Math 6710 – Probability (3 pages) Due: Thursday, September 6, 2012

Homework 2

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1, 2, 3, 4, 5

Problem 1. Suppose $\phi(x)$ is strictly convex, i.e.

$$\phi(tx + (1-t)y) < t\phi(x) + (1-t)\phi(y)$$

for all $x \neq y$ and $t \in [0,1]$. Show that under this assumption Jenson's inequality holds only in the trivial case when X is a constant. In particular, show if X and $\phi(X)$ are integrable and $E[\phi(X)] = \phi E[X]$ then $EX = X$ a.s.

Proof. Let's say $\mathbb{E}[\phi(X)] = \phi\mathbb{E}[X]$ and let's let $c = \mathbb{E}[X]$. Suppose for contradiction that $X \neq EX$. Then we must have $P(A), P(B) > 0$ where $A = \{\omega | X(\omega) > c\}$ and $B = \{\omega \mid X(\omega) < c\}$. At least one event must have probability greater than 0 as X is assumed non-constant, if only one had probability greater than 0, then we could not have $c = \mathbb{E}[X]$, the 'average' value of X.

From the proof of Jensen's inequality I gave in class (I'm glad I had to do that presentation!) we know there exists m so that $\phi(x) \geq \phi(c) + m(x-c)$ and because ϕ is strictly convex, we can only have equality when $x = c$. Taking into account these facts we have:

$$\phi(x) - \phi(c) > m(x-c)$$

whenever $x \neq c$, so in particular, we have $\phi(X(\omega)) - \phi(c) > m(X(\omega) - c)$ for all $\omega \in A \cup B$, and $\phi(X(\omega)) - \phi(c) \geq m(X(\omega) - c)$ for $\omega \in \Omega \setminus (A \cup B)$.

Noting $EX = \int_\Omega X d\mu$, $\int_\Omega X d\mu = \int_{A\cup B} X d\mu + \int_{\Omega \setminus A \cup B} X d\mu$, and $P(A \cup B) > 0$, we can add the two inequalities from the previous paragraph and retain a strict inequality. So, taking expectations we find:

$$E(\phi(X)) - \phi(c) > mE(X(\omega) - c)$$

remembering $c = EX$ we have:

$$E(\phi(X)) - \phi(EX) = \phi(EX) - \phi(EX) = 0 > m(EX - EX) = 0$$

which is a contradiction so our result is proved. ☐

Problem 2. Suppose $X_n \to X$ in probability and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f_n(X) \to f(X)$ in probability. (Don't use the double subsequence trick.)

Proof. We want to show that $P(\{|f(X_n) - f(X)| > \epsilon\}) \to 0$ as $n \to \infty$, so we break the event up into two parts:

$$P(\{|f(X_n) - f(X)| > \epsilon \text{ and } |X| \leq M\}) + P(\{|f(X_n) - f(X)| > \epsilon \text{ and } |X| > M\})$$

For the first event, we know $|X| \leq M$ and $[-M, M]$ is compact so we may choose an ϵ' so that for all $x, y \in [-M, M]$ $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \epsilon'$. In particular, we know that

$P(|X_n - X| > \epsilon'/2) \to 0$ as $n \to \infty$ so $P(|X_n - X| > \epsilon$ and $|X| \leq M) \to 0$ as $n \to \infty$, further for all ϵ we may choose M so that $P(\{|X| > M\}) < \epsilon$, so that

$$P(\{|f(X_n) - f(X)| > \epsilon \text{ and } |X| \leq M\}) + P(\{|f(X_n) - f(X)| > \epsilon \text{ and } |X| > M\}) < \epsilon$$

for all $\epsilon > 0$ for sufficiently large n, M, thus $P(\{|f(X_n) - f(X)| > \epsilon\}) \to 0$ as $n \to \infty$, and $f(X_n) \to f(X)$ in probability. □

Problem 3. Suppose $X_n \to X$ in probability. Show that, almost surely,

$$\liminf_{n\to\infty} X_n \leq X \leq \limsup_{n\to\infty} X_n$$

Show this either directly or using the double subsequence trick.

