3.2 Trigonometric Integrals

Math 1700

University of Manitoba

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Outline

Learning Objectives

1. Solve integration problems involving products of powers of $\sin(x)$ and $\cos(x)$.
2. Integrate products of sines and cosines of different angles.
3. Solve integration problems involving products of powers of $\tan(x)$ and $\sec(x)$.
4. Use reduction formulas to evaluate trigonometric integrals.

A key idea behind the strategy used to integrate combinations of powers of $\sin(x)$ and $\cos(x)$ involves rewriting these expressions as sums and differences of integrals of the form $\displaystyle\int \sin^j(x)\cos(x)\,dx$ or $\displaystyle\int \cos^j(x)\sin(x)\,dx$ that can be evaluated using u-substitution.

Evaluate $\int \cos^3(x)\sin(x)\,dx$.

Solution:

Make a substitution $u = \cos(x)$. In this case, $du = -\sin(x)\,dx$. Thus,

$$\begin{aligned}\int \cos^3(x)\sin(x)\,dx &= -\int u^3\,du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4(x) + C.\end{aligned}$$

Evaluate $\int \sin^4(x)\cos(x)\,dx$.

Answer: $\frac{1}{5}\sin^5(x) + C$

Hint: Take $u = \sin(x)$.

A Preliminary Example: Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When k is Odd

Evaluate $\int \cos^2(x)\sin^3(x)\,dx$.

Solution: To convert this integral into a combination of integrals of the form $\int \cos^j(x)\sin(x)\,dx$, rewrite

$\sin^3(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x)$.

We now make a substitution $u = \cos(x)$, $du = -\sin(x)\,dx$, and obtain

$$\begin{aligned}\int \cos^2(x)\sin^3(x)\,dx &= \int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx\\ &= -\int u^2(1-u^2)\,du = \int (u^4-u^2)\,du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.\end{aligned}$$

Development of the Integral

Given the integral $\int \cos^3(x)\sin^2(x)\,dx$, we rewrite $\cos^3(x)$ as $\cos^2(x)\cos(x)$. Then, using the identity $\cos^2(x) = 1 - \sin^2(x)$, we get:

$$\begin{aligned}\int \cos^3(x)\sin^2(x)\,dx &= \int (1-\sin^2(x))\cos(x)\sin^2(x)\,dx \\ &= \int (\sin^2(x)-\sin^4(x))\cos(x)\,dx.\end{aligned}$$

Now, let's make the substitution $u = \sin(x)$. Then, $du = \cos(x)\,dx$. we have:

$$\begin{aligned}\int \cos^3(x)\sin^2(x)\,dx = \int (u^2-u^4)\,du &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C.\end{aligned}$$

So, the evaluated integral is $\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$.

Integrating an Even Power of $\sin(x)$

Evaluate $\displaystyle\int \sin^2(x)\,dx$.

Solution: To evaluate this integral, let's use the trigonometric identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\begin{aligned}\int \sin^2(x)\,dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

Development of the Integral

Given the integral $\int \cos^2(x)\,dx$, we can use the trigonometric identity $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$. Thus, we have:

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.\end{aligned}$$

So, the evaluated integral is $\frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.

Problem-Solving Strategy: Integrating Products of Powers of $\sin(x)$ and $\cos(x)$

To evaluate $\displaystyle\int \cos^j(x)\sin^k(x)\,dx$, use the following strategies:

1. If k is odd, rewrite $\sin^k(x) = \sin^{k-1}(x)\sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite $\sin^{k-1}(x)$ in terms of $\cos(x)$. Integrate using the substitution $u = \cos(x)$. This substitution makes $du = -\sin(x)\,dx$.
2. If j is odd, rewrite $\cos^j(x) = \cos^{j-1}(x)\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite $\cos^{j-1}(x)$ in terms of $\sin(x)$. Integrate using the substitution $u = \sin(x)$. This substitution makes $du = \cos(x)\,dx$.
3. If both j and k are even, use identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$. After applying these formulas, simplify and reapply strategies 2 and 3 to the combination of powers of $\cos(2x)$ as appropriate.

(Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When j is Odd

Evaluate $A = \displaystyle\int \cos^5(x)\sin^8(x)\,dx$.

