1.2 The Definite Integral

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Outline

Learning Objectives

1. State the definition of the definite integral.
2. Explain the terms integrand, limits of integration, and variable of integration.
3. Explain when a function is integrable.
4. Describe the relationship between the definite integral and net area.
5. Use geometry and the properties of definite integrals to evaluate them.
6. Calculate the average value of a function.

Reminder

In the preceding section, we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative.

Extension of the concept

Real-world problems often do not adhere to these restrictions. In this section, we explore extending the concept of the area under the curve to a wider range of functions using the definite integral.

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists.

If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

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Notation

The function $f(x)$ is the integrand, and the dx called the variable of integration. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral.

Theorem

We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Theorem

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Remark

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities on a closed interval are integrable.

Evaluation of Definite Integral

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Solution:

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}, \quad \text{where } a = 0, b = 2$$

$$x_i = \frac{2i}{n}, \quad \text{for } i = 1, 2, \ldots, n;\ f(x_i) = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

To calculate the definite integral, take the limit as $n \to \infty$:

$$\int_0^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) = \frac{8}{3}$$

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Problem: Evaluate $\int_0^3 (2x-1)\,dx$ using the definition of the definite integral. Utilize a right-endpoint approximation to generate the Riemann sum. **Solution:**

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}, \quad \text{where } a = 0, b = 3; \; x_i = \frac{3i}{n}, \text{ for } i = 1, 2, \ldots, n$$

$$f(x_i) = 2x_i - 1 = 2\left(\frac{3i}{n}\right) - 1 = \frac{6i}{n} - 1$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{18}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\sum_{i=1}^{n} 1 = \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{3}{n}\sum_{i=1}^{n} 1$$

$$= \frac{18}{n^2}\left[\frac{n^2+n}{2}\right] - \frac{3}{n}(n) = \frac{18}{2} + \frac{18}{2n} - 3$$

$$\int_0^3 (2x-1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(\frac{18}{2} + \frac{18}{2n} - 3\right) = 6$$

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$$x_i = \frac{3i}{n}, \quad \text{for } i = 1, 2, \ldots, n$$

$$f(x_i) = e^{x_i} - 1 = e^{\frac{3i}{n}} - 1$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{3}{n}\sum_{i=1}^{n}\left(e^{\frac{3i}{n}} - 1\right)$$

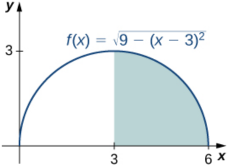
To calculate the definite integral, take the limit as $n \to \infty$:

$$\int_0^3 (e^x - 1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{3}{n}\sum_{i=1}^{n}\left(e^{\frac{3i}{n}} - 1\right)\right]$$

Using Geometric Formulas to Calculate Definite Integrals

Problem: Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution: The function describes a semicircle with radius 3. To find



we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\left(\frac{1}{2}\right)\pi r^2$. The shaded area in the above Figure covers one-half of the semicircle, or $A=\left(\frac{1}{4}\right)\pi r^2$.

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi$$

Using Geometric Formulas to Calculate Definite Integrals

Problem: Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

Solution: The given function represents the height of a trapezoid. To find the area under the curve over the interval $[2,4]$, we can use the formula for the area of a trapezoid:

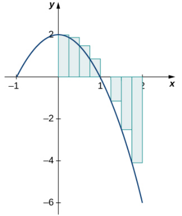
$$A = \frac{1}{2}h(b_1 + b_2)$$

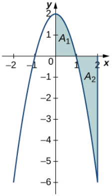
where h is the height and b_1, b_2 are the bases.

Substituting the values:

$$A = \frac{1}{2}(3)(2 + (2 \cdot 4 + 3)) = 18 \text{ square units}$$

Net Area





$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

Net signed and total area

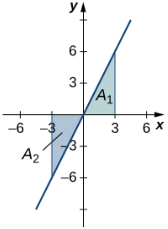
In the case where the function in integrable on $[a, b]$

$$\int_a^b f(x)dx = A_1 - A_2 \text{ and } \int_a^b \left|f\left(x\right)\right| dx = A_1 + A_2.$$

Finding the Net Signed Area

Problem: $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution: The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$,



Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is $A_1 = \frac{1}{2}(3)(6) = 9,$. The area of triangle A_2, below the axis, is $A_2 = \frac{1}{2}(3)(6) = 9$, Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

