Math 6710 – Probability (4 pages)	Due: Thursday, August 30, 2012
Homework 1	
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18.1 : 1, 2, 3, 4, 5

Problem 1. Let X be a random variable with $X \geq 0$ almost surely. Prove: if $\mathbb{E}[X] = 0$ then $X = 0$ almost surely.

Proof. Say $X \neq 0$, but $\mathbb{E}[X] = 0$ and let X_n be a sequence of simple random variables converging monotonically pointwise to X. Then by the monotone convergence theorem we have $\mathbb{E}[X] = \lim \mathbb{E}[X_n]$. Let $A_{X_n} = \cup_{\{A_i|a_i>0\}} A_i$ where $\{a_i\}, \{A_i\}$ are the corresponding finite sets defining X_n. We have $X \neq 0$ so there must be some $A \subseteq \omega$, N, for which $\mathbb{P}[A] > 0$ and $X_j(\omega) > 0$ a.s. for all $j \geq N$, $\omega \in A$, otherwise $X = 0$, a.s. for all n. Let $a_M = \min\{a_i|a_i > 0\}$, then $\mathbb{E}[X_N] \geq a_M \cdot \mathbb{P}[A]$. But then for $j \geq N$, and for all $x \in A_{X_N}$, $X_j(x) \geq a_M$ because $\{X_n\}$ is a monotone sequence, thus $\mathbb{E}[X_j] \geq a_M \cdot \mathbb{P}[A_{X_N}] > 0$ for all $j \geq N$ and $\mathbb{E}[X] = \lim \mathbb{E}[X_n] \neq 0$, a contradiction. □

Problem 2. Let $X \geq 0$ and let $f : [0, \infty) \to [0, \infty)$ be continuous differentiable and monotone increasing.

(a) Show that $\mathbb{E}[f(X)] = \int_0^\infty f'(t)P(X \geq t)dt$. Hint: Write $P(X \geq t)$ as $\mathbb{E}[1_{\{X\geq t\}}]$. Now you have two integrals, one over $[0, \infty)$ and the other over Ω. Use Fubini/Tonelli's theorem.

(b) In particular, $\mathbb{E}[X] = \int_0^\infty P(X \geq t)dt$.

Solution.
(a) First let's see that $\mathbb{P}[X \geq t] = \mathbb{E}[1_{X\geq t}]$. By definition $\mathbb{P}[X \geq t]$ is the probability of the event that $X \geq t$, i.e. $\int 1_{\{X\geq t\}} d\mu = \mathbb{E}[1_{\{X\geq t\}}]$. This justifies writing:

$$\int_0^\infty f'(t)\mathbb{P}[X \geq t]dt = \int_0^\infty f'(t)\mathbb{E}[1_{\{X\geq t\}}] = \int_0^\infty f'(t) \int_\Omega 1_{\{X\geq t\}} d\mu dt = \int_0^\infty \int_\Omega f'(t)1_{\{X\geq t\}} d\mu dt$$

We know that $1_{X\geq t} \geq 0$ and because we are assuming f continuously differentiable and monotone, $f'(t) \geq 0$ and we may apply Tonelli's theorem and interchange the order of integration to get:

$$\int_0^\infty \int_\Omega f'(t)1_{\{X\geq t\}} d\mu dt = \int_\Omega \int_0^t f'(t)1_{\{X\geq t\}} dt d\mu$$

Next we notice that $\int_0^\infty f'(t)1_{\{X\geq t\}}dt = \int_0^{X(\omega)} f'(t)dt$ as for $t > X(\omega)$, $1_{\{X\geq t\}}(\omega) = 0$. This allows us to apply the fundamental theorem of calculus and we have:

$$\int_\Omega \int_0^t f'(t)1_{\{X\geq t\}} dt d\mu = \int_\Omega f(X(\omega))d\mu = \int_\Omega f(X)d\mu = \mathbb{E}[f(X)]$$

thus we have shown $\mathbb{E}[f(X)] = \int_0^\infty f'(t)\mathbb{P}[X \geq t]dt$, as desired.
(b)

This is immediate from part (a). Namely, the function $f(t) = t$ is continuous differentiable and monotone increasing as a map $[0, \infty) \to [0, \infty)$, thus $\mathbb{E}[f(X)] = \int_0^\infty f'(t)\mathbb{P}[X \geq t]dt = \int_0^\infty \mathbb{P}[X \geq t]dt$, as desired.

Problem 3. Let X be a random variable with cumulative distribution function F (i.e. $F(x) := P(X \leq x)$). Show that if F is continuous then $Y = F(X)$ has a uniform distribution on $[0, 1]$, i.e. $P(Y \leq t) = t$ for $t \in [0, 1]$. Give an example to show that this need not be true if F is not continuous. This is a sort of converse to Theorem 1.2.2.

Proof.
The general result.

