7.4 Area and Arc Length in Polar Coordinates

Math 1700

University of Manitoba

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Outline

Learning Objectives

- Derive the formula for the area of a region in polar coordinates.
- Determine the arc length of a polar curve.

Area and Arc Length in Polar Coordinates

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval,

Area between the curve and the x-axis

The area between the curve and the x-axis is given by

$$A = \int_a^b f(x)\,dx.$$

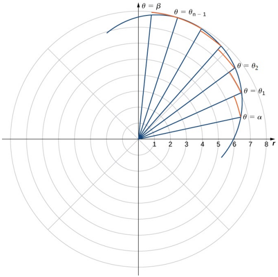
Arc length of this curve

We can also find the arc length of this curve using the formula

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx.$$

Area Bounded by a Polar Curve

Consider a polar curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. Thus $\Delta\theta = \frac{(\beta-\alpha)}{n}$, and the ith partition point $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan(\theta_i)$ passing through the pole as shown in the following graph.



The area of a sector of a circle with angle θ_i can be given as:

$$A_i = \frac{1}{2}(\Delta\theta)\,(f(\theta_i))^2 = \frac{1}{2}(f(\theta_i))^2\Delta\theta.$$

Exact Area Calculation

Summing the areas of sectors for $1 \leq i \leq n$, we obtain a Riemann sum that approximates the polar area:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta.$$

Area of a Region Bounded by a Polar Curve

Formula

Suppose f is continuous and nonnegative on the interval $\alpha \leq \theta \leq \beta$ with $0 < \beta - \alpha \leq 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is:

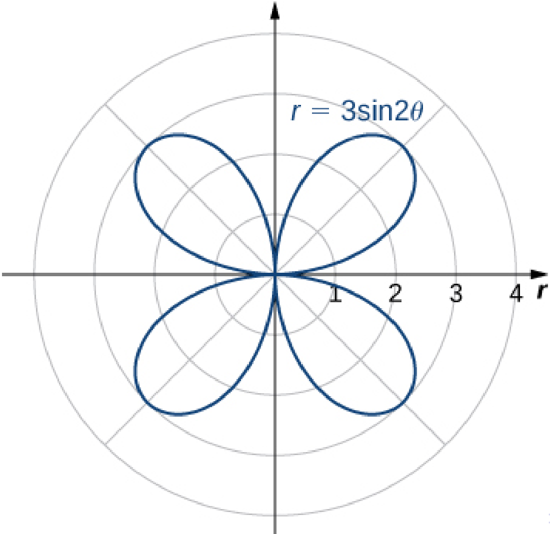
$$(*) \qquad A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Example: Finding the Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Graph

The graph of $r = 3\sin(2\theta)$ is shown below.



Finding the Area Inside the Petal: Solution

It follows that the petal in the first quadrant corresponds to $\theta \in \left[0, \frac{\pi}{2}\right]$. To find the area inside this petal, use (*) from the above theorem with $f(\theta) = 3\sin(2\theta)$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [3\sin(2\theta)]^2 d\theta = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} 9\sin^2(2\theta) d\theta.$$

To evaluate this integral, use the formula $\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$ with $\alpha = 2\theta$:

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} 9\sin^2(2\theta) d\theta = \frac{9}{2}\int_{0}^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2} d\theta = \frac{9}{4}\int_{0}^{\frac{\pi}{2}} (1-\cos(4\theta)) d\theta$$

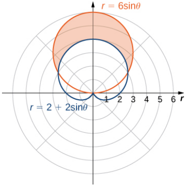
$$= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_{0}^{\frac{\pi}{2}} = \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin(0)}{4}\right) = \frac{9\pi}{8}.$$

Finding the Area Inside the Cardioid

Problem: Find the area inside the cardioid defined by the equation $r = 1 - \cos(\theta)$.
Answer: $A = \frac{3\pi}{2}$.
Hint: Use (*). Be sure to determine the correct limits of integration before evaluating.