Properties of the Definite Integral

Suppose that the functions f and g are integrable over all given intervals.

$$\int_a^a f(x)\,dx = 0; \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Using the Properties of the Definite Integral

Problem: Express $\int_{-2}^{1}(-3x^3+2x+2)\,dx$ as the sum of three definite integrals using the properties of the definite integral.
Solution: Using integral notation, we have

$$\int_{-2}^{1}(-3x^3+2x+2)\,dx.$$

We apply properties 3 and 5 to get

$$\begin{aligned}\int_{-2}^{1}(-3x^3+2x+2)\,dx &= \int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx\\ &= -3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.\end{aligned}$$

Using the Properties of the Definite Integral

Problem: Express $\int_1^3 (6x^3 - 4x^2 + 2x - 3)\, dx$ as the sum of four definite integrals using the properties of the definite integral.

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$$\int_1^3 (6x^3 - 4x^2 + 2x - 3)\,dx.$$

We apply properties to express it as the sum of four definite integrals:

$$\int_1^3 (6x^3 - 4x^2 + 2x - 3)\,dx \quad = 6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$$

Using the Properties of the Definite Integral

Problem: If it is known that $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 5$, find the value of $\int_5^8 f(x)\,dx$.

Solution: By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned} \int_0^8 f(x)\,dx &= \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx \\ 10 &= 5 + \int_5^8 f(x)\,dx \\ 5 &= \int_5^8 f(x)\,dx. \end{aligned}$$

Using the Properties of the Definite Integral

Problem: If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.
Solution: By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned} \int_1^5 f(x)\,dx &= \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx \\ -3 &= \int_1^2 f(x)\,dx + 4 \\ -7 &= \int_1^2 f(x)\,dx. \end{aligned}$$

Comparison Theorem

Suppose that the functions $f(x)$ and $g(x)$ are integrable over the interval $[a, b]$.

If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0.$$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$\begin{aligned} m(b-a) &\leq \int_a^b f(x)\,dx \\ &\leq M(b-a). \end{aligned}$$

Comparing Integrals over a Given Interval

Problem: Compare the integrals of the functions $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.
Solution: Comparing functions $f(x)$ and $g(x)$ when $x \in [0, 1]$. Since $1 + x^2 \geq 0$ and $1 + x \geq 0$ for $x \in [0, 1]$, comparing $\sqrt{1+x^2}$ and $\sqrt{1+x}$ is equivalent to comparing the expressions $(1 + x^2)$ and $(1 + x)$ under the roots on $[0, 1]$. We consider :

$$(1 + x^2) - (1 + x) = 1 + x^2 - 1 - x = x^2 - x = x(x - 1).$$

Since $x \geq 0$ and $x - 1 \leq 0$ on $[0, 1]$, we have that $x(x - 1) \leq 0$ on $[0, 1]$. It follows that $1 + x^2 \leq 1 + x$ on $[0, 1]$, and hence

$$f(x) = \sqrt{1+x^2} \leq \sqrt{1+x} = g(x), \quad x \in [0, 1].$$

Since both functions $f(x)$ and $g(x)$ are continuous on $[0, 1]$,

$$\int_0^1 f(x)\, dx \leq \int_0^1 g(x)\, dx.$$

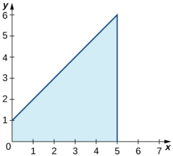
Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the average value of the function $f(x)$ (denoted by f_{ave}) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Finding the Average Value of a Linear Function

Problem: Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.
Solution: First, graph the function on the stated interval, as shown below.



The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 (x+1)\,dx \quad = \tfrac{1}{2}h(a+b) = \tfrac{1}{2}\cdot 5\cdot(1+6) = \tfrac{35}{2}.$$

Thus, the average value of the function is

$$\frac{1}{5}\int_0^5 (x+1)\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

Finding the Average Value of a Linear Function

Problem: Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.
Solution: Use the average value formula and geometry to evaluate the integral. First, note that the function is a linear function, representing a downward-sloping line.
Apply the average value formula:

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\begin{aligned}
\int_0^3 (6-2x)\,dx \quad &= \tfrac{1}{3-0}\int_0^3 (6-2x)\,dx \\
&= \tfrac{1}{3}\left[6x - x^2\right]_0^3 \\
&= \tfrac{1}{3}[(18-9)-(0-0)] \\
&= \tfrac{9}{3} = 3.
\end{aligned}$$