum, we say that f has an **absolute extremum** at c.

Graph for extrema



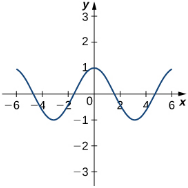
$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)



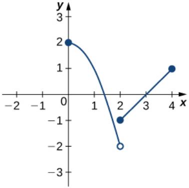
$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)



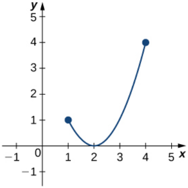
$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$, $\pm 2\pi$, $\pm 4\pi$...
Absolute minimum of -1 at $x = \pm\pi$, $\pm 3\pi$...

(c)



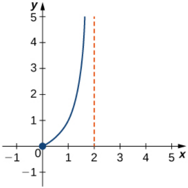
$$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)



$f(x) = (x-2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)



$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

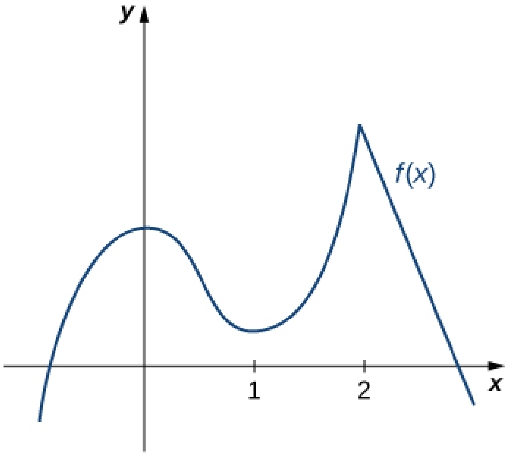
Extreme Value Theorem

Theorem

If f is a continuous function over the closed, bounded interval $[a, b]$, then:

- There is a point in $[a, b]$ at which f has an **absolute maximum**.
- There is a point in $[a, b]$ at which f has an **absolute minimum**.

Illustration local extrema



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Local Extrema

A function f has a local maximum at c if:

- There exists an open interval I containing c.
- I is contained in the domain of f.
- $f(c) \geq f(x)$ for all $x \in I$.

A function f has a local minimum at c if:

- There exists an open interval I containing c.
- I is contained in the domain of f.
- $f(c) \leq f(x)$ for all $x \in I$.

A function f has a local extremum at c if:

- f has a local maximum at c, or
- f has a local minimum at c.

Definition and Fermat's Theorem

Definition

Let c be an interior point in the domain of f. We say that c is a **critical number** of f if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined.}$$

Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then

$$f'(c) = 0.$$

Important

Note this theorem does not claim that a function f must have a local extremum at a critical number. Rather, it states that critical numbers are candidates for local extrema.

Proof Fermat's Theorem

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c.

Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}, \textbf{ and } f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

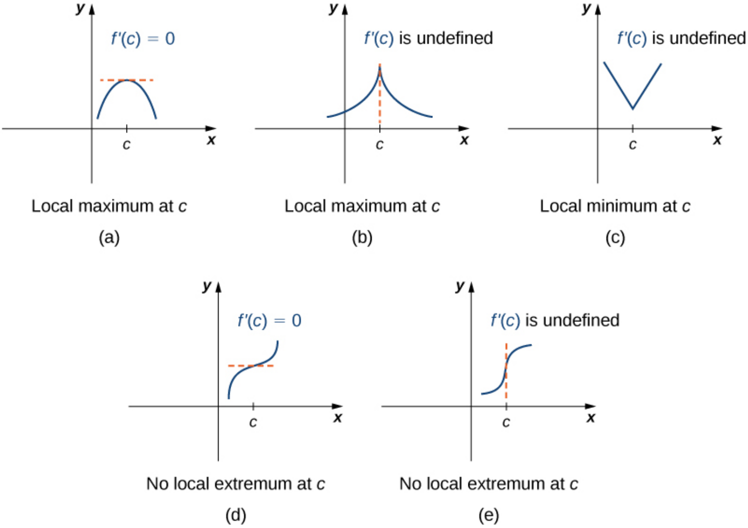
Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x \neq c$, we have

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

From this, we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore,

$$f'(c) = 0.$$

Defined and undefined $f'(c)$



Steps to Find Local Minima and Maxima

Step 1: Find the derivative $f'(x)$

Step 2: Set the derivative equal to zero: Solve $f'(x) = 0$ **critical numbers**

Step 3: Determine where the derivative is undefined

Step 4: Use the First Derivative Test

For each critical point c:

- If $f'(x)$ changes from positive to negative at c, $f(c)$ is a **local maximum**.
- If $f'(x)$ changes from negative to positive at c, $f(c)$ is a **local minimum**.
- If $f'(x)$ does not change sign, $f(c)$ is neither.

Optional and applicable steps

Step 5: Use the Second Derivative Test (optional)

If the second derivative $f''(x)$ exists:

- If $f''(c) > 0$, $f(c)$ is a local minimum.
- If $f''(c) < 0$, $f(c)$ is a local maximum.
- If $f''(c) = 0$, the test is inconclusive.

Step 6: Check the endpoints (if applicable)

If the function is defined on a closed interval, check the function values at the endpoints. These may give the absolute maximum or minimum over the interval.

Locating Critical Numbers: Example a

For the function $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for x to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Locating Critical Numbers: Example b

For the function $f(x) = (x^2 - 1)^3$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for x to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

____

Locating Critical Numbers: Example c

For the function $f(x) = \frac{4x}{1+x^2}$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for x to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Finding Critical Numbers

Find all critical numbers for the function:

$$f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$$

Step 1: Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Step 2: Set the derivative equal to zero and solve for x.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

Step 3: List the critical numbers:

Critical numbers: ___________

Location of Absolute Extrema

Let $I = [a, b]$

If f is a continuous function on a closed interval I, then:

- The **absolute maximum**
- The **absolute minimum**

must happen either at the endpoints of I or at critical points inside I.

Problem-Solving Strategy: Finding Absolute Extrema

Consider a continuous function f defined over the closed interval $[a, b]$.

1. **Step 1: Evaluate** the function f at the endpoints of the interval. Calculate:

$$f(a) \quad \text{and} \quad f(b)$$

2. **Step 2: Find** all critical numbers of the function f within the open interval (a, b). A critical number is where:

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ is undefined}$$

   Then, evaluate the function at each critical number found.

3. **Step 3: Compare** all values obtained from Steps 1 and 2:
   - The largest value is the **absolute maximum**.
   - The smallest value is the **absolute minimum**.

Locating Absolute Extrema for Function a

Function a: $f(x) = -x^2 + 3x - 2$ over $[1, 3]$

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and endpoints $x = 1$ and $x = 3$.

$$f(1) = \_\_\_\_\_\_ \quad f(3) = \_\_\_\_\_\_$$

4. **Step 4:** Compare the values and determine the absolute maximum and minimum.

Max: ________ Min: ________

Locating Absolute Extrema for Function b

Function b: $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and endpoints $x = 0$ and $x = 2$.

$$f(0) = \_\_\_\_\_\_ \quad f(2) = \_\_\_\_\_\_$$

4. **Step 4:** Compare the values and determine the absolute maximum and minimum.

Max: ________ Min: _______

Locating Absolute Extrema

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and the endpoints $x = 1$ and $x = 4$.

$$f(1) = \_\_\_\_\_\_\_ \quad f(4) = \_\_\_\_\_\_\_$$

4. **Step 4:** Compare the values to determine the absolute maximum and minimum.

Max: _________ Min: ________

Formula for the Maximum or Minimum of a Quadratic

Problem: In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was:

$$m = -\frac{b}{2a}$$

Prove this formula using calculus.

