Math 6710 – Probability (3 pages) Due: Thursday, September 18, 2012

Homework 4

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Problem 1. Let μ be a probability measure on $\mathbb{R}$. The **support** of μ is the set S_μ (also denoted $\operatorname{supp}\mu$) defined by

$$S_\mu := \cap\{F \subset \mathbb{R} \,:\, F \text{ closed}, \mu(F) = 1\}$$

i.e. the intersection of all closed subsets of $\mathbb{R}$ which have measure 1. This is in some sense the set where the measure "lives".

(a) Show that $x \in S_\mu$ iff every open set U containing x has $\mu(U) > 0$.

(b) Show that $\mu(S_\mu) = 1$ (or equivalently, $\mu(S_\mu^C) = 0$, which may be easier to think about). Caution: The definition of $\operatorname{supp}\mu$ involves an uncountable intersection.

(c) Suppose that $X_1, \ldots,$ is an iid sequence of random variables with distribution μ. Show that, almost surely, the closure $\overline{\{X_1, \ldots\}}$ is S_μ. That is for P-almost every $\omega \in \Omega$, the countable set of real numbers $\{X_1(\omega), X_2(\omega), \ldots\}$ has S_μ as its closure.

(d) Show that $\limsup_{n\to\infty} X_n = \sup S_\mu$ and $\liminf_{n\to\infty} X_n = \inf S_\mu$ almost sure.

(e) We know from the Kolmogorov zero-one law that the event $\{\lim_{n\to\infty} X_n \text{ exists }\}$ has probability 0 or 1. Show that if it has probability 1 then there is a constant c such that $X_n = c$ almost surely. Hint: Show that S_μ consists of one point. So except in trivial cases, an iid sequence of random variables never converges.

Solution. (a) Say $x \in S_\mu$, but there is an open set containing x with $\mu(U) = 0$. Then $\mu(\overline{U}) = 1$ and $\overline{U}$ is closed, $x \notin \overline{U}$, so $x \notin S_\mu$, a contradiction.

Now say every open set containing x has $\mu(U) > 0$, but $x \notin S_\mu$. Then there is a closed set F with $\mu(F) = 1$ and $x \notin F$. But then $\overline{F}$ is open with measure 0 and $x \in \overline{F}$, a contradiction.

(b) We'll show $\mu(S_\mu^c) = 0$. In particular, S_μ is an intersection of closed sets of measure 1 so $S_\mu^c = (\cap F)^c = \cup F^c$ is a union of open sets of measure 0, but then for any open set $U = \cup U_\alpha$ expressed as an arbitrary union of open sets we can take a countable subset of those sets so that $U = \cup U_i$ (by Real Analysis), so $S_\mu^c = \cup F^c = \cup_i F_i^c$, but each F_i^c has measure 0 and so S_μ^c is a countable union of sets of measure 0, hence $\mu(S_\mu^c) = 0$.

(c) We'll show that any point $x \in S_\mu$ is a limit point of $\{X_i(\omega)\}$ with probability 1. Given ϵ, $\mu(B_\epsilon(x)) > 0$ by part (a) and so $P(X_i(\omega) \in B_\epsilon(x)) = \mu(B_\epsilon(x)) \neq 0$. Then the probability that there is no i such that $X_i(\omega) \in B_\epsilon(x)$ is $\prod_{i=1}^\infty 1 - \mu(B_\epsilon(x)) = 0$ so with probability 1 there is such an i. Since ϵ was arbitrary, there is an i for which $X_i(\omega) \in B_\epsilon(x)$ for all ϵ and x is an accumulation point of $\{X_i\}$, thus is in its closure. Conversely, say x is an accumulation point of $\{X_i\}$ but it is not in S_μ. Then by (a), there is an open set of measure 0 containing x, but then we may take an ϵ so that $B_\epsilon(x)$ is contained in this open set, and then we have $\mu(B_\epsilon(x)) = 0$. But then $P(X_i(\omega) \in B_\epsilon(x)) = 0$ as $\mu(B_\epsilon(x)) = 0$ and all of the X_i have μ as their distribution. This is a contradiction, so $x \in S_\mu$.

