Math 6520 – Differentiable Manifolds 1 (3 pages) Due: Wednesday, September 5, 2012

Homework 1

Instructor: Reyer Sjamaar *Costandino Dufort Moraites*

1-1,1-4,1-7,1-9

Problem 1 (1-1)**.** Let X be the est of all points $(x,y) \in \mathbb{R}^2$ such taht $y = \pm 1$, and let M be the quotient of X by the equivalence relation $(x,-1) \sim (x,1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. This is the *line with two origins.*

Proof. First we show M is locally Euclidean of dimension 1. Namely, for any $x \neq 0$, $(x-\epsilon, x+\epsilon)$ for $\epsilon < |x-0|$ is open as $\pi^{-1}(x-\epsilon, x+\epsilon)$ is a disjoint union of two open intervals. Further, it is itself homeomorphic to an open interval by the identity. For $x = (0,1)$, we notice that $(-\epsilon, +\epsilon)$ is open in M as $\pi^{-1}(-\epsilon,\epsilon) = ((-\epsilon,1),(\epsilon,1)) \cup ((-\epsilon,-1),(0,-1)) \cup ((0,-1),(\epsilon,1))$ which is open in $\mathbb{R}^2$, and it is homeomorphic to $(-\epsilon,\epsilon)$ by the identity. The same argument works for $x = (0,-1) \in M$ and M is locally Euclidean.

For second countability, we may take the image under the projection of a countable basis for $\mathbb{R}^2$. As sets of M are open if and only if they are open in $\mathbb{R}^2$ under π^{-1}, this gives a countable basis for M.

We see M is not Hausdorff by looking at the two origins. Consider any open sets U, V containing $(0,1)$, $(0,-1)$ respectively. Regarding the preimages of these sets under π, we see that $B_\epsilon(0,1) \subseteq \pi^{-1}(U)$ and $B_\epsilon(0,-1) \subseteq \pi^{-1}(V)$ as $\pi^{-1}(U), \pi^{-1}(V)$ are open in $\mathbb{R}^2$. But then $\pi(\epsilon/2,-1) = \pi(\epsilon/2,1)$ and so $\pi(\epsilon/2,-1) \in U, V$ so $U \cap V \neq \varnothing$ and M is not Hausdorff. □

Problem 2 (1-4)**.** If k is an integer between 0 and $\min(m,n)$, show that the set of $m \times n$ matrices whose rank is at least k is an open submanifold of $M(m \times n, \mathbb{R})$. Show that this is not true if "at least k" is replaced by "equal to k."

Proof. Let K denote the set of matrices of rank at least k. We'll show it's open by showing that for every $M \in K$, there exists an $\epsilon > 0$ so that $B_\epsilon(M) \subseteq K$, showing K is open. In particular, M is a matrix of rank at least k so it has a nonsingular minor of at least k. Because the determinant is a continuous function, we may choose ϵ sufficiently small so that all matrices in $B_\epsilon(M)$ cannot have determinant 0 on that minor by continuity, thus all $M' \in B_\epsilon(M)$ have rank at least k as desired.

Let's see why the statement isn't true if "at least k" is replaced by "equal to k". Let $k = 1$. We will give an example of a 2×2 matrix with rank 1 so that every open set containing it contains a matrix of rank 2.

Let $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then for any open set U containing M, there exists an ϵ so that $B_\epsilon(M) \subseteq U$. But then, $M' = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon/2 \end{pmatrix} \in B_\epsilon(M)$ and M' has rank $2 > 1$. □

Problem 3 (1-7)**.** *Complex projective n-space*, denoted by $\mathbb{CP}^n$, is the set of 1-dimensional complex-linear subspaces $\mathbb{C}^{n+1}$ with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topology manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $(x^1 + iy^1, \ldots, x^n + iy^n) \leftrightarrow (x^1, y^1, \ldots, x^n, y^n)$.

