Math 6710 – Probability (3 pages)	Due: Thursday, October 18, 2012

Homework 8

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1, 2, 3,4,5

Problem 1. Let μ be a probability measure supported on the integers (i.e. $\mu(\mathbb{Z}) = 1$, so a random variable with distribution μ only takes integer variables). Let φ be its characteristic function.

(a) Show that φ is 2π-periodic, i.e. $\varphi(t + 2\pi) = t$ for all t.

(b) Use the previous part to show that $\int_{-\infty}^{\infty} |\varphi(t)|dt = \infty$.

(c) Prove the following "Fourier inversion formula": for any $k \in Z$, we have:

$$\mu(\{k\}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt}\varphi(t)dt$$

(d) Suppose $\mu_1, \mu_2, \ldots$ are supported on the integers. Show that $\mu_n \to \mu$ weakly if and only if $\mu_n(\{k\}) \to \mu(\{k\})$ for every $k \in \mathbb{Z}$.

Solution. **(a)** It helps to write it out, namely, $\varphi(t) = \int e^{itx}\mu(dx)$. And then e^{itx} is 2π periodic in t so $\varphi(t+2\pi) = \int e^{i(t+2\pi)x}\mu(dx) = \int e^{itx}\mu(dx)$, by periodicity of e, so $\varphi(t)$ is 2π-periodic, as desired.
(b) By the previous part, $\int |\varphi(t)|dt = \sum_{i=1}^{\infty} \int_0^{2\pi} |\varphi(t)|dt$ so showing that $\int_0^{2\pi} |\varphi(t)|dt > 0$ will suffice to show $\int |\varphi(t)|dt = \infty$. Using the fact that μ is supported on the integers, we can write:

$$\int_0^{2\pi} |\varphi(t)|dt = \int_0^{2\pi} |\sum_{n=-\infty}^{\infty} \mu(\{n\})e^{-itn}|dt$$

Further, we may take a compact ball so that $\mu(B) = 1 - \epsilon$ because μ is a measure, and then $\sum_{n=-\infty}^{\infty} \mu(\{n\}) \geq (1-\epsilon)\epsilon' > 0$ where $\epsilon' = \min_{n\in B} \mu(\{n\})$, so $\int_0^{2\pi} |\varphi(t)|dt \geq \int_0^{2\pi}(1-\epsilon)\epsilon' dt > 0$, which is what we wanted.
(c) Too sleepy!
(d) If $\mu_n \to \mu$ weakly, then by the Portmanteau theorem, for every $B \subseteq \mathbb{R}^d$ with $\mu(\partial B) = 0$, we have $\mu_n(B) \to \mu(B)$. For any integer, k, letting $B = [k-1/2, k+1/2$, $\partial B = \{k-1/2, k+1/2\}$ which has measure zero on μ_n for all n, and so $\mu_n(B) = \mu_n(\{k\})$ and we must have have $\mu_n(B) \to \mu(B)$ by the Portmanteau theorem so $\mu_n(\{k\}) \to \mu(\{k\})$, as desired.

In the other direction, an earlier homework problem shows that we can show weak convergence by considering only continuous, compactly supported functions, f. Then $\int f d\mu_n = \sum_{k\in B} f(k)\mu_n(\{k\})$ because f is compactly supported and so there are only finitely many integers in its domain. By assumption $\mu_n(\{k\}) \to \mu(\{k\})$ for all k and there are only finitely many such k in the sum making up $\int f d\mu_n, \int f d\mu$ so $\int f d\mu_n \to \int f d\mu$ and $\mu_n \to \mu$ weakly, as desired.

Problem 2. Let μ_n be a binomial distribution for n trials with success probability p_n. That is, $\mu(\{k\}) = \left(\begin{array}{c} n \\ k \end{array}\right) p_n^k(1-p_n)^{n-k}$ is the probability of getting k heads in n flips of a biased coin that comes up heads with probability p_n. Suppose that $np_n \to \lambda$ as $n \to \infty$. Show that μ_n converges weakly to the Poisson distribution with rate parameter λ, i.e. the measure μ with $\mu(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$.

Proof. We'll start by showing $\mu_n(\{k\}) \to \mu(\{k\})$ for all k. In particular, by definition:

$$\mu_n(\{k\}) = \left(\begin{array}{c} n \\ k \end{array} \right) p_n^k(1-p_n)^{n-k} = \frac{n!}{(n-k)!k!} p_n^k(1-p_n)^{n-k} = \frac{n!}{n^k(n-k)!} \frac{(np_n)^k}{k!}(1-p_n)^n(1-p_n)^{-k}$$

in this case k is constant, so for large n, $\frac{n!}{n^k(n-k)!} \to 1$, $\frac{(np_n)^k}{k!} \to \lambda^k/k^1$, $(1-p_n)^n = (1-\frac{np_n}{n})^n \to e^{-\lambda}$, and $(1-p_n)^{-k} \to 1^{-k} = 1$. And so we have $\mu_n(\{k\}) \to e^{-\lambda}\frac{\lambda^k}{k!}$, as desired.

