AN INVESTIGATION OF HEMICOMPACTNESS USING π-BASE

CODY MARTIN

1. Historical Background

The notion of hemicompactness was first defined in 1946 by Richard Arens in his work A Topology for Spaces of Transformations. He defines a topological space as hemicompact when the space is a union of countably many compact sets and any compact subset of the space is a subset of some finite collection of those compact sets. He further demonstrates many equivalencies in properties and hemicompactness: locally compact and perfectly separable spaces are hemicompact, hemicompact spaces which are first countable are locally compact, and perfectly separable spaces are hemicompact if and only if the space is also locally compact. He further provides a proof for his theorem that if the space of all real-valued functions with the compact-open topology on an S-space is first countable, then that S-space witnesses hemicompactness. [Are46]

In 1980, R. A. McCoy published two papers, one being Countability Properties of Function Spaces, as well as Function Spaces which are k-Spaces. In the former, he provides more equivalencies for hemicompactness as well as another definition for hemicompactness, "there exists a countable family of compact subsets such that every compact subset of the space is contained in some member of this family". He asserts the corollary that if X is a completely regular space, Y contains a nontrivial path, and $C_k(X,Y)$ is first countable then Y is also first countable and X witnesses hemicompactness. Maintaining the same spaces X and Y, if $C_k(X,Y)$ is metrizable, then it is necessary for Y to be metrizable and X witness hemicompactness. He further provides a theorem stating that if a space X is completely regular, a space Y has a point-countable base while containing a nontrivial path, and $C_k(X,Y)$ is first countable, the the space X must witness hemicompactness. [McC80a]

In the latter publication, McCoy continues with the space of continuous real-valued functions on X which have the compact-open topology, denoted as $C_k(X)$. He asserts that if $C_k(X)$ is first countable as well as metrizable, then $C_k(X)$ is hemicompact. He further proves through a series of propositions that every first countable k-compact space is hemicompact. Two more corollaries are provided, first that if X is first countable, then the following are equivalent: $C_k(X)$ is a k-space, $C_k(X)$ is competely metrizable, and X is hemicompact. Second, if X is locally compact, then the following statement are equivalent: $C_k(X)$ is a k-space, $C_k(X)$ is completely metrizable, $C_k(X)$ has countable tightness, and X is hemicompact. McCoy follows these corollaries with the question, "Is every k-compact k-space hemicompact?" Through a series of examples, McCoy illustrates that this is a false proposition. [McC80b]

Gary Gruenhage and Glenn Hughes in their work Completeness Properties in the Compact-Open Topology on Fans characterize hemicompactness on $S_{\mathfrak{u}}$ by properties on the filter $\mathfrak{u}$. They assert a theorem given in a publication by McCoy and Ntantu that when the space $C_k(X)$ of all continuous real valued functions on X with the compact-open property is completely metrizable, then X is both a k-space and hemicompact. Deeper in the article, it is proposed easy to verify that sequential fans, denoted as S_ω are indeed hemicompact. This will be a point I will cover in my own work if unable to be found elsewhere. They continue to prove that the metric fan, denoted as M, does not witness hemicompactness. To continue, they give a result which shows that a filter-fan, $S_{\mathfrak{u}}$, is also hemicompact under the condition that the filter-fan does not contain a copy of the metric fan. A series of equivalencies are then given for a filter-fan witnessing hemicompactness. [GH15a]

2. Definitions

Below are some definitions that will be used in this paper.

Definition 1. Let (X, τ) be a topological space; a set $F \subseteq X$ is called **closed** in the space if its complement $X \setminus F$ is an open set. [Eng89]

Definition 2. A topological space X is **metrizable** if there exists a metric p on the set X such that the topology induced by the metric p coincides with the original topology of X. [Eng89]

Definition 3. A topological space is called **locally compact** if each point is contained in an open set $U \subseteq K$ for compact K. [SS78]

Definition 4. A set D is **discrete** in X if and only if each $d \in D$ has a neighborhood U in X such that $U \cap D = \{d\}$. [Wil04]

Definition 5. A space is **$2nd$-countable** if it has a countable basis. [SS78]

Definition 6. For a **Metric Fan** M, $M = \omega^2 \cup \{\infty\}$ where each point in ω^2 is isolated ($\{(n, m)\}$ is always open) and $T_n = \{\infty\} \cup \{(i, j) : j \geq n\}$ is a basic open neighborhood of ∞.

