7.2 Calculus of Parametric Curves

Math 1700

University of Manitoba

March 22, 2024

Outline

1. Derivatives of Parametric Equations
2. Second-Order Derivatives
3. Integrals Involving PE
4. Arc Length of a PC
5. Surface Area Generated by a PC

Learning Objectives

1. Determine derivatives and equations of tangents for parametric curves.
2. Find the area under a parametric curve.
3. Determine the arc length of a parametric curve.
4. Apply the formula for the surface area of the surface generated by revolving a parametric curve about the x-axis or the y-axis.

Challenge

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Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Derivative of Parametric Equations

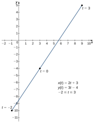
Theorem

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$(*) \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

Derivative of Parametric Equations: Example

Let



$x(t) = 2t + 3,$
$y(t) = 3t - 4, \quad -2 \leq t \leq 3.$

It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$. We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t:

$$x(t) = 2t + 3, \; x - 3 = 2t, \; t = \frac{x-3}{2}.$$

Substituting this into $y(t)$, we obtain:

$$y(t) = 3t - 4, \; y = 3\left(\frac{x-3}{2}\right) - 4, \; y = \frac{3x}{2} - \frac{9}{2} - 4, \; y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$.

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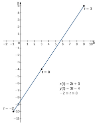

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$$x(t) = 2t + 3,\ x - 3 = 2t,\ t = \frac{x-3}{2}.$$

Substituting this into $y(t)$, we obtain:

$$y(t) = 3t - 4,\ y = 3\left(\frac{x-3}{2}\right) - 4,\ y = \frac{3x}{2} - \frac{9}{2} - 4,\ y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$.
Using Theorem, we calculate $x'(t)$ and $y'(t)$, which gives $x'(t) = 2$ and $y'(t) = 3$. Notice that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}$.

Example 1

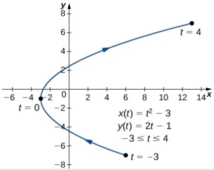
For the parametric equations:

$$x(t) = t^2 - 3,\ y(t) = 2t - 1, \quad -3 \leq t \leq 4,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2.$
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}.$$

Since $\frac{dy}{dt} \neq 0$, there are no points where the tangent line is horizontal. Solving $\frac{dx}{dt} = 2t = 0$, we find $t = 0$, corresponding to the point $(-3, -1)$ on the curve.



Example 2

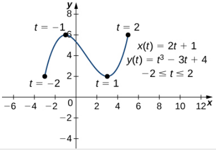
For the parametric equations:

$$x(t) = 2t + 1,\ y(t) = t^3 - 3t + 4, \quad -2 \leq t \leq 2,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 3t^2 - 3.$
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2}.$$

Since $\frac{dx}{dt} \neq 0$, there are no points where the tangent line is vertical. To find where the tangent line is horizontal, we solve $\frac{dy}{dt} = 3t^2 - 3 = 0$, giving $t = \pm 1$. At $t = -1$, the point $(-1, 6)$ is on the curve, and at $t = 1$, the point $(3, 2)$ is on the curve.



Example 3

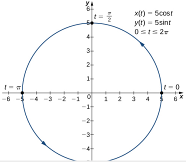
For the parametric equations:

$$x(t) = 5\cos(t),\ y(t) = 5\sin(t), \quad 0 \le t \le 2\pi,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = -5\sin(t), \quad \frac{dy}{dt} = 5\cos(t)$.
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\cot(t).$$

Points where $\frac{dy}{dt} = 0$ occur at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ in the interval $[0, 2\pi]$.
Solving $\frac{dx}{dt} = -5\sin(t) = 0$ yields $t = 0, \pi, 2\pi$. The points corresponding to these values are $(5, 0), (-5, 0), (5, 0)$ respectively.



Derivative and Tangent Lines

Calculate the derivative $\frac{dy}{dx}$ for the curve defined by the parametric equations

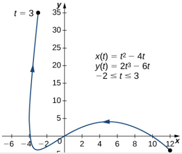
$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and find all points on the curve where the tangent line is horizontal or vertical.

