Math 6710 – Probability (2 pages)	Due: Thursday, September 27, 2012
Homework 5	
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Problem 1. The (open) n-**dimensional simplex** is the set $\Delta^n \subset [0,1]^n$ defined by

$$\Delta^n = \{(x_1, x_2, \dots, x_n) \in [0,1]^n \ : \ x_1, \dots, x_n\}$$

For example, Δ^2 is a triangle, Δ^3 is a tetrahedron, etc. Let m_n denote n-dimensional Lebesgue measure on $[0,1]^n$.

(a) Show that $m_n(\Delta^n) = \frac{1}{n!}$. Induction may be helpful.

(b) Let μ_n be the probability measure on $[0,1]^n$ that spreads a unit of mass uniformly on Δ^n, i.e. $\mu_n(A) = n! m_n(A \cap \Delta^n)$. Show that the sequence $\{\mu_n\}$ is consistent in the sense of the Kolmogorov extension theorem.

(c) Let μ be the limiting measure on $[0,1]^{\mathbb{N}}$ produced by applying the Kolmogorov extension theorem to $\{\mu_n\}$. If $\Delta \subset [0,1]^{\mathbb{N}}$ is the set of all strictly increasing sequence which converge to 1, show that $\mu(\Delta) = 1$.

(d) On the other hand if m is Lebesgue measure on $[0,1]^{\mathbb{N}}$ (i.e. the limiting measure of $\{m_n\}$), show that $m(\Delta) = 0$.

(e) Suppose $U_1, U_2, \dots$ is an iid sequence of uniform $(0,1)$ random variables on some probability space. Use the U_i to directly construct a sequence of random variables $X_1, X_2, \dots$, whose joint distribution is μ.

Problem 2. For each of the following sequence of probability measure on $\mathbb{R}$, determine whether the sequence converges weakly, and if so find its weak limit. m denotes Lebesgue measure on δ_x is the Dirac delta measure at x (i.e. $\delta_x(A) = 1$ if $x \in A$ and 0 if $x \notin A$).

Recall the definition that $\mu_n \to \mu$ weakly iff for every boudned continuous f, we have $\int_{\mathbb{R}} f d\mu_n \to \int_R f d\mu$.

(a) μ_n is uniform measure on $[0, 1/n]$.

(b) $\mu_n = \delta_n$.

(c) $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{1/i}$.

(d) (Bonus) μ_n is uniform measure on $[0,n]$ i.e. $\mu_n(A) = \frac{1}{n} m(A \cap [0,n])$. This may be a bit tricky to prove. Think about it, but don't waste the whole week.

Solution. (a) Converges weakly. The limiting measure is $\mu = \delta_0$. For any continuous and bounded f, $\int f d\mu = f(0)$, and for $\int f d\mu_n$ we have:

$$\inf_{[0,1/n]} f(x) \leq \int f d\mu_n \leq \sup_{[0,1/n]} f(x)$$

But by continuity, $\inf_{[0,1/n]} f(x) \to f(0)$, $\sup_{[0,1/n]} f(x) \to f(0)$ as $n \to \infty$, so for all $\epsilon > 0$ there is an n so that $f(0) - \epsilon \leq \int f d\mu_n \leq f(0) + \epsilon$ and $\int f d\mu_n \to \int f d\mu$, as desired.

(b) Does not converge weakly. Let f be a periodic, bounded continuous function such that $f(n) = 1$ if n if even and $f(n) = -1$ if n is odd. Then $\int f d\mu_n = f(n) = \pm 1$ and the sequence $\int f d\mu_n$ alternates between 1 and -1 so cannot converge.

(c) Yes, does converge weakly. In particular, let's show $\int f d\mu_n \to f(0)$ and $n \to \infty$. Given $\epsilon > 0$ choose i so that $|f(x) - f(0)| < \epsilon$ for all x with $|x - 0| < 1/i$. Then for $n > i$ we have:

$$\begin{aligned}
\int f d\mu_n &= \\
&= \textstyle\sum_{j=1}^n \frac{1}{n} f(1/j) \\
&= \textstyle\sum_{j>i}^n \frac{1}{n} f(1/j) + \sum_{j \leq i} \frac{1}{n} f(1/i) \\
&\leq \textstyle\sum_{j>i}^n \frac{1}{n} f(1/j) + C\frac{i}{n} \\
&\leq \tfrac{n-i}{n}(f(0) + \epsilon) + \delta \\
&= (1-\delta) f(0) + \delta \\
&\to f(0)
\end{aligned}$$

where C demonstrates that f is bounded and $\delta = Ci/n \to 0$ as $n \to \infty$.

Problem 3. (a) Suppose that $X_1, \ldots, X$ are random variables and $X_n \to X$ in probability. Show that $X_n \to X$ weakly, i.e. if μ_n, μ are the distribution of X_n, X, then $\mu_n \to \mu$ weakly.

(b) Suppose $X_1, X_2, \ldots$ are random variables, $X_n \sim \mu_n$, and $\mu_n \to \delta_c$ weakly. Show that $X_n \to c$ in probability.

Solution. (a) Letting $F_X(a) = P(X \leq a)$ it suffices to show that $F_{X_n}(a) \to F_X(a)$ for all a where F_X is continuous at a. Notice that $P(X_n \leq a) \leq P(X \leq a+\epsilon) + P(|X_n - X| < \epsilon)$. Noticing that the union of the events $\{X \leq a+\epsilon\}, \{|X_n - X| < \epsilon)\}$ include the event that $\{X_a \leq a\}$. Similarly, we may write: $P(X_n \leq a) + P(|X_n - X| \geq \epsilon) \geq P(X \leq x - \epsilon)$ or $P(X_n \leq a) \geq P(X \leq x - \epsilon) + P(|X_n - X|)$ by applying the same logic as before. Putting these together we have:

$$P(X \leq a - \epsilon) + P(|X - X_n| > \epsilon) \leq P(X_n \leq a) \leq P(X \leq +\epsilon) + P(|X - X_n| > \epsilon)$$

But $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$ as $X_n \to X$ in probability, so we have $P(X_n \leq a) \to P(X \leq a)$ because $P(X \leq a)$ is assumed continuous at a.

(b) By part 3 of the Portmanteau theorem we have $\limsup \mu_n(E) \leq \mu(E)$ for any closed E. Let $E = B_\epsilon(c)^c$ and then notice that $\delta_c(B_\epsilon(c)^c) = 0$.

We want to show that $P(|X_n - c| \leq \epsilon) \to 1$ as $n \to \infty$. But $P(|X_n - c| < \epsilon) = P(X_n \in B_\epsilon(c))$. Notice that $B_\epsilon(c)^c$ is closed.

Then $\lim P(X_n \in B_\epsilon(c)^c) \leq \limsup P(X_n \in B_\epsilon(c)^c) \leq P(c \in B_\epsilon(c)^c) = 0$ so $\lim P(X_n \in B_\epsilon(c)) = 1$, as desired.