We want to calculate $P(Y \leq t) = P(F(X) \leq t)$. We know that F is continuous and $t \in [0, 1]$, so assuming F assumes the values $0, 1$ we can apply the intermediate value theorem to take an x so that $F(x) = t$. Further, we know the set of all such values y where $F(y) = t$ is an interval because F is monotone increasing (this follows from the fact that F is defined by a probability and as x increases, the event $\{X \leq x\}$ is a superset of the event $\{X \leq x'\}$ for $x' \leq x$). We know it is a closed interval because if it was an open interval (a, b) then we would have $F(b-\epsilon) = x$ and $F(b) = x'$ and F would not be continuous at b. This justifies letting b be the maximal number so that $F(b) = t$.

Now $P(F(X) \leq t) = P(X \leq b) = F(b) = t$, and Y has a uniform distribution on $[0, 1]$ as desired.

A counterexample when F is not continuous.

Let X be the random variable corresponding to the flip of a fair coin. I.e. $X = 1$ with probability $1/2$ and $X = 0$ with probability $1/2$. Then $F(a) = P(X \leq a) = 1/2$ for $a < 1$ and $F(1) = P(X \leq 1) = 1$, so F is not continuous at 1. Now looking at $P(Y \leq 3/4) = P(F(X) \leq 3/4) = P(X \leq 0) = 1/2 \neq 3/4$ and we have our desired counterexample. □

Problem 4. Let (A_i) be a sequence of events. Define:

$$\limsup_{n\to\infty} A_n := \cap_{m=1}^\infty \cup_{n=m}^\infty A_n$$

$\limsup A_n$ is also sometimes denoted $\{A_n \text{ i.o. }\}$ (for "infinitely often"); it is the event that "infinitely many of the events A_n occur".

(a) Show that $1_{\limsup A_n} = \limsup 1_{A_n}$.

(b) Show that $P(\limsup A_n) \geq \limsup P(A_n)$.

(c) Give an example to show that equality need not hold in the previous part. Indeed, try to find an example where $P(\limsup A_n) = 1$ but $\limsup P(A_n) = 0$.

(d) Define $\liminf A_n := \cup_{m=1}^\infty \cap_{n=m}^\infty A_n$. This is also denoted $\{A_n a.a.\}$ and is the event that "all but finitely many of the events A_n occur". By taking complements (or directly) show that $1_{\liminf A_n} = \liminf 1_{A_n}$, $P(\liminf A_n) \leq \liminf P(A_n)$a nd that equality need not hold.

Solution.
(a) $1_{\limsup A_n} = \limsup 1_{A_n}$

Proof. Let's say $1_{\limsup A_n}(\omega) = 1$. Then $\omega \in \limsup A_n$. But this means that for every A_n such that $\omega \in A_n$, $1_{A_n}(\omega) = 1$ and so $1_{A_n}(\omega) = 1$ infinitely often, i.e. $\limsup A_n(\omega) = 1$. On the other hand, say $1_{\limsup A_n}(\omega) = 0$, then $\omega \notin \limsup A_n$ and so there must be some last N so that $\omega \notin A_j$ for all $j \geq N$. But then for all $j \geq N$, $1_{A_j}(\omega) = 0$ and so $\limsup 1_{A_n}(\omega) = 0$. So $1_{\limsup A_n}, \limsup 1_{A_n}$ agree on all values, thus we have $1_{\limsup A_n} = \limsup 1_{A_n}$. □

(b)

Proof. First notice that $\limsup P(A_n)$ is a $\limsup$ of a sequence of real numbers and so we may choose a subsequence B_n of A_n so that $\lim P(B_n) = \limsup P(A_n)$. Let $a(B_n)$ denote the index m of (A_n) so that $B_n = A_m$. Notice $a(B_n) \geq n$.

Now, if we write out:

$$\begin{aligned} P(\limsup A_n) &= \\ &= \textstyle\int 1_{\limsup A_n} d\mu \end{aligned}$$

Which in particular makes us notice the monotone decreasing sequence of functions $1_{\cup_{m=n}^{\infty} A_n} \to 1_{\limsup A_n}$. So we may apply the monotone convergence theorem (well at least a 'standard' corollary of it) and write:

$$\begin{aligned} P(\limsup A_n) &= \\ &= \textstyle\int 1_{\limsup A_n} d\mu \\ &= \lim_{n\to\infty} \textstyle\int 1_{\cup_{m=n}^{\infty} A_n} d\mu \\ &= \lim_{n\to\infty} P(\cup_{m=n}^{\infty} A_n) \end{aligned}$$

Letting $C_n = P(\cup_{m=n}^{\infty} A_n)$ we have $C_n \geq P(B_n)$ for all n as $a(B_n) \geq n$ and so $B_n \subseteq \cup_{m=n}^{\infty} A_n$. Thus we must have

$$\lim_{n\to\infty} P(\cup_{m=n}^{\infty} A_n) \geq \lim P(B_n)$$

So $P(\limsup A_n) \geq \limsup P(A_n)$, as desired. □

(c)

