AN INVESTIGATION OF HEMICOMPACTNESS USING π-BASE

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1. Historical Background

The notion of hemicompactness was first defined in 1946 by Richard Arens in his work A Topology for Spaces of Transformations. He defines a topological space as hemicompact when the space is a union of countably many compact sets and any compact subset of the space is a subset of some finite collection of those compact sets. He further demonstrates many equivalencies in properties and hemicompactness: locally compact and perfectly separable spaces are hemicompact, hemicompact spaces which are first countabile are locally compact, and perfectly separable spaces are hemicompact if and only if the space is also locally compact. He further provides a proof for his theorem that if the space of all real-valued functions with the compact-open topology on an S-space is first countable, then that S-space witnesses hemicompactness. [Are46]

In 1980, R. A. McCoy published two papers, one being Countability Properties of Function Spaces, as well as Function Spaces which are k-Spaces. In the former, he provides more equivalencies for hemicompactness as well as another definition for hemicompactness, "there exists a countable family of compact subsets such that every compact subset of the space is contained in some member of this family". He asserts the corollary that if X is a completely regular space, Y contains a nontrivial path, and $C_k(X, Y)$ is first countable then Y is also first countable and X witnesses hemicompactness. Maintaining the same spaces X and Y, if $C_k(X, Y)$ is metrizable, then it is necessary for Y to be metrizable and X witness hemicompactness. He further provides a theorem stating that if a space X is completely regular, a space Y has a point-countable base while containing a nontrivial path, and $C_k(X, Y)$ is first countable, the the space X must witness hemicompactness. [McC80a]

In the latter publication, McCoy continues with the space of continuous real-valued functions on X which have the compact-open topology, denoted as $C_k(X)$. He asserts that if $C_k(X)$ is first countable as well as metrizable, then $C_k(X)$ is hemicompact. He further proves through a series of propositions that every first countable k-compact space is hemicompact. Two more corollaries are provided, first that if X is first countable, then the following are equivalent: $C_k(X)$ is a k-space, $C_k(X)$ is competely metrizable, and X is hemicompact. Second, if X is locally compact, then the following statement are equivalent: $C_k(X)$ is a k-space, $C_k(X)$ is completely metrizable, $C_k(X)$ has countable tightness, and X is hemicompact. McCoy follows these corollaries with the question, "Is every k-compact k-space hemicompact?" Through a series of examples, McCoy illustrates that this is a false proposition. [McC80b]

Gary Gruenhage and Glenn Hughes in their work Completeness Properties in the Compact-Open Topology on Fans characterize hemicompactness on $S_{\mathfrak{u}}$ by properties on the filter $\mathfrak{u}$. They assert a theorem given in a publication by McCoy and Ntantu that when the space $C_k(X)$ of all continuous real valued functions on X with the compact-open property is completely metrizable, then X is both a k-space and hemicompact. Deeper in the article, it is proposed easy to verify that sequential fans, denoted as S_ω are indeed hemicompact. This will be a point I will cover in my own work if unable to be found elsewhere. They continue to prove that the metric fan, denoted as M, does not witness hemicompactness. To continue, they give a result which shows that a filter-fan, $S_{\mathfrak{u}}$, is also hemicompact under the condition that the filter-fan does not contain a copy of the metric fan. A series of equivalencies are then given for a filter-fan witnessing hemicompactness. [GG15]

2. Definitions

Below are some definitions that will be used in this paper.

Definition 1. Let (X, τ) be a topological space; a set $F \subset X$ is called **closed** in the space if its complement $X \setminus F$ is an open set. [Eng89]

Definition 2. A topological space X is **metrizable** if there exists a metric p on the set X such that the topology induced by the metric p coincides with the original topology of X. [Eng89]

Definition 3. A topological space is called **locally compact** if each point is contained in a compact neighborhood. [SS78]

Definition 4. If every point $x \in X$ has a neighborhood that intersects at most one set of a given family, then that family is **discrete**.

Definition 5. A space is **2*nd*-countable** if it has a countable basis. [SS78]

Definition 6. For a **Metric Fan** M, $M = \omega^2 \cup \{\infty\}$ where each point in ω^2 is isolated ($\{(n, m)\}$ is always open) and $T_n = \{\infty\} \cup \{(i, j) : j \geq n\}$ is a basic open neighborhood of ∞.

Definition 7. For a **Sequential Fan** S, $S = \omega^2 \cup \{\infty\}$ where each point in ω^2 is isolated and $T_f = \{\infty\} \cup \{(i, j) : j \geq f(i)\}$ is a basic open neighborhood of ∞ for $f : \omega \to \omega$.

Definition 8. A topological space A is **hemicompact** whenever $A = H_1 \cup H_2 \cup \cdots \cup H_n \cup \cdots$ where each H_n is compact, and any compact set $K \subset A$ is a subset of some finite collection $H_{n_1} \cup \cdots \cup H_{n_k}$. [Are46]

3. Personal Work

The following is work that I have done to demonstrate topological spaces are hemicompact.