1. **Step 1:** Start with the given quadratic function:

$$y = ax^2 + bx + c$$

2. **Step 2:** Find the derivative of y, $y'(x)$.

$$y'(x) = \_\_\_\_\_\_\_\_$$

3. **Step 3:** Set $y'(x) = 0$ and solve for x to find the critical point.

$$0 = \_\_\_\_\_\_\_\_ \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

4. **Step 4:** Conclude that the critical point $x = \frac{-b}{2a}$ gives the position of the maximum or minimum.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical number. However, a function need not have a local extremum at a critical number.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical number or an endpoint.

Meam Value Thorem

Clotilde Djuikem

Learning Objectives

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

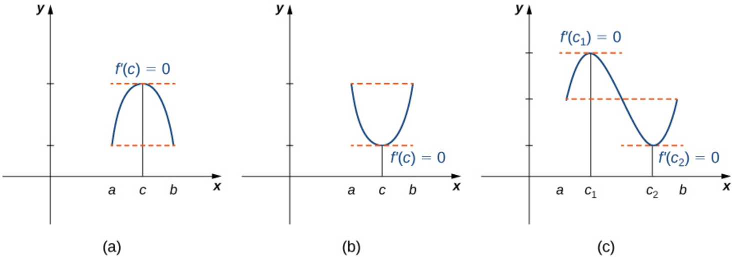
Rolle's Theorem

Definition

If the outputs of a differentiable function f are equal at the endpoints of an interval, there must be an interior point c where $f'(c) = 0$.

Visual Illustration

See the figure with parabolas and sine wave, illustrating different cases of Rolle's theorem.



Formal Statement of Rolle's Theorem

Statement

Let f be a

- Continuous function over the closed interval $[a, b]$ and
- Differentiable over the open interval (a, b) such that $f(a) = f(b)$.

Then, there exists at **least one** $c \in (a, b)$ such that $f'(c) = 0$.

Using Rolle's Theorem

Example

For the function $f(x) = x^2 + 2x$ over $[-2, 0]$, verify the criteria of Rolle's theorem and find the value c where $f'(c) = 0$.

Solution

Since $f(x)$ is continuous and differentiable on the interval, and $f(-2) = f(0) = 0$, there exists a point $c = -1$ where $f'(c) = 0$.

Example : Using Rolle's Theorem

Problem: Let $f(x) = x^2 - 4x + 4$ on the interval $[0, 4]$. Verify that f satisfies the conditions of Rolle's theorem and find the point c such that $f'(c) = 0$.

Solution:

1. **Check continuity and differentiability:**
   $f(x) = x^2 - 4x + 4$ is a polynomial, so it is continuous and differentiable on $[0, 4]$.
2. **Verify endpoint equality:**
   $f(0) = 4$ and $f(4) = 4$. Since $f(0) = f(4)$, the conditions of Rolle's theorem are satisfied.
3. **Differentiate $f(x)$:**
   $f'(x) = 2x - 4$.
4. **Set $f'(c) = 0$ and solve for c:**
   $2c - 4 = 0 \Rightarrow c = 2$.
5. **Conclusion:**
   There exists a point $c = 2$ such that $f'(c) = 0$, which satisfies Rolle's theorem.

Example : Verifying Conditions for Rolle's Theorem

Problem: Let $f(x) = x^3 - 3x + 2$ over $[-2, 2]$. Determine if Rolle's theorem applies and find the value(s) of c if applicable.

Solution:

1. **Continuity and Differentiability:**
   $f(x) = x^3 - 3x + 2$ is a polynomial, so it is continuous and differentiable on $[-2, 2]$.
2. **Check endpoint equality:**
   $f(-2) = 0$ and $f(2) = 0$, so $f(-2) = f(2)$.
3. **Differentiate $f(x)$:** $f'(x) = 3x^2 - 3$.
4. **Solve $f'(c) = 0$:**

$$3c^2 - 3 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1.$$

5. **Conclusion:**
   The points $c = 1$ and $c = -1$ satisfy $f'(c) = 0$, as required by Rolle's theorem.

Exercise

Problem: Let $f(x) = x^2 - 5x + 6$ on $[1, 3]$. Use Rolle's theorem to verify the conditions and find the value(s) of c if possible.

Solution:

1. **Continuity and Differentiability:**

   ---

2. **Verify endpoint equality:**

   ---

3. **Differentiate $f(x)$:**

   ---

4. **Solve $f'(c) = 0$:**

   ---

5. **Conclusion:**

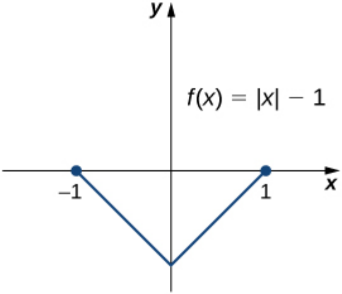
Rolle's Theorem: Importance of Differentiability

Example: Let $f(x) = |x| - 1$ on $[-1, 1]$.

- $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1) = 0$.
- However, f is not differentiable at $x = 0$.

Conclusion

Since f is not differentiable at $x = 0$, Rolle's theorem does not apply. There is no $c \in (-1, 1)$ such that $f'(c) = 0$.



No c such that $f'(c) = 0$

Using Rolle's Theorem

Problem: For each of the following functions, verify that the function satisfies the criteria of Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$

Solution Steps:

1. **Check continuity and differentiability:**

   ____

2. **Verify endpoint equality:**

   $f(-2) =$ __________, $f(0) =$ __________

   Since $f(-2) = f(0)$, the conditions are satisfied.

3. **Differentiate $f(x)$:**

   $f'(x) =$ __________________________

4. **Set $f'(c) = 0$ and solve for c:**

   $f'(c) = 0 \Rightarrow c =$ ________________

Using Rolle's Theorem

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution Steps:

1. **Check continuity and differentiability:**

   ____

2. **Verify endpoint equality:**
   $f(-2) =$ ________, $f(2) =$ ________
   Since $f(-2) = f(2)$, the conditions are satisfied.

3. **Differentiate $f(x)$:**
   $f'(x) =$ ____________________

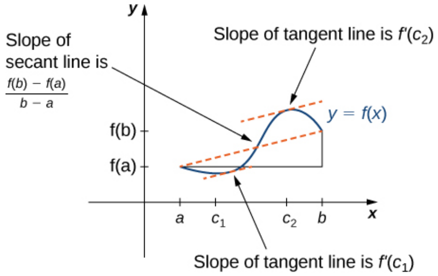
4. **Set $f'(c) = 0$ and solve for c:**
   $f'(c) = 0 \Rightarrow c =$ ______________

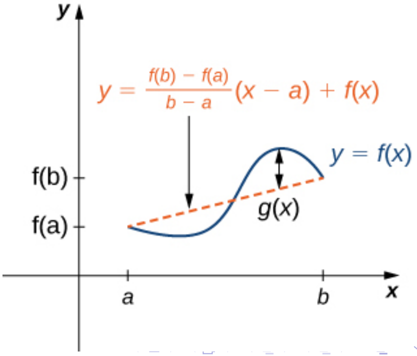
The Mean Value Theorem (MVT)

Statement

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$





Applying the Mean Value Theorem

Example

For the function $f(x) = \sqrt{x}$ over $[0, 9]$, find the point c such that $f'(c)$ equals the slope of the secant line between $(0, f(0))$ and $(9, f(9))$.

Solution

The slope of the secant line is $\frac{3}{9} = \frac{1}{3}$. We find $c = \frac{9}{4}$ such that $f'(c) = \frac{1}{3}$.

Example 2: Applying the Mean Value Theorem

Problem: Let $f(x) = \sqrt{x}$ on the interval $[1, 4]$. Use the Mean Value Theorem to find the value c such that $f'(c)$ equals the slope of the secant line between $(1, f(1))$ and $(4, f(4))$. **Solution:**

1. **Verify continuity and differentiability:** $f(x) = \sqrt{x}$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$.
2. **Calculate the secant slope:**
$$\text{slope} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$
3. **Differentiate $f(x)$:** $f'(x) = \frac{1}{2\sqrt{x}}$.
4. **Set $f'(c) = \frac{1}{3}$ and solve for c:**
$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \Rightarrow \sqrt{c} = \frac{3}{2} \Rightarrow c = \frac{9}{4}.$$
5. **Conclusion:** At $c = \frac{9}{4}$, the tangent slope equals the secant slope, satisfying the Mean Value Theorem.