Proof. Let our probability space be the unit interval with Lebesgue measure. Let $A_{(i,j)}$ be the j-th partition of $[0,1]$ into i equal sized pieces, let A_n denote the sequence $A_{(1,1)}, A_{(2,1)}, A_{(2,2)}, A_{(3,1)}, \ldots$. Then $\limsup P(A_n) = 0$ as the length of the interval A_n goes to 0 as $n \to \infty$, but for any finite m, $\cup_{n=m}^{\infty} A_n$ is the whole unit interval so $\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n = \cap_{m=1}^{\infty}[0,1] = [0,1]$ and so $P(\limsup A_n) = 1$. □

(d)

Proof. For the first two parts, we redo the proofs directly and for the last part we appeal to 'taking complements' to get our counterexample.

Let's say $1_{\liminf A_n}(\omega) = 1$. Then $\omega \in \liminf A_n$. But this means there exists an N so that for all $j \geq N$, $\omega \in N$, so $1_{A_j}(\omega) = 1$ for all $j \geq N$ and so $\liminf 1_{A_n}(\omega) = 1$. On the other hand, say $1_{\limsup A_n}(\omega) = 0$, then $\omega \notin \limsup A_n$ and so for all N there exists a $j > N$ such that $\omega \notin A_j$. But then for all N, $1_{A_j}(\omega) = 0$ and so $\limsup 1_{A_n}(\omega) = 0$ thus $1_{\limsup A_n}, \limsup 1_{A_n}$ agree on all values and we have $1_{\limsup A_n} = \limsup 1_{A_n}$.

For (b) I think it is unnecessary to rewrite the whole proof, I will just point out the places where the two proofs are different. We still choose a subsequence (B_n) so that $\lim P(B_n) = \liminf P(A_n)$, and when we rewrite $P(\liminf A_n)$ as an integral we get a monotone increasing sequence of functions $1_{\cap_{n=m}^{\infty} A_n}$ increasing to $1_{\liminf A_n}$ and so we can use the monotone convergence theorem directly to interchange the integral and the limit. Now though, we find that $\cap_{n=m}^{\infty} A_n \subseteq B_n$ for all n because $a(B_n) \geq n$ and so $P(B_n) \geq P(\cap_{n=m}^{\infty} A_n)$ for all n, thus we get the inequality from part b with the direction reversed, i.e. $\liminf P(A_n) \geq P(\liminf A_n)$.

Now for our counterexample, let our probability space be the unit interval with Lebesgue measure as before. Let $A_{(i,j)}$ be the complement of the j-th partition of $[0,1]$ into i equal sized pieces, let A_n denote the sequence $A_{(1,1)}, A_{(2,1)}, A_{(2,2)}, A_{(3,1)}, \ldots$. Then $\liminf P(A_n) = 1$ as the length of the interval A_n goes to 1 as $n \to \infty$ (it is the complement of an interval whose length goes to 0), but for any finite m, $\cap_{n=m}^{\infty} A_n = 0$ because we can always pick a $j \geq m$ so that way the next k A_l are the complements of the partition of $[0,1]$ into k pieces. Each of these A_l are missing a different piece of the unit interval so their intersection is $\varnothing$ and so $P(\liminf A_n) = 0$. $\square$

Problem 5. Let $1 \leq p \leq \infty$. Suppose X_n is a sequence of random variables, $X_n \to X$ a.s., and Y is another random variable such that $|X_n| \leq Y$ and $\mathbb{E}[Y^p] < \infty$. Show that $\mathbb{E}[|X|^p] \leq \infty$ and that $\mathbb{E}[|X_n - X|^p] \to 0$. This is a strengthening of the dominated convergence theorem.

Proof. Notice that $X \leq Y$ almost everywhere. This follows from the fact that $X_n \to X$ and $X_n \leq Y$. Thus we have $X \leq Y$ so $X^p \leq Y^p$ and so $\mathbb{E}[X^p] \leq \mathbb{E}[Y^p]$, say by theorem 1.4.7 in Durrett.

Now because $X_n \to X$ a.e., $Z_n = |X_n - X| \to 0$ a.e., and similarly $Z_n^p \to 0$ a.e. To be able to apply the dominated convergence theorem, it suffices to show that $|Z_n|^p \leq h$ for h integrable, but we already know that

$$Z_n^p = |X_n - X|^p \leq |X_n|^p + |X|^p \leq 2^p Y^p$$

So we have that $|Z_n^p| \leq 2^p Y^p$ and by assumption $\mathbb{E}[Y^p] < \infty$ so $\mathbb{E}[2^p Y^p] = 2^p \mathbb{E}[Y^p] < \infty$ and the dominated convergence theorem applies. Thus we have $\mathbb{E}[Z_n^p] = \mathbb{E}[|X_n - X|^p] \to \mathbb{E}[0] = 0$, as desired. $\square$