1.1 Approximating Areas

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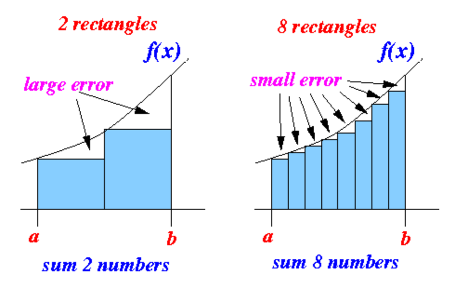
Outline

Motivation

Before

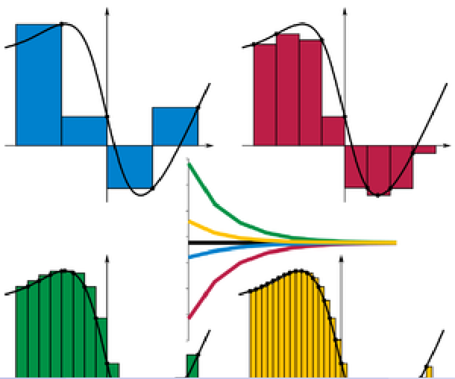
Imagine a bumpy field at a fair. We want to know how much space is there! Long ago, Archimedes used shapes to estimate areas. We do the same with rectangles. More rectangles mean a better guess.



Motivation

Today

Why do we do this? Think of planning a music festival. Calculating areas helps us organize spaces better. It is like having a secret tool for cool designs! We are learning these tricks to solve real-world puzzles someday. Is not that cool?



Learning Objectives

Objective 1

Use the sigma (summation) notation to calculate sums and powers of integers.

Objective 2

Use the sum of rectangular areas to approximate the area under a curve.

Objective 3

Use Riemann sums to approximate the area.

Sigma (Summation) Notation

In calculus, we use **sigma (Σ)** notation to make adding up lots of numbers easier.

Notation

For example, instead of writing $1 + 2 + 3 + \ldots + 19 + 20$,

we simply write $\sum_{i=1}^{20} i$.

Sigma notation looks like $\sum_{i=m}^{n} a_i$, where a_i are the terms to be added, i is the index of summation, and $m \leq n$ are the limits.

Let's try a couple of examples using sigma notation.

Example for Sigma

Using Sigma Notation

1. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
2. Write the sum in sigma notation: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$.
3. Write in sigma notation and evaluate the sum of terms 2^i for i=3,4,5,6.

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3. Write in sigma notation and evaluate the sum of terms 2^i for i=3,4,5,6.

Solution

1. We have $\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363$.
2. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Properties of Sigma Notation

Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers k, with $1 \leq k < n$.

$$\textbf{1.} \sum_{i=1}^{n} c = nc, \qquad \textbf{2.} \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\textbf{3.} \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i, \qquad \textbf{4.} \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\textbf{5.} \sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$$

Sums of Powers of Integers: To keep in mind

The sum of the first n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

The sum of the squares of the first n integers is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The sum of the cubes of the first n integers is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

Evaluation Using Sigma Notation

Write the following sums using sigma notation and then evaluate them.

1. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
2. The sum of the terms $\left(i^3 - i^2\right)$ for $i = 1, 2, 3, 4, 5, 6$.

Solution 1

We expand $(i-3)^2$, and then use properties of sigma notation along with the summation formulas to obtain

$$\begin{aligned}
\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}\left(i^2-6i+9\right) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \quad \text{(properties 3 and 4)} \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \quad \text{(property 2)} \\
&= \frac{200\,(200+1)\,(400+1)}{6} - 6\left[\frac{200\,(200+1)}{2}\right] + 9\,(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

Solution 2

We use sigma notation property 4 and the formulas for the sum of squared terms and the sum of cubed terms to obtain

$$\begin{aligned}
\sum_{i=1}^{6}\left(i^3 - i^2\right) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\
&= \frac{6^2(6+1)^2}{4} - \frac{6\,(6+1)\,(2\,(6)+1)}{6} \\
&= \frac{1764}{4} - \frac{546}{6} \\
&= 350
\end{aligned}$$

Problem

Find the sum of the values of $(4 + 3i)$ for $i = 1, 2, \ldots, 100$.

