Math 6710 – Probability (3 pages)	Due: Thursday, September 13, 2012

Homework 3

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1, 2, 3, 4, 5

Problem 1. Let X be a random variable. Prove that the following are equivalent:

(a) X is independent of every random variable Y.

(b) X is independent of itself.

(c) For all events $A \in \sigma(X)$, $P(A) = 0$ or $P(A) = 1$. We say the σ-field $\sigma(X)$ is *almost trivial.*

(d) There exists a constant $c \in \mathbb{R}$ such that $X = c$ a.s.

Proof. (Dancing with definitions.)

(a) $\implies$ (b) X is a random variable and it is independent of every random variable, therefore it is independent of itself.

(b) $\implies$ (c) X is independent of itself so for any event $A \in \sigma(X)$, we have $p(A) = p(A \cap A) = p(A)p(A)$, but this is only true if $p(A) = 0$ or $p(A) = 1$, as desired.

(c) $\implies$ (d) Say that there is no such constant c such that $X = c$ a.s. Then for the events $\{X < EX\}$, $\{X > EX\}$, $\{X = EX\}$ we must have $P(\{X = EX\}) \neq 1$ as X is non-constant. If $P(\{X = EX\}) \neq 0$ we are done, so say $P(\{X = EX\}) = 0$, then $P(\{X < EX\}) = 1 - P(\{X > EX\})$ and the probability of neither event can be 0 or 1 as then EX would be too small (or too big). So $X = c$ a.s.

(d) $\implies$ (a) Consider any event $A \in \sigma(X)$, $B \in \sigma(Y)$ and $P(A \cap B)$. We have that $A = X^{-1}(A')$ and $B = Y^{-1}(B')$ for Borel sets A', B'. If $c \in A'$ and $c \in B'$ then $P(A) = 1$ as $X = c$ a.s. and so $P(A)P(B) = P(B)$, further A is a probability 1 event so $A \cap B$ differs from B by an event of probability 0 and $P(A \cap B) = P(B) = P(A)P(B)$ and A, B are independent thus making X, Y independent. □

Problem 2 (D 2.1.4)**.** Let $g_1, \ldots, g_n$ be probability density functions (i.e. $g_i : \mathbb{R} \to [0, \infty)$ is measurable and $\int_{\mathbb{R}} g_i dm = 1$), and define $f : \mathbb{R}^n \to [0, \infty)$ by $f(x_1, \ldots, x_n) = g_1(x_1) \cdots g_n(x_n)$. Show that $X = (X_1, \ldots, X_n)$ is a random vector with density f if and only if $X_1, \ldots X_n$ are independent random variables where X_i has density g_i.

Note: In Durrett's version, the g_i are not assumed to be probability densities, i.e. it is not assumed that $\int_{\mathbb{R}} g_i dm = 1$. However, one can just rescale them to achieve this.

Proof. (Dancing with integrals!)

If $X_1, \ldots, X_n$ are independent random variables with X_i having density g_i then X is a random vector with density f.

Assume that $A = \prod A_i$ a box in $\mathbb{R}^n$. First we show the result for $P(X \in A)$ and then apply the monotone convergence theorem to pass to arbitrary measurable events.

Then we have $P((X_1, \ldots, X_n) \in A) = P(\{X_1 \in A_1\}, \{X_2 \in A_2\}, \ldots \{X_n \in A_n\})$, further the X_i are independent so we have $P(\{X_1 \in A_1\}, \{X_2 \in A_2\}, \ldots \{X_n \in A_n\}) = \prod P(\{X_i \in A_i\}) = \prod \int_{\mathbb{R}} g_i(x_i) dx_i$, then because from the definition of A_i each $g_i(x_i)$ is constant with respect to the other variables of integration so we have $\prod \int_{A_i} g_i(x_i)dx_i = \int_{A_1}\int_{A_2} \prod g_i dx_1 \ldots dx_n$ and because each $g_i(x_i)$ is assumed non-negative, we may apply Tonelli's theorem and we have

$$\int_{A_1} \cdots \int_{A_n} \prod g_i dx_1 \ldots dx_n = \int_A \prod g_i dx = \int_A f(x)dx$$

and X has density $f(x)$, as desired. Now for arbitrary A we may write it as an increasing sequence of almost disjoint cubes and break the above integral up into finitely many individual integrals for which the result applies. Because the sequence is increasing, we may apply the monotone convergence theorem to get the result in general.