Definition 7. For a **Sequential Fan** S, $S = \omega^2 \cup \{\infty\}$ where each point in ω^2 is isolated and $T_f = \{\infty\} \cup \{(i, j) : j \geq f(i)\}$ is a basic open neighborhood of ∞ for $f : \omega \to \omega$.

Definition 8. A topological space A is **hemicompact** whenever $A = \bigcup\limits_{n<\omega} H_n$ where each H_n is compact, and any compact set $K \subseteq A$ is a subset of some H_n. [Are46]

3. Data Collected for π-Base

The following properties were found to be missing from π-Base during my research into the hemicompact property and proofs related to it. These items are being entered into π-Base for further review and eventual use for deriving the existence of properties through direct implications and deduction: hemicompact [Are46], semimetrizable [KG09], pointwise countable type [KG09], M space [KG09], q space [KG09], r space [KG09], K analytic [MR06], weakly K analytic [MR06], k$\mathbb{R}$ space [MR06], strictly angelic [CMPT], Ascoli [GGKZ16], Z-compact [Tah], homotopy dense [YYZ17], moving off property [GH15b], $\aleph_0$ [yiyt93], cosmic [yiyt93], pseudo-Polish [Min12], pseudo-metrizable [Min12], submetrizable [KG09], has a compact resolution [GGKZ16], locally Céch complete [KG09], pseudocomplete [KG09], almost Céch complete [KG09], S space [Are46], angelic [CMPT], countable type [KG09].

4. Proofs

Theorem 9. *The Metric Fan is not hemicompact.*

Proof. For a metric fan M, let $K_n \subseteq M$ be compact for $n < \omega$. Note, $D = \omega \times \{n\}$ is infinite, closed and discrete. Since a compact set cannot contain an infinite, closed and discrete subset, choose $a_n \in \omega$ such that $(a_n, n) \in D \setminus K_n$. Let $K = \{\infty\} \cup \{(a_n, n) : n < \omega\}$. Note, $K \not\subseteq K_n$ for any $n < \omega$ since $(a_n, n) \notin K_n$. Let $\mathcal{U}$ be an open cover of K. So let $\infty \in U \in \mathcal{U}$. Pick $N < \omega$ such that $\infty \in T_N \subseteq U$. So $(a_n, n) \in T_N \subseteq U$ for $n \geq N$. For $0 \leq n < N$, pick any $U_n \in \mathcal{U}$ such that $(a_n, n) \in U_n$. So, $\mathcal{F} = \{U\} \cup \{U_n : 0 \leq n < N\}$ is a finite subcover of K, and K is compact. Since for every $\{K_n : n < \omega\}$, there exists compact K with $K \not\subseteq K_n$, M is not hemicompact. □

Lemma 10. *A set $K \subseteq X$ is closed and discrete if and only if for every point $x \in X$ there exists an open set U such that $x \in U$ and $U \cap K \subseteq \{x\}$.*

Proof. To show K is closed and discrete, we will first demonstrate that $X \setminus K$ is open. For each x $\in X \setminus K$ there is an open set U where $x \in U \subseteq X \setminus K$. For $x \in X \setminus K, x \in U$ so $x \in \bigcup\{U : x \in X \setminus K\}$. Thus, $X \setminus K \subseteq \bigcup\{U : x \in U\}$. Now let y $\in U$ for some $x \in X \setminus K$, then $U \subseteq X \setminus K$. Thus $X \setminus K \supseteq \bigcup\{U : x \in X \setminus K\}$. Therefore $X \setminus K = \bigcup\{U : x \in X \setminus K\}$, $X \setminus K$ is therefore open and its complement K is closed. Now to show that K is discrete, since for any $x \in X$, $x \in U \subseteq X$, and $U \cap X \subseteq \{x\}$, then the points in X must be isolated points and therefore points $k \in K$ must also be isolated points with open neighborhoods U_k such that $k \in U_k \subseteq K$. Since they are isolated, $U_k \cap K = \{k\}$ so K is discrete.

