Math 6710 – Probability (3 pages)	Due: Thursday, October 4, 2012

Homework 6

Instructor: Nate Eldredge *Costandino Dufort Moraites*

1, 2, 3

Problem 1. Let $\mu_1, \dots, \mu$ be probability measures on $\mathbb{R}^d$. Suppose that for every continuous function $f : \mathbb{R}^d \to \mathbb{R}$ *with compact support*, we have

$$\int f d\mu_n \to \int f d\mu$$

Then $\mu_n \to \mu$ weakly, i.e. the above equation holds for all bounded continuous functions f. Hint: Start by showing that $\{\mu_n\}$ is tight. Then either proceed directly or use Prohorov's theorem and a double subsequence trick.

Proof. We want to be in a position to apply Prohorov's so we want to start by showing $\{\mu_n\}$ is tight. Since $\mathbb{R}^d = \cup_{m=1}^{\infty} B(0, m)$ we can take m sufficiently large so that $\mu(B(0, m)) > 1 - \epsilon/3$. Let $B(0, m) = K$. Then we may approximate χ_K by an increasing sequence of bump functions $\{f_i\}$, and so, in particular, we may take an i so that $|\int f_i d\mu - \int \chi_B d\mu| < \epsilon/3$. Certainly f_i is compactly supported and continuous, and so $\int f_i d\mu_n \to \int f_i d\mu$ and lopping of finitely many μ_n we can regard $\{\mu_n\}$ as a sequence of measures so that $|\int f_i d\mu_n - \int f_i d\mu| < \epsilon/3$ for all n. So in particular, we have $|int f_i d\mu_n - \int \chi_K d\mu| = |\int f_i d\mu_n - \int f_i d\mu + \int f_i d\mu - \int f d\mu| < \epsilon/3 + \epsilon/3 = 2/3\epsilon$, so in particular, $\int f_i d\mu_n > 1 - \epsilon$. Further $f_i \leq \chi_B$ and so $\int \chi_B d\mu_n > \int f_i d\mu_n > 1 - \epsilon$ for all n and μ_n is tight, as desired.

Notice that the double subsequence trick still works for weak convergence. Namely, if μ_n is a sequence of measures and every subsequence has a further subsequence that converges weakly to μ then $\mu_n \to \mu$. Say not, then there exists a bounded continuous f and a subsequence $\{\mu_{m(n)}\}$ such that $\int f d\mu_{m(n)}$ is bounded away from $\int f d\mu$ for all n. Then certainly this subsequence can have no subsequence converging to $\int f d\mu$ as for all n $\int f d\mu_{m(n)}$ is bounded away from $\int f d\mu$. So we can still apply the double subsequence trick.

Now having established μ_n tight and that the double subsequence trick will work, we use Prohorov's theorem and the double subsequence trick. Namely, for any subsequence $\{\mu_{m(n)}\}$ of $\{\mu_n\}$, $\{\mu_{m(n)}\}$ is still tight (as it is just a subsequence of $\{\mu_n\}$) and thus has a weakly convergent subsequence. Then by the double subsequence trick, this means $\mu_n \to \mu$, weakly, as desired. □

Problem 2. If μ, ν are probability measures on $\mathbb{R}^d$, their **convolution** is the probability measure $\mu * \nu$ defined by

$$(\mu * \nu)(B) = \int \int 1_B(x + y)\mu(dx)\nu(dy)$$

(a) Verify that $\mu * \nu$ is indeed a probability measure.

(b) For any bounded measurable f, $\int f d(\mu * \nu) = \int \int f(x + y)\mu(dx)\nu(dy)$.

(c) If $X \sim \mu$, $Y \sim \nu$, and X, Y are independent, then $X + Y \sim \mu * \nu$.

(d) If $\mu_n \to \mu$ weakly, then $\mu_n * \nu \to \mu * \nu$ weakly.

