7.3 Polar Coordinates

Math 1700

University of Manitoba

March 28, 2024

Outline

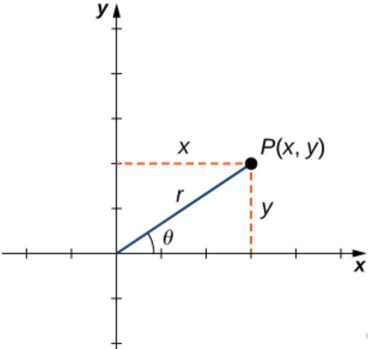
1. Defining Polar Coordinates
2. Polar Curves
3. Symmetry in Polar Coordinates

Learning Objectives

- Locate points in a plane using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves with given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider the Figure below. The point P has Cartesian coordinates (x, y). Consider the line segment connecting the origin to the point P. Its length is equal to the distance from the origin to P and we denote it by r. We also denote the angle between the positive x-axis and the line segment by θ. Then (r, θ) are the polar coordinates of P.



Converting Points between Coordinate Systems

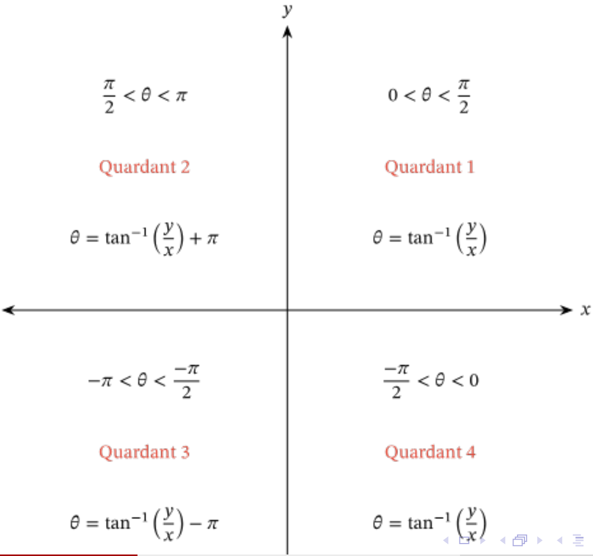
Conversion Formulas

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

(*) $x = r\cos(\theta)$ and $y = r\sin(\theta)$,

(**) $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$.

Quadrants


y
x
$\frac{\pi}{2} < \theta < \pi$
Quardant 2
$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$
$0 < \theta < \frac{\pi}{2}$
Quardant 1
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$-\pi < \theta < \frac{-\pi}{2}$
Quardant 3
$\theta = \tan^{-1}\left(\frac{y}{x}\right) - \pi$
$\frac{-\pi}{2} < \theta < 0$
Quardant 4
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Converting Rectangular to Polar Coordinates: Example 1

Convert the rectangular coordinates $(1, 1)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 1

Convert the rectangular coordinates $(1, 1)$ into polar coordinates:
Solution:

$$\begin{aligned} r^2 &= x^2 + y^2 = 1^2 + 1^2 = 2 \\ r &= \sqrt{2} \\ \tan(\theta) &= \frac{y}{x} = 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

Therefore, the polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Converting Rectangular to Polar Coordinates: Example 2

Convert the rectangular coordinates $(-3, 4)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 2

Convert the rectangular coordinates $(-3, 4)$ into polar coordinates:
Solution:

$$r^2 = x^2 + y^2 = (-3)^2 + 4^2 = 25$$

$$r = 5$$

$$\tan(\theta) = \frac{y}{x} = -\frac{4}{3}$$

$$\theta = \pi - \arctan\left(\frac{4}{3}\right)$$

Therefore, the polar coordinates are $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$.

Converting Rectangular to Polar Coordinates: Example 3

Convert the rectangular coordinates $(0, 3)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 3

Convert the rectangular coordinates $(0, 3)$ into polar coordinates:
Solution:

$$r = 3$$
$$\theta = \frac{\pi}{2}$$

Therefore, the polar coordinates are $(3, \frac{\pi}{2})$.

