3.3 Trigonometric Substitution

Math 1700

University of Manitoba

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Outline

Learning Objectives

Solve integration problems involving the square root of a sum or difference of two squares.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

- It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{x}{\sqrt{a^2-x^2}}\,dx$ and $\int x\sqrt{a^2 - x^2}\,dx$, they can each be integrated directly by a simple substitution.
- Make the substitution $x = a\sin(\theta)$ and $dx = a\cos(\theta)d\theta$.
- Note: This substitution yields $\sqrt{a^2 - x^2} = a\cos(\theta)$.
- Simplify the expression.
- Evaluate the integral using techniques from the section on trigonometric integrals.
- You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

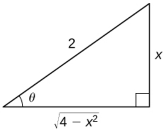
Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \sqrt{4 - x^2}\, dx$.

Solution:

Begin by making the substitutions $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$. Since $\sin(\theta) = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.

$\sin\theta = \frac{x}{2}$



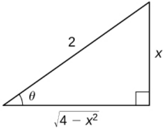
$\int \sqrt{4-x^2}\, dx = \int \sqrt{4-(2\sin(\theta))^2}\, 2\cos(\theta)d\theta$	Substitute $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$.
$= \int \sqrt{4\left(1-\sin^2(\theta)\right)}\, 2\cos(\theta)d\theta$	Simplify.
$= \int \sqrt{4\cos^2(\theta)} 2\cos(\theta)d\theta$	Use the identity $\cos^2(\theta) = 1 - \sin^2(\theta)$.
$= \int 2\lvert\cos(\theta)\rvert 2\cos(\theta)d\theta$	Take the square root.
$= \int 4\cos^2(\theta)d\theta$	Simplify. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos(\theta) \ge 0$ and $\lvert\cos(\theta)\rvert = \cos(\theta)$.
$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$	Use the strategy for integrating an even power of $\cos(\theta)$.
$= 2\theta + \sin(2\theta) + C$	Evaluate the integral.
$= 2\theta + (2\sin(\theta)\cos(\theta)) + C$	Substitute $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$	Substitute $\sin^{-1}\left(\frac{x}{2}\right) = \theta$ and $\sin(\theta) = \frac{x}{2}$. Use the reference triangle to see that $\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$ and make this substitution.
$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C.$	Simplify.

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\displaystyle\int \frac{\sqrt{4 - x^2}}{x}\, dx$.

Solution: First make the substitutions $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$. Since $\sin(\theta) = \frac{x}{2}$, we can construct the reference triangle shown in Figure 3 below.

$\sin\theta = \frac{x}{2}$



$$\int \frac{\sqrt{4 - x^2}}{x}\, dx = \int \frac{\sqrt{4 - (2\sin(\theta))^2}}{2\sin(\theta)} 2\cos(\theta)d\theta$$

Substitute $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$.

$$= \int \frac{2\cos^2(\theta)}{\sin(\theta)} d\theta$$

Substitute $1 - \sin^2(\theta) = \cos^2(\theta)$ and simplify.

$$= \int \frac{2\left(1 - \sin^2(\theta)\right)}{\sin(\theta)} d\theta$$

Substitute $\cos^2(\theta) = 1 - \sin^2(\theta)$.

$$= \int \left(2\csc(\theta) - 2\sin(\theta)\right) d\theta$$

Separate the numerator, simplify, and use $\csc(\theta) = \dfrac{1}{\sin(\theta)}$.

$$= 2\ln|\csc(\theta) - \cot(\theta)| + 2\cos(\theta) + C$$

Evaluate the integral.

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4 - x^2}}{x}\right| + \sqrt{4 - x^2} + C.$$

Use the reference triangle to rewrite the expression in terms of x and simplify.