Proof. First we deal with showing $\mathbb{CP}^n$ is a topological manifold and then we check it has a smooth structure. Notice, that $\mathbb{CP}^n$ is immediately Hausdorff and second countable regarding it as a quotient of $\mathbb{C}^{n+1}$. In particular, given points representing $[x],[y]$ in $\mathbb{C}^{n+1}$, the angle between the lines (or rather planes) $[x],[y]$ is $\theta > 0$. Taking all lines (planes) whose angle to $[x]$ is less $\theta/2$ and calling this A and similarly for $[y]$ calling in B we see there are disjoint open sets containing $[x],[y]$ respectively so $\mathbb{CP}^n$ is Hausdorff. For second countability, we may take a countable basis of $\mathbb{C}^{n+1}$ and project it by π into $\mathbb{CP}^n$. Becuase sets in $\mathbb{CP}^n$ are open if and only if they are the projections of open sets in $\mathbb{C}^{n+1}$ and because we are projecting a basis of $\mathbb{C}^{n+1}$ this gives us a basis of $\mathbb{CP}^n$.

Now we deal with being locally Euclidean. This part is extremely similar to the steps for $\mathbb{RP}^{2n}$ as $\mathbb{R}^{2n+2}$ and $\mathbb{C}^{n+1}$ are topological equivalent via the homeomorphism $(x^1+iy^1,\dots,x^{n+1}+iy^{n+1}) \leftrightarrow (x^1,y^1,\dots,x^n,y^n+1)$. We will call this map ψ.

Now for each i let $\tilde{U}_i \subset \mathbb{C}^{n+1}\setminus\{0\}$ be the set wehre $z^i = x^i + iy^i \neq 0$. We know $\tilde{U}_i$ is open so its projection $\pi(\tilde{U}_i) = U_i$ is also open and $\pi|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a quotient map in the language of Lemma A.10. Then we define $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i[z^1,\dots,z^{n+1}] = (\frac{z^1}{z^i},\dots,\frac{z^{i-1}}{z^i},\frac{z^{i+1}}{z^i},\dots,\frac{z^{n+1}}{z^i})$$

which is well-defined as continuous by the fact it is unchanged by constant multiplication and Lemma A.10 respectively. Then to get a map from $U_i \to \mathbb{R}^n$ we take the composition $\psi\circ\phi_i$. We already mentioned ψ is a homeomorphsim so this map is continuous. In fact it is a homemophism with inverse given by: $(\psi\circ\phi_i)^{-1} = \phi_i^{-1}\circ\psi^{-1}$ where ψ^{-1} is the inverse of ψ and ϕ_i^{-1} is the inverse of ϕ_i given by $\phi_i^{-1}(z^1,\dots,z^n) = [z^1,\dots,z^{i-1},1,z^i,\dots,z^n]$. The sets U_i cover $\mathbb{CP}^n$ and so this shows $\mathbb{CP}^n$ is locally Euclidean.

Next we reason about the maps $\{\psi\circ\phi_i\}$ in greater detail to show they are smoothly compatible giving rise to a smooth structure on $\mathbb{CP}^n$. Let $\psi\circ\phi_i = \rho_i$. First, it is actually easier to reason about $\phi_j\circ\phi_i^{-1}$. Let $i > j$ for convenience (we would need to write three very similar versions of the same equation otherwise), then we get:

$$(\phi_j\circ\phi_i^{-1}(u^1,\dots,u^{2n}) = (\frac{u^1}{u^j},\dots,\frac{u^{j-1}}{u^j},\frac{u^{j+1}}{u^j},\dots,\frac{u^{i-1}}{u^j},\frac{1}{u^j},\frac{u^i}{u^j},\dots,\frac{u^n}{u^j})$$

This is a diffeomorphism from $\phi_i(U_i\cap U_j) \to \phi_j(U_i\cap U_j)$ as it is a polynomial on the factors. Then applying the diffeomorphisms ψ,ψ^-1 we get the desired diffeomorphism from $\psi\circ\phi_i(U_i\cap U_j) \to \psi\circ\phi_j(U_i\cap U_j)$. This establishes the smooth structure on $\mathbb{CP}^n$. □

Problem 4 (1-9)**.** Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary, and that it can be given a natural smooth structure in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point.

