# 7.1 Parametric Equations

Math 1700

University of Manitoba

March 19, 2024

# Outline

1. Parametric Equations and Their Graphs
2. Eliminating the Parameter
3. Cycloids and Other Parametric Curves

# Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

# Definition

If $x$ and $y$ are continuous functions of $t$ on an interval $I$, then the equations

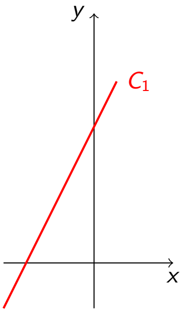
$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called parametric equations and $t$ is called the parameter. The set of points $(x, y)$ obtained as $t$ varies over the interval $I$ is called the graph of the parametric equations. The graph of parametric equations is also called a parametric curve or plane curve, and is denoted by $C$.

# Graphing Parametrically Defined Curves

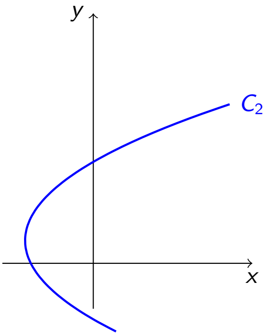
**1st:**

$$x(t) = t - 1$$
$$y(t) = 2t + 4$$
$$-3 \le t \le 2$$



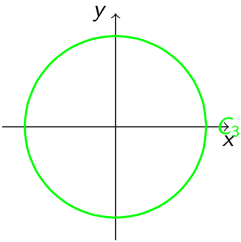

**2nd:**

$$x(t) = t^2 - 3$$
$$y(t) = 2t + 1$$
$$-2 \le t \le 3$$




**3rd:**

$$x(t) = 4\cos t$$
$$y(t) = 4\sin t$$
$$0 \le t \le 2\pi$$



# Example 1

Sketch the curves described by the following parametric equations:

$$x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \le t \le 2$$

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is $t$, let $t$ appear in the first column.

| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| $-3$ | $-4$ | $-2$ |
| $-2$ | $-3$ | 0 |
| $-1$ | $-2$ | 2 |
| 0 | $-1$ | 4 |
| 1 | 0 | 6 |
| 2 | 1 | 8 |

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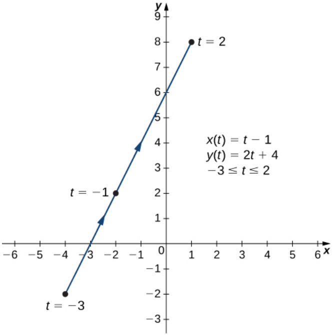
| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| $-3$ | $-4$ | $-2$ |
| $-2$ | $-3$ | $0$ |
| $-1$ | $-2$ | $2$ |
| $0$ | $-1$ | $4$ |
| $1$ | $0$ | $6$ |
| $2$ | $1$ | $8$ |




The arrows on the graph indicate the orientation of the graph, that is, the direction that a point moves on the graph as $t$ varies from -3 to 2.

# Example 2

Sketch the curves described by the following parametric equations:

$$x(t) = t^2 - 3, \; y(t) = 2t + 1, \; -2 \le t \le 3$$

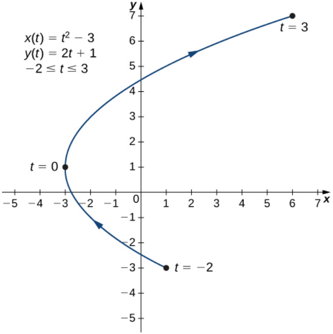
| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| $-2$ | 1 | $-3$ |
| $-1$ | $-2$ | $-1$ |
| 0 | $-3$ | 1 |
| 1 | $-2$ | 3 |
| 2 | 1 | 5 |
| 3 | 6 | 7 |

# Example 2

Sketch the curves described by the following parametric equations:

$$x(t) = t^2 - 3,\ y(t) = 2t + 1,\ -2 \leq t \leq 3$$

| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| $-2$ | $1$ | $-3$ |
| $-1$ | $-2$ | $-1$ |
| $0$ | $-3$ | $1$ |
| $1$ | $-2$ | $3$ |
| $2$ | $1$ | $5$ |
| $3$ | $6$ | $7$ |