Exercise

Problem: Let $f(x) = x^2 + x - 12$ on $[2, 6]$. Use the Mean Value Theorem to find c such that $f'(c)$ equals the secant slope.

Solution:

1. **Continuity and Differentiability:**

   ____

2. **Calculate the secant slope:**

   ____

3. **Differentiate $f(x)$:**

   ____

4. **Set $f'(c)$ equal to the secant slope and solve for c:**

   ____

5. **Conclusion:**

   ____

Problem

Use the Mean Value Theorem to show that if $x > 0$, then $\sin x \leq x$.

Mean Value Theorem and Inequalities

Problem

Use the Mean Value Theorem to show that if $x > 0$, then $\sin x \leq x$.

Solution **Step 1: Define the function and interval.**
$f(x) = \sin x - x$. To show that $f(x) \leq 0$, we consider the interval $[0, x]$.
Step 2: Check continuity and differentiability. The function f is the difference of a trigonometric function and a polynomial. Thus, f is continuous on $[0, x]$ and differentiable on $(0, x)$.
Step 3: Calculate the derivative of $f(x)$. We find that $f'(x) = \cos x - 1$.
Step 4: Apply the Mean Value Theorem. By the Mean Value Theorem, there exists a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Step 5: Analyze $f'(c)$ to conclude.
Since $f'(c) = \cos c - 1 \leq 0$ (because $\cos c \leq 1$ for all c), and $x > 0$, we conclude that $f(x) \leq 0$. Therefore, $\sin x \leq x$.

Problem: Ball Dropped from a Height

Problem: A ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity equals the average velocity.

Hint

1. Find the time it takes for the ball to hit the ground.
2. Calculate the average velocity.

Problem: Ball Dropped from a Height

Problem: A ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity equals the average velocity.

Hint

1. Find the time it takes for the ball to hit the ground.
2. Calculate the average velocity.

Solution **Step 1: Time to hit the ground**

$$s(t) = 0 \Rightarrow -16t^2 + 200 = 0 \Rightarrow t = \frac{5\sqrt{2}}{2}.$$

Step 2: Average velocity $v_{\text{avg}} = \frac{s(t)-s(0)}{t} = -40\sqrt{2}$ ft/sec.
Step 3: Instantaneous velocity
$s'(t) = -32t$. Set $s'(t) = -40\sqrt{2}$:

$$-32t = -40\sqrt{2} \Rightarrow t = \frac{5\sqrt{2}}{2}\text{seconds.}$$

Corollaries of the Mean Value Theorem

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x)$ is constant for all $x \in I$.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then

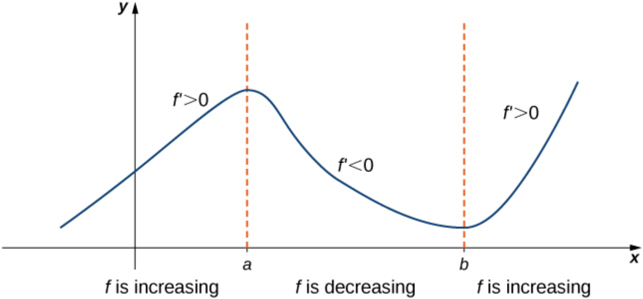
$$f(x) = g(x) + C$$

for some constant C.

Corollaries of the Mean Value Theorem

Key Corollaries

- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If $f'(x) > 0$ over I, then f is increasing over I.
- If $f'(x) < 0$ over I, then f is decreasing over I.



Key Concepts

Rolle's Theorem

If f is continuous over $[a, b]$ and differentiable over (a, b), and $f(a) = f(b) = 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem (MVT)

If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Constant Function Property

If $f'(x) = 0$ over an interval I, then f is constant over I.

Equality of Derivatives Implies Constant Difference

If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over an interval I, then $f(x) = g(x) + C$ for some constant C.

Key Concepts

Monotonicity

If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

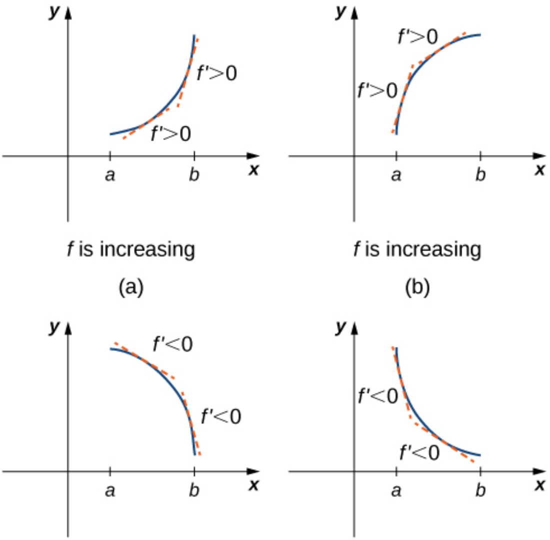
Derivatives and the Shape of a Graph

Clotilde Djuikem

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for identifying critical numbers.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Describe the relationship between a function and its first and second derivatives.
- State the second derivative test for identifying local extrema.

Graph of function and sign of derivative



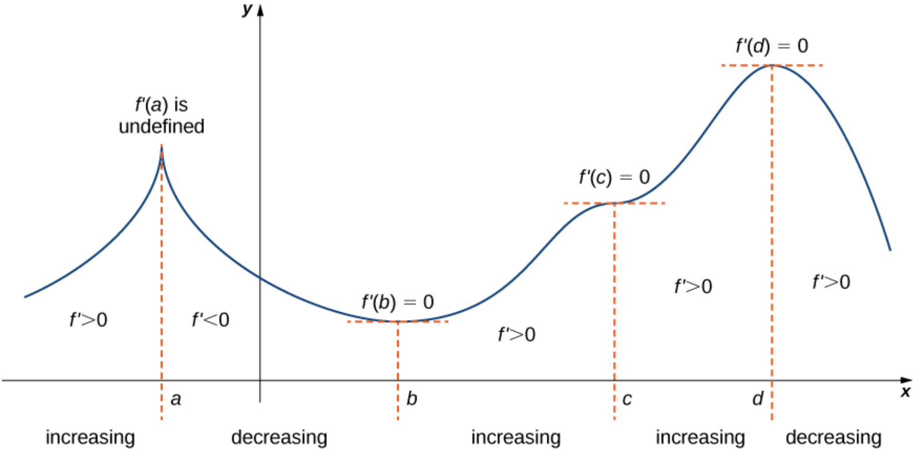
f is increasing (a)

f is increasing (b)

f is decreasing (c)

f is decreasing (d)

Graph



y
x
f'(a) is
undefined
f'(b) = 0
f'(c) = 0
f'(d) = 0
f'>0
f'<0
f'>0
f'>0
f'>0
a
b
c
d
increasing
decreasing
increasing
increasing
decreasing

First Derivative Test

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical number c. If f is differentiable over I, except possibly at c, then $f(c)$ satisfies one of the following conditions:

1. If f' changes sign from positive to negative at c, then $f(c)$ is a local maximum.
2. If f' changes sign from negative to positive at c, then $f(c)$ is a local minimum.
3. If f' has the same sign on both sides of c, then $f(c)$ is not a local extremum.

