Math 4140 – Honors Analysis II (6 pages)	Due: Thursday, February 2, 2012

Homework 1

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9.1.4 Exercises : 4,8,9,10,11,12,15

Problem (4). Prove that $||x||_{\sup} = \lim_{p\to\infty} ||x||_p$ on $\mathbb{R}^n$.

Proof. Let $x_M = \max_{1\le i\le n} x_i = ||x||_{\sup}$. Then we want to prove $||x||_{\sup} = x_M = \lim_{p\to\infty}(\sum |x_i|^p)^{1/p}$. This follows from multiplying by $1 = \frac{x_M^p}{x_M^p}$ appropriately. Namely,

$$\begin{aligned}\lim_{p\to\infty}(\textstyle\sum_{i=1}^n |x_i|^p)^{1/p} &= \lim_{p\to\infty}(\frac{x_M^p}{x_M^p}\cdot \textstyle\sum_{i=1}^n |x_i|^p)^{1/p} \\ &= \lim_{p\to\infty} x_M^{p/p}\cdot (\textstyle\sum_{i=1}^n |(x_i^p/x_M^p)|)^{1/p} \\ &= x_M \cdot \lim_{p\to\infty}(\textstyle\sum_{i=1}^n (|x_i/x_M|)^p)^{1/p}\end{aligned}$$

By assumption, $x_M = \max_{1\le i\le n} x_i$ so $|x_i/x_M| \le 1$ for all i. If $|x_i/x_m| < 1$ then $(x_i/x_M)^p \to 0$ as $p \to \infty$ and those x_i do not effect the sum. Say there are q x_i such that $\left|x_i/x_m\right| = 1$. Then we have

$$x_M \cdot \lim_{p\to\infty}(\sum_{i=1}^n (|x_i/x_M|)^p)^{1/p} = x_M \cdot \lim_{p\to\infty} q^{(1/p)}$$

but q is a fixed positive integer for all p so $q^{1/p} \to 1$ as $p \to \infty$ and so we have

$$\lim_{p\to\infty}(\sum_{i=1}^n |x_i|^p)^{1/p} = x_M \cdot \lim_{p\to\infty} q^{(1/p)} = x_M \cdot 1 = x_M$$

establishing $||x||_{\sup} = \lim_{p\to\infty} ||x||_p$. □

Problem (8). Prove that if a norm $||x||$ on a real vector spaces satisfies the parallelogram law, then the polarization identity defines an inner product and that the norm associated with this inner product is the original norm.

Solution. The tricky condition to verify is bilinearity. First we start with symmetry and positive definiteness. Define our proposed inner product by the polarization identity: $x \cdot y := \frac{1}{4}(\left|\left|x+y\right|\right|^2 - \left|\left|x-y\right|\right|^2)$.

$x \cdot y = y \cdot x$.

Proof. This is a straightforward check.

$$\begin{aligned}x\cdot y &= \tfrac{1}{4}(\left|\left|x+y\right|\right|^2 - \left|\left|x-y\right|\right|^2) \\ &= \tfrac{1}{4}(\left|\left|y+x\right|\right|^2 - \left|\left|-1\cdot(y-x)\right|\right|^2) \\ &= \tfrac{1}{4}(\left|\left|y+x\right|\right|^2 - \left|\left|y-x\right|\right|^2) \\ &= y\cdot x\end{aligned}$$

□

$x \cdot x \geq 0$ with $x \cdot x = 0$ if and only if $x = 0$.

Proof. Consider $x \cdot x = \frac{1}{4}(||x+x||^2 - ||x-x||^2)$, $||\cdot||$ is a norm so $||x-x|| = ||0|| = 0$ and we have

$$x \cdot x = \frac{1}{4}(||x+x||^2) = \frac{1}{4}||2x||^2 = \frac{1}{4}\cdot 4\,||x||^2 = ||x||^2$$

Our norm is positive definite, so the result holds. □

$(ax+by)\cdot z = ax\cdot z + by \cdot z$ and $x\cdot(ay+bz) = ax\cdot y + bx \cdot z$.

