IM equation in $\alpha\beta$ reference frame

Induction motor equation derivation

- stator voltage equation
- stator flux linkage equation
- rotor voltage equation
- rotor flux linkage equation
- torque equation

Stator voltage equation

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix}$$

times $K_{abc2\alpha\beta}$ on both side of the equation and we get

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + K_{abc2\alpha\beta} \frac{d}{dt} \big(K_{\alpha\beta 2abc} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix})$$

$$K_{abc2\alpha\beta} \frac{d}{dt} \big(K_{\alpha\beta 2abc} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix}) =$$

$$K_{abc2\alpha\beta} \frac{d}{dt} \big(K_{\alpha\beta 2abc} \big) \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix} + K_{abc2\alpha\beta} K_{\alpha\beta 2abc} \frac{d}{dt} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix}$$

because $K_{abc2\alpha\beta}$ and $K_{\alpha\beta 2abc}$ are constant matrix and the inverse of each other, we have

$K_{abc2\alpha\beta} \frac{d}{dt} \big(K_{\alpha\beta 2abc} \big) \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix} = 0, K_{abc2\alpha\beta} K_{\alpha\beta 2abc} \frac{d}{dt} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix}$ thus,stator voltage equation obtained $\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & R_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix}$

Stator flux linkage equation

$$\begin{bmatrix}\lambda_a\\\lambda_b\\\lambda_c\end{bmatrix} = \begin{bmatrix}L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms}\\-\frac{1}{2}L_{ms} & L_{ms} & -\frac{1}{2}L_{ms}\\-\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ms}\end{bmatrix}\begin{bmatrix}i_a\\i_b\\i_c\end{bmatrix} + \begin{bmatrix}L_{sleak} & 0 & 0\\0 & L_{sleak} & 0\\0 & 0 & L_{sleak}\end{bmatrix}\begin{bmatrix}i_a\\i_b\\i_c\end{bmatrix} +$$

$$L_{ms}\begin{bmatrix}\cos(\theta_r) & \cos(\theta_r+\frac{2}{3}\pi) & \cos(\theta_r-\frac{2}{3}\pi)\\\cos(\theta_r-\frac{2}{3}\pi) & \cos(\theta_r) & \cos(\theta_r+\frac{2}{3}\pi)\\\cos(\theta_r+\frac{2}{3}\pi) & \cos(\theta_r-\frac{2}{3}\pi) & \cos(\theta_r)\end{bmatrix}\begin{bmatrix}i_{ar}\\i_{br}\\i_{cr}\end{bmatrix}$$

times $K_{abc2\alpha\beta}$ on both sides of the equation

$$\begin{bmatrix}\lambda_\alpha\\\lambda_\beta\\\lambda_0\end{bmatrix} = K_{abc2\alpha\beta}\begin{bmatrix}L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms}\\-\frac{1}{2}L_{ms} & L_{ms} & -\frac{1}{2}L_{ms}\\-\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ms}\end{bmatrix}K_{\alpha\beta2abc}\begin{bmatrix}i_\alpha\\i_\beta\\i_0\end{bmatrix} +$$

$$\begin{bmatrix}L_{sleak} & 0 & 0\\0 & L_{sleak} & 0\\0 & 0 & L_{sleak}\end{bmatrix}\begin{bmatrix}i_\alpha\\i_\beta\\i_0\end{bmatrix} +$$

$$K_{abc2\alpha\beta}L_{ms}\begin{bmatrix}\cos(\theta_r) & \cos(\theta_r+\frac{2}{3}\pi) & \cos(\theta_r-\frac{2}{3}\pi)\\\cos(\theta_r-\frac{2}{3}\pi) & \cos(\theta_r) & \cos(\theta_r+\frac{2}{3}\pi)\\\cos(\theta_r+\frac{2}{3}\pi) & \cos(\theta_r-\frac{2}{3}\pi) & \cos(\theta_r)\end{bmatrix}K_{\alpha\beta2rabc}\begin{bmatrix}i_{\alpha_r}\\i_{\beta_r}\\i_{0_r}\end{bmatrix}$$