Problem-Solving Strategy: First Derivative Test

For a continuous function f over interval I:

1. **Identify Critical Numbers:** Find points where $f'(x) = 0$ or $f'(x)$ is undefined.
2. **Determine f' Sign in Each Subinterval:**
   - Select a test point in each subinterval.
   - Check if $f'(x)$ is positive (increasing) or negative (decreasing).
3. **Conclude Local Behavior at Each Critical Number:**
   - f' changes $+ \to -$: local max.
   - f' changes $- \to +$: local min.
   - f' does not change: no extremum.

Finding Local Extrema Using the First Derivative Test

Problem: Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Confirm your results using a graphing utility. **Solution Steps:**

1. **Find the derivative:** $f'(x) = 3x^2 - 6x - 9$.
2. **Set $f'(x) = 0$ and solve for x:**

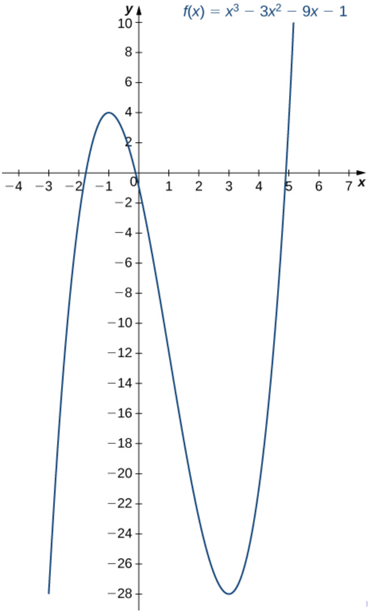
$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$$

   Critical points: $x = 3$ and $x = -1$.
3. **Test the sign of f' around each critical point:**

x Interval	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
Test Point x	-2	0	4
$f'(x)$	$+$ (positive)	$-$ (negative)	$+$ (positive)
$f(x)$ Behavior	Increasing	Decreasing	Increasing

4. **Determine local extrema:**
   - f' changes from positive to negative at $x = -1$: local maximum.
   - f' changes from negative to positive at $x = 3$: local minimum.

Confirm results with a graphing utility.



Problem: Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Solution Steps:

1. **Find the derivative:** $f'(x) = -3x^2 + 3x + 18$
2. **Set $f'(x) = 0$ and solve for x:**

$$-3x^2 + 3x + 18 = 0 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0$$

   Critical points: $x = 3$ and $x = -2$.

3. **Test the sign of f' around each critical point:**

x Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Test Point x	-3	0	4
$f'(x)$	$+$ (positive)	$-$ (negative)	$+$ (positive)
$f(x)$ Behavior	Increasing	Decreasing	Increasing

4. **Determine local extrema:**
   - f' changes from positive to negative at $x = -2$: local maximum.
   - f' changes from negative to positive at $x = 3$: local minimum.

Using the First Derivative Test

Problem: Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Confirm your results using a graphing utility. **Solution Steps:**

1. **Find the derivative:**

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3}\left(x^{-2/3} - x^{2/3}\right) = \frac{5}{3} \cdot \frac{1-x}{x^{2/3}}$$

2. **Set $f'(x) = 0$ and solve for x:** $\frac{5}{3} \cdot \frac{1-x}{x^{2/3}} = 0 \Rightarrow 1 - x = 0$ Critical point: $x = 1$. Note that $f'(x)$ is undefined at $x = 0$, so $x = 0$ is also a critical point.

3. **Test the sign of f' around each critical point:**

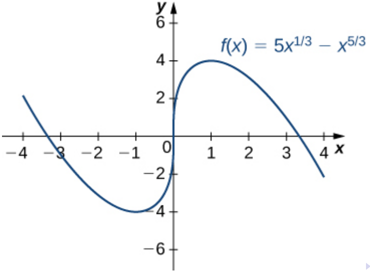
x Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Test Point x	-1	0.5	2
$f'(x)$	$-$ (negative)	$+$ (positive)	$-$ (negative)
$f(x)$ Behavior	Decreasing	Increasing	Decreasing

4. **Determine local extrema:**
   - f' changes from negative to positive at $x = 0$: local minimum.
   - f' changes from positive to negative at $x = 1$: local maximum.

Graph

Confirm results with a graphing utility: Plot $f(x)$ to verify the local minimum at $x = 0$ and local maximum at $x = 1$.

Since f is decreasing over the interval $(-\infty, -1)$ and increasing over $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over both $(-1, 0)$ and $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over $(0, 1)$ and decreasing over $(1, \infty)$, f has a local maximum at $x = 1$. These analytical results are confirmed by the following graph.



Using the First Derivative Test

Problem: Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x} - 1$.

Hint: The only critical number of f is $x = 1$.

Using the First Derivative Test

Problem: Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x} - 1$.

Hint: The only critical number of f is $x = 1$. **Solution Steps:**

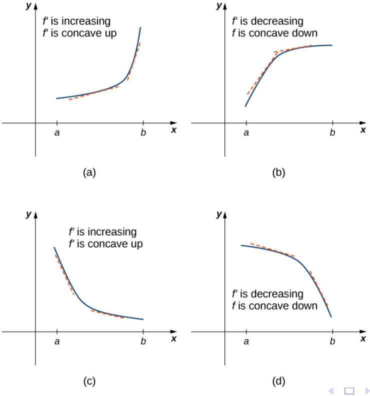
1. **Find the derivative:**
$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$
2. **Identify critical points:** $f'(x)$ is undefined at $x = 0$ and equal to zero at $x = 1$. Thus, the critical number is $x = 1$.
3. **Test the sign of f' around $x = 1$:**
   - For $x = 0.5$ (left of $x = 1$): $f'(0.5) = \frac{1}{3 \cdot \sqrt[3]{0.5^2}} > 0$ (positive).
   - For $x = 2$ (right of $x = 1$): $f'(2) = \frac{1}{3 \cdot \sqrt[3]{2^2}} > 0$ (positive).
4. **Conclusion:** Since $f'(x)$ is positive on both sides of $x = 1$, there is no local extremum at $x = 1$.

Definition

Concavity of a Function

Let f be a function that is differentiable over an open interval I.

- If f' is increasing over I, we say f is **concave up** over I.
- If f' is decreasing over I, we say f is **concave down** over I.



Test for Concavity with Examples

Concavity Test

Let f be a function that is twice differentiable over an interval I.

1. If $f''(x) > 0$ for all $x \in I$, then f is **concave up** over I.
2. If $f''(x) < 0$ for all $x \in I$, then f is **concave down** over I.

Examples:

- **Example 1:** $f(x) = x^2$
  - $f'(x) = 2x$, $f''(x) = 2$
  - Since $f''(x) = 2 > 0$ for all x, $f(x) = x^2$ is **concave up** everywhere.
- **Example 2:** $f(x) = -x^2$
  - $f'(x) = -2x$, $f''(x) = -2$
  - Since $f''(x) = -2 < 0$ for all x, $f(x) = -x^2$ is **concave down** everywhere.

Definition of an Inflection Point

Inflection Point

If f is continuous at a and f changes concavity at a, then the point $(a, f(a))$ is an **inflection point** of f.

Example 1: Determine the inflection points of $f(x) = x^3 - 3x^2 + 4$.

- Find $f''(x)$ and set it equal to zero to find potential inflection points.
- Verify if f changes concavity at these points.

Solution:

- $f'(x) = 3x^2 - 6x$
- $f''(x) = 6x - 6$
- Set $f''(x) = 0 \Rightarrow x = 1$
- Check concavity around $x = 1$:
    - $f''(x) > 0$ for $x > 1$ (concave up)
    - $f''(x) < 0$ for $x < 1$ (concave down)
- Conclusion: $(1, f(1)) = (1, 2)$ is an inflection point.

Second Derivative Test

Suppose $f'(c) = 0$ and $f''(x)$ is continuous over an interval containing c.

1. If $f''(c) > 0$, then f has a **local minimum** at c.
2. If $f''(c) < 0$, then f has a **local maximum** at c.
3. If $f''(c) = 0$, then the test is **inconclusive**.