(d) This is asking us to show that the $\limsup$ of the sequence $\{X_i\}$ is almost surely $\sup S_\mu$, but by (c), S_μ is almost surely the set of $\{X_i\}$ and all of its limit points, hence it would contain $\limsup X_i, \liminf X_i$, almost surely. If we only had $\limsup X_i \in S_\mu$ then we would have $\limsup X_i < \sup S_\mu$, but S_μ is closed so $\sup S_\mu \in S_\mu$ and $\sup S_\mu$ is an accumulation point of X_i greater than $\limsup X_i$, this is impossible (by (c)) so $\limsup X_i = \sup S_\mu$. The same argument holds for $\liminf X_i$.

(e) If $\{\lim_{n\to\infty} X_n$ exists $\}$ exists and has probability 1 then by part (d):

$$\lim_{n\to\infty} X_n = \limsup X_n = \liminf X_n = \inf S_\mu = \sup S_\mu$$

And so $\inf S_\mu = \sup S_\mu$ and S_μ is a point, c, thus the only accumulation point of $\{X_i\}$ is almost surely the point c, thus for all i $X_i = c$ with probability 1.

Problem 2. (a) Let $x_1, x_2, \ldots,$ be a sequence of real numbers. Set $s_n = \sum_{i=1}^n x_i$ and $s_{m,n} = \sum_{i=m}^n x_i$. Show that for any fixed m we have:

$$\lim_{n\to\infty} \left| \frac{s_n - s_{m,n}}{n} \right| = 0$$

Conclude that

$$\limsup_{n\to\infty} \frac{s_n}{n} = \limsup_{n\to\infty} \frac{s_{m,n}}{n}, \ \liminf_{n\to\infty} \frac{s_n}{n} = \liminf_{n\to\infty} \frac{s_{m,n}}{n}$$

(b) Let $X_1, \ldots$ be a sequence of random variables, let $\mathcal{T} = \cap_{m=1}^\infty \sigma(X_m, X_{m+1}, \ldots)$ be their tail σ-field, and let $S_n = X_1 + \ldots + X_n$. Show that $\limsup_{n\to\infty} \frac{s_n}{n}$ and $\liminf_{n\to\infty} \frac{s_n}{n}$ are $\mathcal{T}$-measurable random variables.

(c) If $X_1, \ldots$ are *independent* random variables, then $\frac{s_n}{n}$ converges with probability 0 or 1, and if it converges, its limit is constant. (If they are furthermore identically distributed and integrable, the SLLN tells us this limit is $E[X_1]$.)

Solution. (a) For any fixed m we have $\frac{s_n - s_{m,n}}{n} = \frac{\sum_{i=1}^{m-1} x_i}{n} = C/n$ which certainly $\to 0$ as $n \to \infty$ as C does not depend on n.

We have that

$$\lim_{n\to\infty} \left| \frac{s_n - s_{m,n}}{n} \right| = \limsup_{n\to\infty} \left| \frac{s_n - s_{m,n}}{n} \right| = \liminf_{n\to\infty} \left| \frac{s_n - s_{m,n}}{n} \right| = 0$$

then we may write $\limsup_{n\to\infty} \frac{s_n - s_{m,n}}{n} = \limsup_{n\to\infty} \frac{s_n}{n} - \limsup_{n\to\infty} \frac{s_{m,n}}{n} = 0$ so $\limsup_{n\to\infty} \frac{s_n}{n} = \limsup \frac{s_{n,m}}{n}$. The same argument works for $\liminf$.

(b) We want to show $\limsup_{n\to\infty} \frac{s_n}{n}$ is measurable with respect to $\sigma(X_m, \ldots,)$ for all m. But, $\limsup_{n\to\infty} \frac{s_n}{n} = \limsup_{n\to\infty} \frac{s_{n,m}}{n}$ and certainly $\limsup_{n\to\infty} \frac{s_{n,m}}{n}$ is measurable with respect to $\sigma(X_m, \ldots)$ as $s_{n,m} = \sum_{i=m}^n X_i$.

(c) Let $Y = \limsup_{n\to\infty} \frac{s_n}{n}$ and $Z = \liminf_{n\to\infty} \frac{s_n}{n}$. Then both Y, Z are measurable with respect to the tail field of the independent random $X_1, X_2, \ldots$. By the Kolmogorov $0-1$ law every event in the tail field has probability 0 or 1, and this means that the σ-field of Y, Z is either *almost trivial* in the terminology of last week's homework. This means that by Y, Z are constant a.s., also by last week's homework.

