3.7 Improper Integrals

Math 1700

University of Manitoba

March 15, 2024

Outline

Learning Objectives

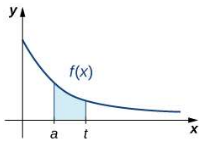
- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

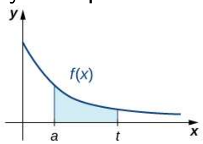
Introduction

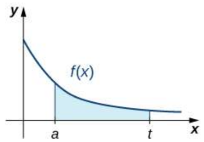
To define the integral $\int_a^\infty f(x)\,dx$, we interpret it as the limit of the definite integral $\int_a^t f(x)\,dx$ as t approaches infinity:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

In the figure below, we visually interpret this definition:







Definition

Let $f(x)$ be continuous over an interval $[a, \infty)$. Then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx, \text{ provided this limit exists.}$$

Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx, \text{ provided this limit exists.}$$

Convergence

If the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge

Definition

Let $f(x)$ be continuous over $(-\infty, \infty)$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx,$$

provided that both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge.

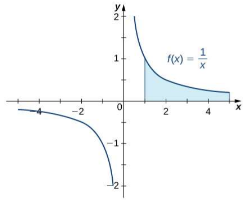
If either of these two integrals is divergent, then $\int_{-\infty}^{\infty} f(x)\,dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$ for any value of a.)

Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ is finite or infinite.

Solution:

- We first do a quick sketch of the region in question, as shown in the following graph.



We can find the area between the curve $f(x) = \frac{1}{x}$ and the x-axis on an infinite interval.

- We can see that the area of this region is given by $A = \int_1^\infty \frac{1}{x}\,dx$.

Solution

Then we have

$$\begin{aligned}
A &= \int_1^{\infty} \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx \quad \text{Rewrite the improper integral as a limit.} \\
&= \lim_{t\to\infty} \ln|x| \Big|_1^t \quad \text{Find the antiderivative.} \\
&= \lim_{t\to\infty} \left(\ln|t| - \ln(1)\right) \quad \text{Evaluate the antiderivative.} \\
&= \infty \quad \text{Evaluate the limit.}
\end{aligned}$$

Since the improper integral diverges to ∞, the area of the region is infinite.

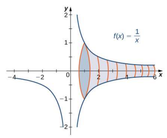
Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ about the x-axis.

Solution:

- The solid is shown in Figure below. Using the disk method, we see that the volume V is

$$V = \int_1^\infty \pi[f(x)]^2 dx = \pi \int_1^\infty \frac{1}{x^2}\, dx.$$



The solid of revolution can be generated by rotating an infinite area about the x-axis.

Solution part 2

- Then we have

$$\begin{aligned}
V = \pi \int_1^\infty \frac{1}{x^2}\,dx &= \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \quad \text{Rewrite as a limit.}\\
&= \pi \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t \quad \text{Find the antiderivative.}\\
&= \pi \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) \quad \text{Evaluate the antiderivative.}\\
&= \pi.
\end{aligned}$$

- The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

Chapter Opener: Traffic Accidents in a City

Probability Theory:

- If accidents occur at a rate of one every 3 months, then the probability that the time between accidents is between a and b is given by

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx,$$

where

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3e^{-3x} & \text{if } x \geq 0 \end{cases}$$

Chapter Opener: Traffic Accidents in a City

Solution:

- We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \geq 8) &= \int_8^\infty 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} \left(-e^{-3x}\right)\Big|_8^t \\ &= \lim_{t\to\infty} \left(-e^{-3t} + e^{-24}\right) \\ &= e^{-24} \approx 3.8 \times 10^{-11}. \end{aligned}$$

- The value 3.8×10^{-11} represents the probability of no accidents in 8

Evaluating an Improper Integral over an Infinite Interval

Evaluate $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx$. State whether the improper integral converges or diverges.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+4}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_{t}^{0} && \text{Find the antiderivative.}\\
&= \frac{1}{2}\lim_{t\to-\infty}\left(\tan^{-1}(0)-\tan^{-1}\left(\frac{t}{2}\right)\right) && \text{Evaluate the antiderivative.}\\
&= \frac{\pi}{4}. && \text{Evaluate the limit and simplify.}
\end{aligned}$$

- The improper integral converges to $\frac{\pi}{4}$.