Using the Second Derivative Test

Problem: Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Using the Second Derivative Test

Problem: Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$. **Solution Steps:**

1. **Find the first derivative:**

$$f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3)$$

   Set $f'(x) = 0$: $5x^2(x^2 - 3) = 0 \Rightarrow x = 0 \quad \text{and} \quad x = \pm\sqrt{3}$ Critical points: $x = 0$, $x = \sqrt{3}$, and $x = -\sqrt{3}$.

2. **Find the second derivative:**

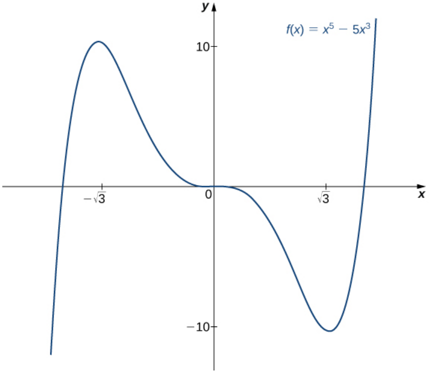
$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3)$$

3. **Evaluate $f''(x)$ at each critical point:**
   - $f''(0) = 10 \cdot 0 \cdot (2 \cdot 0^2 - 3) = 0$ (inconclusive).
   - $f''(\sqrt{3}) = 10\sqrt{3}(2 \cdot 3 - 3) = 30\sqrt{3} > 0$ (local minimum).
   - $f''(-\sqrt{3}) = 10(-\sqrt{3})(2 \cdot 3 - 3) = -30\sqrt{3} < 0$ (local maximum).

4. **Conclusion:**
   - $f(x)$ has a local minimum at $x = \sqrt{3}$.
   - $f(x)$ has a local maximum at $x = -\sqrt{3}$.
   - The second derivative test is inconclusive at $x = 0$.

The second derivative is inconcluse

- **Conclusion:** Since f' is negative on both intervals around $x = 0$, f is decreasing across $x = 0$. Therefore, f does not have a local extremum at $x = 0$.
- The graph confirms these results.



Example 1: Polynomial Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = x^4 - 4x^2$.

Example 1: Polynomial Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = x^4 - 4x^2$.

Solution Steps:

1. **Find the First Derivative:**

$$f'(x) = 4x(x^2 - 2) \Rightarrow x = 0, \pm\sqrt{2}$$

   Critical points: $x = 0$ and $x = \pm\sqrt{2}$.

2. **Find the Second Derivative:**

$$f''(x) = 12x^2 - 8$$

3. **Evaluate $f''(x)$ at Each Critical Point:**
   - $f''(0) = -8$: local maximum at $x = 0$.
   - $f''(\pm\sqrt{2}) = 16$: local minima at $x = \pm\sqrt{2}$.
4. **Conclusion:**
   - Local maximum at $x = 0$.
   - Local minima at $x = \pm\sqrt{2}$.

Example 2: Exponential Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = e^{-x^2}$.

Example 2: Exponential Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = e^{-x^2}$.

Solution Steps:

1. **Find the First Derivative:**

$$f'(x) = -2xe^{-x^2}$$

   Set $f'(x) = 0$: Critical point is $x = 0$.

2. **Find the Second Derivative:**

$$f''(x) = (4x^2 - 2)e^{-x^2}$$

3. **Evaluate $f''(x)$ at the Critical Point $x = 0$:**

$$f''(0) = -2 \Rightarrow \text{local maximum at } x = 0$$

4. **Conclusion:**
   - Local maximum at $x = 0$.

Example 3: Trigonometric Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = \sin(x) + \cos(x)$ over the interval $[0, 2\pi]$.

Example 3: Trigonometric Function

Problem: Use the second derivative test to find the location of all local extrema for $f(x) = \sin(x) + \cos(x)$ over the interval $[0, 2\pi]$.

Solution Steps:

1. **Find the First Derivative:**

   $$f'(x) = \cos(x) - \sin(x)$$

   Set $f'(x) = 0$: Solving $\cos(x) = \sin(x)$ gives critical points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.
2. **Find the Second Derivative:**

   $$f''(x) = -\sin(x) - \cos(x)$$

3. **Evaluate $f''(x)$ at Each Critical Point:**
   - $f''\left(\frac{\pi}{4}\right) = -\sqrt{2}$: local maximum at $x = \frac{\pi}{4}$.
   - $f''\left(\frac{5\pi}{4}\right) = \sqrt{2}$: local minimum at $x = \frac{5\pi}{4}$.
4. **Conclusion:**
   - Local maximum at $x = \frac{\pi}{4}$.
   - Local minimum at $x = \frac{5\pi}{4}$.

Key Concepts

Critical Points and Sign of f'

- If c is a critical number of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical number of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.

Concavity

- If $f''(x) > 0$ over an interval I, then f is concave up over I.
- If $f''(x) < 0$ over an interval I, then f is concave down over I.

Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, use the First Derivative Test or evaluate $f'(x)$ at points around c to determine if f has a local extremum at c.

Applied Optimization Problems

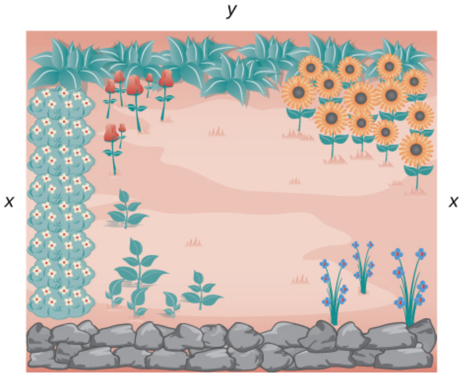
Clotilde Djuikem

Learning Objective

- Set up and solve optimization problems in several applied fields.

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides . Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?



Problem Setup

Objective: Maximize the area of a rectangular garden.

Let:

- x: Length of the side perpendicular to the rock wall.
- y: Length of the side parallel to the rock wall.

Given:

- Total fencing available: 100 ft.
- Area of the garden: $A = x \cdot y$.
- Constraint: $2x + y = 100$.

Step 1: Express Area in Terms of One Variable

From the constraint equation:

$$2x + y = 100 \quad \Longrightarrow \quad y = 100 - 2x.$$

Substitute $y = 100 - 2x$ into $A = x \cdot y$:

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Thus, the area is given by:

$$A(x) = 100x - 2x^2.$$

Step 2: Determine the Domain

To construct a rectangular garden:

- Both x and y must be positive:

$$x > 0 \quad \text{and} \quad y = 100 - 2x > 0.$$

- This implies $x < 50$.

Therefore, the domain for x is:

$$0 < x < 50.$$

To use the Extreme Value Theorem, we extend this to the closed interval:

$$[0, 50].$$

Step 3: Find the Critical Number

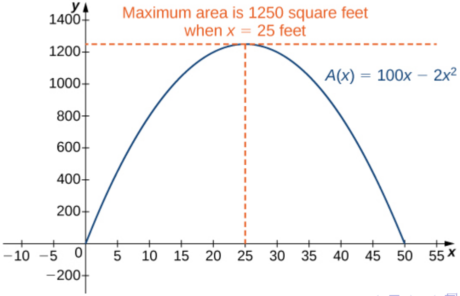
Differentiate $A(x) = 100x - 2x^2$:

$$A'(x) = 100 - 4x.$$

Set $A'(x) = 0$ to find the critical number:

$$100 - 4x = 0 \quad \Longrightarrow \quad x = 25.$$

The critical number is $x = 25$.



Step 4: Evaluate the Area

Evaluate $A(x)$ at the endpoints and critical number:

- At $x = 0$: $A(0) = 100(0) - 2(0)^2 = 0$.
- At $x = 50$: $A(50) = 100(50) - 2(50)^2 = 0$.
- At $x = 25$:

$$A(25) = 100(25) - 2(25)^2 = 2500 - 1250 = 1250\,\text{ft}^2.$$

Maximum Area: $1250\,\text{ft}^2$ occurs at $x = 25$.

