Kinematics of Geometrically Exact Beams Undergoing Large Deformations

The large-deformation of a flexible beam may be decomposed into the deformation of the neutral axis of the beam and the rotation of the cross-section of the beam at any point along the neutral axis. The position of any point ($\vec{r}^P$) on a flexible beam may be described by locating the centroid of a rigid-cross section ($\vec{r}_0$) in the undeformed configuration, the deformation vector of the centroid $\left(\vec{\Delta}\right)$, and the location of the point in the cross-section ($\vec{s}$). The resulting vector is expressed in the Newtonian frame as

$$\left\{\vec{r}^P\right\}_N = \left\{\vec{r}_0\right\}_N + \left\{\vec{\Delta}\right\}_N + \left\{\vec{s}\right\}_N \ . \tag{1}$$

If x is the coordinate along the beam axis, then

$$\left\{\vec{r}_0\right\}_B = \begin{Bmatrix} x \\ 0 \end{Bmatrix} \tag{2}$$

and

$$\left\{\vec{r}_0\right\}_N =^N \mathbf{R}^{B_0} \left\{\vec{r}_0\right\}_B \ , \tag{3}$$

where ${}^N\mathbf{R}^{B_0}$ rotates the beam-basis of the reference configuration to the Newtonian basis. Additionally

$$\left\{\vec{s}\right\}_N =^N \mathbf{R}^B \left\{\vec{s}\right\}_B \ , \tag{4}$$

where ${}^N\mathbf{R}^B$ rotates the beam-basis of the deformed configuration (cross-section after rotation from the starting configuration) to the Newtonian basis.

The velocity of any point ($\vec{v}_P$) can be determined by differentiating $\vec{r}^P$ as

$$\left\{\vec{v}_P\right\}_N = \frac{d}{dt}\left\{\vec{r}^P\right\}_N = \underbrace{\frac{d}{dt}\left\{\vec{r}_0\right\}_N}_{0} + \frac{d}{dt}\left\{\vec{\Delta}\right\}_N + \frac{d}{dt}\left\{{}^N\mathbf{R}^B\right\}\left\{\vec{s}\right\}_B \tag{5}$$

$$= \left\{\dot{\vec{\Delta}}\right\}_N +^N \dot{\mathbf{R}}^B \left\{\vec{s}\right\}_B \tag{6}$$

If Eq. (6) is then left multiplied by the identity and the identity is then replaced with $\underline{\underline{R}}\,\underline{\underline{R}}^T$, the skew-symmetric angular velocity matrix can be extracted from the above expression as follows:

$$\left\{\vec{v}\right\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{I}}\,\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0 \left\{\vec{s}\right\}_I \tag{7}$$

$$\left\{\vec{v}\right\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}\,\underbrace{\underline{\underline{R}}^T \dot{\underline{\underline{R}}}}_{\left\{\tilde{\omega}\right\}_B}\,\underline{\underline{R}}_0 \left\{\vec{s}\right\}_I \tag{8}$$

This results in an expression of the velocity of any point as a function of the time rate of change of the deformation vector and the angular velocity of the beam cross section as

$$\left\{\vec{v}\right\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}\left\{\tilde{\omega}\right\}_B \underline{\underline{R}}_0 \left\{\vec{s}\right\}_I \, , \tag{9}$$

where $\{\tilde{\omega}\}_B$ denotes the skew-symetric angular velocity matrix expressed in the material basis. Equation (9) is different from the simplification given also in equation (16.52) of *Flexible Multibody Dynamics*, which is

$$\underline{v} = \dot{\underline{u}} + \underline{\underline{R}}\,\underline{\underline{R}}_0 \tilde{\omega}^* s^* . \tag{10}$$

An alternate derivation can yield a similar result to Eq. (10), but will not result in an expression matching Eq. (**??**). This is accomplished by replacing the product of the rotation matrices associated with the transformation from the reference configuration to the material configuration $\left(\underline{\underline{R}}\right)$ and the transformation from the inertial frame to the reference configuration $\left(\underline{\underline{R}}_0\right)$ can be replaced by a single matrix $\left(\underline{\underline{R}}_{BI} = \underline{\underline{R}}\,\underline{\underline{R}}_0\right)$. The time derivative of the position of any point on the deformable beam can then be written as

$$\frac{d}{dt}\left\{\vec{X}\left(\alpha_1, \alpha_2, \alpha_3\right)\right\}_I = \frac{d}{dt}\left\{\vec{x}_0\right\}_I + \frac{d}{dt}\left\{\vec{u}\right\}_I + \frac{d}{dt}\left\{\underline{\underline{R}}_{BI}\right\}\left(\left\{\vec{w}\right\}_I + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3\right), \tag{11}$$

again neglecting the warp.

With a similar simplification, the velocity of any point on the beam can be expressed as

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \dot{\underline{\underline{R}}}_{BI}\left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3\right) . \tag{12}$$

This equation can be further simplified as follows:

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \dot{\underline{\underline{R}}}_{BI}\underbrace{\left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3\right)}_{\{\vec{s}\}_I}, \tag{13}$$

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \dot{\underline{\underline{R}}}_{BI}\{\vec{s}\}_I . \tag{14}$$

By left multiplying by the identity and replacing it with $\underline{\underline{R}}_{BI}\underline{\underline{R}}_{BI}^T$, the skew-symmetric angular velocity matrix can be extracted from the above expression as follows:

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{I}}\,\dot{\underline{\underline{R}}}_{BI}\{\vec{s}\}_I \tag{15}$$

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}_{BI}\underbrace{\underline{\underline{R}}_{BI}^T\dot{\underline{\underline{R}}}_{BI}}_{\{\tilde{\omega}\}_B}\{\vec{s}\}_I \tag{16}$$

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}_{BI}\{\tilde{\omega}\}_B\{\vec{s}\}_I \tag{17}$$

This results in an expression of the velocity of any point as a function of the time rate of change of the deformation vector and the angular velocity of the beam element as

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}\,\underline{\underline{R}}_0\{\tilde{\omega}\}_B\{\vec{s}\}_I . \tag{18}$$

This seems to suggest that I am missing something, or perhaps my understanding of the problem description is incorrect. For example, $\dot{\underline{\underline{R}}}_{BI}$ may not necissarily be equal to $\dot{\underline{\underline{R}}}\underline{\underline{R}}_0$, which may account for the descrepancy between Eq. (9) and Eq. (10). However, this would mean that my understanding of the reference configuration (b_i), which is that it is not moving, would be incorrect.

Any advice would be greatly appreciated!