Game Theory Stories

Algorithm and Code Training School

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05.07.2025

Structure

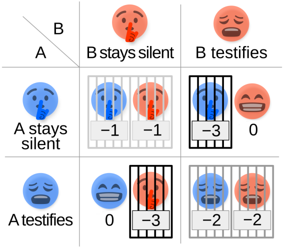
Normal talk: go deep into one topic.

This talk: cover the base (Nash eq., truthfullness, etc), and also see many cool things but quite briefly.

Prisoner’s Dilemma

A \ B	B stays silent	B testifies
A stays silent	−1, −1	−3, 0
A testifies	0, −3	−2, −2

Prisoner's Dilemma



Note that whatever B does, A is better off to defect. Same is B.

Thus, the rational choice for A, B is to defect, getting $-2, -2$ payout.

However, this is worse then if both of them collaborate and stay silent!!

Equilibrium

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Definition

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Definition

A Nash equilibrium is a situation where no player could gain by only changing their own strategy (all other players' strategies remain fixed).

- $(Silent, Silent)$ — is not Nash equilibrium.
- $(Testisfy, Testify)$ — is Nash equilibrium

Pop-Quiz Time

Question

Is there a 2-P game with no Nash Equilibrium?

Pop-Quiz Time

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-1, 1	1, -1
1, -1	-1, 1

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-1, 1	1, -1
1, -1	-1, 1

Note that for all cells, $outcome_1 + outcome_2 = 0$. Those games are called *zero-sum*. For those games we can only write payoff of the first player. I.e.

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We can also consider *Mixed* strategies for both players, i.e. strategy is probability distribution $p_1, \ldots, p_n$ over possible choices.

Pop-Quiz Time

Question

Is there a 2-P game with no **(Pure)** Nash Equilibrium?

-1	1
1	-1

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We can also consider *Mixed* strategies for both players, i.e. strategy is probability distribution $p_1, \ldots, p_n$ over possible choices.

Excercise: What is **Mixed** Nash Equilibrium?

John Nash

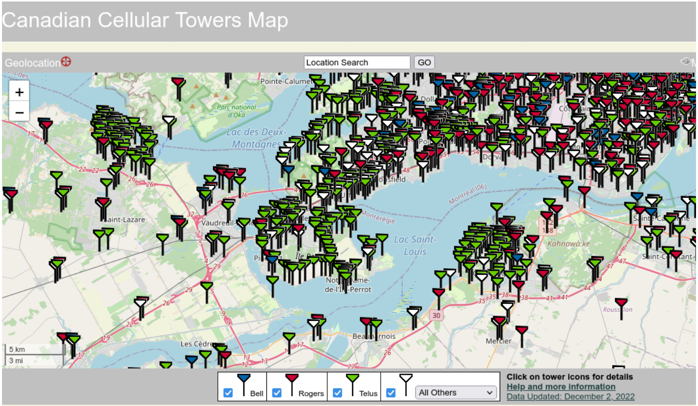
1994 Nobel Prize in Economics for his work on non-cooperative games.

2015 Abel Prize (together w/ Louis Nirenberg) for the contribution to the field of partial differential equations.

Auctions

How we imagine them:

Auctions



Also an auction.

Single Item Auction

Selling one item. Buyers:

- A 50$
- B 80$
- C 200$
- D 1000$

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Usual algorithm: $\implies$ allocate to buyer d, charge them 1000\$.
However, note that the buyer d might then prefer to bid 900\$, or even 300\$, because they still win the auction, but pay less.

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Normally, the utility of the agent i is
$v_i(Outcome(v_i, v_{-i})) - Payment(v_i, v_{-i})$

It makes sense to misrepresent v_i with v_i' if

$$v_i(Outcome(v_i', v_{-i})) - Payment(v_i', v_{-i}) >$$
$$> v_i(Outcome(v_i, v_{-i})) - Payment(v_i, v_{-i})$$

Second price auction

Selling one item. Buyers:

- A 50$
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Second price auction $\Longrightarrow$ Allocate item to buyer d, but charge them 200$.