Proof. First we check that M is a topological manifold with boundary, then we see that charts we constructed are smoothly compatible. Being Hausdorff and second countable are immediate from the fact that M gets its topology as a subspace of $\mathbb{R}^n$, we can just take intersections of the relevant neighborhoods representing these properties in R^n to get them to demonstrate the properties in M.

For our charts, we will follow an approach similar to that used for defining stereographic projection. Namely, we will imagine $M\setminus\{N\}$ sitting on top of $\mathbb{R}^n$. Then our chart on $M\setminus\{N\}$ is the map obtained by stretching $M\setminus\{N\}$ out onto the half plane so the boundary of $M\setminus\{N\}$ is 'matched up' with the boundary of $\mathbb{H}^n$.

Before I describe my maps, let me also note that I'm not going to use stereographic projection as defined in the book. I'm using the version which projects S^{n-1} onto the copy of $\mathbb{R}^{n-1}$ tangent to the 'bottom' of S^{n-1}. This requires changing the constant c from 0 to -1 as mentioned on the planet math article `http://planetmath.org/encyclopedia/NorthPole.html`. If I didn't do this, I would have trouble extending my projection to all of M. I should also note that I am imagining M sitting on top of the hyperplane given by $x^n = 0$ basically 'immersed' in the half plane.

Let $x = (x^1, \ldots, x^n) \in M$, and associate with x, $p_x = (p^1, \ldots, p^n)$ which is the point on the boundary of M on the line through x and N, then the map $\phi_1 : M \setminus \{N\} \to \mathbb{R}^n$ is given by $(x^1, \ldots, x^n) \mapsto |s(x)|\frac{|x|}{|p_x|}(x_1, \ldots, x_n)$ where $s(x) = (p^1, \ldots, p^{n-1})/(1-p^n)$. Notice that $s(x)$ is just the stereographic projection of p_x and $|p_x|$ is a trigonometric function of x, then this map has inverse:

$$(x_1, \ldots, x_n) \mapsto |p_x| \cdot \frac{|x|}{|s(x)|}(x_1, \ldots, x_n)$$

We define $\phi_2 : M \setminus \{S\}$ as $\phi_1 \circ R$ where R is the composition of the reflection of M about the copy of $\mathbb{R}^{n-1}$ we were stereographically projecting onto followed by the translation of M up by 2. These maps are homeomorphisms as they are clearly bijective (for a given line through one of the poles they are just linear maps) and continuous as they are the product of continuous maps.

Now notice that ϕ_i have a natural extension to the open set $\{x \mid 0 < x^n < 2\}$ in $\mathbb{R}^n$ which contains all of $\phi_i(M \setminus \{N\} \cap M \setminus \{S\})$, except for the $\{x \mid x^n = 0\} \setminus (0, 2)$, by using the same formula. Looking at the composition on this extension we have: $\phi_1 \circ \phi_2^{-1} = \phi_1 \circ (\phi_1 \circ R)^{-1} = \phi_1 \circ R^{-1} \circ \phi_1^{-1}$ which is a composition of smooth maps–ϕ_1 is just a product of smooth maps and R is an invertible linear transformation, thus smooth as a map between subsets of $\mathbb{R}^n$. For extending about a neighborhood of points in $\{x \mid x^n = 0\} \setminus (0, 0)$, we define ϕ_1 for x such that $x^n > 2$ to be $R'(\phi_1(R'(x)))$ where R' is the reflection about the hyper plane given by $x^n = 2$. Using this redefinition of ϕ_1 appropriately, we get a smooth extension of $\phi_1 \circ \phi_2^{-1}$ to a neighborhood of all points in $\{x \mid x^n = 0\} \setminus (0, 0)$, notice however we would still be unable to extend the composition to a neighborhood of $(0, 0)$, but we don't this as $(0, 0) \notin \phi_i(M \setminus \{S\} \cap M \setminus \{N\})$.

Finally we note our maps ϕ_i were stereographic projection on the boundary of M, thus making the boundary of M map to the boundary of $\mathbb{H}^n$ and the interior points were mapped to the interior, as desired. $\square$