Answer:

$$\frac{dy}{dx} = \frac{6t^2 - 6}{2t - 4} = \frac{3t^2 - 3}{t - 2}$$

The tangent line is horizontal at $(-3, 4)$ and $(5, 4)$, corresponding to $t = 1$ and $t = -1$ respectively. The tangent line is vertical at $(-4, 4)$, corresponding to $t = 2$.



Slope of the Tangent Line in a Special Case

Determine the slope of the tangent line to the hypocycloid

$$x(t) = 3\cos(t) + \cos(3t), \quad y(t) = 3\sin(t) - \sin(3t)$$

at the point corresponding to $t = 0$.
Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin(t) - 3\sin(3t), \quad y'(t) = 3\cos(t) - 3\cos(3t).$$

We see that $x'(0) = 0$, and so $(*)$ cannot be applied to find $\frac{dy}{dx}$ when $t = 0$. However, $x'(t) \neq 0$ when $t \in [-\frac{\pi}{6}, \frac{\pi}{6}] \setminus \{0\}$, $x'(t) > 0$ when $t \in \left[-\frac{\pi}{6}, 0\right)$ and $x'(t) < 0$ when $t \in (0, \frac{\pi}{6}])$, and so we can consider

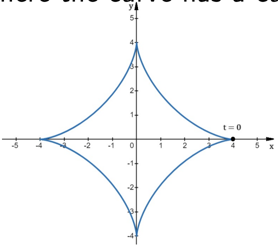
$$\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{y'(t)}{x'(t)} = \lim_{t\to 0} \frac{3\cos(t) - 3\cos(3t)}{-3\sin(t) - 3\sin(3t)}.$$

Since $\lim_{t\to 0}(3\cos(t) - 3\cos(3t)) = 0 = \lim_{t\to 0}(-3\sin(t) - 3\sin(3t))$, we deal with a $\frac{0}{0}$ indeterminate form and can apply L'Hospital's rule:

Part 2

$$\lim_{t\to 0}\frac{dy}{dx} = \lim_{t\to 0}\frac{3\cos(t)-3\cos(3t)}{-3\sin(t)-3\sin(3t)}$$
$$= \lim_{t\to 0}\frac{-3\sin(t)+9\sin(3t)}{-3\cos(t)-9\cos(3t)}$$
$$= \frac{-0+0}{-3-9} = \frac{0}{-12} = 0.$$

Therefore, when $t=0$, the slope of the tangent line is zero, and hence the tangent line to the hypocycloid is horizontal at the point $(4,0)$, corresponding to $t=0$, where the curve has a cusp.



Finding a Tangent Line

Find the equation of the tangent line to the parametric curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \leq t \leq 4$$

at the point corresponding to $t = 2$.
Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t, \quad y'(t) = 2.$$

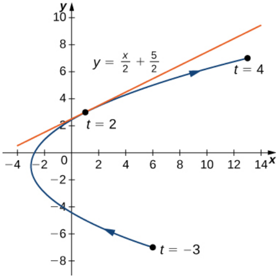
Next we substitute these into (*):

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives $x(2) = 2^2 - 3 = 1$ and $y(2) = 2(2) - 1 = 3$, which corresponds to the point $(1, 3)$ on the curve. We now use the point-slope form of the equation of a line to find the equation of the tangent line at this point:

Part 2

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= \frac{1}{2}(x - 1) \\ y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ y &= \frac{1}{2}x + \frac{5}{2}. \end{aligned}$$



Finding the Equation of the Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

at the point corresponding to $t = 5$.
Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t - 4, \quad y'(t) = 6t^2 - 6.$$

Next, we evaluate $x'(5) = 2(5) - 4 = 6$ and $y'(5) = 6(5)^2 - 6 = 144$. Using the point-slope form of the equation of a line with the point $(x(5), y(5))$ and slope $\frac{dy}{dx}(5)$, we have:

$$\begin{aligned} y - y(5) &= \frac{dy}{dx}(5)(x - x(5)) \\ y - (2(5)^3 - 6(5)) &= \frac{dy}{dx}(5)(x - (5^2 - 4(5))) \\ y - 40 &= \frac{144}{6}(x - 6),\ y - 40 = 24(x - 6),\ y = 24x + 100. \end{aligned}$$

Therefore, the equation of the tangent line is $y = 24x + 100$.

