

# AN INVESTIGATION OF HEMICOMPACTNESS USING PIBASE - LITERATURE REVIEW

CODY MARTIN

## 1. HISTORICAL BACKGROUND

The notion of hemicompactness was first defined in 1946 by Richard Arens in his work *A Topology for Spaces of Transformations*. He defines a topological space as hemicompact when the space is a union of countably many compact sets and any compact subset of the space is a subset of some finite collection of those compact sets. He further demonstrates many equivalencies in properties and hemicompactness: locally compact and perfectly separable spaces are hemicompact, hemicompact spaces which are first countable are locally compact, and perfectly separable spaces are hemicompact if and only if the space is also locally compact. He further provides a proof for his theorem that if the space of all real-valued functions with the compact-open topology on an S-space is first countable, then that S-space witnesses hemicompactness. [Are46]

In 1980, R. A. McCoy published two papers, one being *Countability Properties of Function Spaces*, as well as *Function Spaces which are  $k$ -Spaces*. In the former, he provides more equivalencies for hemicompactness as well as another definition for hemicompactness, "there exists a countable family of compact subsets such that every compact subset of the space is contained in some member of this family". He asserts the corollary that if  $X$  is a completely regular space,  $Y$  contains a nontrivial path, and  $C_k(X, Y)$  is first countable then  $Y$  is also first countable and  $X$  witnesses hemicompactness. Maintaining the same spaces  $X$  and  $Y$ , if  $C_k(X, Y)$  is metrizable, then it is necessary for  $Y$  to be metrizable and  $X$  witness hemicompactness. He further provides a theorem stating that if a space  $X$  is completely regular, a space  $Y$  has a point-countable base while containing a nontrivial path, and  $C_k(X, Y)$  is first countable, then the space  $X$  must witness hemicompactness. [McC80a]

In the latter publication, McCoy continues with the space of continuous real-valued functions on  $X$  which have the compact-open topology, denoted as  $C_k(X)$ . He asserts that if  $C_k(X)$  is first countable as well as metrizable, then  $C_k(X)$  is hemicompact. He further proves through a series of propositions that every first countable  $k$ -compact space is hemicompact. Two more corollaries are provided, first that if  $X$  is first countable, then the following are equivalent:  $C_k(X)$  is a  $k$ -space,  $C_k(X)$  is completely metrizable, and  $X$  is hemicompact. Second, if  $X$  is locally compact, then the following statement are equivalent:  $C_k(X)$  is a  $k$ -space,  $C_k(X)$  is completely metrizable,  $C_k(X)$  has countable tightness, and  $X$  is hemicompact. McCoy follows these corollaries with the question, "Is every  $k$ -compact  $k$ -space hemicompact?" Through a series of examples, McCoy illustrates that this is a false proposition. [McC80b]

Gary Gruenhage and Glenn Hughes in their work *Completeness Properties in the Compact-Open Topology on Fans* characterize hemicompactness on  $S_u$  by properties on the filter  $u$ . They assert a theorem given in a publication by McCoy and Ntantu that when the space  $C_k(X)$  of all continuous real valued functions on  $X$  with the compact-open property is completely metrizable, then  $X$  is both a  $k$ -space and hemicompact. Deeper in the article, it is proposed easy to verify that sequential fans, denoted as  $S_\omega$  are indeed hemicompact. This will be a point I will cover in my own work if unable to be found elsewhere. They continue to prove that the metric fan, denoted as  $M$ , does not witness hemicompactness. To continue, they give a result which shows that a filter-fan,  $S_u$ , is also hemicompact under the condition that the filter-fan does not contain a copy of the metric fan. A series of equivalencies are then given for a filter-fan witnessing hemicompactness. [GG15]

## 2. PERSONAL WORK

This section contains work that I have done to demonstrate which topological spaces do and do not witness hemicompactness. First to start I will provide definitions for the Metric Fan and Sequential Fan.

**Definition 1.** For a Metric Fan  $M$ ,  $M = \omega^2 \cup \{\infty\}$  where each point in  $\omega^2$  is isolated ( $\{(n, m)\}$  is always open) and  $T_n = \{\infty\} \cup \{(i, j) : j \geq n\}$  is a basic open neighborhood of  $\infty$ .

**Definition 2.** For a Sequential Fan  $S$ ,  $S = \omega^2 \cup \{\infty\}$  where each point in  $\omega^2$  is isolated and  $T_f = \{\infty\} \cup \{(i, j) : j \geq f(i)\}$  is a basic open neighborhood of  $\infty$  for  $f : \omega \rightarrow \omega$ .

**Theorem 3.** *The Metric Fan does not witness Hemicompactness*

*Proof.* For a metric fan  $M$ , let  $K_n \subseteq M$  be compact for  $n < \omega$ . Note,  $D = \omega \times \{n\}$  is infinite, closed and discrete. Since a compact set cannot contain an infinite, closed and discrete subset, choose  $a_n \in \omega$  such that  $(a_n, n) \in D \setminus K_n$ . Let  $K = \{\infty\} \cup \{(a_n, n) : n < \omega\}$ . Note,  $K \not\subseteq K_n$  for any  $n < \omega$  since  $(a_n, n) \notin K_n$ . Let  $\mathcal{U}$  be an open cover of  $K$ . So let  $\infty \in U \in \mathcal{U}$ . Pick  $N < \omega$  such that  $\infty \in T_N \subseteq U$ . So  $(a_n, n) \in T_N \subseteq U$  for  $n \geq N$ . For  $0 \leq n < N$ , pick any  $U_n \in \mathcal{U}$  such that  $(a_n, n) \in U_n$ . So,  $\mathcal{F} = \{U\} \cup \{U_n : 0 \leq n < N\}$  is a finite subcover of  $K$ , and  $K$  is compact. Since for every  $\{K_n : n < \omega\}$ , there exists compact  $K$  with  $K \not\subseteq K_n$ ,  $M$  is not hemicompact.  $\square$

In [GG15], it is said to be simple to prove that the sequential fan observes hemicompactness, which I will demonstrate below.

**Theorem 4.** *The Sequential Fan Witnesses Hemicompactness*

*Proof.* For a sequential fan  $S$ , let  $K_n \subseteq S$  be compact for  $n < \omega$ .  $D = \omega \times \{n\}$ . Choose  $a_n \in \omega$  where  $(a_n, n) \in D \setminus K_n$ . Let  $K = \{\infty\} \cup \{(a_n, n) : n < \omega\}$ , where  $K \not\subseteq K_n$ . We want to show that there exists an open covering  $\mathcal{U}$  of  $K$ , such that there is no finite  $\mathcal{F} \subseteq \mathcal{U}$  which also covers  $K$ . At least I think that's what I need to do...  $\square$

## REFERENCES

- [Are46] Richard F Arens. A topology for spaces of transformations. *Annals of Mathematics*, pages 480–495, 1946.
- [GG15] Glenn Hughes Gary Gruenhage. Completeness properties in the compact-open topology on fans. *Houston Journal of Mathematics*, 41(1):321–337, 2015.
- [McC80a] R. A. McCoy. Countability properties of function spaces. *Rocky Mountain Journal of Mathematics*, Vol. 10(4):717–730, 1980.
- [McC80b] R.A. McCoy. Function spaces which are  $k$ -spaces. *Topology Proceedings*, Vol. 5:139–146, 1980.