If X is a random vector with density $f(x) = \prod g_i(x_i)$ then the X_i are independent random variables with density g_i.

First let's see that X_i has density g_i. Consider:

$$\begin{aligned} P(X \in \mathbb{R} \times \ldots \times A_i \times \ldots \times \mathbb{R}) &= \\ &= P(X_1 \in \mathbb{R},\ \ldots,\ X_i \in A_i,\ X_n \in A_n) \\ &= P(X_i \in A_i) \\ &= \textstyle\int_A \prod g_i dx \\ &= \textstyle\prod \int_{A_i} g_i dx_i \end{aligned}$$

Then $\int_{A_j} g_j dx_j = 1$ for $j \neq i$ as $A_j = \mathbb{R}$, so $P(X \in A) = P(X_i \in A_i) = \int_{A_i} g(x_i)dx_i$, and X_i has density $g(x_i)$.

Having established this, for an event A, we let $B = \prod B_i$, a 'box' in $\mathbb{R}^n$, and we reason about $P(X \in \prod B_i)$. This is equivalent to asking about the event $\{\wedge X_i \in A_i\}$, to show the X_i are independent. We may apply Tonelli's theorem and the properties of the integral to see:

$$P(X \in B) = P(\wedge\{X_i \in B_i\}) = \int_B \prod g_i dx = \prod \int_{B_i} g_i dx_i = \prod P(\{X_i \in B_i\})$$

so the X_i are independent. □

Problem 3. (a) Let $\mathcal{G}$ be a σ-field, and let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ be an increasing sequence of σ-fields. Suppose that for each n, $\mathcal{G}$ and $\mathcal{G}_n$ are independent. Let $\mathcal{G}_\infty = \sigma(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ be the σ-field generated by the $\mathcal{G}_n$ (i.e. the smallest σ-field such that $\mathcal{G}_n \subset \mathcal{G}_\infty$ for all n). Show that $\mathcal{G}$ and $\mathcal{G}_\infty$ are independent. Hint: $\cup_{n=1}^{\infty} \mathcal{G}_n$ is a π-system.

(b) Let Y be a random variable, and $X_1, X_2, \ldots$ a sequence of random variables such that Y is independent of $(X_1, \ldots, X_n)$ for all n. Show that Y is independent of $\sup_n X_n$. (It is also independent of $\inf_n X_n, \limsup_n X_n$, etc.) Use Part (a)!

Proof. ($\lambda - \pi$ theorem ftw!)

(a) $\mathcal{G}$ and $\mathcal{G}_\infty$ are independent. Notice that $\sigma(\mathcal{G}_\rangle) = \sigma(\cup_{n=1}^{\infty} G_n)$ basically because the two arguments are really describing the same set. If a σ-field contains $\cup_{n=1}^{\infty} G_n$ then for any $A \in \cup_{n=1}^{\infty} G_n$, A is in that σ-field. A was arbitrary from any G_n so G_n is contained in the σ-field, thus G_n for any n is contained in the σ-field reversing the argument staring with $\sigma(\mathcal{G}_\rangle)$, we see the two are the same.