Answer: 15,550

Hint: Use the properties of sigma notation to solve the problem.

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution:

$$\sum_{i=1}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100 \times 121}{4} = 3025.$$

Finding the Sum of a Linear Function

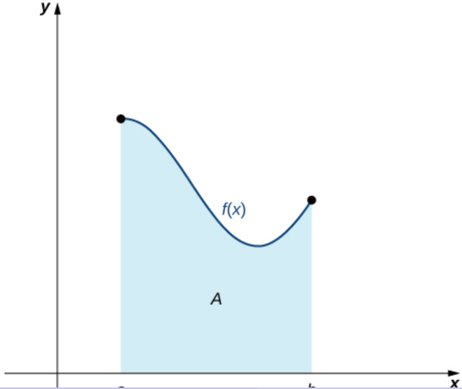
Let $f(x) = 2x + 1$. Evaluate the sum $\sum_{k=1}^{20} f(k)$.

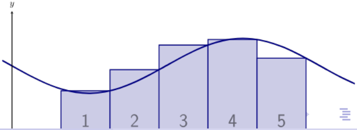
Answer: 440

Hint: Use the rules of sums and formulas for the sum of integers.

Problem

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A of the region under the curve $y = f(x)$, above the x-axis, and between the lines x=a and x=b, as shown on the figure below.





Idea

To approximate the area under the curve, we use a geometric approach. We divide the region into many small shapes, approximate each of them with a rectangle that has a known area formula, and then sum the areas of rectangles to obtain a reasonable estimate of the area of the region. We begin by dividing the interval $[a, b]$ into subintervals.

Definition

Consider an interval $[a, b]$. A set of points $P = \{x_i\}_{i=1}^{n}$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ is called a partition of $[a, b]$. If all the subintervals have the same width, the set of points forms a regular partition of the interval $[a, b]$.

For the regular partition, the width of each subinterval is denoted by Δx, so that

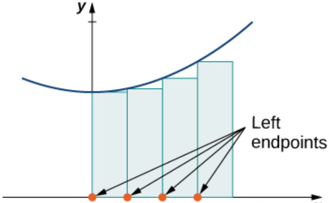
subinterval

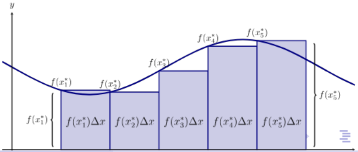
The subinterval $\Delta x = \frac{b-a}{n}$ and then $x_i = x_0 + i\Delta x$ for $i = 1, 2, 3, \ldots, n$

Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ ($i = 1, 2, 3, \ldots, n$), construct a rectangle with a width of Δx and a height of $f(x_{i-1})$, the function value at the left endpoint of the subinterval. This ensures that the left upper corner of the rectangle belongs to the curve $y = f(x)$ (see Figure 2 below). This rectangle approximates the region below the graph of f over the subinterval $[x_{i-1}, x_i]$, and its area is $f(x_{i-1})\Delta x$.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

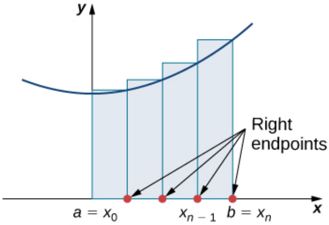


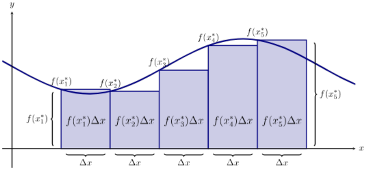


Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$ $(i = 1, 2, 3, \ldots, n)$ with the height of $f(x_i)$, the function value at the right endpoint of the subinterval. This ensures that the right upper corner of the rectangle belongs to the curve $y = f(x)$ (see Figure 3 below).

