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| Math 6710 – Probability (3 pages) | Due: Thursday, October 25, 2012 |

# Homework 9

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## 1,2,3,4,5

**Problem 1.** Let $\{X_n\}$ be an adapted process with state space $(S, \mathcal{S})$. Fix a measurable $B \subset S$. In class we showed that the hitting time $\tau_B = \inf\{n \geq 0 \ : \ X_n \in B\}$ is a stopping time. Let $\tau_B^{(2)} = \inf\{n > \tau_B \ : \ X_n \in B\}$ be the *second* time that $X_n$ hits $B$. Show that $\tau_B^{(2)}$ is also a stopping time. The same argument shows $\tau_B^{(n)}$ is a stopping time for all $n$.

*Proof.* We want to show $\tau_B^{(2)}$ is a stopping time, i.e. we want to show $\{\tau_B^{(2)} \leq n\} \in F_n$ for all $n$. But, we have:

$$\{\tau_B^{(2)} \leq n\} = \cup_{i=0}^{n-1}\{\tau_B = i\} \cap (\{X_{i+1} \in B\} \cup \cdots \cup \{X_n \in B\})$$

And we've already shown $\{\tau_B = i\} \in F_n$ for all $i \leq n$ because $\tau_B$ is a stopping time, and $\{X_i \in B\} \in F_n$ for all $i \leq n$ as $\{X_n\}$ is an adapted process, so we're done. □

**Problem 2.** Suppose $L, M$ are stopping times with $L \leq M$ and $A \in \mathcal{F}_L$ is an event. Set $N = L1_A + M1_{A^c}$. Show that $N$ is a stopping time.

*Proof.* First we want a lemma. **If $L \leq M$ then $F_L \subseteq F_M$.**

*Proof.* We want to show if $A \cap \{L = n\} \in \mathcal{F}_n$ for all $n$ then $A \cap \{M = n\} \in \mathcal{F}_n$ for all $n$. We do this by writing $A \cap \{M = n\}$ as a finite union of elements of $\mathcal{F}_n$, namely:

$$A \cap \{M = n\} = \cup_{i=0}^{m} A \cap \{L = i\} \cap \{M - L = n - i\}$$

Where $A \cap \{L = i\} \in \mathcal{F}_i$ for all $i \leq n$ hence is in $\mathcal{F}_n$ as $\{\mathcal{F}_n\}$ is a filtration, and $\{M - L = n - i\}$ is the finite union of all events of the form $\{M = j\} \cap \{L = j - (n - i)\}$ which is a little awkward, but is certainly in $\mathcal{F}_n$ as $L, M$ are stopping times. □

We want to show $\{N = n\} \in \mathcal{F}_n$ for all $n$. If $N(\omega) = n$ there are two possibilities, either $\omega \in A$ and $L(\omega) = n$ or $\omega \in A^c$ and $M(\omega) = n$, this is represented as:

$$\{N = n\} = (A \cap \{L = n\}) \cup (A^c \cap \{M = n\})$$

We have $A \in F_L$ so $A \cap \{L = n\} \in F_n$ by definition and $A^c \in F_M$ by the lemma so $A^c \cap \{M = n\} \in F_n$ as well and so $\{N = n\} \in F_n$, as it is an intersection of elements of $F_n$. □

**Problem 3.** Show that if $(X', Y')$ has the same joint distribution as $(X, Y)$, then $E[Y'|X'] = f(X')$ a.s. for the same function $f$ gauranteed by the Doob-Dynkin lemma for $X, Y$.

*Proof.* The Doob-Dynkin lemma in the lecture notes was actually showing there exists an $f$ so that $f(X) = Y$ under the hypothesis that $Y \in \sigma(X)$. I am going to interpret the problem statement this way.

Certainly $f(X')$ is $\sigma(X')$ measurable as $f$ is measurable, so we need to show that $\int_A Y'd\mu = \int_A f(X')d\mu$ for all $A \in \sigma(X)$, in fact we have:

$$\int_A Y'd\mu = \int_{Y^{-1}(Y'(A))} Y d\mu = \int_{Y^{-1}(Y'(A))} f(X)d\mu = \int_A f(X')d\mu$$

where the first and last equalities follow because $(X, Y)$ and $(X', Y')$ have the same joint distribution so $P(Y \in Y'(A)) = P(Y' \in Y'(A))$ and similarly for $f(X), f(X')$. The middle equality holds by the Doob-Dynkin lemma which gave that $f(X) = Y$ a.s. □

**Problem 4.** Prove the conditional Markov inequality: if $X$ is a nonnegative random variable and $a > 0$, and $\mathcal{G}$ is a $\sigma$-field, then $P(X \geq a|\mathcal{G}) \leq \frac{1}{a}E[X|\mathcal{G}]$, almost surely.

*Proof.* Theorem 5.1.2 in Durrett establishes linearity of conditional expectation and shows that if $X \leq Y$ then $E(X|\mathcal{F}) \leq E(Y|\mathcal{F})$. With these facts, the proof of the Markov inequality is basically the same as for normal expectation.