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Method 1: Using the substitution $u = 1 - x^2$

Solution: Let $u = 1 - x^2$, hence $x^2 = 1 - u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\begin{aligned}
\int x^3\sqrt{1 - x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1 - x^2}(-2x\,dx) \quad \text{(Make the substitution)} \\
&= -\frac{1}{2}\int (1 - u)\sqrt{u}\,du \quad \text{(Expand the expression)} \\
&= -\frac{1}{2}\int (u^{1/2} - u^{3/2})\,du \quad \text{(Evaluate the integral)} \\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \quad \text{(Rewrite in terms of } x\text{)} \\
&= -\frac{1}{3}(1 - x^2)^{3/2} + \frac{1}{5}(1 - x^2)^{5/2} + C.
\end{aligned}$$

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Method 2: Using trigonometric substitution $x = \sin(\theta)$

Solution: Let $x = \sin(\theta)$. In this case, $dx = \cos(\theta)d\theta$. Using this substitution, we have

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta \\
&= \int (1-\cos^2(\theta))\cos^2(\theta)\sin(\theta)\,d\theta \quad (\text{Let } u = \cos(\theta)) \\
&= \int (u^4 - u^2)\,du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \quad (\text{Substitute } u = \cos(\theta)) \\
&= \frac{1}{5}\cos^5(\theta) - \frac{1}{3}\cos^3(\theta) + C \quad (\text{Use a reference triangle to} \\
&= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.
\end{aligned}$$

Integrating an Expression

Using Trigonometric Substitution

Rewrite the integral:

$$\int \frac{x^3}{\sqrt{25 - x^2}}\, dx$$

Answer:

$$\int 125 \sin^3(\theta)\, d\theta$$

Hint: Substitute $x = 5\sin(\theta)$ and $dx = 5\cos(\theta)d\theta$.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

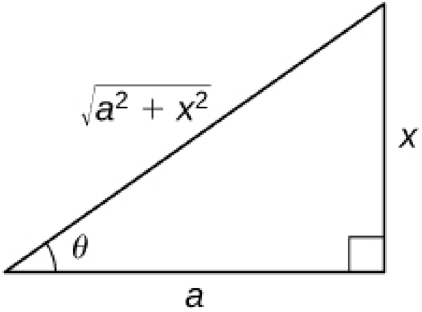
Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method. **Substitute** $x = a\tan(\theta)$ and $dx = a\sec^2(\theta)\,d\theta$. This substitution yields:

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + (a\tan(\theta))^2} = \sqrt{a^2\left(1 + \tan^2(\theta)\right)} \\ &= \sqrt{a^2\sec^2(\theta)} = |a\sec(\theta)| = a\sec(\theta).\end{aligned}$$

(Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec(\theta) > 0$ over this interval, $|a\sec(\theta)| = a\sec(\theta)$.)

- Simplify the expression.
- Evaluate the integral using techniques from the section on trigonometric integrals.
- Use the reference triangle from to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.
- (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

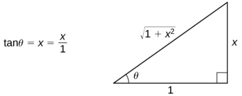
$$\tan\theta = \frac{x}{a}$$



Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$

Solution: Begin with the substitution $x = \tan(\theta)$ and $dx = \sec^2(\theta)\, d\theta$. Since $\tan(\theta) = x$, draw the reference triangle in the following figure.



This figure is a right triangle. It has an angle labeled θ. This angle is opposite the vertical side. The hypotenuse is labeled $\sqrt{1+x^2}$, the vertical leg is labeled x, and the horizontal leg is labeled 1. To the left of the triangle is the equation $\tan(\theta) = x/1$. Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2(\theta)}{\sec(\theta)} d\theta \quad \text{(Substitute } x = \tan(\theta) \text{ and } dx = \sec^2(\theta)d\theta\text{. This substitutio}$$

$$= \int \sec(\theta) d\theta \quad \text{(Evaluate the integral.)}$$

$$= \ln|\sec(\theta) + \tan(\theta)| + C \quad \text{(Use the reference triangle to express the result in}$$

$$= \ln|\sqrt{1+x^2} + x| + C.$$

Checking the Solution by Differentiation

To check the solution, differentiate:

$$\begin{aligned}\frac{d}{dx}\left(\ln\left|\sqrt{1+x^2}+x\right|\right) &= \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right)\\ &= \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\\ &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln\left|\sqrt{1+x^2}+x\right|+C$ as $\ln(\sqrt{1+x^2}+x)+C$, if desired.

Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Using $x = \sinh(\theta)$

Solution: Because $\sinh(\theta)$ has a range of all real numbers, and $1 + \sinh^2(\theta) = \cosh^2(\theta)$, we may also use the substitution $x = \sinh(\theta)$ to evaluate this integral. In this case, $dx = \cosh(\theta)d\theta$. Consequently,

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\cosh(\theta)}{\sqrt{1+\sinh^2(\theta)}}d\theta \quad \text{(Substitute } x = \sinh(\theta) \text{ and } dx = \cosh(\theta)d\theta\text{. This su}\\
&= \int \frac{\cosh(\theta)}{\sqrt{\cosh^2(\theta)}}d\theta = \int \frac{\cosh(\theta)}{|\cosh(\theta)|}d\theta \quad (\sqrt{\cosh^2(\theta)} = |\cosh(\theta)|)\\
&= \int \frac{\cosh(\theta)}{\cosh(\theta)}d\theta \quad (|\cosh(\theta)| = \cosh(\theta) \text{ since } \cosh(\theta) > 0 \text{ for all } \theta)\\
&= \int 1\, d\theta \quad \text{(Simplify.)}\\
&= \theta + C \quad \text{(Evaluate the integral. Since } x = \sinh(\theta)\text{, we know } \theta = \sinh^{-1}(x)\text{.)}\\
&= \sinh^{-1}(x) + C.
\end{aligned}$$

Analysis: Comparison of Solutions

This answer looks quite different from the answer obtained using the substitution $x = \tan(\theta)$. To see that the solutions are the same, set $y = \sinh^{-1}(x)$. Then $\sinh y = x$, that is,

$$\frac{e^y - e^{-y}}{2} = x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation formula to solve for e^y:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$, it must be the case that $e^y = x + \sqrt{x^2 + 1}$. Therefore,

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

At last, we obtain:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Analysis: Comparison of Solutions (Continued)

After we make the final observation that, since $x + \sqrt{x^2 + 1} > 0$,

$$\ln\left(x + \sqrt{x^2 + 1}\right) = \ln\left|\sqrt{1 + x^2} + x\right|,$$

we see that the two different methods produced the same solutions.
Conclusion: The solutions obtained using the substitutions $x = \tan(\theta)$ and $x = \sinh(\theta)$ are equivalent. Although they may appear different at first glance, they lead to the same result after careful analysis and simplification.

Finding an Arc Length

Problem: Find the length of the curve $y = x^2$ over the interval $\left[0, \frac{1}{2}\right]$.
Solution: Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\textbf{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_0^{\frac{1}{2}} \sqrt{1 + (2x)^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan(\theta)$ and $dx = \frac{1}{2}\sec^2(\theta)d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(\theta)} \cdot \frac{1}{2}\sec^2(\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3(\theta)\, d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)|\right)\Bigg|_0^{\frac{\pi}{4}} \\
&= \frac{1}{4}\left(\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right).
\end{aligned}$$

Rewriting the Integral

Problem: Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan(\theta)$.

Rewriting the Integral

Problem: Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan(\theta)$. **Answer:** We use the substitution $x = 2\tan(\theta)$ and $dx = 2\sec^2(\theta)d\theta$. Thus,

$$\begin{aligned}\int x^3\sqrt{x^2+4}\,dx &= \int (2\tan(\theta))^3\sqrt{(2\tan(\theta))^2+4}\cdot 2\sec^2(\theta)\,d\theta\\ &= 32\int \tan^3(\theta)\sec^3(\theta)\,d\theta.\end{aligned}$$

Hint: Use $x = 2\tan(\theta)$ and $dx = 2\sec^2(\theta)d\theta$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