Converting Rectangular to Polar Coordinates: Example 4

Convert the rectangular coordinates $(5\sqrt{3}, -5)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 4

Convert the rectangular coordinates $(5\sqrt{3}, -5)$ **into polar coordinates: Solution:**

$$r = 10$$
$$\theta = -\frac{\pi}{6}$$

Therefore, the polar coordinates are $(10, -\frac{\pi}{6})$.

Converting Polar to Rectangular Coordinates: Example 1

Convert the polar coordinates $\left(3, \frac{\pi}{3}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 1

Convert the polar coordinates $\left(3, \frac{\pi}{3}\right)$ into rectangular coordinates:
Solution:

$$x = r\cos(\theta) = 3\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$$
$$y = r\sin(\theta) = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$

Therefore, the rectangular coordinates are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

Converting Polar to Rectangular Coordinates: Example 2

Convert the polar coordinates $\left(2, \frac{3\pi}{2}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 2

Convert the polar coordinates $\left(2, \frac{3\pi}{2}\right)$ **into rectangular coordinates:**
Solution:

$$x = 0$$
$$y = -2$$

Therefore, the rectangular coordinates are $(0, -2)$.

Converting Polar to Rectangular Coordinates: Example 3

Convert the polar coordinates $\left(6, -\frac{5\pi}{6}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 3

Convert the polar coordinates $\left(6, -\frac{5\pi}{6}\right)$ **into rectangular coordinates:**
Solution:

$$x = -3\sqrt{3}$$
$$y = -3$$

Therefore, the rectangular coordinates are $(-3\sqrt{3}, -3)$.

Converting Rectangular to Polar Coordinates

Convert the rectangular coordinates $(-8, -8)$ into polar coordinates:

Converting Rectangular to Polar Coordinates

Convert the rectangular coordinates $(-8,-8)$ into polar coordinates:
Solution:

$$\begin{aligned} r^2 &= x^2+y^2=(-8)^2+(-8)^2=128 \\ r &= \sqrt{128}=8\sqrt{2} \\ \tan(\theta) &= \frac{y}{x}=\frac{-8}{-8}=1 \\ \theta &= \tan^{-1}(1)-\pi \quad \text{(since in the third quadrant)} \end{aligned}$$

Therefore, the polar coordinates are $\left(8\sqrt{2},-\frac{3\pi}{4}\right)$.

Converting Polar to Rectangular Coordinates

Convert the polar coordinates $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates

Convert the polar coordinates $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates:
Solution:

$$x = r\cos(\theta) = 4\cos\left(\frac{2\pi}{3}\right) = 4 \times \left(-\frac{1}{2}\right) = -2$$
$$y = r\sin(\theta) = 4\sin\left(\frac{2\pi}{3}\right) = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Therefore, the rectangular coordinates are $\left(-2, 2\sqrt{3}\right)$.

Non-Uniqueness of Polar Representation

Example: The point $(1, \sqrt{3})$ in the rectangular system has multiple polar representations.
For instance:

$$\left(2, \frac{\pi}{3}\right) \text{ and } \left(2, \frac{7\pi}{3}\right)$$

both represent the same point.
Solution: Both polar representations correspond to the same point in the rectangular system.

Usage of Negative Radius in Polar Coordinates

Example: The point $(1, \sqrt{3})$ in the rectangular system can also be represented using negative radius in polar coordinates.
For instance:

$$\left(-2, \frac{4\pi}{3}\right)$$

Solution: Using the conversion formulas:

$$x = r\cos(\theta) = -2\cos\left(\frac{4\pi}{3}\right) = 1$$

$$y = r\sin(\theta) = -2\sin\left(\frac{4\pi}{3}\right) = \sqrt{3}$$

Therefore, $\left(-2, \frac{4\pi}{3}\right)$ represents the point $(1, \sqrt{3})$.