Solution Summary

Optimal Dimensions:

- $x = 25\,\text{ft}$ (perpendicular to the rock wall).
- $y = 100 - 2(25) = 50\,\text{ft}$ (parallel to the rock wall).

Maximum Area:

$$A = x \cdot y = 25 \cdot 50 = 1250\,\text{ft}^2.$$

Conclusion: To maximize the area, construct a garden with dimensions $25\,\text{ft} \times 50\,\text{ft}$.

Step-by-Step Approach:

1. **Introduce all variables:** Define variables and, if applicable, draw and label a diagram.
2. **Identify the target quantity:** Determine what needs to be maximized or minimized and specify the range of possible values for other variables.
3. **Write the formula:** Express the quantity to optimize in terms of the variables.
4. **Relate variables:** Use additional equations or constraints to rewrite the formula as a function of one variable.
5. **Determine the domain:** Identify valid values for the variable(s) based on the physical context of the problem.
6. **Find the optimal value:** Differentiate the function, locate critical numbers, and justify the maximum or minimum using appropriate methods.
7. **State the final answer:** Provide a clear sentence with units, ensuring the solution satisfies the problem's constraints.

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

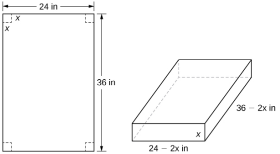
Problem Setup and Volume Function

Objective: Maximize the volume of an open-top box formed by cutting squares from a rectangular sheet.

Setup:

- Dimensions of cardboard: 36 in by 24 in.
- x: Side length of the square cut from each corner (in inches).
- Box dimensions after folding:
  - Height: x,
  - Length: $36 - 2x$,
  - Width: $24 - 2x$.
- Volume formula:

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$



Domain and Critical Numbers

Domain:

- $x > 0$ (side length must be positive).
- $x < 12$ (squares cannot exceed half the shorter side).
- Domain: $x \in [0, 12]$.

Find Critical Numbers:

$$V'(x) = 12x^2 - 240x + 864.$$

Solve $V'(x) = 0$:

$$12x^2 - 240x + 864 = 0 \implies x^2 - 20x + 72 = 0.$$

Using the quadratic formula:

$$x = 10 \pm 2\sqrt{7}.$$

Valid Critical Point: $x = 10 - 2\sqrt{7} \approx 4.708.$

Maximum Volume and Final Answer

Maximum Volume:

$$V(10-2\sqrt{7}) = 4(10-2\sqrt{7})^3 - 120(10-2\sqrt{7})^2 + 864(10-2\sqrt{7}).$$

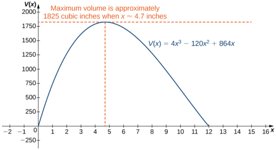
Approximation:

$$V \approx 1825\,\text{in}^3.$$

Optimal Dimensions:

- Height: $x \approx 4.708\,\text{in}$,
- Length: $36 - 2x \approx 26.584\,\text{in}$,
- Width: $24 - 2x \approx 14.584\,\text{in}$.

Final Answer: The maximum volume is approximately $1825\,\text{in}^3$ with optimal dimensions as listed above.



Minimizing Travel Time

An island is located 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore, which is 6 mi west of that closest point. The visitor plans to travel from the cabin to the island.

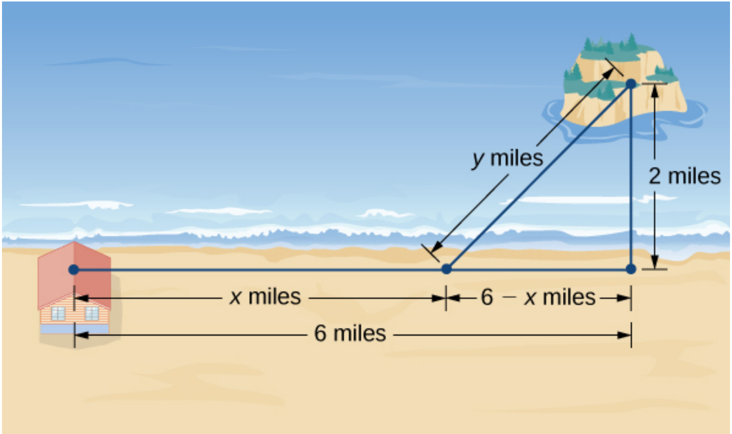
Suppose the visitor:

- Runs at a speed of 8 mph,
- Swims at a speed of 3 mph.

Question: How far should the visitor run along the shoreline before swimming to minimize the total time it takes to reach the island?

Solution

Let x be the distance running and let y be the distance swimming . Let T be the time it takes to get from the cabin to the island.



Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance $=$ Rate $\times$ Time, the time spent running is:

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is:

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is:

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From (Figure), the line segment of y miles forms the hypotenuse of a right triangle with legs of length $2\,\text{mi}$ and $6 - x\,\text{mi}$. Therefore, by the Pythagorean theorem:

$$2^2 + (6 - x)^2 = y^2,$$

and we obtain:

$$y = \sqrt{(6 - x)^2 + 4}.$$

Solution

Thus, the total time spent traveling is given by the function:

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4}}{3}.$$

Step 5: From (Figure), we see that $0 \leq x \leq 6$. Therefore, $[0,6]$ is the domain of consideration.

Step 6: Finding Critical Numbers

Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical numbers of T over the interval $[0,6]$. The derivative is:

$$T'(x) = \frac{1}{8} - \frac{1}{2}\left[(6-x)^2+4\right]^{-1/2} \cdot 2(6-x) = \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2+4}}.$$

If $T'(x) = 0$, then:

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}.$$

Therefore:

$$3\sqrt{(6-x)^2+4} = 8(6-x).$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy:

$$9\left[(6-x)^2+4\right] = 64(6-x)^2,$$

which implies:

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical number, then x satisfies:

$$(6-x)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical numbers are:

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical number. On the other hand, $x = 6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides to arrive at the possible critical numbers, it remains to verify that $x = 6 - \frac{6}{\sqrt{55}}$ satisfies the equation.
Since $x = 6 - \frac{6}{\sqrt{55}}$ does satisfy that equation, we conclude that it is the critical number.

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical number. On the other hand, $x = 6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides to arrive at the possible critical numbers, it remains to verify that $x = 6 - \frac{6}{\sqrt{55}}$ satisfies the equation.

Solution

Since $x = 6 - \frac{6}{\sqrt{55}}$ does satisfy that equation, we conclude that:

$$x = 6 - \frac{6}{\sqrt{55}}$$

is a critical number, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical number $x = 6 - \frac{6}{\sqrt{55}}$.

We find that:

$$T(0) \approx 2.108\,\mathrm{h}, \quad T(6) \approx 1.417\,\mathrm{h}, \quad \text{whereas} \quad T(6 - \frac{6}{\sqrt{55}}) \approx 1.368\,\mathrm{h}.$$

Therefore, we conclude that T has a local minimum at:

$$x \approx 5.19\,\mathrm{mi}.$$

Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function:

$$n(p) = 1000 - 5p.$$

- If they charge $50 perdayorless, theywillrentalltheircars. - Iftheycharge 200$ per day or more, they will not rent any cars.

Assuming the owners plan to charge customers between $50 perdayand 200$ per day to rent a car, how much should they charge to maximize their revenue?

Step 1: Problem Setup

Objective: Maximize the daily revenue of a car rental company.

Variables:

- p: Price charged per car per day ($50 \leq p \leq 200$).
- n: Number of cars rented per day.
- R: Revenue per day.

Model:

- $n(p) = 1000 - 5p$: Linear function modeling the number of cars rented as a function of p.
- Revenue formula: $R = n \times p$.