Solution: Since the power on $\cos(x)$ is odd, use strategy 2.

$$\begin{aligned}
A &= \int \cos^4(x)\sin^8(x)\cos(x)\,dx && \text{Break off } \cos(x).\\
&= \int (\cos^2(x))^2\sin^8(x)\cos(x)\,dx && \text{Rewrite } \cos^4(x) = (\cos^2(x))^2.\\
&= \int (1-\sin^2(x))^2\sin^8(x)\cos(x)\,dx && \text{Substitute } \cos^2(x) = 1-\sin^2(x).\\
&= \int (1-u^2)^2u^8\,du && \text{Let } u=\sin(x) \text{ and } du=\cos(x)\,dx.\\
&= \int (u^8-2u^{10}+u^{12})\,du && \text{Expand.}\\
&= \frac{1}{9}u^9 - \frac{2}{11}u^{11} + \frac{1}{13}u^{13} + C && \text{Evaluate the integral.}\\
&= \frac{1}{9}\sin^9(x) - \frac{2}{11}\sin^{11}(x) + \frac{1}{13}\sin^{13}(x) + C. && \text{Substitute } u=\sin(x).
\end{aligned}$$

Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When k and j are Even

Evaluate $A=\displaystyle\int \sin^4(x)\,dx$.

Solution: Since both the powers of $\sin(x)$ and $\cos(x)$ are even ($k=4$, $j=0$), we must use strategy 3. Thus,

$$A=\int (\sin^2(x))^2\,dx \qquad \text{Rewrite } \sin^4(x)=(\sin^2(x))^2.$$

$$=\int\left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2 dx \qquad \text{Substitute } \sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x).$$

$$=\int\left(\frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)\right)dx \qquad \text{Expand } \left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2.$$

Since $\cos^2(2x)$ has an even power, we use strategy 3 again and $\cos^2(2x)=\frac{1}{2}+\frac{1}{2}\cos(4x)$

$$=\int\left(\frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos(4x)\right)\right)dx$$

$$=\int\left(\frac{3}{8}-\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\right)dx \qquad \text{Simplify.}$$

$$=\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C. \qquad \text{Evaluate the integral.}$$

Problem Statement

Evaluate $\int \cos^3(x)\,dx$.

Hint: Use strategy 2. Write $\cos^3(x) = \cos^2(x)\cos(x)$ and substitute $\cos^2(x) = 1 - \sin^2(x)$.

Problem Statement

Evaluate $\int \cos^3(x)\,dx$.

Hint: Use strategy 2. Write $\cos^3(x) = \cos^2(x)\cos(x)$ and substitute $\cos^2(x) = 1 - \sin^2(x)$.

Answer: $\sin(x) - \dfrac{1}{3}\sin^3(x) + C$

Solution

Problem: Evaluate $\int \cos^2(3x)\,dx$.

Hint: Use strategy 3 and substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$.

Solution

Problem: Evaluate $\int \cos^2(3x)\,dx$.

Hint: Use strategy 3 and substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$.

Solution:

$$\begin{aligned}\int \cos^2(3x)\,dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(6x)\right)\,dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(6x)\,dx \\ &= \frac{1}{2}x + \frac{1}{12}\sin(6x) + C\end{aligned}$$

So, the solution is $\frac{1}{2}x + \frac{1}{12}\sin(6x) + C$.

Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the following identities:

$$\begin{aligned} \sin(ax)\sin(bx) &= \tfrac{1}{2}\cos\left((a-b)x\right) - \tfrac{1}{2}\cos\left((a+b)x\right) \\ \sin(ax)\cos(bx) &= \tfrac{1}{2}\sin\left((a-b)x\right) + \tfrac{1}{2}\sin\left((a+b)x\right) \\ \cos(ax)\cos(bx) &= \tfrac{1}{2}\cos\left((a-b)x\right) + \tfrac{1}{2}\cos\left((a+b)x\right) \end{aligned}$$

These identities are helpful when dealing with integrals involving products of trigonometric functions with different angles.

Evaluating $\int \sin(ax)\cos(bx)\,dx$

To evaluate $\int \sin(ax)\cos(bx)\,dx$, we can use the identity:

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Solution:

$$\begin{aligned}\int \sin(5x)\cos(3x)\,dx &= \int \left(\frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\right)\,dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C\end{aligned}$$

So, $\int \sin(5x)\cos(3x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C$.

Evaluating $\int \cos(6x)\cos(5x)\,dx$

To evaluate $\int \cos(6x)\cos(5x)\,dx$, we can use the hint provided:

$$\cos(6x)\cos(5x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)$$

Evaluating $\int \cos(6x)\cos(5x)\,dx$

To evaluate $\int \cos(6x)\cos(5x)\,dx$, we can use the hint provided:

$$\cos(6x)\cos(5x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)$$

Solution:

$$\begin{aligned}\int \cos(6x)\cos(5x)\,dx &= \int \left(\frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)\right)\,dx \\ &= \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C\end{aligned}$$

So, $\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C$.