Second Derivative of Parametric Functions

To understand how to take the second derivative of a function defined parametrically, we start by considering the second derivative of a function $y = f(x)$. The second derivative of $y = f(x)$ is defined to be the derivative of the first derivative, which can be represented as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we can replace y on both sides of this equation with $\frac{dy}{dx}$. This substitution leads us to:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

If we know $\frac{dy}{dx}$ as a function of t, then this formula is straightforward to apply.

Finding a Second Derivative

Calculate the second derivative $\frac{d^2y}{dx^2}$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3, y(t) = 2t - 1$.

Solution:

Using $(*)$, we find that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$.

Applying $(**)$, we obtain

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Calculating the Second Derivative

Calculate the second derivative $\frac{d^2y}{dx^2}$ for the plane curve defined by the equations

$$x(t) = t^3 + 2t, \quad y(t) = 1 - t + t^2.$$

Solution:

Using the parametric equations, we first find the first derivative $\frac{dy}{dx}$ using the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

The first derivatives are: $\frac{dx}{dt} = 3t^2 + 2, \quad \frac{dy}{dt} = -1 + 2t.$

So, the first derivative $\frac{dy}{dx}$ is given by:

$$\frac{dy}{dx} = \frac{-1 + 2t}{3t^2 + 2}.$$

Next, to find the second derivative $\frac{d^2y}{dx^2}$, we differentiate $\frac{dy}{dx}$ with respect to t and then divide by $\frac{dx}{dt}$:

Part 2

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Differentiating $\frac{dy}{dx}$, we get:

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{-1+2t}{3t^2+2}\right) = \frac{2(3t^2+2) - 2(-1+2t)(6t)}{(3t^2+2)^2}.$$

So, the second derivative $\frac{d^2y}{dx^2}$ is given by:

$$\frac{d^2y}{dx^2} = \frac{4+6t-12t^2}{(3t^2+2)^3}.$$

Answer:

$$\frac{d^2y}{dx^2} = \frac{4+6t-12t^2}{(3t^2+2)^3}.$$

Examining Concavity of a Parametric Curve

Determine where the parametric curve $x(t) = 4t - t^2$, $y(t) = t^3 + 2$ is concave upward and where it is concave downward.

Solution:

Applying (*), we find that $\frac{dy}{dx} = \frac{3t^2}{4-2t}$. Using (**), together with the quotient rule, we obtain

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{3t^2}{4-2t}\right)'}{4-2t} = \frac{\frac{6t(4-2t)-3t^2(-2)}{(4-2t)^2}}{4-2t} = \frac{\frac{24t-6t^2}{(4-2t)^3}}{4-2t}.$$

We rewrite $\frac{d^2y}{dx^2}$ as

$$\frac{d^2y}{dx^2} = \frac{24t-6t^2}{(4-2t)^3} = \frac{6t(4-t)}{(2(2-t))^3} = \frac{6t(4-t)}{2^3(2-t)^3} = \frac{3t(4-t)}{4(2-t)^3}.$$

The numerator has zeros $t = 0$ and $t = 4$, while the denominator has a zero $t = 2$ of multiplicity 3. Using sample points or any other appropriate method, we find that $\frac{d^2y}{dx^2} > 0$, and hence the parametric curve is concave upward, when $t \in (0, 2)$ and $t \in (4, \infty)$, and $\frac{d^2y}{dx^2} < 0$, implying that the curve is concave downward, when $t \in (-\infty, 0)$ and $t \in (2, 4)$.