Important

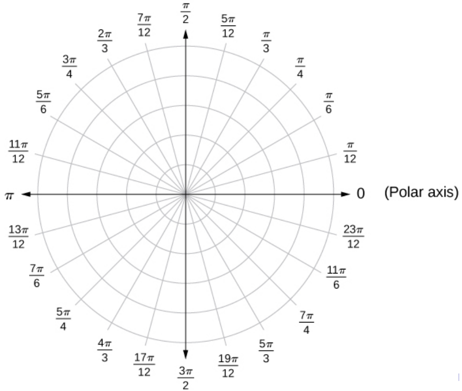
Geometrically, when we plot a point with a negative radial coordinate, we measure the distance of $|r|$ along the halfline that is in the opposite direction to the one that makes the angle of θ with the positive x-axis, so basically the minus reverses the direction, the same way as with angles.)

Infinite number of polar coordinates

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Polar Coordinate System

- r is the directed distance that the point lies from the origin and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis.
- Positive angles are measured in a counterclockwise direction, and negative angles are measured in a clockwise direction.

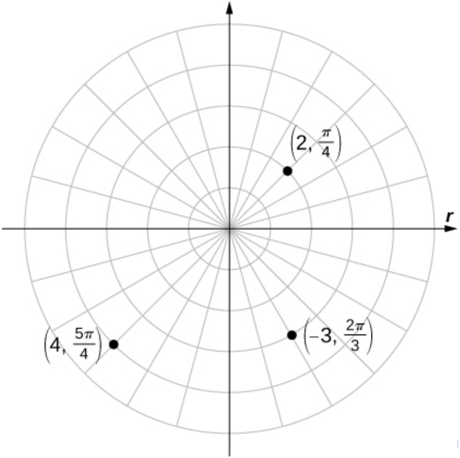


Plotting Points on Polar Plane

Solution

Plot each of the following points on the polar plane:

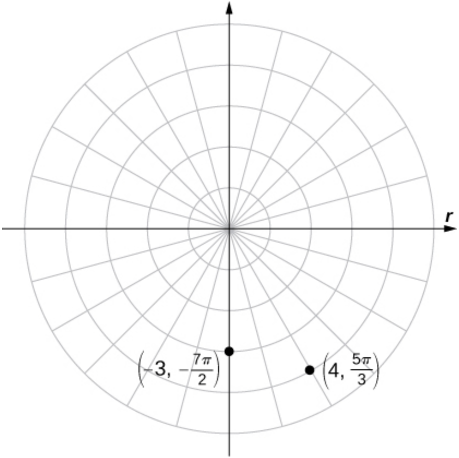
- $(2, \frac{\pi}{4})$, $(-3, \frac{2\pi}{3})$ and $(4, \frac{5\pi}{4})$



Plotting Points on the Polar Plane

Plot the points $\left(4, \frac{5\pi}{3}\right)$ and $\left(-3, -\frac{7\pi}{2}\right)$ on the polar plane.

Solution: The points are plotted in the following figure.



Plotting Curves in the Polar Coordinate System

Now that we know how to plot points in the polar coordinate system, let's discuss how to plot curves.

In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. Similarly, in the polar coordinate system, we can graph a curve that is generated by a function $r = f(\theta)$.

- In this context, r represents the distance from the origin to a point on the curve, and θ represents the angle that the line segment from the origin to that point makes with the positive x-axis.
- To plot a curve given by $r = f(\theta)$, we evaluate r for various values of θ, and then plot the corresponding points in the polar plane.
- Connecting these points with smooth lines or curves gives us the graph of the polar function.

We'll explore this concept further with examples in the upcoming slides.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

To plot a curve in polar coordinates, follow these steps:

Five Steps

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

This strategy helps in visualizing and understanding the behavior of curves in the polar coordinate system.

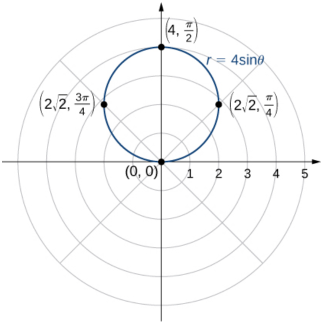
Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin(\theta)$. Identify the curve and rewrite the equation in rectangular coordinates.