Proof. First I want to show the distributive part, i.e., $x+y\cdot z = x\cdot z + y\cdot z$. First, using the parallelogram law for the vectors $x+z$ and y, we get:

$$\left|\left|(x+z)+y\right|\right|^2 + \left|\left|(x+z)-y\right|\right|^2 = 2(||x+z||^2 + \left|\left|y\right|\right|^2)$$

and similarly for $y+z$ and x:

$$\left|\left|(y+z)+x\right|\right|^2 + \left|\left|(y+z)-x\right|\right|^2 = 2(\left|\left|y+z\right|\right|^2 + ||x||^2)$$

we can then solve both for $\left|\left|x+y+z\right|\right|^2$ and get two different expressions for it:

$$\begin{aligned}\left|\left|x+y+z\right|\right|^2 &= 2\,||x+z||^2 + 2\left|\left|y\right|\right|^2 - \left|\left|x+z-y\right|\right|^2\\ &= 2\left|\left|y+z\right|\right|^2 + 2\,||x||^2 - \left|\left|y+z-x\right|\right|^2\end{aligned}$$

Next we 'average' these two expressions. I.e. if $M = N$ and $M = 0$, then $M = \frac{1}{2}(N+O)$. Applying this for $M = \left|\left|x+y+z\right|\right|^2$ and N, O, the RHS of the equalities derived above, we get:

$$\left|\left|x+y+z\right|\right|^2 = ||x+z||^2 + \left|\left|y\right|\right|^2 - \frac{1}{2}\left|\left|x+z-y\right|\right|^2 + \left|\left|y+z\right|\right|^2 + ||x||^2 - \frac{1}{2}\left|\left|y+z-x\right|\right|^2$$

substituting $-z$ instead of z in the above equality we get:

$$\left|\left|x+y-z\right|\right|^2 = ||x-z||^2 + \left|\left|y\right|\right|^2 - \frac{1}{2}\left|\left|x-z-y\right|\right|^2 + \left|\left|y-z\right|\right|^2 + ||x||^2 - \frac{1}{2}\left|\left|y-z-x\right|\right|^2$$

Consider the term $\frac{1}{2}\left|\left|x-z-y\right|\right|$ in the latter equality and $\frac{1}{2}\left|\left|y+z-x\right|\right|$ in the former. We have $\frac{1}{2}\left|\left|y+z-x\right|\right| = \frac{1}{2}\left|\left|-(x-y-z)\right|\right| = \frac{1}{2}\left|\left|x-y-z\right|\right|$, and these terms are equivalent. The same argument applies to the terms $\frac{1}{2}\left|\left|y-z-x\right|\right|^2$ and $\frac{1}{2}\left|\left|x+z-y\right|\right|^2$.

Combing all of these, we are ready to calculate $x+y\cdot z$, namely:

$$\begin{aligned}x+y\cdot z &=\\ &= \tfrac{1}{4}(\left|\left|x+y+z\right|\right|^2 - \left|\left|x+y-z\right|\right|^2)\\ &= \tfrac{1}{4}(||x+z||^2 + \left|\left|y\right|\right|^2 - \tfrac{1}{2}\left|\left|x+z-y\right|\right|^2 + \left|\left|y+z\right|\right|^2 + ||x||^2 - \tfrac{1}{2}\left|\left|y+z-x\right|\right|^2\\ &\quad -(||x-z||^2 + \left|\left|y\right|\right|^2 - \tfrac{1}{2}\left|\left|x-z-y\right|\right|^2 + \left|\left|y-z\right|\right|^2 + ||x||^2 - \tfrac{1}{2}\left|\left|y-z-x\right|\right|^2))\\ &= \tfrac{1}{4}(||x+z||^2 + \left|\left|y+z\right|\right|^2 - (||x-z||^2 + \left|\left|y-z\right|\right|^2))\\ &= \tfrac{1}{4}(||x+z||^2 - ||x-z||^2) + \tfrac{1}{4}(\left|\left|y+z\right|\right|^2 - \left|\left|y-z\right|\right|^2)\\ &= x\cdot z + y\cdot z\end{aligned}$$