Integrating Products and Powers of $\tan(x)$ and $\sec(x)$

Before discussing the integration of products of powers of $\tan(x)$ and $\sec(x)$, it is useful to recall the integrals involving $\tan(x)$ and $\sec(x)$ we have already learned:

$$\int \sec^2(x)\, dx = \tan(x) + C,$$

$$\int \sec(x)\tan(x)\, dx = \sec(x) + C,$$

$$\int \tan(x)\, dx = \ln|\sec(x)| + C,$$

$$\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C.$$

Evaluating $\int \sec^j(x)\tan(x)\,dx$

To evaluate $\displaystyle\int \sec^j(x)\tan(x)\,dx$, we can rewrite $\sec^5(x)\tan(x)$ as $\sec^4(x)\sec(x)\tan(x)$. If we let $u = \sec(x)$, then $du = \sec(x)\tan(x)\,dx$, and so

$$\begin{aligned}\int \sec^5(x)\tan(x)\,dx &= \int \sec^4(x)\sec(x)\tan(x)\,dx \\ &= \int u^4\,du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}\sec^5(x) + C.\end{aligned}$$

So, $\displaystyle\int \sec^5(x)\tan(x)\,dx = \frac{1}{5}\sec^5(x) + C$.

Evaluating $\int \tan^5(x)\sec^2(x)\,dx$

To evaluate $\int \tan^5(x)\sec^2(x)\,dx$, we can use the hint provided:
Let $u = \tan(x)$ and $du = \sec^2(x)\,dx$.
Solution:

$$\begin{aligned}\int \tan^5(x)\sec^2(x)\,dx &= \int u^5\,du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}\tan^6(x) + C\end{aligned}$$

So, $\int \tan^5(x)\sec^2(x)\,dx = \frac{1}{6}\tan^6(x) + C$.

Problem-Solving Strategy: Evaluating $\int \tan^k(x)\sec^j(x)\,dx$

To evaluate $\int \tan^k(x)\sec^j(x)\,dx$, use the following strategies:

- If j is even and $j \geq 2$, rewrite $\sec^j(x) = \sec^{j-2}(x)\sec^2(x)$ and use $\sec^2(x) = \tan^2(x) + 1$ to rewrite $\sec^{j-2}(x)$ in terms of $\tan(x)$. Let $u = \tan(x)$ and $du = \sec^2(x)$.
- If k is odd and $j \geq 1$, rewrite $\tan^k(x)\sec^j(x) = \tan^{k-1}(x)\sec^{j-1}(x)\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ to express $\tan^{k-1}(x)$ in terms of $\sec(x)$. Let $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.
- If k is even and j is odd, then use $\tan^2(x) = \sec^2(x) - 1$ to express $\tan^k(x)$ in terms of $\sec(x)$. Use integration by parts to integrate odd powers of $\sec(x)$.

Evaluating $\int \tan^6(x)\sec^4(x)\,dx$ When j is Even

Since the power on $\sec(x)$ is even, rewrite $\sec^4(x) = \sec^2(x)\sec^2(x)$ and use $\sec^2(x) = \tan^2(x) + 1$ to express the first $\sec^2(x)$ in terms of $\tan(x)$. We now make a substitution $u = \tan(x)$, in which case $du = \sec^2(x)\,dx$, and we obtain

$$\begin{aligned}
\int \tan^6(x)\sec^4(x)\,dx &= \int \tan^6(x)(\tan^2(x)+1)\sec^2(x)dx \\
&= \int u^6(u^2+1)du \\
&= \int (u^8+u^6)du \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C \\
&= \frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + C.
\end{aligned}$$

So, $\int \tan^6(x)\sec^4(x)\,dx = \frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + C.$

Evaluating $\int \tan^5(x)\sec^3(x)\,dx$ When k is Odd

Since the power of $\tan(x)$ is odd, we begin by rewriting $\tan^5(x)\sec^3(x) = \tan^4(x)\sec^2(x)\sec(x)\tan(x)$. We then notice that $\tan^4(x) = (\tan^2(x))^2 = (\sec^2(x)-1)^2$, and make a substitution $u = \sec(x)$ with $du = \sec(x)\tan(x)\,dx$. With this, we obtain

$$\begin{aligned}
\int \tan^5(x)\sec^3(x)\,dx &= \int (\sec^2(x)-1)^2\sec^2(x)\sec(x)\tan(x)\,dx \\
&= \int (u^2-1)^2u^2\,du \\
&= \int (u^6-2u^4+u^2)\,du \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C.
\end{aligned}$$

So, $\int \tan^5(x)\sec^3(x)\,dx = \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C.$

