Geometrically Exact Beams Undergoing Large Deformations and Rotations

Kinematics of A Generic Point

$$\left\{\mathbf{r}^{Q}\right\}_N = \left\{\mathbf{r}^{OP}\right\}_N + \underbrace{\left\{\mathbf{r}^{PP'}\right\}_N}_{\boldsymbol{\Delta}} + {}^N\underline{\underline{C}}^B \left\{\mathbf{r}^{P'Q}\right\}_B \tag{1}$$

$$\left\{{}^N\mathbf{v}^{Q}\right\}_N = \frac{d}{dt}\left\{\mathbf{r}^{Q}\right\}_N \tag{2}$$

$$= \underbrace{\frac{d}{dt}\left\{\mathbf{r}^{OP}\right\}_N}_{\mathbf{0}} + \underbrace{\frac{d}{dt}\left\{\mathbf{r}^{PP'}\right\}_N}_{\dot{\boldsymbol{\Delta}}} + \frac{d}{dt}\left({}^N\underline{\underline{C}}^B\right)\left\{\mathbf{r}^{P'Q}\right\}_B$$

$$+ \underbrace{{}^N\underline{\underline{C}}^B \frac{d}{dt}\left\{\mathbf{r}^{P'Q}\right\}_B}_{\mathbf{0}} \tag{3}$$

$$= \left\{\dot{\boldsymbol{\Delta}}\right\}_N + {}^N\dot{\underline{\underline{C}}}^B \left\{\mathbf{r}^{P'Q}\right\}_B \tag{4}$$

$$= \left\{\dot{\boldsymbol{\Delta}}\right\}_N + {}^N\dot{\underline{\underline{C}}}^B \; \underline{\underline{I}} \; \left\{\mathbf{r}^{P'Q}\right\}_B \tag{5}$$

$$= \left\{\dot{\boldsymbol{\Delta}}\right\}_N + \underbrace{{}^N\dot{\underline{\underline{C}}}^B \left({}^N\underline{\underline{C}}^B\right)^T}_{{}^N\tilde{\omega}^B} \underbrace{{}^N\underline{\underline{C}}^B \left\{\mathbf{r}^{P'Q}\right\}_B}_{\left\{\mathbf{r}^{P'Q}\right\}_N} \tag{6}$$

$$\left\{{}^N\mathbf{v}^{Q}\right\}_N = \left\{\dot{\boldsymbol{\Delta}}\right\}_N + {}^N\tilde{\omega}^B \left\{\mathbf{r}^{P'Q}\right\}_N \tag{7}$$

$$\left\{{}^N\mathbf{a}^{Q}\right\}_N = \frac{d}{dt}\left\{\mathbf{v}^{Q}\right\}_N \tag{8}$$

$$= \frac{d}{dt}\left\{\dot{\boldsymbol{\Delta}}\right\}_N + \frac{d}{dt}\left({}^N\tilde{\omega}^B\right)\left\{\mathbf{r}^{P'Q}\right\}_N + {}^N\tilde{\omega}^B \underbrace{\frac{d}{dt}\left\{\mathbf{r}^{P'Q}\right\}_N}_{{}^N\tilde{\omega}^B\left\{\mathbf{r}^{P'Q}\right\}_N} \tag{9}$$

$$\left\{{}^N\mathbf{a}^{Q}\right\}_N = \left\{\ddot{\boldsymbol{\Delta}}\right\}_N + {}^N\tilde{\alpha}^B \left\{\mathbf{r}^{P'Q}\right\}_N + {}^N\tilde{\omega}^B\,{}^N\tilde{\omega}^B \left\{\mathbf{r}^{P'Q}\right\}_N \tag{10}$$

Interpolation of Deformations and Rotations

$$\boldsymbol{\Delta}(x) = \underline{\underline{H}}(x) \begin{Bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{Bmatrix} \tag{11}$$

$${}^N\underline{\underline{C}}^B(x) = \underline{\underline{H}}(x) \begin{bmatrix} {}^N\underline{\underline{C}}^1 \\ {}^N\underline{\underline{C}}^2 \end{bmatrix} \tag{12}$$

Developing the Equations of Motion

$$\pi \int_0^L \int_0^r \int_0^r {}^N\mathbf{a}^Q \cdot {}^N\mathbf{v}_r^Q \, dr\, dr\, dx - Q_e \cdot {}^N\mathbf{v}_r^Q \tag{13}$$