Math 6710 – Probability (3 pages)	Due: Thursday, November 1, 2012

Homework 10

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1,2,3,4,5

Problem 1. Prove the conditional dominated convergence theorem: suppose $\mathcal{G}$ is a σ-field, X_n, X are random variables with $X_n \to X$ almost surely, and there is an integrable Z with $|X_n| < Z$ a.s. Show that $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ almost surely in L^1.

Proof. First let's see that $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ almost surely.

By assumption, $X_n \to X$ almost surely. So given ϵ we may take N so that $P(X_n = X) > 1 - \epsilon$ for all $n > N$. And then for all $n > N$ we have $X_n - X = 0$ on a set A with $\mu(A) \geq 1 - \epsilon$. So we can write:

$$E[X_n - X|\mathcal{G}] = E[1_A X_n - 1_A X + 1_{A^c} X_n - 1_{A^c} X|\mathcal{G}] = E[1_A X_n - 1_A X|\mathcal{G}] + E[1_{A^c} X_n - 1_{A^c} X|\mathcal{G}]$$

Where $1_A X_n - 1_A X = 0$ so $E[1_A X_n - 1_A x|\mathcal{G}] = 0$ and we may make $E[1_{A^c} X_n - 1_{A_c} X|\mathcal{G}] \neq 0$ arbitrarily rarely by choosing ϵ so that A^c is sufficiently small. Thus $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ almost surely.

Now, $|E[X_n|\mathcal{G}]| \leq E[|X_n||\mathcal{G}] \leq E[Z|\mathcal{G}]$ as $|X_n| \leq Z$ and the triangle inequality and we've just shown that $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ a.s. so applying the (not conditional) dominated convergence theorem to $E[X_n|\mathcal{G}], E[X|\mathcal{G}]$ we have convergence in L^1. □

Problem 2. Let X, Y be iid itegrable random variables. Compute $E[X|X+Y]$. As in last week's homework the answer will be $f(X+Y)$ for some measurable function f; try to find f explicitly. Hint: Consider also $E[Y|X+Y]$.

Ugh. I'm not sure. I'll write down my thoughts.

First, observe that $E[X|X+Y] + E[Y|X+Y] = E[X+Y|X+Y] = X+Y$ as $X+Y$ is $X+Y$ measurable. My guess is that $E[X|X+Y] = E[Y|X+Y] = \frac{X+Y}{2}$ which satisfies this equation. I feel like this should work out either by the "guess and check" technique for calculating conditional probabilities in general with the information that X, Y are iid, but I feel like there is some consequence of X, Y being iid that I'm not thinking of to make this work out.

Other approaches I considered: maybe I could use the problem from last homework to conclude that $(X, X+Y), (Y, X+Y)$ have the same joint distribution, then $E[X|X+Y], E[Y|X+Y]$ would have to be the same function of $X+Y$, which would be enough to get my desired conclusion. The fact that they are iid should give that they have the same joint distribution.

I tried a bunch of other things too, and lots of symbol pushing, but for some reason I couldn't get the intuition for what was going on here. My last thought leaving off was if I want to calculate $E[1_A X; X+Y]$ for $A \in \sigma(X+Y)$ then this is saying calculate $E[1_A X]$ using only information from $\sigma(X+Y)$. Somehow this leads me back to the same argument that because X, Y are iid, $X + Y/2$ should behave like X closely enough on A so that $E[1_A X] = E[1_A (X+Y)/2]$.

Ugh....

Problem 3. The Giants and the Tigers are playing in the World Series. This is a 7 game series and the first team to win 4 games wins the series. Let $\zeta_n, n = 1, \dots, 7$ be the winner of the nth team–so G for Giants and T for Tigers–and let $\mathcal{F}_n = \sigma(\zeta_1, \dots, \zeta_n)$. Assume that the σ_n are iid and that the teams are evenly mathed so that $P(\sigma_n = G) = P(\zeta_n = T) = 1/2$.

Suppose that we bet \$1 on the Giants to win the series. Let M_n be the amount of money we have after the n-th game $n \leq 7$. Thus $M_n = 1$ on the event taht the Giants win 4 of the first n games, $M_n = -1$ if the Tigers do, and $M_n = 0$ if neither team has won 4 games yet. Show that $E[M_n] = 0$ for all n, so this is a "fair" bet. Is $\{M_n\}$ a martingale with respect to the filtration $\mathcal{F}_\backslash$?

Proof. First, we see that $E[M_n] = 0$ for all n by symmetry. For any outcome where $M_n = 1$ there is a distinct outcome where $M_n = -1$–the outcome where the Giants lose all the games they won and the Tigers win those games instead, so $\mu\{\omega : M_n(\omega) = 1\} = \mu\{\omega : M_n(\omega) = -1\}$ so $E[M_n] = 0 + 1 \cdot \mu\{\omega : M_n(\omega) = 1\} - 1 \cdot \mu\{\omega : M_n(\omega) = -1\} = 0$, as desired.