(e) (Bonus) If $\mu_n \to \mu$ weakly and $\mu_n \to \nu$ weakly, then $\mu_n * \nu_n \to \mu * \nu$.

Solution. (a) First we'll show that $\mu * \nu(\varnothing) = 0$ and $\mu * \nu(\mathbb{R}^d) = 1$. First for $\varnothing$, $\mu * \nu(\varnothing) = \int \int 1_\varnothing(x + y)\mu(dx)\nu(dy) = \int \int 0\mu(dx)\nu(dy) = 0$, as desired. Similarly for $\mathbb{R}^d$, $\mu * \nu(\mathbb{R}^d) = \int \int 1_{\mathbb{R}^d}(x + y)\mu(dx)\nu(dy) = \int \int 1\mu(dx)\nu(dy) = 1$ as μ, ν are probability measures. Further, for any B we have $\mu * \nu(\varnothing) = \int \int 0\mu(dx)\nu(dy) \leq \int \int 1_B(x + y)\mu(dx)\nu(dy) \leq \int \int 1\mu(dx)\nu(dy) = 1$ so $\mu * nu(B) \in [0, 1]$ for all B.

Additivity is immediate from the additivity of the integral with respect to any measure which itself is defined in terms of a limit of sums, so we have $\mu * \nu$ is a probability measure.

(b) Assume f is nonnegative. This follows from writing out $\int f d(\mu * \nu)$ as a limit of simple functions f_n, in particular,

$$\begin{aligned}
\int f(x + y)d(\mu * \nu) &= \\
&= \lim \sum_n a_n \mu * \nu(x + y \; : \; f(x + y) \in [a_n, a_{n+1})) \\
&= \lim \sum_n a_n \int \int 1_{\{x+y \, : \, f(x+y) \in [a_n, a_{n+1})\}} \mu(dx)\nu(dy) \\
&= \lim \int \int \sum a_n 1_{\{x \, : \, f(x+y) \in [a_n, a_{n+1})\}} \mu(dx)\nu(dy) \\
&= \int \int \lim a_n 1_{\{x \, : \, f(x+y) \in [a_n, a_{n+1})\}} \mu(dx)\nu(dy) \\
&= \int \int f(x + y)\mu(dx)\nu(dy)
\end{aligned}$$

where the last line is our desired inclusion. We were able to exchange our sum with our integrals because for all n the sum involved was finite and we were able to exchange our sum and our limit by the monotone convergence theorem.

(c) We want to reason about $P(X + Y \in A)$. This is precisely the setting of problem 2 on homework 3. Namely, we consider the random vector (X, Y) where X has density μ and Y has density ν. Let $A' = \{(x, y) \; : \; x + y \in A\}$, then we have:

$$\begin{aligned}
P(X + Y \in A) &= \\
&= P((X, Y) \in A') \\
&= \int \int_{A'} \mu\nu dxdy \\
&= \int \int_{\mathbb{R}^2} 1_A(x + y)\mu\nu dxdy \\
&= \int \int 1_A(x + y)\mu(dx)\nu(dy) \\
&= \mu * \nu(A)
\end{aligned}$$

which is the desired equality.

(d) Let's consider continuous f with compact support K (this suffices by problem 1). Then $\int f(x+y)\mu_n(dx)$ is bounded by the integrable $C\chi_K(y)$ where C demonstrates f bounded and so we if we can show $\int f(x+y)\mu_n d(x) \to \int f(x+y)d\mu$ a.e., then we will have $\int \int f(x+y)\mu_n(dx)\nu(dy) \to \int f(x+y)\mu(dx)\nu(dy)$ by the dominated convergence theorem, which shows $\mu_n * \nu \to \mu * \nu$ weakly, but $f(x+y)$ is still a continuous, compactly supported function for any fixed y, so $\int f(x+y)\mu_n(dx) \to \int f(x + y)\mu(dx)$ by the assumption $\mu_n \to \mu$ weakly.