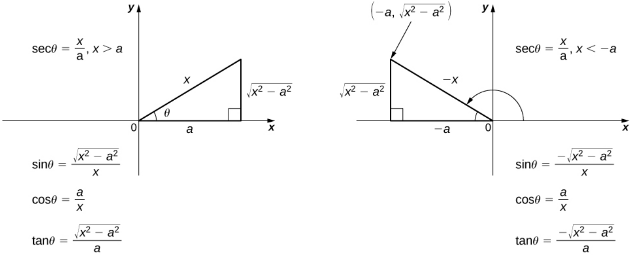
Step 1: Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
Step 2: Substitute $x = a\sec(\theta)$ and $dx = a\sec(\theta)\tan(\theta)d\theta$. This substitution yields

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{(a\sec(\theta))^2 - a^2} = \sqrt{a^2\left(\sec^2(\theta) - 1\right)} = \sqrt{a^2\tan^2(\theta)}\\ &= a|\tan(\theta)|.\end{aligned}$$

For $x \geq a$, we have $\theta \in \left[0, \frac{\pi}{2}\right)$, which implies that $\tan(\theta) \geq 0$, and so $a|\tan(\theta)| = a\tan(\theta)$ while for $x \leq -a$, $\theta \in \left(\frac{\pi}{2}, \pi\right]$, implying that $\tan(\theta) \leq 0$, and hence $a|\tan(\theta)| = -a\tan(\theta)$.
Step 3: Simplify the expression.
Step 4: Evaluate the integral using techniques from the section on trigonometric integrals.
Step 5: Use the reference triangles to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$.
Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x \geq a$ or $x \leq -a$.

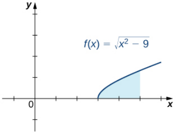
Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$ (continued)

Note (continued): There are also the equations $\sin(\theta) = \frac{\sqrt{x^2-a^2}}{x}$, $\cos(\theta) = \frac{a}{x}$, and $\tan(\theta) = \frac{\sqrt{x^2-a^2}}{a}$. The second triangle is in the second quadrant, with the hypotenuse labeled $-x$. The horizontal leg is labeled $-a$ and is on the negative x-axis. The vertical leg is labeled $\sqrt{x^2 - a^2}$. To the right of the triangle is the equation $\sec(\theta) = \frac{x}{a}$.



Finding the Area of a Region

Problem: Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.
Solution: First, sketch a rough graph of the region described in the problem.



We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}\, dx$. To evaluate this definite integral, substitute $x = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec(\theta)$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$.

Finding the Area of a Region (continued)

After making these substitutions and simplifying, we have:

$$\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2 - 9}\,dx = \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\tan^2(\theta)\sec(\theta)d\theta \quad (\text{since } \tan^2(\theta) = 1 - \sec^2(\theta)) \\
&= \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\left(\sec^2(\theta) - 1\right)\sec(\theta)d\theta \quad (\text{expand}) \\
&= \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\left(\sec^3(\theta) - \sec(\theta)\right)d\theta \quad (\text{evaluate the integral}) \\
&= \left(\frac{9}{2}\ln|\sec(\theta) + \tan(\theta)| + \frac{9}{2}\sec(\theta)\tan(\theta)\right) - 9\ln|\sec(\theta) + \tan(\theta)|\Big|_0^{\sec^{-1}\left(\frac{5}{3}\right)} \quad (\text{simplif} \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) - \frac{9}{2}\ln|\sec(\theta) + \tan(\theta)| + \tan(\theta)|\Big|_0^{\sec^{-1}\left(\frac{5}{3}\right)} \quad (\text{evaluate}) \\
&= 10 - \frac{9}{2}\ln 3.
\end{aligned}$$

Solution (continued): The final area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$ is $10 - \frac{9}{2}\ln 3$.

Evaluating $\int \frac{dx}{\sqrt{x^2-4}}$

Problem: Evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Assume that $x > 2$.

Answer: $\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$

Hint: Substitute $x = 2\sec(\theta)$ and $dx = 2\sec(\theta)\tan(\theta)d\theta$.

Key Concepts

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin(\theta)$ and $dx = a\cos(\theta)d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a\tan(\theta)$ and $dx = a\sec^2(\theta)d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a\sec(\theta)$ and $dx = a\sec(\theta)\tan(\theta)d\theta$.