Then no agent is interested in misrepresenting their internal value.

Extending to Matchings

Suppose there are multiple buyers, and multiple items, and each buyers wants to buy at most one item, but different items have different valuation by different buyers (houses; advertisement slots; radio frequencies; etc).

Question

How to design a good mechanism?

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Vickrey–Clarke–Groves auction:

Algorithm

- Compute max-weight matching, and use it to sell items
- Charge each buyer the amount of harm it did to other buyers. I.e. if buyer b_i wins t_j, charge $OPT(B \setminus \{b_i\}, T) - OPT(B \setminus \{b_i\}, T \setminus \{T_j\})$

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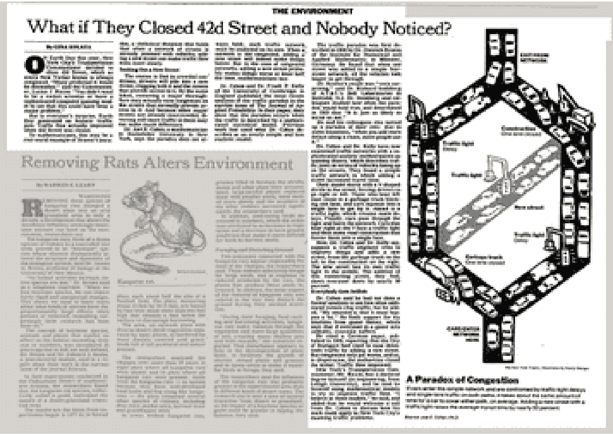
Excercise 1: Payment is at most item's valuation (i.e. utility is always non-negative).

Note: Payment doesn't depend on the agent's valuation (apart from the item being sold or not).

Excercise 2: The mechanism is truthful (it doesn't make sense to lie about valuations).

What if They Closed 42d Street and Nobody Noticed?



THE ENVIRONMENT

What if They Closed 42d Street and Nobody Noticed?

Removing Rats Alters Environment

A Paradox of Congestion

New York Times, Dec. 25, 1990, Gina Kolata.

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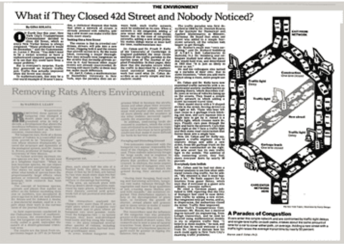
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A Paradox of Congestion

On Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

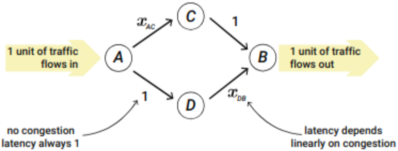
New York Times, Dec. 25, 1990, Gina Kolata.

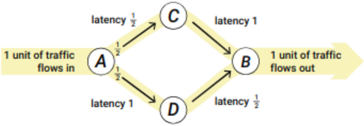
Congestion Games

Drivers(Players) go from vertex A to vertex B trying to minimize their commute time. The time to traverse the edge might depend on the amount of drivers going through it — the more drivers the longer it takes.

In different variations of the problem, drivers can be considered discrete (non-divisible) or continuous (divisible, the amount of drivers can be divided among different path in any real proportion).

Braess Paradox





Excerpt from Game Theory, Alive by Anna R. Karlin and Yuval Peres, 2016.

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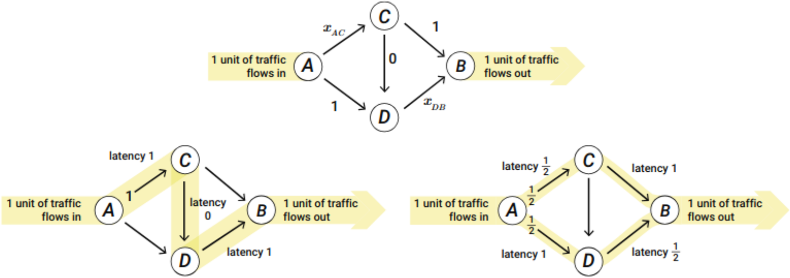

Nash equilibrium flow

optimal flow

So adding one edge makes total commute time $4/3$ times worse!!