Step 2: Revenue Function

The revenue (per day) is given by:

$$R(p) = n(p) \times p.$$

Substituting $n(p) = 1000 - 5p$:

$$R(p) = (1000 - 5p)p = -5p^2 + 1000p.$$

Goal: Maximize $R(p)$ for p in the interval $[50, 200]$.

Step 3: Critical Numbers

To maximize $R(p)$, find its derivative and solve $R'(p) = 0$:

$$R'(p) = -10p + 1000.$$

Setting $R'(p) = 0$:

$$-10p + 1000 = 0 \quad \Longrightarrow \quad p = 100.$$

Critical Point: $p = 100$.

Step 4: Evaluate Revenue at Endpoints

Evaluate $R(p)$ at the critical point and the endpoints of the interval $[50, 200]$:

- At $p = 100$:

$$R(100) = -5(100)^2 + 1000(100) = \$50,000.$$

- At $p = 50$:

$$R(50) = -5(50)^2 + 1000(50) = \$37,500.$$

- At $p = 200$:

$$R(200) = -5(200)^2 + 1000(200) = \$0.$$

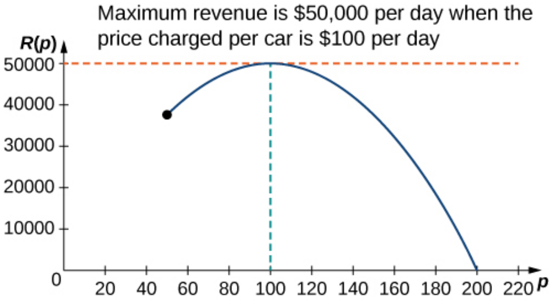
Step 5: Conclusion

Result:

- The revenue is maximized at $p = 100$.
- Maximum revenue: $R(100) = \$50,000$.

Recommendation:

- The car rental company should charge \$100 per day per car to maximize daily revenue.



Problem Statement

Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse:

$$\frac{x^2}{4} + y^2 = 1$$

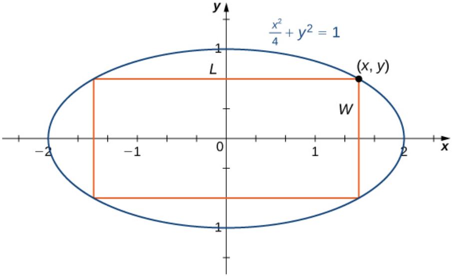
Determine:

- The dimensions of the rectangle that maximize its area.
- The maximum area.

Step 1: Geometry of the Problem

- The ellipse $\frac{x^2}{4} + y^2 = 1$ has x-intercepts ± 2 and y-intercepts ± 1.
- An inscribed rectangle has:
  - Length $L = 2x$ (horizontal).
  - Width $W = 2y$ (vertical).
- The area of the rectangle is:

$$A = L \cdot W = 2x \cdot 2y = 4xy$$



Step 2: Substituting Constraints

- From the ellipse equation:

$$\frac{x^2}{4} + y^2 = 1 \implies y = \sqrt{1 - \frac{x^2}{4}}$$

- Substitute y into the area formula:

$$A = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 3: Domain and Critical Points

- $x \in [0, 2]$ (rectangle in the first quadrant).
- The derivative of $A(x)$ is:

$$A'(x) = \frac{d}{dx}\left(2x\sqrt{4-x^2}\right)$$

  Simplifying:

$$A'(x) = \frac{8-4x^2}{\sqrt{4-x^2}}$$

- Solve $A'(x) = 0$ to find critical points:

$$8 - 4x^2 = 0 \implies x^2 = 2 \implies x = \sqrt{2}$$

Step 4: Dimensions and Maximum Area

- At $x = \sqrt{2}$:

$$y = \sqrt{1 - \frac{x^2}{4}} = \sqrt{1 - \frac{2}{4}} = \frac{1}{\sqrt{2}}$$

- Dimensions of the rectangle:

$$L = 2x = 2\sqrt{2}, \quad W = 2y = \sqrt{2}$$

- Maximum area:

$$A = L \cdot W = (2\sqrt{2})(\sqrt{2}) = 4$$

Optimization Example: Box Surface Area

Problem: Minimize the surface area of an open-top rectangular box with volume 216 in^3.

Surface Area:

$$S(x) = 4xy + x^2$$

Volume Constraint:

$$x^2 y = 216 \implies y = \frac{216}{x^2}$$

Substitute y into $S(x)$:

$$S(x) = 864/x + x^2$$

PSS: Justify a Maximum or Minimum on an Open Interval

Step 1: Analyze the Limits

- Take the limit of the function as the variable approaches the endpoints of the interval.
- If both limits are less than the function value at the critical number, the largest value at the critical points is the **absolute maximum** (similar for minimum).
- If at least one limit is larger (or smaller) than the critical values or diverges to infinity, then no maximum (or minimum) exists.

Step 2: Check Monotonicity

- Verify if the function is **increasing** to the left of the critical number and **decreasing** to the right (or vice versa).
- If true, the critical number corresponds to an **absolute maximum** (or minimum).

Step 3: Verify with Critical Numbers

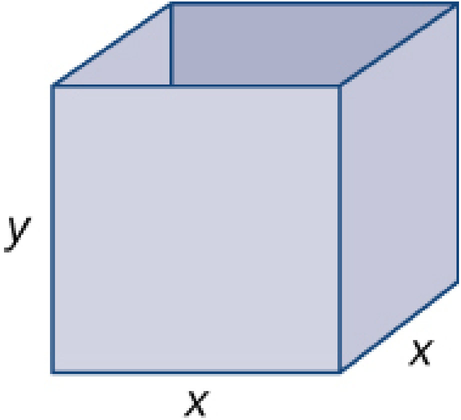
- If there is only **one critical number** on the interval, and the function has a **local maximum** (or minimum) at this value, it is also the **absolute maximum** (or minimum).

Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in. 3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution: Step 1

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.



Solution: Steps 2-4

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base.

- The area of each of the four vertical sides is $x \cdot y$.
- The area of the base is x^2.

Therefore, the surface area of the box is:

$$S = 4xy + x^2$$

Step 4: Since the volume of this box is $x^2 y$ and the volume is given as $216\,\text{in.}^3$, the constraint equation is:

$$x^2 y = 216$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2$$

Solution: Steps 5 and 6

Step 5: Domain Analysis

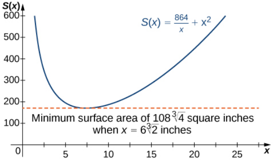
- Since $x^2y = 216$, $x > 0$ and x is unbounded. Domain: $(0, \infty)$.
- As $x \to 0^+$ or $x \to \infty$, $S(x) \to \infty$. Therefore, $S(x)$ must have an absolute minimum on $(0, \infty)$.

Step 6: Finding the Minimum

- Derivative: $S'(x) = -\frac{864}{x^2} + 2x$.
- Solve $S'(x) = 0$: $x^3 = 432 \implies x = 6\sqrt[3]{2}$.
- At $x = 6\sqrt[3]{2}$, $y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$.

Results:

$$x = 6\sqrt[3]{2}\,\text{in.}, \ \ y = 3\sqrt[3]{2}\,\text{in.}, \quad S = 108\sqrt[3]{4}\,\text{in.}^2$$



Key Concepts for Solving Optimization Problems

1. **Draw a Picture:** Begin by drawing a diagram to visualize the problem and introducing variables to represent key quantities.
2. **Relate the Variables:** Find an equation that relates the variables in the problem.
3. **Define the Function:** Write the quantity to be minimized or maximized as a function of a single variable.
4. **Find Critical Numbers:** Compute the derivative, find critical numbers, and determine the local extrema.

Curve Sketching (omit oblique asymptotes)

Clotilde Djuikem

Learning Objectives

- **Objective 1:** Analyze a function and its derivatives to draw its graph.
- **Objective 2:** Integrate the use of the first and second derivatives with other features of a function to create an accurate graph of $f(x)$.