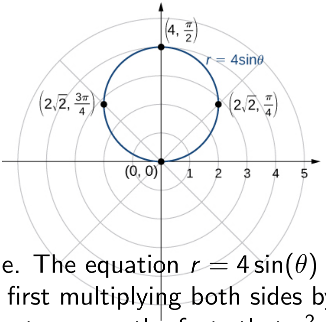
Solution: Because the function is a multiple of a sine function, it is periodic with period 2π. We will use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table:

θ	$r = 4\sin(\theta)$	θ	$r = 4\sin(\theta)$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
2π	0		

Graph and center



Graph and center



This is the graph of a circle. The equation $r = 4\sin(\theta)$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin(\theta)$. Next, we use the facts that $r^2 = x^2 + y^2$ and $y = r\sin(\theta)$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, we subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + (y^2 - 4y) &= 0 \\ x^2 + (y^2 - 4y + 4) &= 0 + 4 \\ x^2 + (y-2)^2 &= 4. \end{aligned}$$

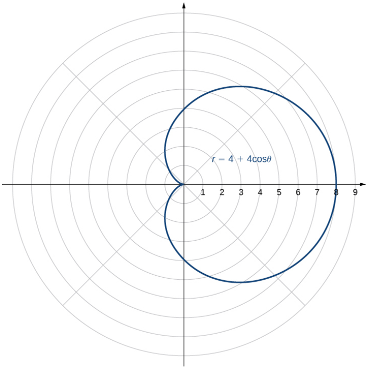
Graph of $r = 4 + 4\cos(\theta)$

Hint: Follow the problem-solving strategy for creating a graph in polar coordinates.

Graph of $r = 4 + 4\cos(\theta)$

Hint: Follow the problem-solving strategy for creating a graph in polar coordinates.

Solution:



The name of this shape is a cardioid, which we will study further later in this section.

Transforming Polar Equations to Rectangular Coordinates

Example 1: Rewrite $\theta = \frac{\pi}{3}$ in rectangular coordinates and identify the graph.
Solution: Take the tangent of both sides. This gives $\tan(\theta) = \tan(\frac{\pi}{3}) = \sqrt{3}$. Since $\tan(\theta) = \frac{y}{x}$, we can replace the left-hand side with $\frac{y}{x}$, resulting in $\frac{y}{x} = \sqrt{3}$. This equation represents a straight line passing through the origin with slope $\sqrt{3}$. Therefore, the graph represents a line passing through the origin with a slope of $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan(K)$.

Transforming Polar Equations to Rectangular Coordinates

Example 2: Rewrite $r = 3$ in rectangular coordinates and identify the graph.
Solution: First, square both sides of the equation. This gives $r^2 = 9$. Next, replace r^2 with $x^2 + y^2$, resulting in $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3.
In general, any polar equation of the form $r = k$, where k is a constant, represents a circle of radius $|k|$ centered at the origin. (Note: when squaring both sides of an equation, it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case, we do not introduce new points. For example, $(-3, \frac{\pi}{3})$ is the same point as $(3, \frac{4\pi}{3})$.)

Transforming Polar Equations to Rectangular Coordinates

Example 3: Rewrite $r = 6\cos(\theta) - 8\sin(\theta)$ in rectangular coordinates and identify the graph.

Solution: Multiplying both sides by r gives $r^2 = 6r\cos(\theta) - 8r\sin(\theta)$. Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get $x^2 + y^2 = 6x - 8y$. Completing the square yields $(x-3)^2 + (y+4)^2 = 25$, which is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

Rewriting Polar Equation in Rectangular Coordinates

To rewrite the given polar equation $r = \sec(\theta)\tan(\theta)$ in rectangular coordinates.

Rewriting Polar Equation in Rectangular Coordinates

To rewrite the given polar equation $r = \sec(\theta)\tan(\theta)$ in rectangular coordinates.
Solution:
The trigonometric identities we'll use are:

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Substituting these identities into the equation $r = \sec(\theta)\tan(\theta)$, we get:

$$r = \frac{1}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)}$$

Now, let's express r in terms of x and y. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have:

$$r^2 = x^2 + y^2$$

Therefore, our equation becomes: $x^2 + y^2 = \frac{y}{x}$
Multiplying both sides by x to clear the fraction, we obtain:

