Math 6520 – Differentiable Manifolds 1 (3 pages) Due: Wednesday, September 12, 2012

Homework 2

Instructor: Reyer Sjamaar *Costandino Dufort Moraites*

1-4,1-10,1-11,1-12

Problem 1 (1-4)**.** Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Such a map is called *homogeneous of degree d*. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}[x] = [P(x)]$ is well-defined and smooth.

Proof. First we want to see that the map is well-defined. In particular, we've defined $\tilde{P}[x] = [P(x)]$, so we must check that $\tilde{P}[x]$ is independent of the representative x of $[x]$ chosen. Say we have two representatives, x, y of $[x]$. Then $x = \lambda y$ for $\lambda \neq 0$ by the definition of the equivalence relation giving our equivalence classes. Then $[P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)]$, so the map is well-defined because P is homogeneous of degree d.

Now for smoothness, we must show that for any $p \in \mathbb{RP}^n$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ so that $F(U) \subseteq V$ and the composite map $\psi \circ \tilde{P} \circ \phi^{-1}$ is smooth. We will show $\tilde{P}$ to be smooth by writing it, locally as a composition of smooth maps. In particular, let (U, ϕ) be some chart containing p, with $\phi[p^1, \ldots, p^{n+1}] = (p^1/p_i, \ldots, p^{n+1}/p_i)$ with $p_i \neq 0$. Notice that $\phi[p]$ is a representative of the equivalence class $[p]$ and that ϕ is smooth as it is a smooth polynomial on the coordinates. Similarly, take a chart (V, ψ) of $F(p)$ and notice ψ^{-1} is smooth as it is a diffeomorphism. We know $\tilde{P}(U) \cap \psi^{-1}(V) \neq \varnothing$ as $p \in \tilde{P}(U) \cap \psi^{-1}(V)$, so restrict ϕ to $\phi^{-1}(\phi(U) \cap \psi^{-1}(V))$, then on this restriction $\tilde{P}[x] = \psi^{-1} \circ P \circ \phi[x]$ which is a composition of smooth maps, hence smooth, thus $\tilde{P}$ is smooth. □

Problem 2 (1-10)**.** Let $\mathbb{CP}^n$ denote n-dimensional complex projective space, as defined on last homework.

(a) Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is smooth.

(b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

(a) The quotient map is smooth.

Proof. This is similar to example 2.5(c). We defined the smooth structure on $\mathbb{CP}^n$ completely analogously to that on $\mathbb{RP}^n$, so using charts over $\mathbb{C}^n$ we have that:

$$\begin{aligned} \hat{\pi}(z^1, \ldots, z^{n+1}) &= \phi_i \circ \pi(z^1, \ldots, z^{n+1}) \\ &= \phi_i[z^1, \ldots, z^{n+1}] \\ &= (\tfrac{z^1}{z^i}, \ldots, \tfrac{z^{i-1}}{z_i}, \tfrac{z^{i+1}}{z_i}, \ldots, \tfrac{z^{n+1}}{z^i}) \end{aligned}$$

This suffices to show that π is smooth, as to get a coordinate representation over $\mathbb{R}$ we use the diffeomorphism from last homework, namely $(x^1 + y^1 i, \ldots, x^n + y^n i) \leftrightarrow (x^1, y^1, \ldots, x^n, y^n)$, compositions of smooth maps is smooth, so this works. □

(b) $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Proof. I'm going to apply the same trick I used in part (a) and only think about local coordinates in $\mathbb{C}$, equating it with $\mathbb{R}^2$ whenever convenient.

We notice that $\mathbb{CP}^1 = (z_1, z_2)$, $z_i \in \mathbb{C}$, at least one nonzero, modulo the equivalence relation $(z_1, z_2) \sim (z_3, z_4)$ iff $(z_1, z_2) = c(z_3, z_4)$ for $c \in \mathbb{C}$. We may think of our coordinate charts $\phi_1 : U_1 \to \mathbb{C}$, where $U_1 = \{(z_1, z_2)|z_1 \neq 0\}$ as $(z_1, z_2) \mapsto z_2/z_1$, and similarly for (ϕ_2, U_2).

Correspondingly, let $\psi_1 : S_2 \setminus \{N\} \to \mathbb{R}^2$ be stereographic project from $\{N\}$ onto $\mathbb{R}^2$ and similarly for ψ_2 and $\{S\}$.

We're ready to define our diffeomorphism, $f : \mathbb{CP}^1 \to \mathbb{S}^2$. Namely $f([z_1, z_2]) = \psi_1^{-1}(\phi_1(z_1, z_2))$ if $z_1 \neq 0$ and $\{N\}$ if $z_1 = 0$. This map is well-defined and a bijection. First notice that for $[z_1, z_2]$, $z_1 \neq 0$, this is a smooth bijection from $\mathbb{CP}^1 \setminus [0, z_2] \to S^2 \setminus \{N\}$. This is because both ϕ_1, ψ_1 are diffeomorphisms of $\mathbb{CP}^1, S^2$ and $\mathbb{C}$ (i.e. $\mathbb{R}^2$), respectively. Further as we approach the bad point $(0, z_2)$ with (w_1, w_2), $w_2/w_1 \to \infty$ so $\phi_1(w_1, w_2) \to \infty$ and $\psi_1^{-1}(\phi_1(w_1, w_2) \to \{N\}$, thus our definition of f on $\{N\}$ makes our map continuous.