To show the converse, by definition of discrete, for all $x' \in K$ there is a U' such that $x' \in U'$ and $U' \cap K = \{x'\}$. For any point $x \in X$, choose an open set U such that $x \in U \subseteq X$ and $U \cap X = \{x\}$. K is discrete, so choose U' such that $x' \in U' \subseteq K$ and $U' \cap K = \{x'\}$. Either $x' = x$ and $U \cap K = \{x\}$ or $x' \neq x$ and $U \cap K = \emptyset$. Therefore $U \cap K \subseteq \{x\}$. □

Theorem 11. *The Sequential Fan is hemicompact.*

Proof. For a sequential fan S, let $K'_n = \{(n, m) : m < \omega\} \cup \{\infty\} \subseteq S$, and $K_n = \bigcup_{i \leq n} K'_n$. Let $\mathcal{U}$ be an open cover of K'_n. So $\infty \in U \in \mathcal{U}$ for some U. So $\infty \in T_f \subseteq U$ for some $f : \omega \to \omega$. Choose U such that $T_f \subseteq U$, so there are only finitely many $k_m \in K'n \setminus U$. For remaining k_m, choose U'_m such that $k_m \in U'_m$. So there is a finite $\mathcal{F} = \{U, U'_1, U'_2, ..., U'_m\} \subseteq \mathcal{U}$ which also covers K'_n. Thus, K'_n is compact and $K_n = \bigcup K'_n$ is compact. Let $K \subseteq S$. If for all $n < \omega$, $K \not\subseteq K_n$, choose $(a_n, b_n) \in K \setminus K_n$ and let $D = \{(a_n, b_n) : n < \omega\}$. Because $D \not\subseteq K_n$, $\infty \notin D$. We need to find U such that $\infty \in U$ yet $U \cap D = \emptyset$. So for $n < \omega$, $a_m > m$ for $m \geq n$ so D is infinite. So if $(a_m, b_m) = (n, b_m)$ then

$m < n$. $\{b_m : a_m = n\}$ is finite so if $\{b_m : a_m = n\} = \emptyset$ let $f(n) = 0$ otherwise let $f(n) = max\{b_m : a_m = n\} + 1$. So for $T_f = \{(i,j) : j \geq f(i)\} \cup \{\infty\}$, $\infty \in T_f$ and $T_f \cap D = \emptyset$. Now for the remaining $(x,y) \in \omega^2 \setminus T_f$, (x,y) are isolated. Choose an open set U such that $U = \{(x,y)\} \subseteq \omega^2 \setminus T_f$. Observe that $(a_n, b_n) \in D$ are also isolated since $D \cap T_f = \emptyset$. Thus, when $(a_n, b_n) = (x,y)$, $U \cap D = \{(x,y)\}$. Otherwise $U \cap D = \emptyset$, thus $U \cap D \subseteq \{(x,y)\}$. Since $D \subseteq K$ is infinite, and by the above lemma also closed and discrete, K is not compact. Therefore, if K is compact then there exists $n < \omega$ such that $K \subseteq K_n$. Thus S is hemicompact. □

Lemma 12. *Every hemicompact space is Lindelöf.*

Proof. For $K \subseteq X$, let $K_n \subseteq K$ witness hemicompactness, of course $\bigcup_{n\in\mathbb{N}} K_n \subseteq K$, since $K_n \subseteq K$. Let $k \in K$, so $\{k\}$ is compact, thus there is some $n \in \mathbb{N}$ such that $\{k\} \subseteq K_n$. Therefore $k \in K_n \subseteq K$, so $K = \bigcup_{n\in\mathbb{N}} K_n$, and K is σ-compact. Now let $\mathcal{F}_n = \{U_{n0}, U_{n1}, ..., U_{nm}\}$ be a finite subcover of $\mathcal{U}$ for K_n. So, $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a countable subcover of $\mathcal{U}$ for K. Therefore K is Lindelöf. □

Theorem 13. *For $T_{3\frac{1}{2}}$ space X, the following are equivalent.*

(1) *X is locally compact and $2nd$-countable*
(2) *X is hemicompact and metrizable*