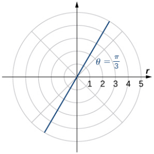
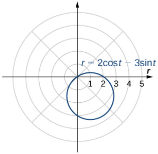
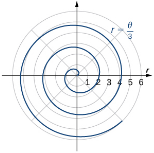
$$x^3 + xy^2 = y$$

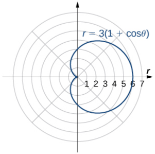
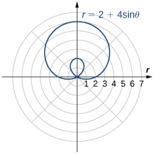
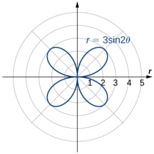
This equation represents a curve in rectangular coordinates. Specifically, it's the

Summary of Common Curves Defined by Polar Equations

Polar Equation	Description
$\theta = K$	Line
$r = a\cos\theta + b\sin\theta$	Circle
$r = a\sin(\theta)$	Circle with radius a centered on x-axis
$r = a\cos(\theta)$	Circle with radius a centered on y-axis
$r = a + b\theta$	Spiral
$r = a \pm b\sin(\theta)$	Cardioid if $a = b$
$r = a \pm b\cos(\theta)$	Cardioid if $a = b$
$r = a + b\sin(\theta)$	Limaçon with a loops if $b > a$
$r = a + b\cos(\theta)$	Limaçon with a loops if $b > a$
$r = a\sin(2\theta)$	Rose with a petals
$r = a\cos(2\theta)$	Rose with a petals

Figures

Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

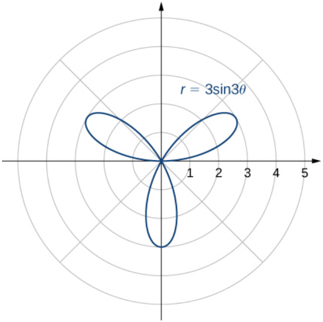
Cardioid and Rose Curves

Cardioid:

- A cardioid is a special case of a limaçon where $a = b$ or $a = -b$.

Rose Curve:

- The graph of $r = 3\sin(2\theta)$ has four petals.
- The graph of $r = 3\sin(3\theta)$ has three petals.
- **If the coefficient is irrational, then the curve never closes,**



Cardioid and Rose Curves

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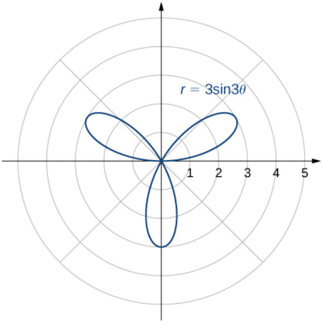
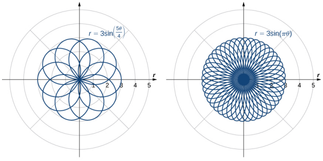




(a) (b)

Calculus with Polar Curves

Find the slope of the tangent line to the spiral with polar equation $r = \pi - \theta$ at the point corresponding to $\theta = \frac{2\pi}{3}$.

Solution:

$$x = r\cos(\theta) = (\pi - \theta)\cos(\theta)$$
$$y = r\sin(\theta) = (\pi - \theta)\sin(\theta)$$

Next, find $\frac{dy}{dx}$ as a function of θ:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}\left((\pi - \theta)\sin(\theta)\right)}{\frac{d}{d\theta}\left((\pi - \theta)\cos(\theta)\right)} \\ &= \frac{-\sin(\theta) + (\pi - \theta)\cos(\theta)}{-\cos(\theta) + (\pi - \theta)(-\sin(\theta))}\end{aligned}$$

Part 2

The slope m of the tangent line at $\theta = \frac{2\pi}{3}$ is:

$$
\begin{aligned}
m &= \frac{dy}{dx}\left(\frac{2\pi}{3}\right) \\
&= \frac{-\sin\left(\frac{2\pi}{3}\right) + \left(\pi - \frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)}{-\cos\left(\frac{2\pi}{3}\right) + \left(\pi - \frac{2\pi}{3}\right)\left(-\sin\left(\frac{2\pi}{3}\right)\right)} \\
&= \frac{-\frac{\sqrt{3}}{2} + \frac{\pi}{3}\cdot\left(-\frac{1}{2}\right)}{-\left(-\frac{1}{2}\right) + \frac{\pi}{3}\cdot\left(-\frac{\sqrt{3}}{2}\right)} \\
&= \frac{3\sqrt{3} + \pi}{-3 + \sqrt{3}\pi}
\end{aligned}
$$

Calculus with Polar Curves

Find the slope of the tangent line to the polar curve $r = 1 + \sin(\theta)$ at the point corresponding to $\theta = -\frac{\pi}{4}$.

