# 1 Prelude - A Simple Proof Goes Awry

The Jordan Curve Theorem is a geometrically intuitive and simple statement, so simple that a necessity to prove it is easily overlooked. Below we will show how easily one would assume the Jordan Curve Theorem without recognizing it as a Theorem needing proof. Subsequently, we will show how significantly more complicated the proof of the Jordan Curve Theorem itself is. But first a couple of definitions and the theorem:

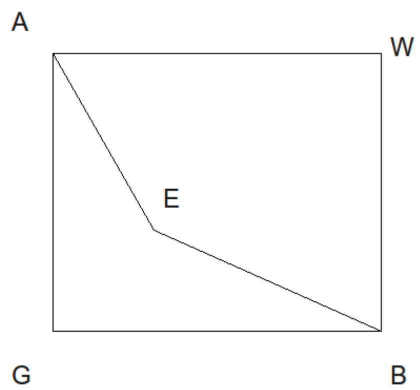
**Definition.** An **arc** is a space homeomorphic to the unit interval [0,1]. A **simple closed curve** is a space homeomorphic to the unit circle $S^1$. [M]

**The Jordan Curve Theorem.** Let C be a simple closed curve in $\mathbb{R}^2$. Then C separates $\mathbb{R}^2$into precisely two components $W_1$ and $W_2$, and each of these sets $W_1$ and $W_2$ has C as its boundary.

As a consequence of the Jordan Curve Theorem it is also important to note that any path connecting a point of $W_1$ to a point of $W_2$will intersect the simple closed curve C at at least one point. [WIKIJ]

We will now proceed with a seemingly innocent proof of the The Utilities Problem, see [B].

**The Three Utilities Problem.** Consider 3 Houses: A, B, C and 3 Utilities: **W**ater, **E**lectricity and **G**as, then there is no way to put lines to all 3 houses from each utility along the ground without at least 2 lines intersecting.




*Proof.* We start by noticing that we will have to give houses A and B Water and Gas. Then we consider where to place the 3rd utility and the 3rd house. The 3rd utility must be placed either inside or outside the simple closed curve made by AWBG. Without loss of generality, place the 3rd utility inside the square.[1] Consider the closed curves formed. There are now 3 possible places to add the 3rd house: inside AEBG or AEBW, or outside AWBG. Placing the 3rd house inside AEBG or AEBW leaves Water or Gas in the outside region of either AEBG or AEBW, respectively. Thus a path between the 3rd house and one utility would be a path connecting the inside region of the simple closed curve to the outside, forcing them to intersect. Placing the 3rd house outside the square leaves E inaccessible by the same logic. ■

In the course of this short and informal proof, the Jordan Curve Theorem, the proof for which is much more complicated than this one, was assumed many times. Not surprisingly, a theorem both as subtle and as seemingly obvious as the Jordan Curve Theorem has a rich history.

## 2 Background

The Jordan Curve Theorem and the Schoenflies Theorem are two well-recognized theorems relating to separations of the plane. We will prove the Jordan Curve Theorem following an elegant proof by Maehara in 1984, which relies heavily on Brouwer's Fixed Point Theorem, see [MR]. After which we will transition to a discussion of the Schoenflies Theorem and discuss various generalizations which have been suggested taking into account for the counterexample provided by the Alexander horned Sphere. Before going further, we will also explain a little about how these proofs are related and the history surrounding them.

The Jordan Curve Theorem is famous for being proved incorrectly by many mathematicians, including Camille Jordan, the first mathematician to submit a proof and the one for whom the theorem is named! The first correct proof was presented by Veblen in 1905, 18 years after Jordan presented his proof.

[1](Recall the homeomorphism between $S^2$with a point removed and $\mathbb{R}^2$, and consider what the difference between "inside" or "outside" the square would mean on the sphere.)

While the statement of the Jordan Curve Theorem is incredibly simple and geometrically intuitive the proof is quite involved.

The Schoenflies Theorem, which states that the 2 regions formed in the Jordan Curve Theorem are homeomorphic to the inside and outside regions formed by the unit circle, also called the Jordan-Schoenflies Theorem, is particularly interesting because, unlike the Jordan Curve Theorem, while true for $\mathbb{R}^2$ it does not extend to $\mathbb{R}^n$. This is shown by the Alexander horned Sphere, a famous counterexample. There are however generalizations of the Schoenflies Theorem, which add additional conditions to the theorem to exclude *wild* spaces such as the Alexander horned sphere. [WIKIJ] Proofs of the Schoenflies Theorem and its generalizations tend to be somewhat less accessible than the proof of the Jordan Curve Theorem, but they are mentioned in the references and the theorems are discussed in section 4.

