3.4 Partial Fractions

Math 1700

University of Manitoba

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Outline

Learning Objectives

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

Informations

We have seen some techniques for integrating specific rational functions:

- Integration of $\frac{du}{u}$ leads to $\ln|u| + C$, which yields:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C \quad (a \neq 0)$$

- Integration of $\frac{dx}{x^2+a^2}$ using trigonometric substitution results in:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$$

Problem!

- However, we still lack a technique for arbitrary polynomial quotients, such as $\int \frac{3x}{x^2-x-2}\, dx$.
- Partial fraction decomposition allows us to decompose such rational functions into simpler forms.
- It's essential to understand the form of decomposition, dependent on the factorization of the denominator.
- Which approach when $\deg(P(x)) > \deg(Q(x))$
- Remember, partial fraction decomposition applies only when $\deg(P(x)) < \deg(Q(x))$. If not, perform long division first.

Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2+3x+5}{x+1}\,dx$.

Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2+3x+5}{x+1}\,dx$. **Solution:** Since $\deg(x^2+3x+5) = 2 > 1 = \deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1}\,dx &= \int \left(x+2+\frac{3}{x+1}\right)\,dx\\ &= \frac{1}{2}x^2+2x+3\ln|x+1|+C.\end{aligned}$$

Evaluate $\int \frac{x-3}{x+2}\,dx$

Answer:

$$\int \frac{x-3}{x+2}\,dx = x - 5\ln|x+2| + C$$

Hint: Use long division to obtain $\frac{x-3}{x+2} = 1 - \frac{5}{x+2}$.

Integrating Rational Functions $\deg(P(x)) < \deg(Q(x))$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors:

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct and no factor is a constant multiple of another, then it is possible to find constants $A_1, A_2, \ldots, A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$. **Solution:** Since $\deg(3x+2) = 1 < 3 = \deg(x^3 - x^2 - 2x)$, we begin by factoring the denominator of the integrand. We can see that $x^3 - x^2 - 2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by bringing the right-hand side to a common denominator. We have:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2).$$

Method of Equating Coefficients

Expand the right-hand side of (1) and then group the terms by the powers of x to rewrite it as:

$$3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x + (-2A)$$

Equating coefficients produces the system of equations:

$$\begin{aligned} A + B + C &= 0 \\ -A + B - 2C &= 3 \\ -2A &= 2 \end{aligned}$$

Method of Equating Coefficients (cont'd)

To solve this system, we first observe that $-2A = 2 \rightarrow A = -1$. Substituting this value into the first two equations gives us the system:

$$\begin{aligned} B + C &= 1 \\ B - 2C &= 2 \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces:

$$-3C = 1$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1$, $B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy (1) for all values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, If we substitute

- $x = 0$, the equation reduces to $2 = A(-2)(1)$, yields $A = -1$.
- $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = \frac{4}{3}$.
- $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Then $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$. **Solution:** Since $\deg(x^2+3x+1) = 2 = \deg(x^2-4)$, we must perform long division of polynomials. This results in:

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on:

$$\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}.$$

We have:

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus:

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = \frac{11}{4}$ and $B = \frac{1}{4}$.

Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos(x)}{\sin^2(x)-\sin(x)}\,dx$.

Solution: Let's begin by letting $u = \sin(x)$. Consequently, $du = \cos(x)\,dx$. After making these substitutions, we have:

$$\int \frac{\cos(x)}{\sin^2(x)-\sin(x)}\,dx = \int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives:

$$\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}.$$

Therefore:

$$\begin{aligned}\int \frac{\cos(x)}{\sin^2(x)-\sin(x)}\,dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin(x)| + \ln|\sin(x)-1| + C.\end{aligned}$$

Evaluate $\int \frac{x+1}{(x+3)(x-2)}\,dx$.

Answer:

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Hint:

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Repeated Linear Factors $\int \frac{P(x)}{(ax+b)^n}$

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, there is at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the corresponding terms in the decomposition are:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$. **Solution:** We have $\deg(x-2) = 1 < 3 = \deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, the corresponding terms in the decomposition are going to be $\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$, and hence

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

Bringing to a common denominator and equating the numerators, we have:

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2.$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C).$$

Equating coefficients yields $2A+4C = 0$, $-3A+B-4C = 1$, and $A-B+C = -2$. Solving this system we obtain that $A = 2$, $B = 3$, and $C = -1$.

Partial Fractions with Repeated Linear Factors (Contd.)

Alternatively, we can use the method of strategic substitution. In this case, substituting $x = 1$ and $x = 1/2$ into the equation easily produces the values $B = 3$ and $C = -1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any x that we haven't used yet. The value $x = 0$ is a good option since it's very easy to substitute. This way, we obtain:

$$-2 = A(-1)(-1) + 3(-1) + (-1)(-1)^2,$$

and solving for A we get $A = 2$. Now that we have the values for A, B, and C, we rewrite the original integral:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1} \right) dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C.\end{aligned}$$

To integrate $\frac{3}{(2x-1)^2}$, we make a substitution $u = 2x - 1$, yielding $du = 2dx$, and then use the power formula to evaluate: $\int \frac{3}{(2x-1)^2}\,dx = \frac{3}{2}\int u^{-2}\,du$.

