Math 6710 – Probability (4 pages)	Due: Thursday, November 15, 2012
Homework 12	
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1,2,3,4,5

Problem 1. Hat guessing. Parts in-line.

Solution. (a) Show A_n is independent of G_n, $P(A_n) = 1/2$ and for $n > m$, $P(A_n|\mathcal{F}_m) = 1/2$ a.s.

Proof. First let's see that $P(A_n) = 1/2$. Because X_n is indepdent of X_i for all $i \neq n$ X_i is independent of Y_i and so:

$$\begin{aligned} P(A_n) &= P\{Y_n = X_n\} = P\{Y_n = H, X_n = H\} + P\{Y_n = T, X_n = T\} \\ &= P(X_n = H)P(Y_n = H) + P(X_n = T)P(Y_n = T) \end{aligned}$$

And we know $P(X_n = H) = P(X_n = T) = 1/2$ so:

$$P(X_n = H)P(Y_n = H) + P(X_n = T)P(Y_n = T) = 1/2(P(Y_n = H) + P(Y_n = T)) = 1/2$$

so $P(A_n) = 1/2$ as desired. Now for showing that A_n is independent of $\mathcal{G}_n$ take any $B \in \mathcal{G}_n$ and calculate $P(A_n, B)$ in a similar fashion using in independence of X_n from:

$$\begin{aligned} P(A_n, B) &= P(\{X_n = Y_n\},\ B) \\ &= P(\{X_n = H, Y_n = H\},\ B) + P(\{X_n = T, Y_n = T\},\ B) \\ &= P(\{X_n = H\})P(Y_n = H,\ B) + P(X_n = T)P(Y_n = T,\ B) \\ &= 1/2P(Y_n = H,\ B) + 1/2P(Y_n = T,\ B) \\ &= 1/2P(B) \\ &= P(A_n)P(B) \end{aligned}$$

which establishes the desired independence. Because A_n is independent of $\mathcal{G}_n$, it is independent of $\mathcal{G}_n$ as $\mathcal{F}_n \subseteq \mathcal{G}_n$. Then by the properties of conditional expectation: $P(A_n|\mathcal{F}_m) = E[1_A|\mathcal{F}_m] = E[1_A] = 1 \cdot P(A) + 0 \cdot P(A^c) = 1/2$ a.s., as desired. □

(b) Show that $P(A|\mathcal{F}_m) \le 1/2$ a.s.

Proof. By definition $P(A|\mathcal{F}_m) = E[1_A|\mathcal{F}_m]$. By homework 1, $1_A = \liminf 1_{A_n}$ and by conditional Fatou's lemma we have: $E[\liminf 1_{A_n}|\mathcal{F}_m] \le \liminf E[1_{A_n}|\mathcal{F}_m] = 1/2$ by the previous part. □

(c) Show $P(A) = 0$.

Proof. First we have to notice that $A \in \mathcal{F}_\infty$. Because $A = \liminf A_n$ it suffices to notice that $A_n \in \mathcal{F}_\infty$ for all n as $Y_n \in \mathcal{G}_n$ and $X_n \in \mathcal{F}_n$ and $\mathcal{G}_n, \mathcal{F}_n \in \mathcal{F}_\infty$ and $A_n = \{Y_n = X_n\}$. So $A_n \in \mathcal{F}_\infty$ for all n and so $\liminf A_n = A \in \mathcal{F}_\infty$. Now the Lévy 0-1 law applies and so $P(A|\mathcal{F}_n) \to P(A)$ a.s. This implies $P(A)$ is 0 or 1, but $P(A|\mathcal{F}_n) \le 1/2$ a.s. so it cannot converge to 1, so we must have $P(A) = 0$, as desired. □

(d) For all n describe a strategy where all the prisoners guess correctly with probability $1/2$.