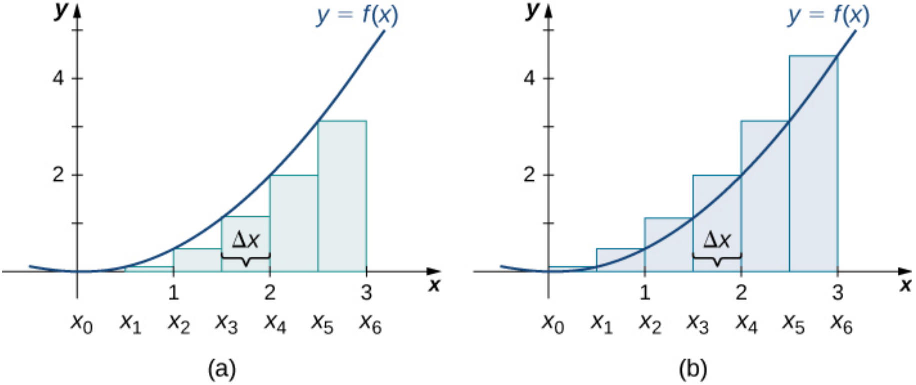
$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$





Frame Title

In this Figure, the area of the region below the graph of the function $f(x) = \frac{x^2}{2}$ over the interval $[0, 3]$ is approximated using left- and right-endpoint approximations with six rectangles.



Left-Endpoint Approximation

In this case, $\Delta x = \dfrac{3-0}{6} = 0.5$, and the subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$, that is, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, and $x_6 = 3$. Using the left-approximation formula for L_n, we obtain

$$\begin{aligned}
A &\approx L_6 = \sum_{i=1}^{6} f(x_{i-1})\Delta x \\
&= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&= f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5) \\
&= 0\cdot 0.5 + 0.125\cdot 0.5 + 0.5\cdot 0.5 + 1.125\cdot 0.5 + 2\cdot 0.5 + 3.125\cdot 0.5 \\
&= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\
&= 3.4375.
\end{aligned}$$

Right-Endpoint Approximation

Using the right-approximation formula for R_n, we obtain

$$
\begin{aligned}
A &\approx R_6 = \sum_{i=1}^{6} f(x_i)\Delta x \\
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\
&= f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 + f(3) \\
&= 0.125\cdot 0.5 + 0.5\cdot 0.5 + 1.125\cdot 0.5 + 2\cdot 0.5 + 3.125\cdot 0.5 + 4.5\cdot 0.5 \\
&= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\
&= 5.6875.
\end{aligned}
$$

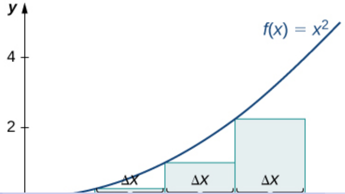
Approximating the Area Under a Curve

Use both left- and right-endpoint approximations to approximate the area under the graph of $f(x) = x^2$ over the interval $[0, 2]$ using $n = 4$.

Solution - Left-Endpoint Approximation

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ are shown in Figure 5. Using the left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, $f(1.5) = 2.25$. Then,

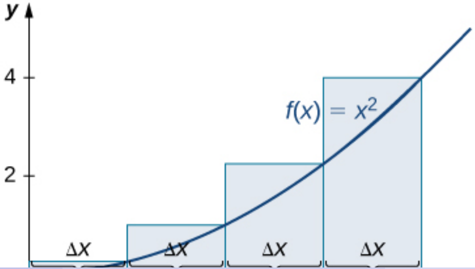
$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0 \cdot 0.5 + 0.25 \cdot 0.5 + 1 \cdot 0.5 + 2.25 \cdot 0.5 \\ &= 1.75. \end{aligned}$$



Solution: Right-Endpoint Approximation

The right-endpoint approximation is shown in Figure 6. The intervals are the same, $\Delta x = 0.5$, but now we use the right endpoints to calculate the heights of the rectangles. We have