Namely, we notice that $a1_{\{X\geq a\}} \leq X$ as $X$ is non-negative and for any $\omega$, $a1_{\{X\geq a\}} \leq a$ with equality only achieved when $X(\omega) \geq a$. Then we have $E(a1_{\{X\geq a\}}|\mathcal{G}) \leq E(X|\mathcal{G})$ by Durrett theorem 5.1.2 and so $E(1_{\{X\geq a\}}|\mathcal{G}) \leq \frac{1}{a}E(X|\mathcal{G})$ by linearity of conditional expectation. Finally, $E(1_{\{X\geq a\}}|\mathcal{G}) = P(\{X \geq a\}|\mathcal{G})$ by definition so we have $P(\{X \geq a\}|\mathcal{G}) \leq \frac{1}{a}E[X|\mathcal{G}]$, as desired. □

**Problem 5.** Suppose $X$ is a nonnegative random varible which is not necessarily integrable, and let $\mathcal{G}$ be a $\sigma$-field. Show that there exists a unique random variable $Y$ with $Y \in G$ and for every $A \in \mathcal{G}$, $E[Y; A] = E[X; A]$.

*Proof.* I think the trick is to go with the hint and hit it with the hammer known as the 'monotone convergence theorem'.

In particular let $X_m = X \wedge M$, and $Y_m = E(X_m|\mathcal{G})$. Then for all $m$ and any $A$,

$$\int_A X_m d\mu = \int_A Y_m d\mu$$

and $Y_m$ is $\mathcal{G}$ measurable. We know $X_m$ is monotone increasing to $X$ so $\lim \int_A X_m d\mu = \int_A X d\mu$ by the monotone convergence theorem. By theorem 5.1.2 in Durrett, $Y_m = E(X_m|\mathcal{G}) \geq E(X_{m-1}|\mathcal{G}) = Y_{m-1}$ so $Y_m$ is a monotone increasing sequence of random variables and the limit $Y = \lim Y_m$ makes sense. Again by monotone convergence we have

$$\int_A X d\mu = \lim \int_A Y_m d\mu = \int_A \lim Y_m d\mu = \int_A Y d\mu$$

So $\int AX d\mu = \int_A Y d\mu$ and $Y$ is $\mathcal{G}$-measurable, as desired.

We know have constructed our desired $Y$. For uniqueness, there are two cases, either there exists some $A$ so that $E[Y; A] = E[Y'; A] \neq \infty$ or not. If there is no such $A$ then $E[Y; A] = E[Y'; A] = \infty$ for all $A$ and $Y = Y' = \infty$ a.s. To see this, notice that if $|Y|, |Y'| \leq a$ for any $\infty > a > 0$ on any subset $A \in \mathcal{G}$ with $\mu(A) \neq 0$, then $\int_A Y d\mu \leq a < \infty$.

In the other case, we can use the same proof as in the lecture notes. □

### My notes for my presentation on October 25 in class.

I was asked to prove Durrett's Theorem 5.1.7 in class on October 25. I figured I would need to have my homework with me to hand in anyways, so I just put my notes here too.

**Theorem 1.** If $X \in \mathcal{F}$ and $E|Y|, E|XY| < \infty$, then:

$$E(XY|\mathcal{F}) = XE(Y|F)$$

*Proof.* By definition $E(Y|\mathcal{F}) \in \mathcal{F}$ and by assumption $X \in \mathcal{F}$ so to check $E(XY|F) = XE(Y|F)$, we just need to check that for all $A \in \mathcal{F}$, $E(XY; A) = E(X(E(Y|F)); A)$. I think it's going to get a little ugly writing a bunch of nested $E$'s so I am going to switch over to writing integrals where appropriate. Given $A \in \mathcal{F}$ we should the desired equality using the usual four-step procedure of indicator functions, simple functions, non-negative functions and then any integrable $X$. Showing the result for indicator functions is really the meat of it, then it's the usual programme. Say $X = 1_B$ an indicator function. We have $E(X(E(Y|\mathcal{F})); A) = \int_A 1_B E(Y|\mathcal{F})dP$ and so:

$$\int_A 1_B E(Y|\mathcal{F})dP = \int_{A\cap B} E(Y|\mathcal{F})dP = \int_{A\cap B} Y dP = \int_A 1_B Y dP = E(XY; A)$$

The first equality holds because $X$ is an indicator function and the second holds by the definition of $E(Y|\mathcal{F})$ and the fact that $A \cap B \in \mathcal{F}$ as $A, B \in \mathcal{F}$.

Conditional expectation is linear, so the result for simple $X$ is the same. Next we may take simple $X_n$ converging upwards to $X$ and use the monotone convergence theorem to get the result for non-negative $X$. Finally, split $X$ into its positive and negative parts and we get the result for arbitrary $X$. $\square$