As $t$ progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

# Example 3

Sketch the curves described by the following parametric equations:

$$x(t) = 4\cos t,\ y(t) = 4\sin t,\ 0 \le t \le 2\pi$$

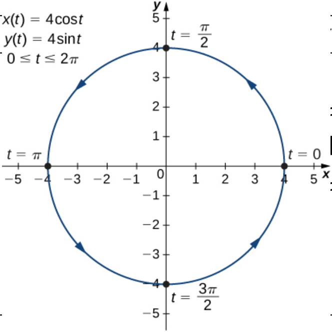
| $t$ | $x(t)$ | $y(t)$ | $t$ | $x(t)$ | $y(t)$ |
|---|---|---|---|---|---|
| $0$ | $4$ | $0$ | $\frac{7\pi}{6}$ | $-2\sqrt{3} \approx -3.5$ | $2$ |
| $\frac{\pi}{6}$ | $2\sqrt{3} \approx 3.5$ | $2$ | $2$ | $\frac{4\pi}{3}$ | $-2\sqrt{3} \approx -3.5$ |
| $\frac{\pi}{3}$ | $2$ | $2\sqrt{3} \approx 3.5$ | $\frac{3\pi}{2}$ | $0$ | $-4$ |
| $\frac{\pi}{2}$ | $0$ | $4$ | $\frac{5\pi}{3}$ | $2$ | $-2\sqrt{3} \approx -3.5$ |
| $\frac{2\pi}{3}$ | $-2$ | $2\sqrt{3} \approx 3.5$ | $\frac{11\pi}{6}$ | $2\sqrt{3} \approx 3.5$ | $2$ |
| $\frac{5\pi}{6}$ | $-2\sqrt{3} \approx -3.5$ | $2$ | $2\pi$ | $4$ | $0$ |
| $\pi$ | $-4$ | $0$ | | | |

# Example 3

Sketch the curves described by the following parametric equations:

$$x(t) = 4\cos t,\ y(t) = 4\sin t,\ 0 \le t \le 2\pi$$

| $t$ | $x(t)$ | $y(t)$ | $t$ | | $y(t)$ |
|---|---|---|---|---|---|
| $0$ | $4$ | $0$ | $\frac{7\pi}{6}$ | | |
| $\frac{\pi}{6}$ | $2\sqrt{3} \approx 3.5$ | $2$ | $2$ | | $-3.5$ |
| $\frac{\pi}{3}$ | $2$ | $2\sqrt{3} \approx 3.5$ | $\frac{3\pi}{2}$ | | |
| $\frac{\pi}{2}$ | $0$ | $4$ | $\frac{5\pi}{3}$ | | $-3.5$ |
| $\frac{2\pi}{3}$ | $-2$ | $2\sqrt{3} \approx 3.5$ | $\frac{11\pi}{6}$ | | |
| $\frac{5\pi}{6}$ | $-2\sqrt{3} \approx -3.5$ | $2$ | $2\pi$ | | |
| $\pi$ | $-4$ | $0$ | | | |

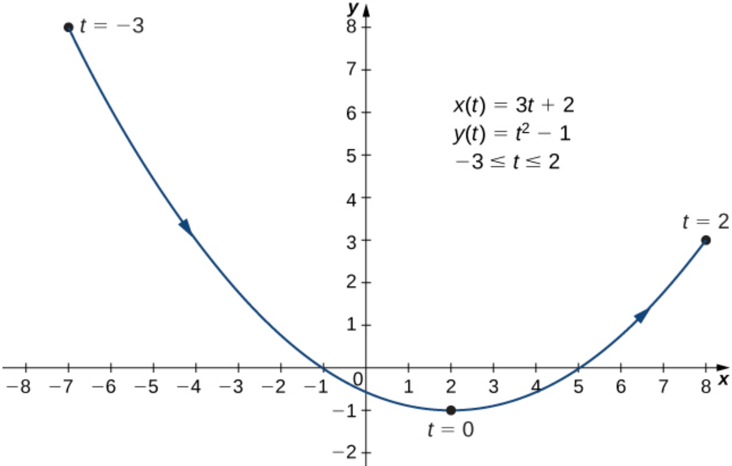



This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending point of the curve both have coordinates $(4, 0)$.

# Sketching the Curve

Sketch the curve described by the parametric equations:

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad -3 \le t \le 2$$



# Rewriting Parametric Equations

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables $x$ and $y$. For example, consider the parametric equations:

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3$$

Solving the second equation for $t$ gives:

$$t = \frac{y-1}{2}$$

This can be substituted into the first equation:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{(y^2 - 2y + 1)}{4} - 3 = \frac{y^2 - 2y - 11}{4}$$

This equation describes $x$ as a function of $y$.

# Eliminating the Parameter: Example 1

**Eliminate the parameter** for each of the plane curves described by the following parametric equations and describe the resulting graph:

$$x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad -2 \leq t \leq 6$$

**Solution:** To eliminate the parameter, we can solve either of the equations for $t$. For example, solving the first equation for $t$ gives:

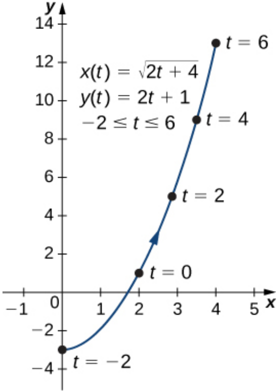
$$x = \sqrt{2t+4} \Rightarrow t = \frac{x^2-4}{2}$$

Note that when we square both sides, it is important to observe that $x \geq 0$. Substituting $t = \frac{x^2-4}{2}$ into $y(t)$ yields:

$$y(t) = 2t+1 = 2\left(\frac{x^2-4}{2}\right)+1 = x^2-4+1 = x^2-3$$

# Part 2 example 1

$y = x^2 - 3$, this is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter $t$. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.