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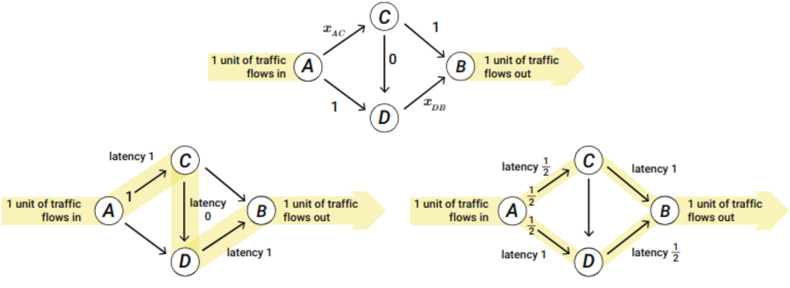

Nash equilibrium flow

optimal flow

One more term used here:

Definition

Price of Anarchy is $\frac{BestSolution}{\min NashEquilibrium}$

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Assets division

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240	?	?	?

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Equal share method:

Total \ Creditors	100	200	300
240	80	80	80

Proportional method:

Total \ Creditors	100	200	300
240	40	80	120

Talmud's salution

Total \ Creditors	100	200	300
100	33.3	33.3	33.3
200	50	75	75
300	50	100	150

Principle of Fairness

The Talmud describes procedure how to fair divide among two claimants:

Contested Garment

Two hold a garment. One claims it all, the other claims a half. How to divide fairly?

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Two hold a garment. One claims it all, the other claims a half. How to divide fairly?

The ownership of one half of the garnment is not contested. The contested part should be divided equally, i.e. resulting division is 3/4, 1/4.

Consistency rule

Let $A(d_1, \ldots, d_n; E)$ be an allocation rule of E with creditors d_1, $\ldots$, d_n.

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Consistency (CG-consistency) rule

Let $A(d_1, \ldots, d_n; E) = (a_1, \ldots, a_n)$
Then for all $\{i, j\}$, (a_i, a_j) is division of $(d_i, d_j; a_i + a_j)$ as per CG rule.

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Then for all $\{i, j\}$, (a_i, a_j) is division of $(d_i, d_j; a_i + a_j)$ as per CG rule.

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300	50	100	150

Observe that Talmud's solution is consistent!!

Bankruptcy problem

Theorem (Robert Aumann and Michael Maschler, 1984)

Each bankruptcy problem has a unique consistent solution

Bankcruptcy problem

But what about game theory?

Cooperative games

A cooperative cooperative game with transferable utilities

Is defined by function $v : 2^S \to R$, where S is the set of players, where $v(S)$ is a payoff that subset S of players can achieve on their own regardless of what the remaining players do. This value can then be split among the players in any way that they agree on.
$v(\varnothing) = 0$, $v(S) \leqslant v(T)$ if $S \subseteq T$

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$v(\emptyset) = 0$, $v(S) \leqslant v(T)$ if $S \subseteq T$

Note: different ways to find fair solution: i.e. Shapley value, Nucleolus.

Nucleolus

Suppose $x_1, x_2, \ldots, x_{|S|}$ is a payoff vector to players.

Definition

For coalition C, $excess(C) = \sum_{p \in C} x_p - v(C) = x(c) - v(C)$

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Let $\Theta = \{excess(C) \mid C \subseteq 2^S\}$ (multiset).

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Theorem (David Schmeidler in 1969)

There is a unique solution which maximizes Θ lexicographically (maximize smallest excess, then the second-smallest, etc.).

Bankruptcy Problem

Let $(d_1, \ldots, d_n; E)$ be a bancruptcy problem.

Bankruptcy problem (cooperative game)

Let $v : 2^{[n]} \to \mathbb{R}$ be cooperative game defined as:

$$v(S) = max(0, E - \sum_{p \in [n] \setminus S} d_p) = (E - d([n] \setminus S))_+$$

I.e. $v(S)$ is the amount of money S can guarantee after creditors in $[n] \setminus S$ are paid.