Now we prove the homogeneity-like condition on the inner product, namely: $(ax)\cdot y = a(x\cdot y)$ for all $a \in \mathbb{R}$. For $a = 1$, the result is trivial. Then for all $n > 1$, $n \in \mathbb{N}$, we argue by induction. Namely, given $(ax)\cdot y$, we have $(ax)\cdot y = ((a-1)x+x)\cdot y = ((a-1)x)\cdot y + x\cdot y = (a-1)(x\cdot y)+x\cdot y = a(x\cdot y)$ by applying the linearly condition we just proved for $\cdot$ and our inductive hypothesis. The result holds trivial for 0 as well so now we have the result for all $n \in \mathbb{N}$. By inspection, $-1x\cdot y = \frac{1}{4}(\left|\left|-x+y\right|\right|^2 - \left|\left|-x-y\right|\right|^2) = \frac{1}{4}(-1(\left|\left|x+y\right|\right|^2 - \left|\left|x-y\right|\right|)) = -1(x\cdot y)$, as desired. The same inductive proof as for positive n, with our new inductive hypothesis shows the result holds for all $n \in \mathbb{Z}$.

Now say $n = \frac{p}{q} \in \mathbb{Q}$. Then $q * (\frac{p}{q}x\cdot y) = (q * \frac{p}{q}x\cdot y) = (px\cdot y) = p(x\cdot y)$, diving both sides of the equality by q, we get: $(\frac{p}{q}x\cdot y) = \frac{p}{q}(x\cdot y)$, and we have the result for all $n \in \mathbb{Q}$.

We've shown the result for all rational numbers. Consider $r \in \mathbb{R}$, finally, it suffices to show $\left|r(x\cdot y) - (rx\cdot y)\right| = 0$. Let $q \in \mathbb{Q}$, and using the facts we've proved about our proposed norm so far, we have:

$$\begin{aligned}\left|r(x\cdot y)-(rx\cdot y)\right| &= \\ &= \left|r(x\cdot y) - q(x\cdot y) + q(x\cdot y) - (rx\cdot y)\right| \\ &= \left|(r-q)(x\cdot y) + ((q-r)x\cdot y)\right|\end{aligned}$$

Now we can use the Cauchy-Schwartz inequality (theorem 9.1.1 in the book) to get: $\left|r(x\cdot y)-(rx\cdot y)\right| = \left|(r-q)(x\cdot y)+((q-r)x\cdot y)\right| \le 2 * \left|q-r\right| ||x|| \left|\left|y\right|\right|$. $r \in \mathbb{R}$, so it is representable by a Cauchy sequence of rational numbers. Then for all $\epsilon > 0$, we can choose a rational number such that $\left|q-r\right| \le \epsilon/(2||x||\left|\left|y\right|\right|)$ so that $0 \le \left|r(x\cdot y)-(rx\cdot y)\right| \le \epsilon$ for all $\epsilon > 0$ proving $\left|r(x\cdot y)-(rx\cdot y)\right| = 0$ and $r(x\cdot y) = (rx\cdot y)$ for all $r \in \mathbb{R}$.

Combining these two facts, (i.e. linearity and homogeneity) we get the desired: $(ax+by)\cdot z = a(x\cdot z)+b(y\cdot z)$. For $x\cdot(ay+bz) = a(x\cdot y)+b(x\cdot z)$, use symmetry to get $x\cdot(ay+bz) = (ay+bz)\cdot x = a(y\cdot x)+b(z\cdot x) = a(x\cdot y)+b(x\cdot z)$. This completes verifying $\cdot$ is an inner product. □

The induced norm from our inner product is our original norm.