This is not a martingale though, as for $n = 1, 2, 3$, $M_n = 0$, i.e. is constant as it is impossible for the Giants to win the series until at least 4 games have been played, then for $n = 4$, $E[M_4|\mathcal{F}_3] \neq 0$ as there certainly exist events $A \in \mathcal{F}_3$ such that the Giants can win the world series in the fourth game, such as the event A that the Giants win the first three games of the series–in which case $E[1_A M_4|\mathcal{F}_3] = 1/2 \neq 0 = M_3$. $\square$

Problem 4. (5.2.6) Let $\zeta_1, \zeta_2, \dots$ be independent with $E\zeta_i = 0$ and $var(\zeta_i) = \sigma_i^2 < \infty$, and let $s_n^2 = \sum_{m=1}^n \sigma_m^2$ and $S_n^2 = \sum \zeta_m^2$ show that $S_n^2 - s_n^2$ is a martingale.

Proof. So we want to show $S_n^2 - s_n^2 = E[S_{n+1}^2 - s_{n+1}^2|\mathcal{F}_n]$. To start we'll write it out and appeal to linearity of expectation:

$$E[S_{n+1}^2 - s_{n+1}^2|\mathcal{F}_n = E[S_n^2 - s_n^2|\mathcal{F}_n] + E[\zeta_{n+1}^2 - \sigma_{n+1}^2|\mathcal{F}_n] = S_n^2 - s_n^2 + E[\zeta_{n+1}^2 - \sigma_{n+1}^2|\mathcal{F}_n]$$

where the last equality follows because $S_n^2 - s_n^2$ is $\mathcal{F}_n$ measurable. But then ζ_{n+1} is independent of $\mathcal{F}_n$ and ζ_{n+1}^2 is $\sigma(\zeta_{n+1})$ measurable so ζ_{n+1}^2 is also independent of $\mathcal{F}_n$ and so $E[\zeta_{n+1}^2|\mathcal{F}_n] = E[\zeta_{n+1}^2]$. Further $\sigma_{n+1}^2 = E[\zeta_{n+1}^2] - (E[\zeta_{n+1}])^2$ is a constant, so we have:

$$E[\zeta_{n+1}^2 - \sigma_{n+1}^2|\mathcal{F}_n] = E[\zeta_{n+1}^2] - E[\zeta_{n+1}]^2 - (E[\zeta_{n+1}])^2 = (E[\zeta_{n+1}])^2 = 0$$

as we are assuming $E[\zeta_i] = 0$ for all i. Thus we have that $S_n^2 - s_n^2$ is a martingale. $\square$

Problem 5. (5.2.13) Suppose X_n^1 and X_n^2 are supermartingales with respect to $\mathcal{F}_n$ and N is a stopping time so that $X_N^1 \geq X_N^2$. Then show:

$$Y_n = X_n^1 1_{(N>n)} + X_n^2 1_{(N \leq n)} \text{ is a supermatingle}$$

$$Z_n = X_n^1 1_{(N \geq n)} + X_n^2 1_{(N<n)} \text{ is a supermatingle}$$

Proof. Z_n, Y_n are adapted as X_n^1, X_n^2 are supermartingales, hence adapted and Z_n, Y_n are just simple combinations of X_n^1, X_n^2.

Now we notice that $Z_n \geq Y_n$ which we'll then use to prove that both Y_n, Z_n are supermartingales alongside each other.

In particular, we have $X_N^1 \geq X_N^2$ and so:

$$Z_n = X_n^1 1_{(N \geq n)} + X_n^2 1_{(N<n)} = X_n^1 1_{N>n} + X_n^1 1_{N=n} + X_n^2 1_{N<n} \geq X_n^1 1_{N>n} X_n^2 1_{N \leq n} = Y_n$$

Now we have the following string of inequalities to establish that $Z_n \geq E[Z_{n+1}|\mathcal{F}_n]$ and to get us most of the way there for Y_n, namely:

$$\begin{aligned} Z_n &\geq Y_n \\ &\geq E[X_{n+1}^1|\mathcal{F}_n]1_{N>n} + E[X_{n+1}^2|\mathcal{F}_n]1_{N\leq n} && \text{because } Y_n \text{ is a supermartingale} \\ &= E[X_{n+1}^1 1_{N>n} + X_{n+1}^2 1_{N\leq n}|\mathcal{F}_n] \\ &= E[Z_{n+1}|\mathcal{F}_n] && \text{by linearity of expectation} \end{aligned}$$

So now we have that Z_n is a supermartingale. As an intermediary step, we showed $Y_n \geq E[X_{n+1}^1 1_{N>n} + X_{n+1}^2 1_{N\leq n}]$, but applying the same trick we used to get $Z_n \geq Y_n$ we get that $X_{n+1}^1 1_{N>n} + X_{n+1}^2 1_{N\leq n} \geq X_{n+1}^1 1_{N>n+1} + X_{n+1}^2 1_{N\leq n+1} = Y_{n+1}$, so taking expectations conditional to $\mathcal{F}_n$ gives that $Y_n \geq E[Y_{n+1}|\mathcal{F}_n$, and Y_n is also a supermartingale. $\square$