Proof. Let's try the double subsequence trick.

In particular applying this to the subsequence $\{X_n\}$ of $\{X_n\}$ we know that $\{X_n\}$ must have a subsequence $\{X_{n(m))}\}$ converging to X almost surely. Now we have a subsequence of X so that, a.s.,:

$$X = \lim_n X_{n(m)}$$

but $X_{n(m)}$ is a subsequence of X_n, $\liminf$ is the least limit point of X_n and $\limsup$ is the greatest limit point of X_n, so in particular, a.s. $\liminf X_n \leq \lim_n X_{n(m)} \leq \limsup X_n$, substituting this in to our equality above:

$$\liminf X_n \leq X \leq \limsup X_n$$

as desired. □

Problem 4. A set of random variables, $\mathcal{S}$ is called uniformly integrable or u.i. if for all $\epsilon > 0$ there exists an $M \geq 0$ such that for all $X \in \mathcal{S}$

$$\mathbb{E}[|X|1_{|X|\geq M}] < \epsilon$$

Prove the crystal ball condition: Let $\mathcal{S}$ be a set of random variables. If for $p > 1$ we have $\sup_{X\in\mathcal{S}} \mathbb{E}[|X|^p] < \infty$ then $\mathcal{S}$ is uniformly integrable.

Proof. I admit that I was a little confused about the definition of uniformly integrable so I looked it up on the notes on the webpage. There was something similar to this problem in Bruce Driver's notes.

Let $C = \sup_{X\in\mathcal{S}} \mathbb{E}[|X|^p] < \infty$.

First notice that $p > 1$ so $|X|/|X|^p = 1/|X|^{p-1} \to 0$ as $X \to \infty$. Now:

$$\mathbb{E}[|X|1_{X\geq M}] = \mathbb{E}[(|X|/|X|^p)|X|^p 1_{X\geq M}] \leq \mathbb{E}[(|M|/|M|^p|)|X|^p 1_{X\geq M}] \leq (|M|/|M|^p)\mathbb{E}[|X|^p] \leq |M|/|M|^p C$$

and for given $\epsilon > 0$, we may choose M so that $0 < |M|/|M|^p < \epsilon/C$ so for any $X \in \mathcal{S}$ we have:

$$\mathbb{E}[|X|1_{X\geq M}] \leq (|M|/|M|^p)C < \epsilon$$

showing $\mathcal{S}$ is uniformly integrable, as desired. □

Problem 5. Let $\mathcal{S}, \mathcal{S}'$ be two sets of random variables. Suppose that $\mathcal{S}'$ is u.i. and for every $X \in \mathcal{S}$ there exists a $Y \in \mathcal{S}'$ with $|X| \leq |Y|$ a.s. Show that $\mathcal{S}$ is also ui.

Proof. Given $\epsilon > 0$ we want to find an $M \geq 0$ so that for all $X \in \mathcal{S}$ $\mathbb{E}[|X|1_{|X|\geq M}] < \epsilon$. Let M be the M for $\mathcal{S}'$ given ϵ and let $Y \in \mathcal{S}'$ so that $|X| \leq |Y|$ a.s., then it suffices to show that $|X|1_{|X|\geq M} \leq |Y|1_{|Y|\geq M}$. If we have $|X|1_{|X|\geq M} > |Y|1_{|Y|\geq M}$ then either both $|X|, |Y| \geq M$ and $|X| \geq |Y|$, call this event A, or $|X| \geq M$ and $|Y| < M$, call this event B. Because $|X| \leq |Y|$ a.s. these are both probability 0 events, so $|X|1_{|X|\geq M} \leq |Y|1_{|Y|\geq M}$ a.s. and we have:

$$\mathbb{E}[|X|1_{|X|\geq M}] \leq \mathbb{E}[|Y|1_{|Y|\geq M}] < \epsilon$$

and so $\mathcal{S}$ is ui.

$\square$