Concavity of Parametric Curve

Determine where the parametric curve $x(t) = t^2 + 1$, $y(t) = t^2 + t$ is concave upward.

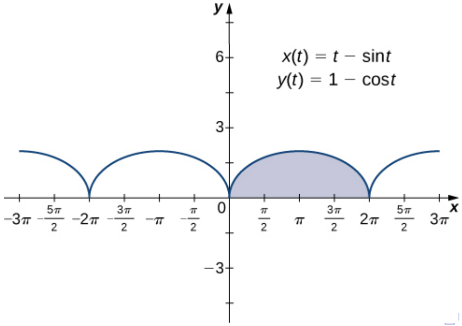
Answer: The curve is concave upward when $t \in (-\infty, 0)$.

Finding the Area under a Parametric Curve

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically?

Recall the cycloid defined by the equations

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t).$$



Area under a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t) \geq 0, \quad a \leq t \leq b$$

and assume that $x(t)$ is differentiable.
If $x(t)$ is increasing, then the area under this curve is given by

$$A = \int_a^b y(t) \frac{dx}{dt}\, dt.$$

If $x(t)$ is decreasing, then the area under this curve is given by

$$A = \int_b^a y(t) \frac{dx}{dt}\, dt.$$

Finding the Area under a Parametric Curve

Find the area under one arc of the cycloid defined by the equations

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t), \quad 0 \leq t \leq 2\pi.$$

Solution: To determine whether $x(t)$ is increasing or decreasing, we look at the sign of $x'(t)$. We have that $x'(t) = 1 - \cos(t) \geq 0$, and hence $x(t)$ is increasing. Applying the above theorem, we have

$$\begin{aligned}
A &= \int_a^b y(t)\frac{dx}{dt}\,dt = \int_0^{2\pi} (1-\cos(t))\,(1-\cos(t))\;dt \\
&= \int_0^{2\pi} \left(1 - 2\cos(t) + \cos^2(t)\right)\;dt = \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1+\cos(2t)}{2}\right)\;dt \\
&= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{\cos(2t)}{2}\right)\;dt \\
&= \left(\frac{3t}{2} - 2\sin(t) + \frac{\sin(2t)}{4}\right)\Bigg|_0^{2\pi} \\
&= 3\pi.
\end{aligned}$$

Finding the Area under a Parametric Curve

Find the area under the upper half of the hypocycloid defined by the equations

$$x(t) = 3\cos(t) + \cos(3t), \quad y(t) = 3\sin(t) - \sin(3t), \quad 0 \leq t \leq \pi.$$

Answer:

$$A = 3\pi$$

Hint: Use the above theorem, along with the identities

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

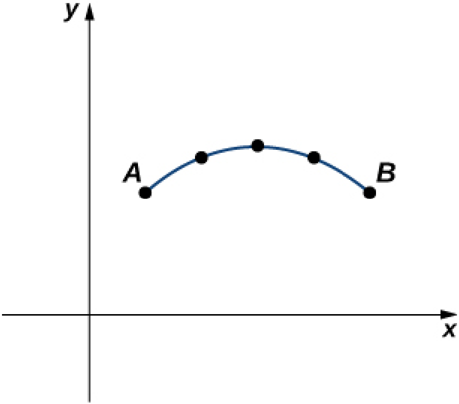
and

$$\sin^2(t) = \frac{1 - \cos(2t)}{2}.$$

Note that $x(t)$ is decreasing.

Approximating the Arc Length of a Parametric Curve

The same way we did for a regular curve with explicit equation $y = f(x)$ or $x = g(y)$, to derive a formula for the arc length of a parametric curve, we approximate it by a union of line segments as shown in the figure above.



Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leq t \leq t_2$$

and assume that $x(t)$ and $y(t)$ are smooth, that is, their derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length Formula for a Regular Curve: Proof

Now suppose that the parameter can be eliminated, leading to a function $y = F(x)$.