Evaluating $\int \tan^3(x)\,dx$

Although there is no $\sec(x)$ under the integral, we can still use the strategy outlined above for the case when the power k of $\tan(x)$ is odd. For this, we will need to multiply and divide the integrand by $\sec(x)$:

$$\begin{aligned}\tan^3(x) &= \frac{\sec(x)\tan^3(x)}{\sec(x)} = \frac{1}{\sec(x)}\tan^3(x)\sec(x)\\ &= \frac{1}{\sec(x)}\tan^2(x)\sec(x)\tan(x) = \frac{\sec^2(x)-1}{\sec(x)}\sec(x)\tan(x).\end{aligned}$$

Hence, using the substitution $u = \sec(x)$, we obtain

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \frac{\sec^2(x)-1}{\sec(x)}\sec(x)\tan(x)\,dx\\ &= \int \frac{u^2-1}{u}\,du = \int \left(u - \frac{1}{u}\right)\,du\\ &= \frac{1}{2}u^2 - \ln|u| + C = \frac{1}{2}\sec^2(x) - \ln|\sec(x)| + C.\end{aligned}$$

Evaluating $\int \sec^3(x)\, dx$

Integrate $\int \sec^3(x)\, dx$**. Solution:** This integral requires integration by parts. Let $u = \sec(x)$ and $dv = \sec^2(x)\, dx$. These choices make $du = \sec(x)\tan(x)\, dx$ and $v = \tan(x)$. Thus,

$$\begin{aligned}
\int \sec^3(x)\, dx &= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\, dx \\
&= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\, dx \quad \text{(Simplify)} \\
&= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\, dx \quad \text{(Substitute } \tan^2(x) = \sec^2(x) - \\
&= \sec(x)\tan(x) + \int \sec(x)\, dx - \int \sec^3(x)\, dx \quad \text{(Rewrite)} \\
&= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\, dx \quad \text{(Evaluate } \int \sec(x)\, d
\end{aligned}$$

Evaluating $\int \sec^3(x)\,dx$ (continued)

We now have

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx.$$

We see that the last integral is the same as the original one. Let I be a particular antiderivative of $\sec^3(x)$. Substituting I instead of $\int \sec^3(x)\,dx$ into the above equality:

$$I = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - I.$$

Adding I to both sides, we obtain

$$2I = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|.$$

Dividing by 2, we arrive at

$$I = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|.$$

we obtain that $\int \sec^3(x)\,dx = I + C = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C.$

Reduction Formulas for $\int \sec^n(x)\,dx$ and $\int \tan^n(x)\,dx$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

Revisiting $\int \sec^3(x)\,dx$

Apply a reduction formula to evaluate $\int \sec^3(x)\,dx$.

Revisiting $\int \sec^3(x)\,dx$

Apply a reduction formula to evaluate $\int \sec^3(x)\,dx$.

Solution:

By applying the first reduction formula with $n = 3$, we obtain

$$\begin{aligned}
&\int \sec^3(x)\,dx \\
&= \frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx \\
&= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C.
\end{aligned}$$

Using a Reduction Formula

Evaluate $\int \tan^4(x)\,dx$.

Using a Reduction Formula

Evaluate $\int \tan^4(x)\,dx$. **Solution:**
Applying the second reduction formula with $n = 4$, we obtain

$$\int \tan^4(x)\,dx = \frac{1}{4-1}\tan^{4-1}(x) - \int \tan^{4-2}(x)\,dx.$$

To evaluate $\int \tan^2(x)\,dx$, we apply the second reduction formula with $n = 2$, which allows us to continue the chain of equalities as follows:

$$\begin{aligned}\int \tan^4(x)\,dx &= \frac{1}{3}\tan^3(x) - \int \tan^2(x)\,dx \\ &= \frac{1}{3}\tan^3(x) - \left(\frac{1}{2-1}\tan^{2-1}(x) - \int \tan^{2-2}(x)\,dx\right) \\ &= \frac{1}{3}\tan^3(x) - \tan(x) + \int 1\,dx = \frac{1}{3}\tan^3(x) - \tan(x) + x + C.\end{aligned}$$

Applying the Reduction Formula

Apply the reduction formula to $\int \sec^5(x)\,dx$.

Applying the Reduction Formula

Apply the reduction formula to $\int \sec^5(x)\,dx$.

Answer:

$$\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) - \frac{3}{4}\int \sec^3(x)\,dx$$

Key Concepts

Integrals of trigonometric functions can be evaluated using various strategies. These strategies include the following:

1. Applying trigonometric identities to rewrite the integrand so that it may be evaluated via an appropriate substitution.
2. Using integration by parts.
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions.
4. Applying reduction formulas.

Understanding and mastering these techniques enables one to effectively evaluate integrals involving trigonometric functions and solve a wide range of mathematical problems.

Key Equations

Sine Products

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

Sine and Cosine Products

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Cosine Products

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

Power Reduction Formula for Secant

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-1}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

Power Reduction Formula for Tangent

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$