Partial Fraction Decomposition Setup

Set up the partial fraction decomposition for $\frac{x+2}{(x+3)^3(x-4)^2}$. (Do not solve for the coefficients or perform integration.)

Answer:

$$\frac{x+2}{(x+3)^3(x-4)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{x-4} + \frac{E}{(x-4)^2}$$

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $\deg(P(x)) < \deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
    - If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)...(a_nx+b_n)$, where each linear factor is distinct and no factor is a constant multiple of another, then it is possible to find constants $A_1, A_2, ...A_n$ satisfying
    $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$
    - If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain.
    $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Problem-Solving Strategy: Partial Fraction Decomposition

- For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include
$$\frac{Ax + B}{ax^2 + bx + c}.$$
- For each repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, the decomposition must include
$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}.$$
- After the appropriate decomposition is determined, solve for the constants.
- If using the decomposition to evaluate an integral, rewrite the integrand in its decomposed form and evaluate it using previously developed techniques or integration formulas.
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Rational Expressions with an Irreducible Quadratic Factor

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

Evaluate $\int \frac{2x-3}{x^3+x} dx$.

Solution

Since $\deg(2x-3) = 1 < 3 = \deg(x^3 + x)$, factor the denominator and proceed with partial fraction decomposition. Because $x^3 + x = x(x^2 + 1)$ contains irreducible quadratic factor $x^2 + 1$, include $\frac{Ax+B}{x^2+1}$ as a part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x-3}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}.$$

Rational Expressions with an Irreducible Quadratic Factor

After bringing to a common denominator and equating the numerators, we obtain the equation

$$2x - 3 = (Ax + B)x + C(x^2 + 1).$$

Solving for A, B, and C, we get $A = 3$, $B = 2$, and $C = -3$. Therefore,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}
\int \frac{2x-3}{x^3+x} dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x} \right) dx \\
&= 3 \int \frac{x}{x^2+1} dx + 2 \int \frac{1}{x^2+1} dx - 3 \int \frac{1}{x} dx \\
&= \frac{3}{2} \ln |x^2 + 1| + 2 \tan^{-1} x - 3 \ln |x| + C.
\end{aligned}$$

In order to evaluate $\int \frac{x}{x^2+1} dx$, we perform a substitution $u = x^2 + 1$. Note: We may rewrite $\ln |x^2 + 1| = \ln(x^2 + 1)$, if we wish to do so, since $x^2 + 1 > 0$.

Partial Fractions with an Irreducible Quadratic Factor 1

Evaluate $\int \frac{dx}{x^3-8}$. **Solution** Since the numerator is 1 and $\deg(1) = 0 < 3 = \deg(x^3 - 8)$, we can proceed with partial fraction decomposition. We start by factoring $x^3 - 8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor $x^2 + 2x + 4$ is irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After bringing to a common denominator and equating the numerators, this becomes

$$1 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$. Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

Partial Fractions with an Irreducible Quadratic Factor 2

We can see that

$$\int \frac{1}{x-2}\,dx = \ln|x-2| + C,$$

but $\int \frac{x+4}{x^2+2x+4}\,dx$ requires a bit more effort. Let's begin by completing the square in x^2+2x+4 to obtain

$$x^2+2x+4 = (x+1)^2+3.$$

By letting $u = x+1$ and consequently $du = dx$, we see that

$$\begin{aligned}\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{u+3}{u^2+3}\,du \\ &= \int \frac{u}{u^2+3}\,du + \int \frac{3}{u^2+3}\,du.\end{aligned}$$

Splitting the numerator apart, we get

$$\int \frac{u}{u^2+3}\,du + \int \frac{3}{u^2+3}\,du = \frac{1}{2}\ln|u^2+3| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C.$$

End

Replace $u = x + 1$

$$\frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3 - 8} = \frac{1}{12}\ln|x - 2| - \frac{1}{24}\ln|x^2 + 2x + 4| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

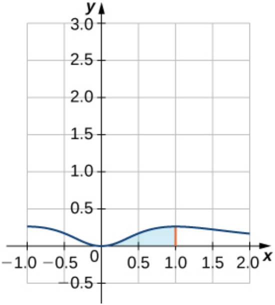
Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x.

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0, 1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved. From the sketch, we see that the shell method is a good choice for solving this problem.



Solution (Contd.)

The volume is given by

$$V = 2\pi \int_0^1 x \cdot \frac{x^2}{(x^2+1)^2}\, dx = 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\, dx$$

Since $\deg(x^3) = 3 < 4 = \deg((x^2+1)^2)$, we can proceed with partial fraction decomposition.

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

After finding $A = 1,\ B = 0,\ C = -1$, and $D = 0$, we substitute back into the integral:

$$V = \pi(\ln(2) - \frac{1}{2})$$

Partial Fraction Decomposition Setup

We aim to find the partial fraction decomposition for the expression:

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}$$

We express it as the sum of simpler fractions:

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Now, we need to determine the values of coefficients A, B, C, D, E, F, and G.

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must ensure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. These types include:
  - Nonrepeated linear factors
  - Repeated linear factors
  - Nonrepeated irreducible quadratic factors
  - Repeated irreducible quadratic factors