

Kinematics of Geometrically Exact Beams

The position of any point (\vec{x}) on a flexible beam may be described in terms of the location (α_1) of a rigid cross-section in the undeformed configuration ($\vec{x}_0(\alpha_1)$) and its translation under deformation ($\vec{u}(\alpha_1)$) expressed in the inertial (Newtonian) reference frame (N) as

$$\{\vec{x}\}_N = \{\vec{x}_0\}_N + \{\vec{u}\}_N \quad (1)$$

From equation (16.43) of *Flexible Multibody Dynamics*,

$$\vec{X}(\alpha_1, \alpha_2, \alpha_3) = \vec{X}_0 + w_1 \bar{B}_1 + (w_2 + \alpha_2) \bar{B}_2 + (w_3 + \alpha_3) \bar{B}_3 \quad (2)$$

the displacement of any point on the beam may be written as

$$\left\{ \vec{X}(\alpha_1, \alpha_2, \alpha_3) \right\}_I = \{\vec{x}_0\}_I + \{\vec{u}\}_I + (\underline{R}\underline{R}_0) \left(\{\vec{w}\}_I + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right), \quad (3)$$

which is given by equation (16.44) of *Flexible Multibody Dynamics*. In Eq. (3), the subscript B denotes vector quantities expressed in the material basis and the subscript I denotes quantities expressed in the inertial basis. To obtain the velocity of any point on the beam the time derivative of Eq (3) is taken, resulting in

$$\frac{d}{dt} \left\{ \vec{X}(\alpha_1, \alpha_2, \alpha_3) \right\}_I = \frac{d}{dt} \{\vec{x}_0\}_I + \frac{d}{dt} \{\vec{u}\}_I + \frac{d}{dt} \{\underline{R}\underline{R}_0\} \left(\{\vec{w}\}_I + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right). \quad (4)$$

After simplification the velocity of any point on the beam can be expressed as

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \dot{\underline{R}}\underline{R}_0 \left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right), \quad (5)$$

if the contribution of the warp vector is neglected. This equation can be further simplified as follows:

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \dot{\underline{R}}\underline{R}_0 \underbrace{\left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right)}_{\{\vec{s}\}_I}, \quad (6)$$

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \dot{\underline{R}}\underline{R}_0 \{\vec{s}\}_I. \quad (7)$$

Equation (7) agrees with equation (16.52) of *Flexible Multibody Dynamics*, which is

$$\underline{v} = \underline{\dot{u}} + \dot{\underline{R}}\underline{R}_0 \underline{s}^* \quad (8)$$

If Eq. (7) is then left multiplied by the identity and the identity is then replaced with $\underline{R}\underline{R}^T$, the skew-symmetric angular velocity matrix can be extracted from the above expression as follows:

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{I}\dot{\underline{R}}\underline{R}_0 \{\vec{s}\}_I \quad (9)$$

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underbrace{\underline{R}\underline{R}^T \dot{\underline{R}}\underline{R}_0}_{\{\tilde{\omega}\}_B} \{\vec{s}\}_I \quad (10)$$

This results in an expression of the velocity of any point as a function of the time rate of change of the deformation vector and the angular velocity of the beam cross section as

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R} \{\tilde{\omega}\}_B \underline{R}_0 \{\vec{s}\}_I, \quad (11)$$

where $\{\tilde{\omega}\}_B$ denotes the skew-symmetric angular velocity matrix expressed in the material basis. Equation (11) is different from the simplification given also in equation (16.52) of *Flexible Multibody Dynamics*, which is

$$\underline{v} = \dot{\underline{u}} + \underline{R} \underline{R}_0 \tilde{\omega}^* s^*. \quad (12)$$

An alternate derivation can yield a similar result to Eq. (12), but will not result in an expression matching Eq. (8). This is accomplished by replacing the product of the rotation matrices associated with the transformation from the reference configuration to the material configuration (\underline{R}) and the transformation from the inertial frame to the reference configuration (\underline{R}_0) can be replaced by a single matrix ($\underline{R}_{BI} = \underline{R} \underline{R}_0$). The time derivative of the position of any point on the deformable beam can then be written as

$$\frac{d}{dt} \left\{ \vec{X}(\alpha_1, \alpha_2, \alpha_3) \right\}_I = \frac{d}{dt} \{\vec{x}_0\}_I + \frac{d}{dt} \{\vec{u}\}_I + \frac{d}{dt} \{\underline{R}_{BI}\} \left(\{\vec{w}\}_I + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right), \quad (13)$$

again neglecting the warp.

With a similar simplification, the velocity of any point on the beam can be expressed as

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \dot{\underline{R}}_{BI} \left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right). \quad (14)$$

This equation can be further simplified as follows:

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R}_{BI} \underbrace{\left(\vec{0} + \alpha_2 \hat{i}_2 + \alpha_3 \hat{i}_3 \right)}_{\{\vec{s}\}_I}, \quad (15)$$

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R}_{BI} \{\vec{s}\}_I. \quad (16)$$

By left multiplying by the identity and replacing it with $\underline{R}_{BI} \underline{R}_{BI}^T$, the skew-symmetric angular velocity matrix can be extracted from the above expression as follows:

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{I} \dot{\underline{R}}_{BI} \{\vec{s}\}_I \quad (17)$$

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R}_{BI} \underbrace{\underline{R}_{BI}^T \dot{\underline{R}}_{BI}}_{\{\tilde{\omega}\}_B} \{\vec{s}\}_I \quad (18)$$

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R}_{BI} \{\tilde{\omega}\}_B \{\vec{s}\}_I \quad (19)$$

This results in an expression of the velocity of any point as a function of the time rate of change of the deformation vector and the angular velocity of the beam element as

$$\{\vec{v}\}_I = \left\{ \dot{\vec{u}} \right\}_I + \underline{R} \underline{R}_0 \{\tilde{\omega}\}_B \{\vec{s}\}_I. \quad (20)$$

This seems to suggest that I am missing something, or perhaps my understanding of the problem description is incorrect. For example, $\underline{\dot{R}}_{BI}$ may not necessarily be equal to $\dot{\underline{R}}_0$, which may account for the discrepancy between Eq. (11) and Eq. (12). However, this would mean that my understanding of the reference configuration (b_i), which is that it is not moving, would be incorrect.

Any advice would be greatly appreciated!