Math 6310 – Algebra (3 pages) Due: September 5, 2011

Homework 1 : Groups

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3.4 (p. 106): 8,11,12

Problem (3.4.8). *Prove the following equivalent of a finite group G:*

1. *G is solvable.*
2. *G has a chain of subgroups: $1 = H_0 \trianglelefteq \ldots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is cyclic, $0 \leq i \leq s-1$.*
3. *All composition factors are of prime order.*
4. *G has a chain of subgroups: $1 = N_0 \trianglelefteq \ldots \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N_i is abelian, $0 \leq i \leq t-1$.*

Proof. **(1) $\implies$ (2)**

Because G is solvable, there is a chain of groups,

$$G = G_0 \geq \ldots \geq G_n$$

such that G_i/G_{i+1} is abelian. So because G_i/G_{i+1} is abelian, it decomposes as a direct sum of p-groups. So $G_i/G_{i+1} \approxeq H_i/H_{i+1} \oplus \mathbb{Z}_p$. Then using some isomorphism theorem we move $\mathbb{Z}_p$ inside so we have $G_i/G_{i+1} \approxeq H_i \oplus \mathbb{Z}_p/H_{i+1}$. Then we add H_i to the quotient so we have $\mathbb{Z}_p \approxeq \mathbb{Z}_p \oplus H_i/H_i$ and we set $Y_i = G_i$ and $Y_{i+1} = H_i \approxeq G_{i+1} \oplus H_i/G_{i+1}$.

Because G_i/G_{i+1} is an abelian group. It decomposes as a direct sum of p groups. Because these p-groups are cyclic, what we want to is successively quotient G by bigger and bigger pieces to get slices which enumerate each of the p-groups making up the direct sum of G_i/G_{i+1}. This will allow us to form our new chain as is desired for condition (2) to hold.

(2) $\implies$ (3)

Using the cyclic slices, we refine them to slices of prime order by applying the same process as in the step above only now pulling out groups P of order such that G_i/P is a group of prime order.

(3) $\implies$ (4)

For this step it will help to use the results of 6.1

(4) $\implies$ (1)

(4) implies (1) trivially as (1) only requires that the slices are abelian while (4) adds the additional requirement that elements of the chain are normal subgroups of G.

□

Problem (3.4.11). *Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrivial subgroup A of H with $A \trianglelefteq G$ and A is abelian.*

Proof. Namely, take the sequence of N_i such that each N_i is normal in G with slices, N_{i+1}/N_i abelian. Then either N_1 is a subgroup of H, and we're done as $N_1/N_0 = N_1/\{1\} = N_1$ is abelian. Otherwise, if N_1 is not a subgroup of H then either H is a subgroup of N_1 which means H is abelian because it is a subgroup of an abelian group and we're done. If H is not a subgroup of N_1 and N_1

is not a subgroup of H, then H contains no N_i in the chain because N_1 is the smallest nontrivial group in the chain and all groups in the chain are ordered by inclusion.

□

Problem (3.4.12)**.** *Prove the following equivalent:*

1. *every group of odd order is solvable.*
2. *the only simple groups of odd order are those of prime order.*

Proof. To see (1) $\implies$ (2). Assume there are simple groups of odd, but not prime order. Take some such group, G. G doesn't have prime order, so there is a prime p dividing $|G|$ and by proposition 21 of chapter 3.4 there exists an element g of order p so G has a subgroup that is not normal because G is simple. So G can't be abelian. Because G is simple, the only possible chain of subgroups one could construct would be $\{1\} \triangleleft G$. Because any subgroup immediately to the left of G would have to be normal in G and we know no such groups exist. But $G/\{1\} = G$ and we have shown G is not abelian so G can't be solvable.

To see (2) $\implies$ (1) assume not every group of odd order is solvable, but the only simple groups of odd order are of prime order. Trivially, every group of prime order must be solvable because groups of prime order are abelian thus the chain $\{1\} \trianglelefteq G$ suffices to solve it. Then there must be a group, D, of odd, but not prime order that is not solvable. D cannot be abelian otherwise the chain $\{1\} \trianglelefteq G$ shows it is solvable.

So D has a nontrivial normal subgroup D_1. Consider the pieces D/D_1 and D_1. They must both be of odd order because $|D|$ is odd and their orders must divide that of D. Either both D_1 and D/D_1 have prime order, and we are done or they have odd, but not prime order. In this case, they must also not be simple (or we have a contradiction) so they have nontrivial normal subgroups. We repeat the process we applied to D until we have slices with prime order, thus making the slices abelian. This makes D solvable, which is a contradiction. □

6.1 (pp. 198-201): 31, 32

Problem (6.1.31)**.** *Prove every minimal normal subgroup of a finite solvable group is an elementary abelian p-group for some prime p.*

Proof. □

Problem (6.1.32)**.** *Prove that every maximal subgroup H of a finite solvable group has prime power index.*

Proof. We proceed by induction. First the result holds vacuously for $|G| = 1, 2, 3$ as for these orders G has no nontrivial subgroups. Consider some minimal normal subgroup M.

There are exactly two cases, either $M \leq H$ or $M \nleq H$.

In the case where $M \leq H$, consider G/M and $H/M \leq G/M$. By the fourth isomorphism theorem, H/M is maximal in G/M and our induction hypothesis applies so $[H/M : G/M]$ is a prime power. This means

$$[H/M : G/M] = |G||M|/|H||M| = |G|/|H| = [G : H]$$

is a prime power, and we are done.

In the case when $M \nleq H$ there is some element $m \in M$, $m \notin H$. Because H is maximal and M is normal in G, HM is a subgroup of G properly containing H, so it must be $HM = G$. Then by the fourth isomorphism theorem we get $HM/M = G/M \approxeq H/H \cap M$. This means $|G|/|M| = |H|/|H \cap M|$. Solving for $|M|$, we find $|M| = |G||H \cap M|/|H|$. $|M|$ is a prime power by exercise 31 so $|G||H \cap M|/|H| = [G : H]|H \cap M|$. Now $[G : H]$ must be a prime power as it divides a prime power, $|M|$.

□

Additional Problems

Problem (1). *Let G be an Ω-group with an Ω-composition series. If H is a normal Ω-subgroup of G, show that G has an Ω-composition series in which H is one of the terms. Deduce that H and G/H have Ω-composition series.*

Proof. □

Problem (2). *If G is an Ω-group with a composition series, show*

$$l(G) = l(H) + l(G/H)$$

.

If H_1 and H_2 are two normal Ω subgroups of G, show that

$$l(H_1 H_2) = l(H_1) + l(H_2) - l(H_1 \cap H_2)$$

.

Proof. □

Problem (3). *Let G have two composition series $(G_i)_{0 \le i \le n}$ and $(H_j)_{0 \le j \le n}$. Show that the proof given in class of the Jordan-Hölder theorem yields yields an explicit permutation, π, of $\{0, \ldots, n\}$ such that*

$$G_i/G_{i+1} \approxeq H_j/H_{j+1}$$

if $j = \pi(i)$.

Proof. □

Problem (4). *Assume G is any Ω-group satisfying both ACC and DCC.*

1. *Show that any Ω-series in G can be refined to an Ω-composition series.*

2. *Let $f : G \to G$ be an endomorphism. Prove that for n sufficiently large, the Ω-subgroups $N := \ker(f^n)$ and $H := \operatorname{im}(f^n)$ satisfy $G = NH$ and $N \cap H = \{1\}$, so that G is the semidirect product of N and H. Moreover, f induces an automorphism of H.*

Proof. □