Evaluating an Improper Integral over $(-\infty, \infty)$

Evaluate $\int\limits_{-\infty}^{\infty} xe^x\,dx$. **State whether the improper integral is convergent or divergent.**

Solution:

- Start by splitting up the integral:

$$\int\limits_{-\infty}^{\infty} xe^x\,dx = \int\limits_{-\infty}^{0} xe^x\,dx + \int\limits_{0}^{\infty} xe^x\,dx.$$

- If either $\int\limits_{-\infty}^{0} xe^x\,dx$ or $\int\limits_{0}^{\infty} xe^x\,dx$ diverges, then $\int\limits_{-\infty}^{\infty} xe^x\,dx$ diverges. Compute each integral separately.

Continued

Solution (continued): For the first integral,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx = \lim_{t\to-\infty}(xe^x - e^x)\Big|_t^0$$
$$= \lim_{t\to-\infty}(-1 - te^t + e^t) = -1.$$

The first improper integral converges. For the second integral,

$$\int_{0}^{\infty} xe^x\,dx = \lim_{t\to\infty}\int_{0}^{t} xe^x\,dx = \lim_{t\to\infty}(xe^x - e^x)\Big|_0^t$$
$$= \lim_{t\to\infty}(te^t - e^t + 1) = \infty.$$

Thus, $\int_0^\infty xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{\infty} xe^x\,dx$ diverges as well.

Evaluating an Improper Integral

Evaluate $\int_{-3}^{\infty} e^{-x}\,dx$**. State whether the improper integral converges or diverges.**

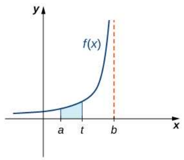
Answer: e^3, converges

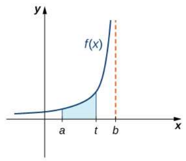
Hint: $\int_{-3}^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_{-3}^{t} e^{-x}\,dx$

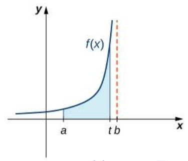
Understanding Integrals with Infinite Discontinuities

In mathematical analysis, the concept of integration is vital for understanding the accumulation of quantities over intervals. However, when dealing with functions that exhibit infinite discontinuities within the interval of integration, a nuanced approach is required.

- Consider an integral of the form: $\int_a^b f(x)\,dx$
- $f(x)$ is continuous over $[a, b)$ but discontinuous at b.
- Let's examine the behavior of this integral as t, the upper limit of integration, approaches b.
- Since $f(x)$ remains continuous over $[a, t]$ for all t satisfying $a < t < b$, the integral $\int_a^t f(x)\,dx$ is well-defined for such values of t.
- We define: $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$ provided this limit exists.







Definition

Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

If the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

Definition

If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then we define $\int_a^b f(x)\,dx$ as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these two integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$**, if possible. State whether the integral converges or diverges.**

Solution: The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0,4)$ and discontinuous at 4. Using the above definition, we rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a:

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.}\\
&= 4. && \text{Evaluate the limit.}
\end{aligned}$$

Because the limit exists, the improper integral converges.

Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x\ln(x)\,dx$. State whether the integral converges or diverges.

Solution: Since $f(x) = x\ln(x)$ is continuous over $(0,2]$ and is discontinuous at zero, we can rewrite :

$$\begin{aligned}
\int_0^2 x\ln(x)\,dx &= \lim_{t\to 0^+}\int_t^2 x\ln(x)\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 0^+}\left(\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right)\Big|_t^2 && \text{Evaluate using integr}\\
&= \lim_{t\to 0^+}\left(2\ln(2) - 1 - \frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2\right). && \text{Evaluate the antideri}\\
&= 2\ln(2) - 1.
\end{aligned}$$

The improper integral converges.

Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^{1} \frac{1}{x^3}\,dx$**. State whether the improper integral converges or diverges.**

Solution: Since $f(x) = \frac{1}{x^3}$ is continuous at every point of $[-1, 1]$ except zero, we use the corresponding definition to write

$$\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{1} \frac{1}{x^3}\,dx.$$

Our integral converges if both integrals on the right converge. If either of the two integrals on the right diverges, then the original integral diverges as well. Begin with $\int_{-1}^{0} \frac{1}{x^3}\,dx$:

Solution

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{x^3}\,dx &= \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 0^-} \left(-\frac{1}{2x^2}\right)\Big|_{-1}^{t} && \text{Find the antiderivative.}\\
&= \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2}\right) && \text{Evaluate the antiderivative.}\\
&= -\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Therefore, $\int_{-1}^{0} \frac{1}{x^3}\,dx$ diverges, and hence $\int_{-1}^{1} \frac{1}{x^3}\,dx$ diverges regardless of the behavior of $\int_{0}^{1} \frac{1}{x^3}\,dx$.

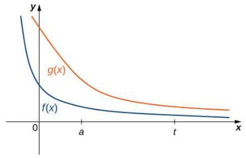
Evaluating an Improper Integral

Evaluate $\int_0^1 \frac{1}{(1-x)^{3/2}}\,dx$**. State whether the integral converges or diverges.**

Answer: ∞, diverges

Comparison Property for Integrals

To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \leq f(x) \leq g(x)$ for $x \geq a$. In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas. By the comparison property for definite integrals, we have the relationship: $0 \leq \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx \quad$ for $t \geq a$.



If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then for $t \geq a$,

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

Thus, if $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$, then

$$\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx \geq \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$$

as well.

Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, \infty)$. Assume that $0 \leq f(x) \leq g(x)$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$, then $\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = \infty$.
- If $\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = L$, where L is a real number, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = M$ for some real number $M \leq L$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{xe^x}\,dx$ converges.
Solution:

- The integrand is continuous over $[1,\infty)$ and for $x > 1$:

$$0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$$

- So if $\int_1^\infty e^{-x}\,dx$ converges, then so does $\int_1^\infty \frac{1}{xe^x}\,dx$.
- To evaluate $\int_1^\infty e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^\infty e^{-x}\,dx &= \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(-e^{-x})\Big|_1^t \\ &= \lim_{t\to\infty}(-e^{-t}+e^1) = e.\end{aligned}$$

- Since the limit is finite, $\int_1^\infty e^{-x}\,dx$ converges, and hence, by the comparison theorem, so does $\int_1^\infty \frac{1}{xe^x}\,dx$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution:

- First, note that $\frac{1}{x^p}$ is continuous over $[1, \infty)$.
- If $p < 1$, then $\frac{1}{x} \leq \frac{1}{x^p}$ for all $x \in [1, \infty)$.
- We already showed that $\int_1^\infty \frac{1}{x}\,dx = \infty$.
- Therefore, by the comparison theorem, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_e^\infty \frac{\ln(x)}{x}\,dx$ **diverges.**
Hint:

$$\frac{1}{x} \le \frac{\ln(x)}{x} \quad \text{on} \quad [e, \infty)$$

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution:

- First we note that $\frac{1}{x^p}$ is continuous over $[1, \infty)$. If $p < 1$, then $\frac{1}{x} \leq \frac{1}{x^p}$ for all $x \in [1, \infty)$.
- We already showed that $\int_1^\infty \frac{1}{x}\,dx = \infty$. Therefore, by the comparison theorem, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Key Concepts and Equations

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$