# 3 The Jordan Curve Theorem

At the outset we recall the Brouwer Fixed Point Theorem:

**Theorem.** *Every continuous map from a disc to a disc has a fixed point.*

From here on, C will refer to a simple closed curve, and we make the following observations given C:

1. $\mathbb{R}^2$- C has exactly one unbounded component
2. Each component of $\mathbb{R}^2$- C is path-connected and open.

1 follows from the boundedness of C. See [CV]. 2 from the local path-connectedness of $\mathbb{R}^2$ and the closedness of C.

**Lemma 1.** *If* $\mathbb{R}^2$- *C is not connected, then each component has C as its boundary.*

*Proof.* By assumption, $\mathbb{R}^2$- C has at least 2 components. Consider one of those components, U and any other component V. Then because U and V are disjoint and open, the boundary of U, denoted by, $\delta$(U) is disjoint from V. Being disjoint from all components V and, by the openness of U, disjoint from U, $\delta$(U) must be a subset of C. Assume, for contradiction, that $\delta$(U) is a strict subset of C. By this observation, there is an arc A such that $\delta$(U) $\subset$A $\subset$C.

By 1, $\mathbb{R}^2$- C must have at least 1 bounded component. Let x be a point in one of the bounded components, in U if U is a bounded component. Let D be a disc with center at x, such that C $\subset$ int(D) . Since A is homeomorphic to [0, 1], the identity map A $\longrightarrow$A has a continuous extension to s: D $\longrightarrow$S by the Tietze Extension Theorem (See [M]). Let p: D - {x} $\longrightarrow$ S be a deformation retract of D - {x} to S, and t: S $\longrightarrow$S by the antipodal map. Then we define y: D - {x} as follows depending on whether U is bounded or not:

y(z) := $\begin{cases} s(z) & for \quad z \in U \cup \delta(U) \\ z & for \quad z \in U^c \end{cases}$ , or y (z) := $\begin{cases} z & for \quad z \in U \cup \delta(U) \\ s(z) & for \quad z \in U^c \end{cases}$

respectively. Finally the composition t $\circ$ p $\circ$ y : D $\longrightarrow$S $\subset$D has no fixed point. A contradiction of the Brouwer's fixed point theorem. ■

Let R(a, b; c, d) := $\{(x, y) \in \mathbb{R}^2 |$ a $\leqslant x \leqslant$b, c $\leqslant$x $\leqslant$d$\}$. a $\lneq$b, c $\lneq$d.

**Lemma 2.** *Let $h(t) = (h_1(t)\ ,\ h_2(t))$ and $v(t) = (v_1(t),\ v_2(t))$ ( -1 $\leqslant t \leqslant$1) be paths in R(a, b; c, d) satisfying*

*$h_1(-1) = a$, $h_1(1)= b$, $v_2(-1) = c$, $v_2(1) = d$.*

*Then for some s, x $\in$[-1, 1], h(s) = v(t).*

*Proof.* In short, both h and v are paths thus continuous. They are restricted to R(a, b; c, d) $\cong$D$^2$. We define a composition of these functions and show that it has no fixed point, which forces a contradiction of the Brouwer's fixed point theorem.

We assume for contradiction that h and v never meet.

Consider N and F:

N(s,t) := Max$\{|h_1(s) - v_1(s)|, |h_2(s) - v_2(s)|\}$, F(s, t) = $(\frac{v_1(t)-h_1(s)}{N(s,t)}, \frac{v_2(t)-h_2(s)}{N(s,t)})$.

Note: N(s,t) never equals 0, and the image of F is the boundary of R(-1 , 1; -1, 1) which we will consider below.

Assume F(y,g) = (y, g) then either |y| = 1 or |g|= 1. Without loss of generality, suppose y= -1. Then F(-1, g) gives $h_1(-1)$= -1. So F(-1,g)= (q, g). Where q is non-negative. Thus F(-1, g) $\neq$(q, g). The same occurs when y=1 and |g|=1. Hence F(s,t) has no fixed point, a contradiction. ■

The proof of the Jordan Curve Theorem now comes down to 3 steps: establishing notation and defining a point $z_0$ in $\mathbb{R}^2$- C, showing the component containing $z_0$to be bounded and showing it is the only bounded component.

Consider rectangle R(d , e; f, g) and a simple closed curve C. Let a, b$\in$C such that ||a - b || $\geqq$|| x - y || for all x, y $\in$C. Then let d, e, f and g be such that R contains C and intersects C exactly at a and b on the opposite left-right sides of R. Let n be the middle point on top of R and m the middle point below.