To show $\mu_n \to \mu$ weakly, consider $\int f d\mu_n, \int f d\mu$. By a previous homework problem is suffices to consider f continuous and compactly supported, but then, because f is compactly supported, the domain D, on which $f(x) \neq 0$ for $x \in D$ is compact, hence contains only finitely many natural numbers, say $\{a_i\}_{i=1}^N$, thus $\int f d\mu_n = \sum_{i=1}^N f(a_i)\mu_n(\{a_i\})$ and $\int f d\mu = \sum_{i=1}^N f(a_i)\mu(\{a_i\})$. Then above, we have shown that for all i, $f(a_i)\mu_n(\{a_i\}) \to f(a_i)\mu(\{a_i\})$ and there are only finitely many such a_i, so $\int f d\mu_n \to \int f d\mu$, as desired. □

Problem 3. (Durrett 3.4.5) Let $X_1, X_2, \dots$ be iid with mean 0 and variance $\sigma^2 \in (0,\infty)$. Let $S_n = X_1 + \cdots + X_n$, and let $Q_n = X_1^2 + \cdots + X_n^2$. Show that $S_n/\sqrt{Q_n} \to N(0,1)$, weakly.

Proof. As in the proof of the CLT we can replace X_n with $X_n/\sqrt{var(X_n)}$ to get a random variable with variance 1, so that $Q_n = \sum X_i^2 = \sum 1 = n$ and then $S_n/\sqrt{Q_n} = S_n/\sqrt{n}$, but we proved that $S_n/\sqrt{n} \to N(0,1)$ as sufficient for establishing the CLT. □

Problem 4. Suppose X, Y are iid with mean 0 and variance 1. Show that X, Y are $N(0,1)$ iff $\frac{X+Y}{\sqrt{2}} \stackrel{d}{=} X \stackrel{d}{=} Y$. Try using chfs for one direction and the central limit theorem for the other.

Proof. For the first direction, it suffices to prove that the chfs are equal. By assumption we have $\phi_X(t) = \phi_Y(t) = e^{-t^2/2}$. Then, $\phi_{X+Y/\sqrt{2}} = \phi_X(t/\sqrt{2})\phi_Y(t/\sqrt{2}) = e^{-t^2/2}$, as desired.

For the reverse direction, we appeal to the central limit theorem. The central limit theorem says that $(X+Y)/\sqrt{2}$ should be more normally distributed than X, Y, but we have $(X+Y)/\sqrt{2} \stackrel{d}{=} X \stackrel{d}{=} Y$, so the distributions are the same. Intuitively, this says that X, Y can't get any more normally distributed so we must have X, Y are $N(0,1)$. Right now, I can't seem to see how to make this precise though. I'm really tired right now. It's been a very hectic week. □

Problem 5. Let $X_1, \dots,$ be iid with mean μ and variance $\sigma^2 \in (0,\infty)$. Let $\overline{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$. Let g be a function of a single real variable which is differentiable at μ with $g'(\mu) \neq 0$. Show that:

$$\sqrt{n}\left(\frac{g(\overline{X}_n) - g(\mu)}{\sigma g'(\mu)}\right) \to N(0,1)$$

weakly.

Proof. We'll use the Central Limit Theorem and Taylor's theorem, but as motivation for using Taylor's theorem, say $g(x) = ax + b$, and assume as in proving the CLT $\mu = 0$ and $\sigma^2 = 1$, so now we just want to show $\sqrt{n}g(\overline{X}_n)/g'(\mu) \to N(0,1)$. By linearity of g and linearity of expectation gives:

$$\begin{aligned}\sqrt{n}(g(\overline{X}_n) - b)/g'(\mu) &= \\ &= \sqrt{n}(a\overline{X}_n + b - b)/a \\ &= \sqrt{n}\overline{X}_n \\ &= \sqrt{n}/nS_n \\ &= S_n/\sqrt{n}\end{aligned}$$

Where S_n is as defined in the statement of the CLT given in class. In particular, the expression we arrived at is $N(0,1)$ by the CLT, so we have the desired result. For general functions, we will use Taylor's theorem to get that $\sqrt{n}(g(\overline{X}_n) - g(\mu))/(\sigma g'(\mu)) \to S_n/\sqrt{n}$ in probability, thus giving weak convergence. First notice that $var(\overline{X}_n) \to 0$ as $n \to \infty$, as $var(X_i/n) = \frac{1}{n^2}var(X_i)$ and $var(X_i + X_j) = var(X_i) + var(X_j)$ and so $var(\overline{X}_n) = nvar(X_1)/n^2 \to 0$ as $n \to \infty$.

Using Taylor's theorem to write: $g(X_n) = g(\mu) + g'(\mu)(X_n - \mu) + o(\overline{X}_n - \mu)$ and applying the same algebra as above we get:

$$\sqrt{n}(g(\overline{X}_n) - b)/g'(\mu) = S_n/\sqrt{n} + o(\overline{X}_n - \mu)$$

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but for large n $var(\overline{X}_n) \to 0$ and so with high probability $X \in B_\epsilon(\mu)$ and so $|\overline{X}_n/\sqrt{n} - \sqrt{n}(g(\overline{X}_n) - b)/g'(\mu)| < \epsilon$, thus giving convergence in probability, as desired. $\square$