Finding the Area between Two Polar Curves

Problem: Find the area outside the cardioid $r = 2 + 2\sin(\theta)$ and inside the circle $r = 6\sin(\theta)$.
Solution: First draw a graph containing both curves as shown below.



$$6\sin(\theta) = 2 + 2\sin(\theta) \Rightarrow 4\sin(\theta) \quad = 2 \ \Rightarrow \sin(\theta) = \frac{1}{2}.$$

Then $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ in the interval $(-\pi, \pi]$, which are the limits of integration since from the picture we see that $6\sin(\theta) \geq 2 + 2\sin(\theta)$ on $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$. The circle $r = 6\sin(\theta)$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin(\theta)$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

Part 2

$$\begin{aligned}
A &= \text{circle} - \text{cardioid} \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [6\sin(\theta)]^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2+2\sin(\theta)]^2 d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 36\sin^2(\theta) d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4+8\sin(\theta)+4\sin^2(\theta)) d\theta \\
&= 18\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+2\sin(\theta)+\frac{1-\cos(2\theta)}{2}) d\theta \\
&= 9\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - 2\left(\frac{3\theta}{2} - 2\cos(\theta) - \frac{\sin(2\theta)}{4}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
&= 9\left(\frac{5\pi}{6} - \frac{\sin\left(\frac{5\pi}{3}\right)}{2}\right) - 9\left(\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right) \\
&\quad - \left(3\left(\frac{5\pi}{6}\right) - 4\cos\frac{5\pi}{6} - \frac{\sin\left(\frac{5\pi}{3}\right)}{2}\right) + \left(3\left(\frac{\pi}{6}\right) - 4\cos\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right) = 4\pi.
\end{aligned}$$

Finding the Area Inside and Outside Circles

Problem: Find the area inside the circle $r = 4\cos(\theta)$ and outside the circle $r = 2$.
Answer: $A = \frac{4\pi}{3} + 2\sqrt{3}$.
Hint: Use (*) and take advantage of symmetry.

Arc Length of a Curve in Polar Coordinates

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos(\theta) = f(\theta)\cos(\theta) \quad \text{and} \quad y = r\sin(\theta) = f(\theta)\sin(\theta).$$

Differentiating, we obtain

$$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta).$$

Second part

Applying the known arc length formula, we get

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\right)^2 + \left(f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\right)^2} d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2\left(\cos^2(\theta) + \sin^2(\theta)\right) + (f(\theta))^2\left(\cos^2(\theta) + \sin^2(\theta)\right)} d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

Formula

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Finding the Arc Length of a Polar Curve

Problem: Find the arc length of the cardioid $r = 2 + 2\cos(\theta)$.

Solution:

$$\begin{aligned}
L &= \int_{-\pi}^{\pi} \sqrt{[2 + 2\cos(\theta)]^2 + [-2\sin(\theta)]^2}\, d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{4 + 8\cos(\theta) + 4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{8 + 8\cos(\theta)}\, d\theta \\
&= 2\int_{-\pi}^{\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta = 2\int_{-\pi}^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\
&= 2\int_{-\pi}^{\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta = 4\int_{-\pi}^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta = 4\left(2\sin\left(\frac{\theta}{2}\right)\right)\Bigg|_{-\pi}^{\pi} \\
&= 8(1 - (-1)) = 16.
\end{aligned}$$

Finding the Arc Length of $r = 3\sin(\theta)$

Problem: Find the total arc length of $r = 3\sin(\theta)$.
Answer: 3π
Hint To determine the correct limits, make a table of values.
Solution: To determine the correct limits, make a table of values for θ and r, then observe the behavior of r as θ varies.

θ	r
0	0
$\pi/2$	3
π	0
$3\pi/2$	-3
2π	0

As θ goes from 0 to 2π, the curve traces out a single wave of the sine function from $r = 0$ to $r = 3$ and back to $r = 0$. Hence, the total arc length is $s = 3\pi$.

Key Concepts

- The area of the region bounded by the polar curve $r = f(\theta)$ and between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by the integral

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta.$$

- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + \left[\frac{df}{d\theta}\right]^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

Key Equations

Area of a region bounded by a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

Arc length of a polar curve:

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + \left[\frac{df}{d\theta}\right]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$