Description:

In this section, we explore a comprehensive approach to graphing functions. By combining knowledge of derivatives with other analytical tools, you will be able to sketch the shape and key features of any given function.

Problem-Solving Strategy: Drawing the Graph of a Function

Steps:

1. **Determine the Domain:** Identify all x-values for which the function is defined.
2. **Intercepts:** Locate x- and y-intercepts.
3. **End Behavior:** Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to find horizontal or oblique asymptotes.
4. **Vertical Asymptotes:** Check for x-values where the function approaches $\pm\infty$.
5. **First Derivative Analysis ($f'(x)$):**
   - Identify critical points.
   - Determine intervals of increase and decrease.
   - Locate local extrema.
6. **Second Derivative Analysis ($f''(x)$):**
   - Determine concavity (up or down).
   - Find inflection points.
   - Confirm or verify extrema.

Example: Sketch a graph of $f(x) = (x-1)^2(x+2)$

Step 1: Determine the Domain

Function: $f(x) = (x-1)^2(x+2)$

- $f(x)$ is a polynomial, so it is defined for all real numbers.
- **Domain:** $\mathbb{R}$ (all real numbers).

Step 2: Locate the Intercepts

- *y*-**Intercept:**

$$f(0) = (0-1)^2(0+2) = 2 \quad \Longrightarrow \text{Intercept: } (0,2).$$

- *x*-**Intercepts:** Solve $(x-1)^2(x+2) = 0$:

$$x = 1 \quad (\text{multiplicity } 2), \quad x = -2.$$

- **Intercept Points:** $(1,0), (-2,0)$.

Step 3: End Behavior

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$:

- As $x \to \infty$:

$$(x-1)^2 \to \infty, \quad (x+2) \to \infty \implies \lim_{x\to\infty} f(x) = \infty.$$

- As $x \to -\infty$:

$$(x-1)^2 \to \infty, \quad (x+2) \to -\infty \implies \lim_{x\to-\infty} f(x) = -\infty.$$

Step 4: Vertical Asymptotes

Observation:

- $f(x)$ is a polynomial function.
- Polynomial functions do not have vertical asymptotes.
- **Conclusion:** No vertical asymptotes exist for $f(x)$.

Step 5: First Derivative Analysis

First Derivative:

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1).$$

Critical Numbers: Solve $f'(x) = 0$:

$$x = 1, \quad x = -1.$$

Divide the domain into intervals and test $f'(x)$:

- $(-\infty, -1) : f'(x) > 0 \quad$ (increasing).
- $(-1, 1) : f'(x) < 0 \quad$ (decreasing).
- $(1, \infty) : f'(x) > 0 \quad$ (increasing).

Local Extrema:

$$f(-1) = 4 \quad \text{(local max)}, \quad f(1) = 0 \quad \text{(local min)}.$$

Step 6: Second Derivative and Concavity

Second Derivative:

$$f''(x) = 6x.$$

Concavity Analysis:

- $(-\infty, 0) : f''(x) < 0 \quad$ (concave down).
- $(0, \infty) : f''(x) > 0 \quad$ (concave up).

Inflection point at $x = 0$.

Summary of Key Points:

- Intervals of increase: $(-\infty, -1), (1, \infty)$.
- Intervals of decrease: $(-1, 1)$.
- Local maximum: $(-1, 4)$, local minimum: $(1, 0)$.
- Concave up: $(0, \infty)$, concave down: $(-\infty, 0)$.
- Inflection point: $(0, f(0)) = (0, 2)$.

Graph of $f(x) = (x-1)^2(x+2)$

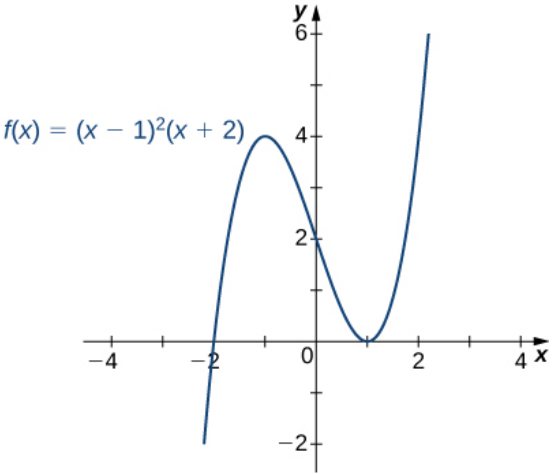


Figure:

Exercice

Sketching the Graph of $f(x) = (x-1)^3(x+2)$

Step 1: Determine the Domain

Function: $f(x) = (x-1)^3(x+2)$

- $f(x)$ is a polynomial, so it is defined for all real numbers.
- **Domain:** $\mathbb{R}$ (all real numbers).

Step 2: Locate the Intercepts

- y-**Intercept:**

$$f(0) = (0-1)^3(0+2) = (-1)^3(2) = -2.$$

  y-**Intercept:** $(0,-2)$**.**

- x-**Intercepts:** Solve $(x-1)^3(x+2) = 0$:

$$x = 1 \quad (\text{multiplicity } 3), \quad x = -2 \quad (\text{multiplicity } 1).$$

  x-**Intercepts:** $(1,0), (-2,0)$**.**

Step 3: Evaluate End Behavior

- The degree of $f(x)$ is 4 (even degree), and the leading coefficient is positive.
- As $x \to \infty$:

$$(x-1)^3 \to \infty, \quad (x+2) \to \infty \implies \lim_{x\to\infty} f(x) = \infty.$$

- As $x \to -\infty$:

$$(x-1)^3 \to -\infty, \quad (x+2) \to -\infty \implies \lim_{x\to-\infty} f(x) = \infty.$$

- **Conclusion:** Both ends of the graph go to ∞.

Step 4: Check for Vertical Asymptotes

Observation:

- $f(x)$ is a polynomial function.
- Polynomial functions do not have vertical asymptotes.
- **Conclusion:** No vertical asymptotes exist for $f(x)$.

Step 5: Analyze $f'(x)$

First Derivative:

$$f'(x) = 3(x-1)^2(x+2) + (x-1)^3 = (x-1)^2(4x+5).$$

Critical Numbers: Solve $f'(x) = 0$:

- $(x-1)^2 = 0 \implies x = 1.$
- $4x + 5 = 0 \implies x = -\frac{5}{4}.$

Divide the domain into intervals and test the sign of $f'(x)$, the derivative signed change only with $4x - 5$ because $(x-1)^2 \geq 0$:

- $(-\infty, -\frac{5}{4}) : f'(x) > 0 \quad$ (decreasing).
- $(-\frac{5}{4}, 1) : f'(x) > 0 \quad$ (increasing).
- $(1, \infty) : f'(x) > 0 \quad$ (increasing).

Step 6: Analyze $f''(x)$

Second Derivative:

$$f''(x) = 6(x-1)(2x+1).$$

Concavity Analysis:

- $(-\infty, -\frac{1}{2}) : f''(x) > 0 \quad \text{(concave up)}.$
- $(-\frac{1}{2}, 1) : f''(x) < 0 \quad \text{(concave down)}.$
- $(1, \infty) : f''(x) > 0 \quad \text{(concave up)}.$

Inflection pointx at $x = -\frac{1}{2}$ and $x = 1$.

Summary of Key Points:

- Intervals of increase: $(-\frac{5}{4}, \infty)$.
- Intervals of decrease: $(-\infty, -\frac{5}{4})$.
- Local minimum: At $x = -\frac{5}{4}$
- Concave up: $(-\infty, -\frac{1}{2})$, $(1, \infty)$ concave down: $(-\infty, 0)$.
- Inflection points: $(-\frac{1}{2}, f(-\frac{1}{2}))$ and $(1, f(1))$.

Graph of $f(x) = (x-1)^3(x+2)$