Proof. The induced norm would be defined $(x\cdot x)^{\frac{1}{2}}$. We check that this gives us what we want.

$$(x\cdot x)^{\frac{1}{2}} = (\frac{1}{4}(||x+x||^2) = \frac{1}{4}||2x||^2)^{\frac{1}{2}} = (\frac{1}{4}\cdot 4||x||^2)^{\frac{1}{2}} = (||x||^2)^{\frac{1}{2}} = ||x||$$

and we have the desired property. □

Problem (9)**.** Prove that if $||x||$ is any norm on $\mathbb{R}^n$, then there exists a positive constant M such that $||x|| \le M|x|$ for all x in R^n where $|x|$ is the Euclidean norm. Hint: $M = (\sum_{j=1}^n \left|\left|e_j\right|\right|^2)^{1/2}$ will do.

Proof. First let $x = \sum a_i e_i$ and notice that $||x||_\infty \le ||x||_2$ for all x. To see this, consider cases. First if $||x||_\infty = M \ge 1$, then $||x||_2 = (\sum a_i^2)^{1/2}$. $\sum a_i^2 \ge M^2$ so $||x||_2 \ge M$. If $M < 1$ consider the case where $\sum a_i^2 > 1$, then $(\sum a_i^2)^{1/2} > 1 > M$ so $||x||_2 > ||x||_\infty$. Finally consider the case where $M < 1$ and $(\sum a_i^2)^{1/2} < 1$, then $\sum a_i^2 > M^2$ and so we still have $(\sum a_i^2)^{1/2} > M$. So we have $||x||_\infty \le ||x||_2$ for all $x \in \mathbb{R}^n$.

Now consider $||x||$ for any norm $||\cdot||$. $||x|| = \left|\left|\sum a_i x_i\right|\right| \le \sum \left|\left|a_i x_i\right|\right|$ by the triangle inequality. Once again letting $||x||_\infty = M$ we have $||x|| \le \sum \left|\left|a_i e_i\right|\right| \le \sum M \left|\left|e_i\right|\right| = M\cdot \sum \left|\left|e_i\right|\right|$. We've already shown that $||x||_2 \ge M$ and now we've shown that $M \sum \left|\left|e_i\right|\right| \ge ||x||$ for any norm. Combing these, we get $||x|| \leq (\sum ||e_i||) ||x||_2$ and letting $N = \sum ||e_i||$ means we have found a constant N such that $||x|| \leq N ||x||_2$, as desired. □

Problem (10)**.** Prove that the norm $||x||_1$ on $\mathbb{R}^n$ for $n > 1$ is not associated with an inner product by violating the parallelogram law. Do the same for $||x||_{\text{sup}}$.

Solution.
Norms induced by an inner product satisfy the parallelogram law.

Proof. We want to show that every norm induced by an inner product satisfies the parallelogram law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2).$$

This is really just iterated application of bilinearity of the inner product. The norm written in terms of the inner product is $||x|| = \langle x, x\rangle$ and so we rewrite our expression for the parallelogram law accordingly.

$$\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\langle x, x\rangle + 2\langle y, y\rangle$$

Now we use bilinearity (and symmetry where it makes things look cleaner) on the left hand side to further rewrite:

$$\begin{aligned}
\langle x+y, x+y\rangle + \langle x-y, x-y\rangle &= \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle + \langle -y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle x, x\rangle + \langle x, -y\rangle + \langle x, -y\rangle + \langle -y, -y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle + 2\langle x, y\rangle - 2\langle x, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle + 0 \\
&= 2(||x||^2 + ||y||^2)
\end{aligned}$$

And we have the desired equality making satisfying the parallelogram identity a necessary property of a norm induced by an inner product. □

In both cases, the parallelogram law, $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$, fails for a simple square defined by $x = (1,0)$, $y = (0,1)$. This suffices to show that these norms do not arise from inner products, as if they did, they would satisfy the parallelogram law as shown above. We only give an example for $\mathbb{R}^2$, as the same example holds in $\mathbb{R}^n$ for all $n \geq 2$ by using the same vectors with zero's in all of the coordinates after the first 2.