Proof. First I will describe two versions of the strategy for $n = 2$ and $n = 3$ then I will generalize to arbitrary n by induction. For $n = 2$ we have two prisoners. There are two strategies where both prisoners guess correctly with probability $1/2$. The first strategy is that both prisoners act as those they have the same hat and the second is where they act as though they have different hats. The former strategy wins in the hats given out are $\{RR, BB\}$ and the latter strategy works if the hats given out are $\{RB, BR\}$, both are events with probability $1/2$. Call the former strategy the R-strategy and the latter strategy the B-strategy.

Now for $n = 3$ we describe the solution in terms of $n = 2$, namely if the last prisoner in line sees hats in the first two prisoners that can be won by the R-strategy–i.e. he sees the first two prisoners have hats RR or BB he guess R. If he sees the first two prisoners have a configuration of hats that can be won by the B-strategy he plays B. If the first two prisoners see the third prisoner has a red hat they play the R-strategy from $n = 2$ amongst themselves and if they see a blue hat they play the B-strategy from $n = 2$. Call this the R-strategy for $n = 3$. It will win on the following events: $\{RRR, BBR, RBB, BRB\}$, thus with probability $1/2$. Let the B-strategy be the same thing only with the third prisoner playing B when he hopes the first two prisoners will play an R-strategy and vice versa when we hopes they play the B-strategy. This strategy will win on events $\{RRB, BBB, RBR, BRR\}$ so also with probability $1/2$.

Now for arbitrary n, if the last player sees a configuration that can be won by an R-strategy in the first $n - 1$ players, he guesses R and the first $n - 1$ prisoners play the $n - 1$ R-strategy if they see the last person has a red hat. If the last player sees a configuration that can be won by a B strategy, he plays B and the first $n - 2$ players play the B-strategy among themselves if the last player has a blue hat. Again this gives $2^{n-2} \cdot 2 = 2^{n-1}$ possible events on which all the prisoners guess correctly so they have probability $1/2$ of all guessing correctly. □

Problem 2. Let $Z_1, Z_2, \ldots$ be iid integrable random variables with $E[Z_i] = 0$. Let θ be an integrable random varible which is independent of all of the Z_n. Let $Y_n = \theta + Z_n$ and show that $E[\theta|Y_1, \ldots, Y_n] \to \theta$ a.s. and in L^1.

Proof. By assumption θ is integrable so $E[\theta|Y_1, \ldots, Y_n]$ converges almost surely and in L^1 to some M_∞. We just need to show that $M_\infty = \theta$. By theorem 15.28 in the notes $M_\infty = E[\theta|Y_1, Y_2, Y_3, \ldots]$ and so it suffices to express θ as an almost sure limit of functions measurable in $\sigma(Y_1, Y_2, \ldots)$ so that we can conclude their limit, θ is in $\sigma(Y_1, \ldots)$. In particular let $V_n = (\sum Y_i)/n = \theta + (\sum Z_i)/n$. By the strong law of large numbers $(\sum Z_i)/n \to 0$ as $n \to \infty$ as $EZ_i = 0$, so we have $\theta + (\sum Z_i)/n \to \theta$ a.s. and θ is in $\sigma(Y_1, \ldots)$. By the uniqueness of conditional expectation $\theta = M_\infty$, as desired. □

Problem 3. Let ζ_i be iid with arbitrary, nonconstant, integrable distributions. Let $S_n = \sum \zeta_i$ and suppose there exists a $\theta > 0$ so that $E \exp(-\theta\zeta_i) = 1$.

Solution. (a) Show $E\zeta_i > 0$.

Proof. Letting $\phi(x)$ be $exp(x)$ we can apply Jensen's inequality, noting that exp is strictly convex and we assumed ζ_i were nonconstant so we get strict inequality:

$$\begin{array}{lll}\exp(E-\theta\zeta_i) & < E(\exp\theta\zeta_i) & \text{this is Jensen's inequality}\\ \log\exp E(-\theta\zeta_i) & < \log 1 & \\ E(-\theta\zeta_i) & < 0 & \\ E(\zeta_i) & > 0 & \end{array}$$

where the last line is our desired inequality. □

(b) Show $P(\tau < \infty) \leq \exp(-a\theta)$.