This means we can apply lemma 2.3 from the text and show that f is smooth by considering the maps $\psi_i \circ f \circ \phi_j(z)$, i.e. the maps $\psi_1 \circ f \circ \phi_1^{-1}(z), \psi_1 \circ f \circ \phi_2^{-1}(z), \psi_2 \circ f \circ \phi_1^{-1}(z), \psi_2 \circ f \circ \phi_2^{-1}(z)$. In particular we have:

$$\begin{aligned}
\psi_1 \circ f \circ \phi_1^{-1}(z) &= \psi_1(f(1, z)) &&= \psi_1(z) &&= z \\
\psi_1 \circ f \circ \phi_2^{-1}(z) &= \psi_1(f(z, 1)) &&= \psi_1(1/z) &&= z^{-1} \\
\psi_2 \circ f \circ \phi_1^{-1}(z) &= \psi_1(f(1, z)) &&= \psi_2(z) &&= z^{-1} \\
\psi_2 \circ f \circ \phi_2^{-1}(z) &= \psi_2 \circ (f(z, 1)) &&= \psi_2(1/z) &&= z
\end{aligned}$$

and all of these maps are certainly smooth, so f is smooth as desired. Thus f is a smooth bijection with smooth inverse given by $f'(z) = \phi_1(\psi_1^{-1}(z))$, $\{N\} \mapsto [0, 1]$, whose smoothness follows from the same proof as we did for showing f smooth. Thus we have that f is a diffeomorphism, as desired. □

Problem 3 (1-11)**.** Let G be a connected Lie group, and let $U \subset G$ be any neighborhood of the identity. Show that every element of G can be written as a finite product of elements of U. In particular, U generates G.

Proof. Because U is open, the subgroup it generates can be written can be written as a union of open sets, hence making it open.

First notice that for all $g \in G$, the map $f_g(a) = ga$ is a diffeomorphism with inverse $f_{g^{-1}}$. For all $g \in G$, $f_g(x)$ is a smooth map because multiplication is smooth because G is a Lie group. Further the map is injective as if we have $f_g(a) = f_g(b)$ then we have $ga = gb$ and so $a = b$. Finally the map is surjective as for all $a \in G$, $f_g(g^{-1}a) = gg^{-1}a = a$. Thus $f_g(x)$ is a diffeomorphism.

Note that a similar proof shows that $a \mapsto a^{-1}$ is also a diffeomorphism with inverse $a \mapsto a^{-1}$.

Now let $U' = U \cap U^{-1} \subseteq U$. This set is also open as it is an intersection of open sets. We have $1 \in U'$ as $1^{-1} = 1$ and $\{1\} \subset U'$ as if $\{1\} = U'$ then $\{1\}$ is clopen and because G is assumed connected $\{1\} = G$. Now we will show that every element of G can be written as a finite product of elements in U'. Notice now that U' has the property that for every element $a \in U'$, $a^{-1} \in U'$.

Let H be the set all all finite products of elements of U'. Notice that H is open. In particular, we give the following recursive definition for H, Let $H_0 = U'$ and let $H_i = \cup_{g \in U'} gH_{i-1}$ where gA for $A \subseteq G$ is the set $\{g \cdot a \mid a \in A\}$. Each set gA is open if A is open as we've already shown multiplication by g is a diffeomorphism so the image of an open set under the map must also be open, thus H_i is a union of open sets, hence open. Then $H = \cup H_i$ is also a union of open sets so it is open. Further, we ensured that U' contained the inverses of all of its elements, so H is the set of all finite products of elements of U' and their inverses, hence H is the subgroup generated by U'. In particular, this means $G \setminus H$ is open as $G \setminus H = \cup_{g \in G} gH$, the set of cosets of H. Because each coset of H is the image of H under multiplication of an element of G, it is open, thus $G \setminus H$ is open, hence H is closed. We already have shown H is open, so it is clopen. G was assumed connected so we must have $G = H$ and every element of G can be written as a finite product of elements of $U' \subseteq U$, as desired. □

Problem 4 (1-12)**.** Let G be a Lie group, and let G_0 denote the connected component of G containing the identity (called the *identity component* of G).

(a) Show that G_0 is the only connected open subgroup of G.

(b) Show that each connected component of G is diffeomorphic to G_0.

(a) G_0 is the only open connected subgroup of G.

Proof. By proposition 1.8 in the text, G_0 is open as it is a connected component of G. By the previous exercise, because G_0 is an open neighborhood of e, it generates a subgroup of G which is the connected component of G containing e–this is a consequence of how I proved the previous question. I actually proved that the subgroup a neighborhood of the identity generates is clopen, in any case, this means G_0 must be an open connected subgroup of G.

Say there was another open, connected subgroup of G, H. Then $H \subseteq G_0$ as G_0 is a connected component of G and both H, G_0 must contain 1. But then H is an open neighborhood of e, hence, by the previous exercise, it must generate the entire connected component in which e is contained, so $H = G_0$. □

(b) Each connected component of G is diffeomorphic to G_0.

Proof. Say we have another connected component H of G. Say $h \in H$ then $hG_0 \cap H \neq \varnothing$ as $h \cdot e = h$. In particular, we showed in the last exercise that multiplication by h is a diffeomorphism so hG_0 is clopen. If $H \setminus hG_0 \neq \varnothing$, then there is another coset of G_0, say $h'G_0$ which contains $h' \in H \setminus hG_0$. By assumption, H is a connected component of G and since $h'G_0$ is a connected set with nontrivial intersection with H, $h'G_0 \subseteq H$. If $h'G_0 \neq H \setminus hG_0$, then we may continue this process to find $H = \sqcup_{h \in A} hG_0$ for some $A \subseteq G$. The cosets of G_0 partition G, so there must be some set A for which this equality holds. But, then we have given a decomposition of H into a disjoint union of open sets, since H is assumed connected, we must have $A = \{h\}$ and $H = hG_0$, thus H is diffeomoprhic to G_0 under the diffeomorphism multiplication by h. □