Solution: To find the slope of the tangent line, we first need to find the derivative of r with respect to θ, denoted as $\frac{dr}{d\theta}$.

Given the polar equation $r = 1 + \sin(\theta)$, we differentiate it with respect to θ using the chain rule:

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(1 + \sin(\theta)) = \cos(\theta)$$

Now, evaluate $\frac{dr}{d\theta}$ at $\theta = -\frac{\pi}{4}$:

$$\left.\frac{dr}{d\theta}\right|_{\theta=-\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

The slope of the tangent line at $\theta = -\frac{\pi}{4}$ is the negative reciprocal of $\frac{dr}{d\theta}$:

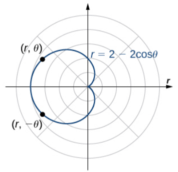
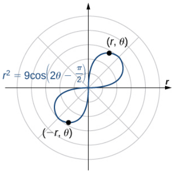
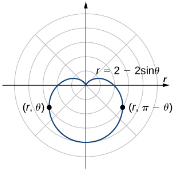
$$\text{Slope} = -\frac{1}{\frac{1}{\sqrt{2}}} = -\sqrt{2} + 1$$

Symmetry in Polar Curves and Equations

Consider a polar curve with equation $r = f(\theta)$.

- The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. This happens if $f(-\theta) = f(\theta)$ or $f(\pi - \theta) = -f(\theta)$.
- The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. This happens if $f(\pi + \theta) = f(\theta)$.
- The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. This happens if $f(\pi - \theta) = f(\theta)$ or $f(-\theta) = -f(\theta)$.

Examples of each type of symmetry

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Using Symmetry to Graph a Polar Equation

Determine all symmetries of the rose

The rose is defined by the equation $r = 3\sin(2\theta)$.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$. Let $f(\theta) = 3\sin(2\theta)$. We first substitute $-\theta$ instead of θ into f:

$$f(-\theta) = 3\sin(-2\theta) = -3\sin(2\theta) = -f(\theta)$$

since sine is an odd function. According to iii in the statement above, this implies symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$.
To test for symmetry with respect to the polar axis, we consider $f(\pi - \theta)$:

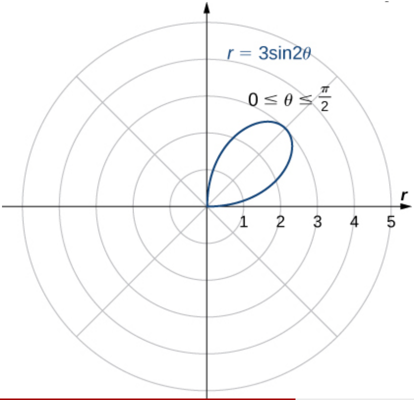
$$f(\pi - \theta) = 3\sin(2\pi - 2\theta) = 3\sin(-2\theta) = -3\sin(2\theta)$$

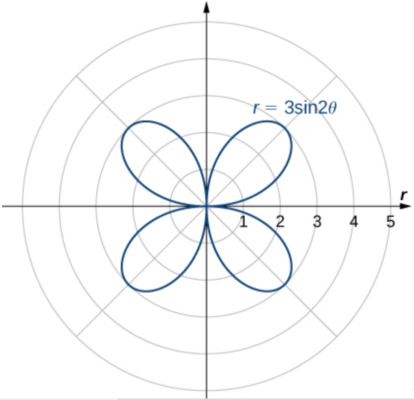
since sine function is 2π-periodic and odd. Hence, by i, we have that the

Graphs : Reflecting into the other three quadrants

Table of Values

θ	r
0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
π/[illegible]	0





Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$
$$\tan(\theta) = \frac{y}{x}$$

- To sketch a polar curve, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.