Proof. First notice that $X_n = \exp(-\theta S_n)$ is a martingale as

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n \cdot \exp(-\theta\zeta_i)|\mathcal{F}_n] = X_n \cdot 1$$

because ζ_i are independent and θ was chosen so that $E\exp(-\theta\zeta_i) = 1$. Furthermore because X_n is a martinagle, $EX_{n\wedge\tau} = EX_0 = 1$ and so we have

$$1 = EX_{n\wedge\tau} = \exp\theta a P(\tau < \infty) + \int_{\tau=\infty} X_{n\wedge\tau} d\mu \geq \exp\theta a P(\tau < \infty)$$

noting $X_n \geq 0$ to justify the inequality and using arithmetic to rearrange we have $P(\tau < \infty) \leq \exp^{-\theta a}$, as desired. □

(c) Show $\liminf S_n > -\infty$ a.s.

Proof. It suffices to show $P\{\liminf S_n \to -\infty\} = 0$. Say we have ω so that $\liminf S_n(\omega) \to -\infty$, then for all $a > 0$, $\tau_a < \infty$ and so $P(\liminf S_n = -\infty) \leq P(\tau_a < \infty) \leq \exp(-a\infty)$. Further, $\exp(-a\infty) \to 0$ as $a \to \infty$ so this is the desired result. □

(d)

Proof. I think we were supposed to notice that because ζ_i has a normal distribution, $1 = \int_\Omega \frac{1}{\sqrt{2\pi\sigma}} e^{(x-\mu)^2)/2\sigma^2} dx = \int e^{-\theta\zeta_i} dx$. I'm not really sure how to finish it off though. It seems like I want to solve for θ by completing the square on the left hand side, but I'm not really sure how this helps me. □

(e)

Proof. By (d) I should have wound up with an expression $\phi(\sigma,\mu) = \theta$. Then considering the event $\{\tau_{10} < \infty)\}$ coresponds to ever losing more than 10 million dollars, which was the amount you started with, you would want to bound the probability $P(\{\tau_{10} < \infty\}$ which by part (b) is $\leq \exp(a\phi(\sigma,\mu))$. □

Problem 4. Let ζ_i be iid with uniform distribution on $\{1,\ldots,N\}$.

Let $X_n = |\{\zeta_1,\ldots,\zeta_n\}|$ be the number of distinct values observed up to time n. Show that X_n is Markov chain.

Proof. As in Lemma 6.2.1 in the text, it suffices to define a transition probability function $p(i,j)$ and show $p(i,j) = P(X_{n+1} = j | X_i = n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$.

The transition probability function is:

$$p(i,j) = \begin{cases} 0 & \text{if } j < i,\ j > i+1 \\ \frac{N-i}{N} & \text{if } j = i+1 \\ \frac{i}{N} & \text{if } j = i \end{cases}$$

It is immediate that $P(X_{n+1} = j | X_n = i, \ldots, X_0 = i_0) = p(i,j)$. The ξ_i are iid with uniform distribution on $\{1, \ldots, N\}$ so which specific values have been taken earlier in the process does not help in determining the probability of a new or repeated value being obtained. □

Problem 5. Let S_n be symmetric simple random walk and let $X_n = \max\{S_i : 0 \le i \le n\}$. Show that X_n is not a Markov chain.

Proof. We just given an example where having the values of X_i for $i \le n+1$ gives a better estimate of $P(X_{n+1} = j)$ than only having the value at time n.

Let $j = 2$ and consider $P(X_8 = 2 | X_1 = -1, X_2 = -1, X_3 = -1, X_4 = -1, X_5 = -1, X_6 = 0, X_7 = 1) = 1/2$, because we know that $X_7 = 1$ and X_i for $i < 7$ is < 1, we know that $S_7 = 1$ so $X_8 = 2$ with probability $1/2$.

But if we only have $P(X_8 = 2 | X_7 = 1)$ we don't know that $S_7 = 1$, we only know $S_7 \le 1$ (with strict inequality with a sequence like $1 - 1 - 1 - 1 \ldots$) so $P(X_8 = 2 | X_7) < 1/2$ thus this isn't a Markov chain. □