Kinematics of a Highly-Flexible Beam

From equation (16.43) of *Flexible Multibody Dynamics*,

$$\vec{X}\left(\alpha_1,\alpha_2,\alpha_3\right)=\vec{X}_0+w_1\bar{B}_1+\left(w_2+\alpha_2\right)\bar{B}_2+\left(w_3+\alpha_3\right)\bar{B}_3 \tag{1}$$

the displacement of any point on the beam may be written as

$$\left\{\vec{X}\left(\alpha_1,\alpha_2,\alpha_3\right)\right\}_I=\left\{\vec{x}_0\right\}_I+\left\{\vec{u}\right\}_I+\left(\underline{\underline{R}}\,\underline{\underline{R}}_0\right)\left(\left\{\vec{w}\right\}_I+\alpha_2\hat{i}_2+\alpha_3\hat{i}_3\right), \tag{2}$$

which is given by equation (16.44) of *Flexible Multibody Dynamics.* In Eq. (2), the subscripts B denotes vector quantities expressed in the material basis and the subscript I denotes quantities expressed in the inertial basis. To obtain the velocity of any point on the beam the time derivative of Eq (2) is taken, resulting in

$$\left\{\vec{v}\right\}_I=\left\{\dot{\vec{u}}\right\}_I+\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0\left(\vec{0}+\alpha_2\hat{i}_2+\alpha_3\hat{i}_3\right) \tag{3}$$

after application of the chain rule, if the contribution of the warp ($\vec{w}$) is neglected. This equation can be further simplified as follows:

$$\begin{aligned}\left\{\vec{v}\right\}_I&=\left\{\dot{\vec{u}}\right\}_I+\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0\underbrace{\left(\vec{0}+\alpha_2\hat{i}_2+\alpha_3\hat{i}_3\right)}_{\left\{\vec{s}\right\}_I}, &(4)\\ &=\left\{\dot{\vec{u}}\right\}_I+\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0\left\{\vec{s}\right\}_I. &(5)\end{aligned}$$

Equation (5) agrees with equation (16.52) of *Flexible Multibody Dynamics*, which is

$$\underline{v}=\dot{\underline{u}}+\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0\underline{s}^* \tag{6}$$

If Eq. (5) is then left multiplied by the identity and the identity is then replaced with $\underline{\underline{R}}\,\underline{\underline{R}}^T$, the skew-symmetric angular velocity matrix can be extracted from the above expression as follows:

$$\begin{aligned}\left\{\vec{v}\right\}_I&=\left\{\dot{\vec{u}}\right\}_I+\underline{\underline{I}}\,\dot{\underline{\underline{R}}}\,\underline{\underline{R}}_0\left\{\vec{s}\right\}_I &(7)\\ &=\left\{\dot{\vec{u}}\right\}_I+\underline{\underline{R}}\underbrace{\underline{\underline{R}}^T\dot{\underline{\underline{R}}}}_{\left\{{}^b\tilde{\omega}^B\right\}_{BB}}\underline{\underline{R}}_0\left\{\vec{s}\right\}_I &(8)\\ &=\left\{\dot{\vec{u}}\right\}_I+\underline{\underline{R}}\left\{{}^b\tilde{\omega}^B\right\}_{BB}\underline{\underline{R}}_0\left\{\vec{s}\right\}_I, &(9)\end{aligned}$$

where $\left\{{}^b\tilde{\omega}^B\right\}_{BB}$ denotes the skew-symmetric angular velocity matrix of the material frame (B) with respect to the reference frame (b), expressed in the material basis (B). Additionally, since $\underline{\underline{R}}_0$ is constant, ${}^I\tilde{\omega}^b=0$. Therefore,

$${}^I\tilde{\omega}^B=\underbrace{{}^I\tilde{\omega}^b}_{0}+{}^b\tilde{\omega}^B. \tag{10}$$

Thus, the expression for velocity becomes

$$\left\{\vec{v}\right\}_I=\left\{\dot{\vec{u}}\right\}_I+\underline{\underline{R}}\left\{{}^I\tilde{\omega}^B\right\}_{BB}\underline{\underline{R}}_0\left\{\vec{s}\right\}_I\ . \tag{11}$$

Equation (11) is different from the simplification given also in equation (16.52) of *Flexible Multibody Dynamics*, which is

$$\underline{v} = \dot{\underline{u}} + \underline{\underline{R}}\,\underline{\underline{R}}_0 \tilde{\omega}^* s^* . \tag{12}$$

An alternate derivation can yield a similar result to Eq. (12), but will not result in an expression matching Eq. (6). This is accomplished by replacing the product of the rotation matrices associated with the transformation from the reference configuration to the material configuration $\left(\underline{\underline{R}}\right)$ and the transformation from the inertial frame to the reference configuration $\left(\underline{\underline{R}}_0\right)$ can be replaced by a single matrix $\left({}^I\underline{\underline{R}}^B = \underline{\underline{R}}\,\underline{\underline{R}}_0\right)$. The time derivative of the position of any point on the deformable beam can then be written as

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \ {}^I\dot{\underline{\underline{R}}}^B \{\vec{s}\}_I . \tag{13}$$

again neglecting the contribution of the warp. Again inserting the identity allows the skew-symmetric angular velocity matrix to be isolated in the expression for velocity of any point as follows

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \ \underline{\underline{I}}\ {}^I\dot{\underline{\underline{R}}}^B \{\vec{s}\}_I \ , \tag{14}$$

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \ {}^I\underline{\underline{R}}^B \underbrace{\left({}^I\underline{\underline{R}}^B\right)^T {}^I\dot{\underline{\underline{R}}}^B}_{\left\{{}^I\tilde{\omega}^B\right\}_B} \{\vec{s}\}_I \ , \tag{15}$$

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \ {}^I\underline{\underline{R}}^B \left\{{}^I\tilde{\omega}^B\right\}_{BB} \{\vec{s}\}_I \ . \tag{16}$$

This results in an expression of the velocity of any point as a function of the time rate of change of the deformation vector and the angular velocity of the beam element as

$$\{\vec{v}\}_I = \left\{\dot{\vec{u}}\right\}_I + \underline{\underline{R}}\,\underline{\underline{R}}_0 \left\{{}^I\tilde{\omega}^B\right\}_{BB} \{\vec{s}\}_I . \tag{17}$$

This equation does agree with Eq. (12). Also if the angular velocity matrix in Eq. (11) replaced with $\underline{\underline{R}}_0\ {}^I\tilde{\omega}^B\ \underline{\underline{R}}_0^T$, then the two equations agree. However, there is no real justification for this, which seems to suggest that I am missing something, or perhaps my understanding of the problem description is incorrect.

Any advice would be greatly appreciated!