For $||\cdot||_1$.

Proof. In this norm, we have $||(1,0)||_1 = 1$, $||(0,1)||_1 = 1$, $||x-y||_1 = ||(1,-1)||_1 = 2$, and $||x+y||_1 = ||(1,1)||_1 = 2$.

Then plugging in for the LHS of the equality we would have: $2^2 + 2^2 = 8$, but plugging in for the RHS we would have $2 \cdot (1+1) = 4 \neq 8$, so this norm is not induced by an inner product. □

For $||\cdot||_{\text{sup}}$.

Proof. In this norm, we have $||(1,0)||_\infty = 1$, $||(0,1)||_\infty = 1$, $\left|\left|x-y\right|\right|_\infty = ||(1,-1)||_\infty = 1$, and $\left|\left|x+y\right|\right|_\infty = ||(1,1)||_\infty = 1$.

Plugging in for the LHS we get: $1+1=2$ while for the RHS we get $2\cdot 2 = 4 \neq 2$, so this norm is also not induced by an inner product. ☐

Problem (11)**.** Prove that a real $n\times n$ matrix A satisfies $Ax\cdot Ay = x\cdot y$ for all x and y in $\mathbb{R}^n$ if and only if $|Ax| = |x|$ for all x in $\mathbb{R}^n$. Such matrices are called *orthogonal*.

Solution.
If $Ax\cdot Ay = x\cdot y$ for all $x,y\in\mathbb{R}^n$ then $|Ax| = |x|$ for all $x\in\mathbb{R}^n$.

Proof. Because $Ax\cdot Ay = x\cdot y$ for all $x,y\in\mathbb{R}^n$, we have $|Ax|^2 = Ax\cdot Ax = x\cdot x = |x|^2$ and so we have $|Ax|^2 = |x|^2$. Taking square roots and noting $|\cdot|$ is always nonnegative, we have $|Ax| = |x|$. ☐

If $|Ax| = |x|$ for all x then $Ax\cdot Ay = x\cdot y$ for all x,y.

Proof. Becuase we are dealing with a norm induced by an inner product, the polarization inequality applies.

$x\cdot y = \frac{1}{4}(\left|x+y\right|^2 - \left|x-y\right|^2)$ and $Ax\cdot Ay = \frac{1}{4}(\left|Ax+Ay\right|^2 - \left|Ax-Ay\right|^2)$. Notice, by the distrbutive property of linear transformations we have $Ax+Ay = A(x+y)$, in particular this means $\left|Ax+Ay\right| = \left|A(x+y)\right| = \left|x+y\right|$ applying our assumption that $|Az| = |z|$ for all $z\in\mathbb{R}^n$. This allows us to see:

$$\begin{aligned} Ax\cdot Ay &= \tfrac{1}{4}(\left|Ax+Ay\right|^2 - \left|Ax-Ay\right|^2) \\ &= \tfrac{1}{4}(\left|A(x+y)\right|^2 - \left|A(x-y)\right|^2) \\ &= \tfrac{1}{4}(\left|x+y\right|^2 - \left|x-y\right|^2) \\ &= x\cdot y \end{aligned}$$

This establishes our desired result. ☐

Problem (12)**.** Prove that a real $n\times n$ matrix is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.

Solution.
If A is orthogonal, its vectors form an orthonormal basis of $\mathbb{R}^n$.

Proof. Say some vector making up a column of A is not of unit length, say it is column i, denote this column y_i. Then $Ae_i = y_i$. $\left|y_i\right| \neq 1$ by assumption, but $\left|e_i\right| = 1$, this contradicts that $|Ax| = |x|$ for all $x\in\mathbb{R}^n$ so all columnns of A are of unit length.

Now say that there are two non-orthogonal columns of A, y_i, y_j. $e_i\cdot e_j = 0$ because e_i, e_j are ortogonal. But then $Ae_i\cdot Ae_j = y_i\cdot y_j \neq 0$ because by assumption both y_i, y_j are of unit length and not orthogonal. This is a contradiction so the columns of A are orthogonal.