Further $\cup_{n=1}^{\infty} G_n$ is a π-system as for any $A_1, A_2 \in \cup G_n$, $A_1 \in G_n$ and $A_2 \in G_m$, because $\{G_i\}$ is an increasing sequence of sets, we may say $m \geq n$ and so $A_1, A_2 \in G_m$. Then $A_1 \cap A_2 \in G_m$ as G_m is a σ-field, thus $A_1 \cap A_2 \in \cup G_n$ and $\cup G_n$ is π-system. Further, $\mathcal{L}_G \subseteq \cup G_n$ as for any $A \in \cup G_n$,

$A \in G_n$ for some n and G_n and G are assumed independent. Then $\sigma(\cup G_n) \subseteq \mathcal{L}_G$ by the $\lambda - \pi$ lemma. This means $\sigma(\cup G_n) = \mathcal{G}_\infty$ is independent of G (it is a subset of all the sets independent from G).

(b) Y and $\sup_n X_n$ are independent. We need to show $\sigma(Y)$ and $\sigma(\sup_n X_n)$ are independent. Recall that for any X, the sets $\{X > a\}$ generated $\sigma(X)$ because the sets (a, ∞) generate the Borel sets on $\mathbb{R}$. Then the set $\{\sup_n X_n > a\} = \cup_n \cup_{a \in \mathbb{R}} \{X_n > a\} = \cup_n \cup m < n \cup_{a \in \mathbb{R}} \{X_m > a\}$, where in the last equality we wrote the set as an increasing union. So we have $\sigma(\sup_n X_n) = \sigma(\cup_n \cup m < n \cup_{a \in \mathbb{R}} \{X_m > a\})$. In particular we have an increasing sequence of σ-algebras, $\sigma(\cup m < n \cup_{a \in \mathbb{R}} \{X_m > a\})$ each independent of Y, by assumption. Then by part (a), the $\sigma(\cup_{n=1}^{\infty} \cup_{m<n} \cup_{a \in \mathbb{R}} \{X_m > a\})$ is independent of $\sigma(Y)$ and so Y and $\sup_n X_n$ are independent. □

Problem 4 (D 2.1.13)**.** Show that if X, Y are independent discrete random variables, then

$$P(X + Y = n) = \sum_m P(X = m)P(Y = n - m)$$

Proof. (Um...)

First we may write $P(X+Y = n) = \sum_m P(X = m,\ Y = n-m)$ because this is just a partition of the event $X + Y = n$ into a disjoint union of events. By assumption X, Y are independent so for each m we have $P(X = m,\ Y = n - m) = P(X = m)P(Y = n - m)$, putting these together: $P(X + Y = n) = \sum_m P(X = m,\ Y = n - m) = \sum_m P(X = m)P(Y = n - m)$, as desired. □

Problem 5 (D 2.1.14)**.** Recall that a random variable Z has the Poisson distribution with parameter λ if $P(Z = k) = \lambda^k e^{-\lambda}/k!$ for $k = 0, 1, \ldots$; we write $Z \sim$ Poisson(λ). Suppose X, Y are independent with $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ). Use the previous exercise to show that $X + Y \sim$ Poisson($\lambda + \mu$).

Proof. (Binomial theorem!)

To show $X + Y \sim$ Poisson($\lambda + \mu$) we need to calculate $P(X + Y = n)$. This is precisely the setting of the last problem so we may write:

$$\begin{aligned} P(X+Y=n) &= \\ &= \textstyle\sum_{m=0}^{n} P(X=m)P(Y=n-m) \\ &= \textstyle\sum_{m=0}^{n} \lambda^m e^{-\lambda}/m!(\mu^{n-m}e^{-\mu}/(n-m)!) \\ &= e^{-(\mu+\lambda)}/n! \textstyle\sum_{m=0}^{n} \lambda^m \mu^{n-m} \frac{n!}{m!(n-m)!} \\ &= e^{-(\mu+\lambda)} \frac{(\lambda+\mu)^n}{n!} & (*) \end{aligned}$$

where (*) is from the Binomial Theorem–the sum from the previous step is just the binomial expansion of $(\lambda + \mu)^n$–and this is what it means for $X + Y \sim$ Poisson($\mu + \lambda$). □