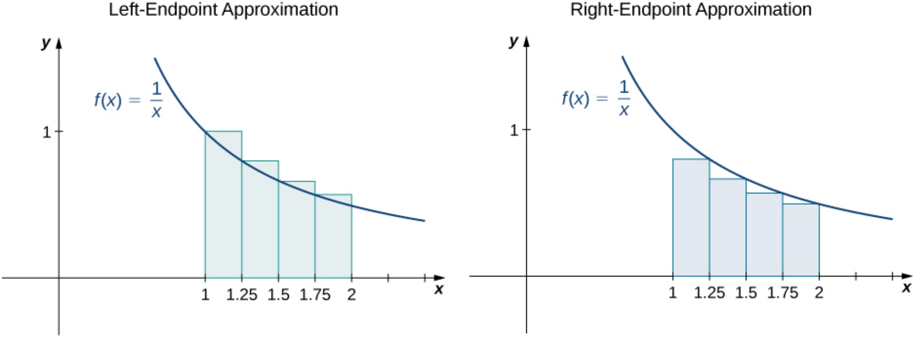
$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25 \cdot 0.5 + 1 \cdot 0.5 + 2.25 \cdot 0.5 + 4 \cdot 0.5 \\ &= 3.75. \end{aligned}$$



Sketch Left- and Right-Endpoint Approximations

Sketch left- and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$ using $n = 4$. Approximate the area using both methods.

Solution The left-endpoint approximation is 0.7595. The right-endpoint approximation is 0.6345. See the figure below.



(a) (b)

Generalizing Approximations

So far, to approximate the area under a curve, we have been using rectangles with the heights determined by evaluating the function at either the left or the right endpoint of the subinterval $[x_{i-1}, x_i]$. However, we could evaluate the function at any point x_i^* in $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of the approximating rectangle. This would result in an estimate $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Riemann Sum

Let the function $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$ with the subinterval width Δx. For each $1 \leq i \leq n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. The numbers $x_1^*, x_2^*, \ldots, x_n^*$ are called the sample points. Then the Riemann sum for $f(x)$ that corresponds to the partition P and the set of sample points $\{x_i^*\}_{i=1}^n$ is defined as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. Then, the area under the curve $y = f(x)$ over $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Finding Lower Sums

Problem: Find the lower sum for $f(x) = 10 - x^2$ over $[1, 2]$ with $n = 4$

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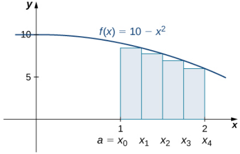


Solution:

$$\Delta x = \frac{2-1}{4} = \frac{1}{4},$$

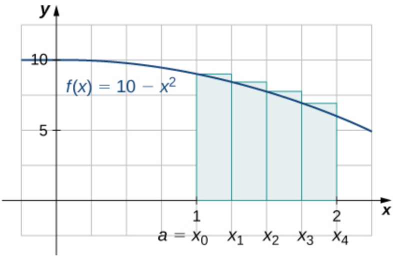
$$R_4 = \sum_{k=1}^{4} (10 - x_i^2) \cdot 0.25$$

$$= 0.25\,[8.4375 + 7.75 + 6.9375 + 6]$$

$$= 7.28.$$

Hence, the lower sum is 7.28.

Finding Upper Sums

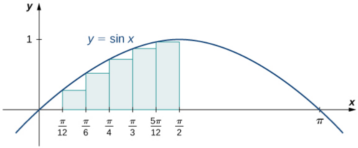


Hence, the upper sum is 8.0313.

Hint: $f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Finding Lower Sums

Problem: Find the lower sum for $f(x) = \sin(x)$ over $[0, \pi/2]$ with $n = 6$ subintervals.



Solution:

$$\begin{aligned}\Delta x &= \frac{\pi/2 - 0}{6} = \frac{\pi}{12},\\ L_6 &= \frac{\pi}{12}\left[0 + \sin\left(\frac{\pi}{12}\right) + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \sin\left(\frac{5\pi}{12}\right)\right]\\ &= \frac{\pi(1 + \sqrt{2} + \sqrt{3} + \sqrt{6})}{24}.\end{aligned}$$

Finding Upper Sums

Problem: Find the upper sum for $f(x) = \sin(x)$ over $[0, \pi/2]$ with $n = 6$ subintervals.
Solution:

$$\Delta x = \frac{\pi/2 - 0}{6} = \frac{\pi}{12},$$
$$R_6 = \frac{\pi(3 + \sqrt{2} + \sqrt{3} + \sqrt{6})}{24}.$$

Hint: Compare the expressions for the upper and lower sums.