Then $\liminf \frac{s_n}{n} \leq \lim \frac{s_n}{n} \leq \limsup \frac{s_n}{n}$. There are only two cases $\liminf \frac{s_n}{n} = \limsup \frac{s_n}{n}$ and the above limit exists with probability 1 or the are not equal and the above limit does not exist. In the former case $\lim \frac{s_n}{n} = \liminf \frac{s_n}{n} = c$, as desired.

Problem 3. Let μ be a probability measure on $\mathbb{R}^n$. A set $A \subset \mathbb{R}^n$ is said to be *μ-regular* if for every $\epsilon > 0$, there exists a closed set F and on open set U such that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. (The idea is that if A is μ-regular we can approximate it from outside by open sets, and from inside by closed sets, missing only a small amount of mass.) Show that every Borel set is μ-regular. Hint: Try showing that the μ-regular sets form a σ-field which contains every open ball.

Proof. Following the hint, we notice that it suffices to show that the μ-regular sets form a σ-field which contains every open ball. This suffices as the open balls generate the Borel sets on $\mathbb{R}^n$, and since they are a subset of the μ-regular sets, the set they generate (i.e. the set of Borel sets) must also be contained in the set of μ-regular sets.

Let $\mathcal{M}$ denote the set of μ-regular sets. We wish to show it is a σ-field. First $\mathcal{M}$ is non-empty, as it contains the open balls. To see an open ball is μ-regular, consider an open ball of radius r, B. Then given ϵ, we can take the closed ball of radius $r - \delta$ and call it F, the open ball of radius $r + \delta$ and call it U and then we have $F \subset B \subset U$. The difference $\mu(U \setminus F)$ can be made smaller than a given ϵ by choosing δ appropriate for $\mathbb{R}^n$, so B is μ-regular.

Next we check the complement of a μ-regular set is μ-regular. Say we have $F \subset A \subset U$ for F closed, U open and A μ-regular. We want to show that $\overline{A}$ is also μ-regular. Given ϵ and $F \subset A \subset U$ showing A μ-regular, we notice $\overline{U}$ is a closed set contained in $\overline{A}$ and $\overline{F}$ is an open set containing $\overline{A}$, so it suffices to show $\mu(\overline{F} \setminus \overline{U}) < \epsilon$, but $\overline{F} \setminus \overline{U}$ is the set of points not in F that are in U, which is precisely $U \setminus F$ and by assumption $\mu(U \setminus F) = \mu(\overline{F} \setminus \overline{U}) < \epsilon$ so $\overline{A}$ is μ-regular.

Finally we check that a countable intersection, $\cap A_i$, of μ-regular A_i is μ-regular. Applying DeMorgan's law and the fact that complements of μ-regular sets are μ-regular this is enough to show that countable unions of μ-regular sets are also μ-regular. Given ϵ, let $F_i \subset A_i \subset U_i$ so that $\mu(U_i \setminus F_i) < \epsilon/2^i$. Then we propose that for $F = \cap F_i$, $U = \cap U_i$, $\mu(U \setminus F) < \epsilon$. We have $\cap U_i \setminus \cap F_i = \cup U_i \setminus F_i$ and we know $\mu(U_i \setminus F_i) < \epsilon/2^i$ for all i and so $\mu(U \setminus F) = \mu(\cup U_i \setminus F_i) \leq \sum \epsilon/2^i = \epsilon$ as desired. The only problem now is that $\cap U_i$ might not be open because it is a countable intersection of open sets. This is not a problem though, as because $\cap U_i$ is an intersection of open sets, it is a Borel set, hence measurable. This means we may take an open set U' containing $\cap U_i$ such that $\mu(U' \setminus \cup U_i) \leq \delta$ for arbitrary δ. Letting $\mu(U \setminus F) = \delta < \epsilon$ and choosing $\delta < \epsilon - \gamma$ we will have that $\mu(U' \setminus F) = \mu((U \setminus F) \cup (U' \setminus U)) \leq \mu(U \setminus F) + \mu(U' \setminus U) = \delta + \gamma < \epsilon$, and so $\cup A_i$ is μ-regular.

Thus we have shown the μ-regular sets form a σ-algebra, which sufficed to show that all Borel sets are μ-regular. $\square$