# Eliminating the Parameter Creatively

Sometimes it is necessary to be a bit creative in eliminating the parameter.

$$x(t) = 4\cos(t) \quad \text{and} \quad y(t) = 3\sin(t).$$

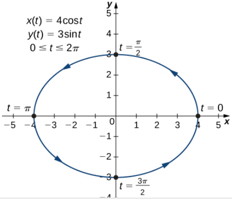
Solving either equation for $t$ directly is not advisable because sine and cosine are not one-to-one functions.

$$\cos(t) = \frac{x}{4} \quad \text{and} \quad \sin(t) = \frac{y}{3}.$$

Using the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$ , we obtain:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and seminor axis 3.



# Eliminating the Parameter

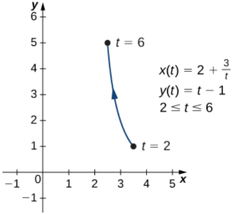
To eliminate the parameter for the plane curve defined by the parametric equations:

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad 2 \le t \le 6$$

We can express $t$ in terms of $x$ and $y$: $x = 2 + \frac{3}{t} \implies t = \frac{3}{x-2}$. Then substitute $t$ into the equation for $y$:

$$y = \frac{3}{x-2} - 1 \implies y = -1 + \frac{3}{x-2}$$

So, we have $x = 2 + \frac{3}{y+1}$ or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



# Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$:

1. **First Parametric Equations:**

$$x(t) = t, \quad y(t) = 2t^2 - 3$$

   Since there is no restriction on the domain in the original graph, there is no restriction on the values of $t$.

2. **Second Parametric Equations:**

$$x(t) = 3t - 2, \quad y(t) = 18t^2 - 24t + 6$$

   We have complete freedom in the choice for the second parameterization. We can choose $x(t) = 3t - 2$ since there are no restrictions imposed on $x$, and then substitute it into the equation $y = 2x^2 - 3$.

Therefore, the second parameterization of the curve can be written as:

$$x(t) = 3t - 2, \quad y(t) = 18t^2 - 24t + 6$$

# Parameterizing a Curve

**Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$:**

1. **First Parametric Equations:**

$$x(t) = t, \quad y(t) = t^2 + 2t$$

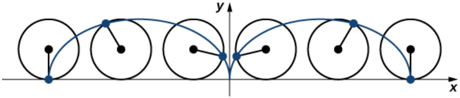
2. **Second Parametric Equations:**

$$x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3$$

There are, in fact, an infinite number of possibilities.

# The Cycloid: Nature's Artistry

- Imagine embarking on a tranquil bicycle ride through the countryside, where every rotation of the tire leaves a rhythmic mark on the road.
- Picture a determined ant seeking its way home after a long day's journey, hitchhiking along the tire's edge for a free ride.
- The path traced by this intrepid ant on a straight road is what we call a cycloid.



# The Cycloid: Parameterizing

## Parametric equations

A cycloid generated by a circle (or bicycle wheel) of radius $a$ is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t)$$

## Proof

If the radius is $a$, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

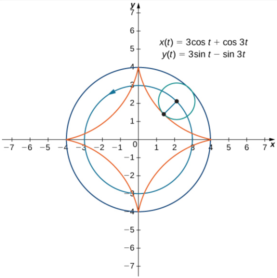
$$x(t) = -a \sin t, \quad y(t) = -a \cos t$$

Adding these equations together gives the equations for the cycloid.

# The hypocycloid.: Large wheel

## Visualizing

Suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.



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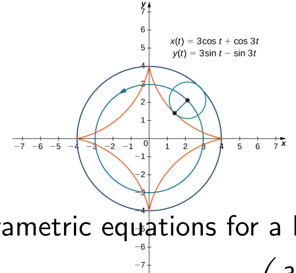


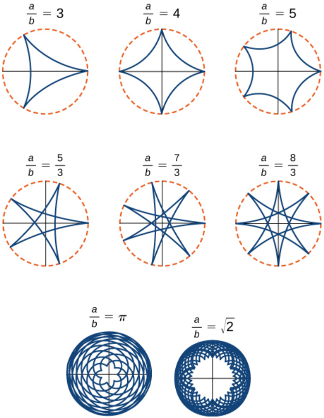

The general parametric equations for a hypocycloid are:

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$$

$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$$

# Examples Hypocycloid

The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$. The ratio $\frac{a}{b}$ is related to the number of cusps (corners or pointed ends) on the graph, as illustrated in the following Figure



# Key Concepts: Parametric Equations

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.