Together these two facts along with the fact that A is an $n\times n$ matrix makes the columns of A an orthonomal basis of $\mathbb{R}^n$. ☐

If the columns of A form an orthonomal basis of $\mathbb{R}^n$ then A is orthogonal.

Proof. We'll show $|Ax| = |x|$ for all x. First write $x = \sum a_i e_i$ fixing some orthonormal basis $\{e_i\}$ of $\mathbb{R}_n$. Notice that $e_i \mapsto y_i$ where y_i is the i-th column of A, by assumption this is a unit vector perpendicular to the other vectors making up columns of A. Applying the distrbutive property and the fact that linear transformations commute with constants we have:

$$\begin{aligned} |Ax| &= \\ &= \left|A(\sum a_i e_i)\right| \\ &= \left|\sum a_i A e_i\right| \\ &= \left|\sum a_i A e_i\right| \\ &= \left|\sum a_i y_i\right| \\ &= (\sum a_i^2)^{1/2} \\ &= |x| \end{aligned}$$

The equality $\left|\sum a_i y_i\right| = (\sum a_i^2)^{1/2}$ comes from the fact that $\{y_i\}$ is, by assumption an orthonormal basis of $\mathbb{R}^n$. ☐

Problem (15). Verify that $d(x,y) = \left|x-y\right| / (1 + \left|x-y\right|)$ defines a metric on $\mathbb{R}^n$ but this metric is not induced by any norm because homogeneity fails.

Solution.
Checking we have a metric.

Proof. $d(x,y) = \left|x-y\right| / (1+\left|x-y\right|)$ is a quotient of nonnegative number by a positive number hence always positive. Say $d(x,y) = 0$, then our numerator must be zero so $\left|x-y\right| = 0$. $|\cdot|$ is just the metric induced by the norm $||\cdot||_1$ so $\left|x-y\right| = 0$ implies $x = y$, as desired.

$d(x,y) = \left|x-y\right| / (1+\left|x-y\right|) = \left|-(y-x)\right| / (1+\left|-(y-x)\right|) = \left|y-x\right| / (1+\left|y-x\right|) = d(y,x)$, once again using the fact that $\left|x-y\right|$ is a valid metric.

Finally, we check the triangle inequality. We want to show $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathbb{R}^n$. Consider the right hand side of the equation:

$$\begin{aligned} d(x,y) + d(y,z) &= \\ &= \frac{\left|x-y\right|}{1+\left|x-y\right|} + \frac{\left|y-z\right|}{1+\left|y-z\right|} \\ &= \frac{\left|x-y\right|(1+\left|y-z\right|) + \left|y-z\right|(1+\left|x-y\right|)}{(1+\left|x-y\right|)(1+\left|y-z\right|)} \\ &= \frac{\left|x-y\right| + \left|y-z\right|\left|x-y\right| + \left|y-z\right| + \left|y-z\right|\left|x-y\right|}{\left|x-y\right| + \left|y-z\right| + 2\left|y-z\right|\left|x-y\right|} \\ &= 1 \end{aligned}$$

So we have $d(x,y) + d(y,z) = 1$, but $d(x,z) = |x-z| / (1+|x-z|)$, whose denominator is greater than its numerator, hence the quotient is less than one, so we have $d(x,z) < 1 = d(x,y) + d(y,z)$ and the triangle inequality holds as desired. ☐

Homogeneity fails.

Proof. Given a norm, we defined our metric by $d(x,y) = \left|\left|x-y\right|\right|$, in particular, this means that $||x|| = d(x,0)$. Checking homogeneity:

$||ax|| = d(ax,0) = |ax| / (1+ax)$, this fails even in $\mathbb{R}$, letting $a = 2$ and $x = 1$, we have $2/3 = ||ax||$ but $a \cdot ||x|| = 2 \cdot (1/2) = 1$, thus this metric is not induced by a norm. ☐