Problem 3. Suppose X_n, Y_n are random variables (not necessarily independent) and we have $X_n \to X$ weakly and $Y_n \to c$ in probability.

(a) Show that $X_n + Y_n \to X + c$ weakly. Sometimes called Slutsky's theorem. Use Problem 1 and the fact that compactly supported continuous functions are uniformly continuous.

(b) Show that $X_nY_n \to cX$ weakly.

(c) Suppose instead that $X_n \to X$ weakly and $Y_n \to Y$ where Y need not be constant. Show that we need not have $X_n + Y_n \to X + Y$ weakly.

Solution. (a) I think I'm going to use a slightly different approach as was recommended in the hint, basically I'm appealing to the dominated convergence theorem instead of problem 1. I'll still assume f is continuous and compactly supported, I just think I use it in a different way...

Namely, let $X_n \sim \mu_n$, $Y_n \sim \nu_n$, $X \sim \mu$ and $c \sim \delta_c$ and consider continuous and compactly supported f. Then for any y, $\int f(x+y)\delta_c(dx) = f(c-y)$. On the other hand, $\int f(x+y)\nu_n(dx) = \int_{B(c-y,\epsilon)} f(x)\nu_n(dx) + \int_{B(c-y,\epsilon)^c} f(x)\nu_n(dx) \to (1-\epsilon)f(c-y) + \epsilon C \to f(c-y)$. The first arrow follows from the fact that $Y_n \to c$ in probability, so all by ϵ of the mass of μ_n must be within a radius of ϵ of $c-y$ for sufficiently large n and by continuity of f where C is the bound of f (f is a continuous function on a compact set, hence bounded). But this establishes $\int f(x+y)\nu_n(dx) \to \int f(x+y)\delta_c(dx)$ pointwise. Further $\int f(x+y)\nu_n(dx)$ is bounded by the integrable function $C \cdot \chi_K(y)$ where K is the support of f, so the dominated convergence theorem applies, and we have: $\int\int f(x+y)\nu_n(dx)\mu_n(dy) \to \int\int (f+x)\delta_c(dx)\mu(dy)$ for all continuous, compactly supported f, so $X_n + Y_n \to X + c$ weakly, as desired.

(b) We adapt our approach from (a) slightly. I'll highlight the differences.

Namely, let $X_n \sim \mu_n$, $Y_n \sim \nu_n$, $X \sim \mu$ and $c \sim \delta_c$ and consider continuous and compactly supported f. Then for any y, $\int f(xy)\delta_c(dx) = f(c/y)$ (we may assume $y \neq 0$ a.s. if $c = 0$ then this integral is just equal to $f(0)$ trivially). On the other hand, $\int f(xy)\nu_n(dx) = \int_{B(c/y,\epsilon)} f(x)\nu_n(dx) + \int_{B(c/y,\epsilon)^c} f(x)\nu_n(dx) \to (1-\epsilon)f(c/y) + \epsilon C \to f(c/y)$. From here everything is really the same. It was just before we had the linear function $x - y$ to deal with and now we have the function x/y to deal with. Using my approach from (a) it works out the same.

(c) Let X_n be the uniform distribution on $\{[0, 1/n] \cup [1 - 1/n, 1]\}$ and let $Y_n = -X_n$. Let $Y = -1$ with probability $1/2$ and 0 with probability $1/2$, and $X = 1$ with probability $1/2$ and 0 with probability $1/2$. Then $X_n \to X$ weakly and $Y_n \to Y$ weakly, but $X_n + Y_n = 0$ for all n while $X + Y$ has distribution:

$$X + Y = \begin{cases} 0 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/4 \\ -1 & \text{with probability } 1/4 \end{cases}$$

Letting $f = x^2$, $\int f d(\mu_{X_n+Y_n}) = 0$ for all n while $\int f d(\mu_{X+Y}) = 1/2$, so $X_n + Y_n$ does not converge to $X + Y$ weakly.