Theorem 9. *The Metric Fan is not hemicompact*

Proof. For a metric fan M, let $K_n \subseteq M$ be compact for $n < \omega$. Note, $D = \omega \times \{n\}$ is infinite, closed and discrete. Since a compact set cannot contain an infinite, closed and discrete subset, choose $a_n \in \omega$ such that $(a_n, n) \in D \setminus K_n$. Let $K = \{\infty\} \cup \{(a_n, n) : n < \omega\}$. Note, $K \not\subseteq K_n$ for any $n < \omega$ since $(a_n, n) \notin K_n$. Let $\mathcal{U}$

be an open cover of K. So let $\infty \in U \in \mathcal{U}$. Pick $N < \omega$ such that $\infty \in T_N \subseteq U$. So $(a_n, n) \in T_N \subseteq U$ for $n \geq N$. For $0 \leq n < N$, pick any $U_n \in \mathcal{U}$ such that $(a_n, n) \in U_n$. So, $\mathcal{F} = \{U\} \cup \{U_n : 0 \leq n < N\}$ is a finite subcover of K, and K is compact. Since for every $\{K_n : n < \omega\}$, there exists compact K with $K \not\subseteq K_n$, M is not hemicompact. $\square$

In [GG15], it is said to be simple to prove that the sequential fan is hemicompact, which I will demonstrate below.

Lemma 10. *A set $K \subseteq X$ is closed and discrete $\iff$ for every point $x \in X$ there exists an open set U such that $x \in U$ and $U \cap K \subseteq \{x\}$.*

Theorem 11. *The Sequential Fan is hemicompact*

Proof. For a sequential fan S, let $K'_n = \{(n, m) : m < \omega\} \cup \{\infty\} \subseteq S$, and $K_n = \bigcup_{i \leq n} K'_n$. Let $\mathcal{U}$ be an open cover of K'_n. For some $k_n < \omega$, choose $U'_n = \bigcup\{(n, m) : m > k_n, (n, m) \in K'_n\}$. Choose $U''_n = \{\infty\}$ and choose $U'''_n = \{(n, m) : 0 \leq m \leq k_n\}$. Let $\mathcal{F} = \{U'_n, U''_n, U'''_n\}$. So K'_n is compact, and therefore $K_n = \bigcup_{i \leq n} K'_n$ is also compact. Let $K \subseteq S$. If for all $n < \omega$, $K \not\subseteq K_n$, choose $D = \{(a_n, b_n) \in K \setminus K_n : n < \omega\}$. Since $a_n > n$, D is infinite. Each point in ω^2 is isolated so for $(a, b) \in \omega^2$ there is an open set $U = \{(a, b)\}$. Observe that because these open sets contain a single point, $U \cap D = \{(a, b)\}$ or $\emptyset$. I am tasked to consider $\{\infty\}$ now but am unsure why as it has already been accounted for in the K_n which we are strictly choosing points in $K \setminus K_n$ so wouldn't that miss $\{\infty\}$? Since D is infinite, closed and discrete, K is not compact. Therefore if K is compact then there exists $n < \omega$ such that $K \subseteq K_n$. Thus S is hemicompact. $\square$

Theorem 12. *For $T_{3\frac{1}{2}}$ space X, the following are equivalent.*

1. *X is locally compact and 2nd-countable*
2. *X is hemicompact and metrizable*

Proof. (1) $\Rightarrow$ (2) : We know from literature that every $2nd$-countable space is Lindelöf [SS78]. We also know that any regular $2nd$-countable space is metrizable, thus $T_{3\frac{1}{2}}$ $2nd$-countable space is also metrizable. [SS78] Should X be locally compact, then for $x \in X$ there is an open set U_x and a compact K_x with $x \in U_x \subseteq K_x$. Since X is Lindelöf, choose a countable subcover of $\{U_x : x \in X\}$ by picking $x_n \in X$ for $n < \omega$ such that $\bigcup\{U_{x_n} : n < \omega\} = X$. So, $\{K_{x_n} : n < \omega\}$ is a countable collection of compact sets which cover X. So $X = bigcup\{K_{x_n}\}$ and for any $K \subseteq X$, $K \subseteq \{K_{x_n}\}$ for finitely many n. Thus X is hemicompact.

(2) $\Rightarrow$ (1) : All hemicompact spaces are Lindelöf [Clo15] and all Lindelöf metrizable spaces are $2nd$-countable. [SS78] Let $\{K_n : n < \omega\}$ witness hemicompactness, and let $\{U_n : n < \omega\}$ be a countable basis. If for all points $x \in X$, there exists $n < \omega$ such that $B_{2^{-n}}(x) \subseteq K_n$, X is locally compact. Otherwise, there exists $x \in X$ such that for all $n < \omega$, $B_{2^{-n}}(x) \not\subseteq K_n$. Choose $x_n \in B_{2^{-n}}(x) \setminus K_n$. Claim: $K = \{x\} \cup \{x_n : n < \omega\}$ is compact but $K \not\subseteq K_n$ for any $n < \omega$, contradiction. By definition of hemicompact, $X = \bigcup_{n<\omega}\{K_n\}$ and for any compact subset K' of X, $K' \subseteq \bigcup\{K_i\}$ for $i < n$. So, for any point $x \in X$, $x \in K_n$ for some n which is a compact neighborhood around x. Thus X is locally compact. $\square$

References

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