Proof. (1) $\Rightarrow$ (2) : We know from the literature that every $2nd$-countable space is Lindelöf [SS78]. We also know that any regular $2nd$-countable space is metrizable, thus a $T_{3\frac{1}{2}}$ $2nd$-countable space is also metrizable. [SS78] Should X be locally compact, then for $x \in X$ there is an open set U_x and a compact K_x with $x \in U_x \subseteq K_x$. Since X is Lindelöf, choose a countable subcover of $\{U_x : x \in X\}$ by picking $x_n \in X$ for $n < \omega$ such that $\bigcup\{U_{x_n} : n < \omega\} = X$. So, $\{K_{x_n} : n < \omega\}$ is a countable collection of compact sets which cover X. Let $L_n = \bigcup_{m\leq n} K_{x_m}$. Suppose $K \not\subseteq L_n$ for all $n < \omega$. So let $V_n = \bigcup_{m\leq n} U_{x_m}$, therefore $\{V_n : n < \omega\}$ is an open cover of X. If K is compact, there is a finite $\mathcal{F} \subseteq \{V_n : n < \omega\}$ with $K \subseteq \bigcup \mathcal{F}$. But $\bigcup \mathcal{F} = V_n \subseteq L_n$ for some $n < \omega$, thus $K \subseteq L_n$ for some $n < \omega$. Contradiction, so K is not compact and therefore X is hemicompact.

(2) $\Rightarrow$ (1) : By the above lemma, all hemicompact spaces are Lindelöf. We also know that Lindelöf metrizable spaces are $2nd$-countable. [SS78] Let $\{K_n : n < \omega\}$ witness hemicompactness, and let $\{U_n : n < \omega\}$ be a countable basis. If for all points $x \in X$, there exists $n < \omega$ such that $B_{2^{-n}}(x) \subseteq K_n$, X is locally compact. Otherwise, there exists $x \in X$ such that for all $n < \omega$, $B_{2^{-n}}(x) \not\subseteq K_n$. Choose $x_n \in B_{2^{-n}}(x) \setminus K_n$. Claim: $K = \{x\} \cup \{x_n : n < \omega\}$ is compact but $K \not\subseteq K_n$ for any $n < \omega$. As n grows, K_n contains more space, yet $B_{2^{-n}}(x)$ shrinks to contain only $\{x\}$. Eventually K will only contain a finite number of points that will be contained in K_n for some n. So $K \subseteq K_n$ for some $n < \omega$ and X is locally compact. □

References

[Are46] Richard F Arens. A topology for spaces of transformations. *Annals of Mathematics*, pages 480–495, 1946.

[CMPT] M. J. Chasco, E. Martin-Peinador, and V. Tarieladze. A class of angelic sequential non-fréchet-urysohn topological groups. *Topology and its Applications*, Vol. 154.

[Eng89] Ryszard Engelking. *General topology*. Heldermann Verlag, 1989.

[GGKZ16] Saak Gabriyelyan, Jan Grebik, Jerzy Kakol, and Lyubomyr Zdomskyy. The ascoli property for function spaces. *Topology and its Applications*, Vol. 214, 2016.

[GH15a] Gary Gruenhage and Glenn Hughes. Completeness properties in the compact-open topology on fans. *Houston Journal of Mathematics*, 41(1):321–337, 2015.

[GH15b] Gary Gruenhage and Glenn Hughes. completeness properties in the compact-open topology on fans. *Houston Journal of Mathematics*, Vol. 41, 2015.

[KG09] S. Kundu and Pratibha Garg. The compact-open topology: A new perspective. *Topology and its Applications*, Vol. 156, 2009.

[McC80a] R. A. McCoy. Countability properties of function spaces. *Rocky Mountain Jornal of Mathematics*, Vol. 10(4):717–730, 1980.

[McC80b] R.A. McCoy. Function spaces which are k-spaces. *Topology Preceedings*, Vol. 5:139–146, 1980.

[Min12] E. Minguzzi. Topological conditions for the representation of preorders by continuous utilities. *Applied General Topology*, Vol. 13:81–89, 2012.

[MR06] S. Moll and L. M. Sanchez Rulz. A note on a theorem of talagrand. *Topology and its Applications*, Vol. 153, 2006.

[SS78] Steen and Seebach. *Counterexamples In Topology*. Springer-Verlag, 1978.

[Tah] Ali Taherifar. On a question of kaplansky. *Topology and its Applications*, Vol. 232.

[Wil04] Stephen Willard. *General Topology*. Dover Publications, 2004.

[yiyt93] yoshito ikeda and yoshio tanaka. spaces having star-countable k-networks. *Topology Preceedings*, Vol. 18, 1993.

[YYZ17] Hanbaio Yang, Zhongqiang Yang, and Yanmei Zheng. Topological classification of function spaces with the fell topology iv. *Topology and its Applications*, Vol. 228, 2017.