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Claim

Talmud solution to bankruptcy problem is exactly the nucleolus.

Talmud example

Total \ Creditors	100	200	300
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$$v(\{1,2,3\}) = 200,\ v(\{2,3\}) = 100,\ v = 0 \text{ otherwise}.$$

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Total \ Creditors	100	200	300
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$v(S) = (E - d([n] \setminus S))_+ = ?$. $excess(S) = x(C) - v(C) = ?$

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Talmud example

Total \ Creditors	100	200	300
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$v(\{1,2,3\}) = 200,\ v(\{2,3\}) = 100,\ v = 0$ otherwise.

$excess(\{1,2,3\}) = 0,\ excess(\{2,3\}) = 50$

$excess(\{1,2\}) = 125,\ excess(\{1,3\}) = 125$

$excess(\{1\}) = 50,\ excess(\{2\}) = 75,\ excess(\{3\}) = 75$

Talmud example

Total \ Creditors	100	200	300
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$v(S) = (E - d([n] \setminus S))_+ =?$. $excess(S) = x(C) - v(C) =?$

$v(\{1,2,3\}) = 200$, $v(\{2,3\}) = 100$, $v = 0$ otherwise.

$excess(\{1,2,3\}) = 0$, $excess(\{2,3\}) = 50$

$excess(\{1,2\}) = 125$, $excess(\{1,3\}) = 125$

$excess(\{1\}) = 50$, $excess(\{2\}) = 75$, $excess(\{3\}) = 75$

Smallest excess: $excess(\{1\}) = excess(\{2,3\})$ so division of 1 vs 2+3 is optimal.

Next smallest excess: $excess(\{2\}) = excess(\{3\})$ so division of 2 vs 3 is optimal.

Elections

Elections — are everywhere.

Can you give some examples?

Elections

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Can you give some examples?

- Political elections (president, parliament, etc.)
- Work Council (Netherlands)
- Which social projects should a City budget fund?
- etc, etc.

Different mechanisms

- First-Past-the-post
- Run-off elections
- Proportional representation
- Approval voting

Condorcet

Suppose there is candidate set S, and voter set V. Each voter has a complete preference over S (a permutation).

Condorcet Paradox (Elections are always rigged)

There exists an election (S, V) such that no matter what outcome is more than half of the people believe switching to other outcome better.

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Let $S = \{A, B, C\}$, $V = \{v_1, v_2, v_3\}$

- $v_1 : A \succ B \succ C$
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- $v_2 : B \succ C \succ A$
- $v_3 : C \succ A \succ B$

B is more preferable than C, A is more preferable than B, yet C is more preferable than A.

Six Flags (2025 paper)

Six Candidates Suffice to Win a Voter Majority

Moses Charikar
Stanford University

Alexandra Lassota
Eindhoven University of Technology

Prasanna Ramakrishnan
Stanford University

Adrian Vetta
McGill University

Kangning Wang
Rutgers University

Abstract

A cornerstone of social choice theory is Condorcet's paradox which says that in an election where n voters rank m candidates it is possible that, no matter which candidate is declared the winner, a majority of voters would have preferred an alternative candidate. Instead, can we always choose a small *committee* of winning candidates that is preferred to any alternative candidate by a majority of voters?

Elkind, Lang, and Saffidine raised this question and called such a committee a *Condorcet winning set*. They showed that winning sets of size 2 may not exist, but sets of size logarithmic in the number of candidates always do. In this work, we show that Condorcet winning sets of size 6 always exist, regardless of the number of candidates or the number of voters. More generally, we show that if $\frac{\alpha}{1-\ln\alpha} \geq \frac{2}{k+1}$, then there always exists a committee of size k such that less than an α fraction of the voters prefer an alternate candidate. These are the first nontrivial positive results that apply for all $k \geq 2$.

Secretary

Let's discuss secretary problem and prophet inequalities (unfortunately, no slides).

End

Thank you for attention!

We only touched a surface here, there are so many things left to do!