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt.$$

Here we have assumed that $x'(t) > 0$, and the case when $x'(t) < 0$ is analogous (the extra minus is going to disappear when the limits of integration are interchanged). Using a substitution $x = x(t)$, we have that $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is exactly the one we had before.

Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

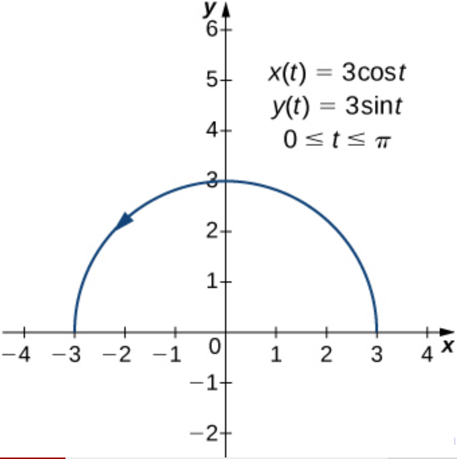
$$x(t) = 3\cos(t), \quad y(t) = 3\sin(t), \quad 0 \leq t \leq \pi.$$

Solution: The parametric curve is shown in Figure 9 below. To determine its length, we use the formula:

$$\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\, dt \\
&= \int_0^{\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt \\
&= \int_0^{\pi} \sqrt{9(\sin^2(t) + \cos^2(t))}\, dt \\
&= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} = 3\pi.
\end{aligned}$$

Note on the Arc Length of a Semicircle

Note that the formula for the arc length of a semicircle is πr, and the radius of this circle is 3. This is a great example of using calculus to derive a known geometric formula.



Finding the Arc Length of a Parametric Curve

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \leq t \leq 3.$$

Answer:

$$s = 2\left(10^{3/2} - 2^{3/2}\right).$$

Surface Area of a Surface of Revolution

Recall the problem of finding the surface area of a surface of revolution. In Section 2.4, we derived a formula for the surface area of a surface generated by revolving the curve $y = f(x) \geq 0$ from $x = a$ to $x = b$ around the x-axis:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

We now consider a surface of revolution generated by revolving a parametrically defined curve $x = x(t),\ y = y(t),\ a \leq t \leq b$ around the x-axis as shown in Figure 11 below.
The formula for its surface area is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

provided that $y(t)$ is non-negative on $[a, b]$.

Finding Surface Area of a Sphere

Find the surface area of a sphere of radius r centered at the origin.
Solution: We start by parametrizing the upper semicircle with center at the origin and radius r:

$$x(t) = r\cos(t), \quad y(t) = r\sin(t), \quad 0 \le t \le \pi.$$

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use the formula:

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= 2\pi \int_0^\pi r\sin(t)\sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt \\ &= 2\pi \int_0^\pi r\sin(t)\sqrt{r^2(\sin^2(t) + \cos^2(t))}\, dt = 2\pi \int_0^\pi r^2\sin(t)\, dt \\ &= 2\pi r^2(-\cos(t))\Big|_0^\pi = 2\pi r^2(-\cos(\pi) + \cos(0)) = 4\pi r^2. \end{aligned}$$

This agrees with the geometric formula you might have seen before.

Finding the Area of the Surface of Revolution

Find the area of the surface generated by revolving the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \le t \le 1$$

around the x-axis.

Answer:

$$A = \frac{\pi(494\sqrt{13} + 128)}{1215}$$

Hint: When evaluating the integral, use a u-substitution.

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- If $y(t) \geq 0$, the area under the parametric curve can be determined by using the formula $A = \pm \int_a^b y(t)x'(t)\,dt$, where the choice of sign depends on whether $x(t)$ is increasing or decreasing over $[a, b]$.
- The arc length of a parametric curve can be calculated by using the formula $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.
- The area of a surface obtained by revolving a parametric curve around the x-axis is given by $S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$, provided $y(t) \geq 0$ when $t \in [a, b]$. If the curve is revolved around the y-axis, then the formula is $S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$, provided $x(t) \geq 0$ when $t \in [a, b]$.