Then the line segment $n\bar{m}$ intersects C by Lemma 2. Let t be the point in $n\bar{m} \cap$ C with the maximum y-value, and l the point in $n\bar{m} \cap$ C with the minimum y-value. Then a and b divide C into 2 arcs, call the one containing t, $C_t$ and the other one $C_w$. (Note: $C_w$might not contain l.) $l\bar{m}$ must meet $C_w$ otherwise $n\bar{t}$ * $t\tilde{l}$ * $l\bar{m}$, where * is the concatenation of paths and $t\tilde{l}$ is the arc created in $C_t$ with end points t and l, could not meet $C_w$ by Lemma 2. Let b and c be the points with the maximum and minimum y values in $C_w \cap$ $l\bar{m}$. Let $z_0$ be the middle point of$\bar{bl}$.

Consider U, the component containing $z_0$. Since U is path connected, there exists a path $\gamma$ in U from $z_0$ to the point r outside of R. Consider just the path between $z_0$ and the point $z_1$ where $\gamma$ meets the boundary of R and call it $\gamma_r$. If $z_1$ is on the boundary of R in the lower of the two arcs formed in C by a and b, then we can find a path $m\tilde{z}_1$from m to $z_1$ containing neither a nor b. But then the path $n\bar{t}$ * $t\tilde{l}$ * $t\bar{z}_0$* $\gamma_r$ * $m\tilde{z}_1$does not meet $C_t$, contradicting Lemma 2. The same occurs when $z_1$ is in the upper of the two arcs formed in C by a and b. Thus U must be bounded.

Finally, assume there exists another bounded component, W. Consider the path $\rho = n\bar{t}$ * $t\tilde{l}$ * $\bar{lb}$* $\tilde{bc}$ * $c\bar{m}$. $\rho$ has no point in W, and a and b are not in $\rho$. Consider open neighborhoods of a and b, $V_a$ and $V_b$, disjoint from $\rho$. By Lemma 1, a and b are in the closure of W, thus there exists a', b' in both W and

$V_a$and $V_b$ respectively. Then the path $\bar{aa'} * \tilde{a'b'} * \bar{b'b}$ fails to meet $\rho$ contradicting Lemma 2 and completing the proof.

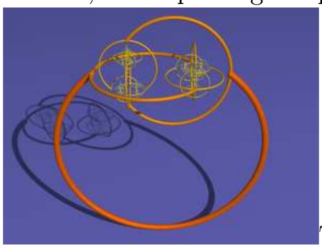
# 4 Schoenflies Theorem and the Alexander Horn Sphere

We will first state our theorems.

**The Schoenflies Theorem.** If C $\subset \mathbb{R}^2$ is a closed simple curve, then there is a homeomorphism f : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$such that f(C) = $S^1$.

**The generalized Schoenflies Theorem.** If $S^{n-1}$ is embedded into $\mathbb{R}^n$as S in a locally flat way, then there is a homeomorphism f : $\mathbb{R}^n \rightarrow \mathbb{R}^n$ such that f(S) = $S^{n-1}$.

**Definition.** The Alexander horned sphere is the space obtained by cutting a radial slice out of the torus, attaching 2 punctured tori, one on each side, interlinked, and repeating the process for the tori added.


The Alexander Horned Sphere [WIKIA]

Discovered in 1924 by J. W. Alexander, the Alexander horned sphere is similar to Antoine's horned sphere, in that they are both an imbedding of the Cantor set into the 3-Sphere. The problem we run into with generalizing the Schoenflies theorem is that $\mathbb{R}^3$- A, where A is an Alexander horned sphere, has a different fundamental group than $\mathbb{R}^3$ - $S^2$ making the homeomorphism mentioned in the Schoenflies Theorem impossible. In the generalized version of the theorem, it is the added condition of "local flatness" that allows the theorem to be extend to higher dimensions, excluding pathological examples like the Alexander horned sphere.

# References


[WIKIA] "Alexander horned Sphere." Wikipedia. 14 Nov 2009. Web. 27 Nov 2009.

[BR] Brown, Morton (1960), "A proof of the generalized Schoenflies theorem", Bull. Amer. Math. Soc., vol. 66, pp. 74–76. (Available Online here: http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.bams/11835

[WIKIJ] "Jordan Curve Theorem." Wikipedia. 26 Oct 2009. Web. 27 Nov 2009.

[CV] Klee, Victor (1964), "Connectedness in Topological Linear Spaces", Israel Journal of Mathematics. vol. 2, pp. 127-131.

[MR] Maehara, Ryuji (1984), "The Jordan Curve Theorem Via the Brouwer Fixed Point Theorem", The American Mathematical Monthly (Mathematical Association of America) 91 (10): 641–643. (Available Online here: http://www.maths.ed.ac.uk/~aar/jordan/index.htm)

[M] Munkres, James (1999). Topology (2nd ed.). Prentice Hall. ISBN 0-13-181629-2.

[B] "Why is it Impossible? The Three Utilities/Three Houses Problem." Math Boy's Page. Web. 28 Nov 2009. <www.pballew.net/3util.doc>.