Here,$K_{\alpha\beta2rabc}$ is not the same as $K_{\alpha\beta2abc}$, the first one converts $\alpha\beta$ variables to rotor abc variables while the second one transforms $\alpha\beta$ variables to stator abc variables. Calculate these tranformation, we get

$$\begin{bmatrix}\lambda_\alpha\\\lambda_\beta\\\lambda_0\end{bmatrix} = \begin{bmatrix}\frac{3}{2}L_{ms} & 0 & 0\\0 & \frac{3}{2}L_{ms} & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}i_\alpha\\i_\beta\\i_0\end{bmatrix} + \begin{bmatrix}L_{sleak} & 0 & 0\\0 & L_{sleak} & 0\\0 & 0 & L_{sleak}\end{bmatrix}\begin{bmatrix}i_\alpha\\i_\beta\\i_0\end{bmatrix} +$$

$$\begin{bmatrix}\frac{3}{2}L_{ms} & 0 & 0\\0 & \frac{3}{2}L_{ms} & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}i_{\alpha r}\\i_{\beta r}\\i_{0r}\end{bmatrix}$$

Stator flux linkage equation is done.

Rotor voltage equation

Now we devrive rotor vlotage equation

$$\begin{bmatrix}v_{ar}\\v_{br}\\v_{cr}\end{bmatrix} = \begin{bmatrix}R_r & 0 & 0\\0 & R_r & 0\\0 & 0 & R_r\end{bmatrix}\begin{bmatrix}i_{ar}\\i_{br}\\i_{cr}\end{bmatrix} + \frac{d}{dt}\begin{bmatrix}\lambda_{ar}\\\lambda_{br}\\\lambda_{cr}\end{bmatrix}$$

Times $K_{rabc2\alpha\beta}$ on both sides

$$\begin{bmatrix} v_{\alpha r} \\ v_{\beta r} \\ v_{0r} \end{bmatrix} = \begin{bmatrix} R_r & 0 & 0 \\ 0 & R_r & 0 \\ 0 & 0 & R_r \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} + K_{rabc2\alpha\beta} \frac{d}{dt} \big(K_{\alpha\beta 2rabc} \begin{bmatrix} \lambda_{\alpha r} \\ \lambda_{\beta r} \\ \lambda_{0r} \end{bmatrix})$$

Calculate this with matlab, we obtain

$$\begin{bmatrix} v_{\alpha r} \\ v_{\beta r} \\ v_{0r} \end{bmatrix} = \begin{bmatrix} R_r & 0 & 0 \\ 0 & R_r & 0 \\ 0 & 0 & R_r \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} + \omega_r \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{\alpha r} \\ \lambda_{\beta r} \\ \lambda_{0r} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{\alpha r} \\ \lambda_{\beta r} \\ \lambda_{0r} \end{bmatrix}$$

Rotor flux linkage equation

$$\begin{bmatrix} \lambda_{ar} \\ \lambda_{br} \\ \lambda_{cr} \end{bmatrix} =$$

$$\begin{bmatrix} L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ms} \end{bmatrix} \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix} + \begin{bmatrix} L_{rleak} & 0 & 0 \\ 0 & L_{rleak} & 0 \\ 0 & 0 & L_{rleak} \end{bmatrix} \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix} +$$

$$L_{ms} \begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - \frac{2}{3}\pi) & \cos(\theta_r + \frac{2}{3}\pi) \\ \cos(\theta_r + \frac{2}{3}\pi) & \cos(\theta_r) & \cos(\theta_r - \frac{2}{3}\pi) \\ \cos(\theta_r - \frac{2}{3}\pi) & \cos(\theta_r + \frac{2}{3}\pi) & \cos(\theta_r) \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix}$$ Times$K_{rabc2\alpha\beta}$ on both sides, we get

$$\begin{bmatrix} \lambda_{\alpha r} \\ \lambda_{\beta r} \\ \lambda_{0r} \end{bmatrix} = K_{rabc2\alpha\beta} \begin{bmatrix} L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ms} \end{bmatrix} K_{\alpha\beta 2rabc} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} +$$

$$\begin{bmatrix} L_{rleak} & 0 & 0 \\ 0 & L_{rleak} & 0 \\ 0 & 0 & l_{rleak} \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} +$$

$$L_{ms} K_{rabc2\alpha\beta} \begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - \frac{2}{3}\pi) & \cos(\theta_r + \frac{2}{3}\pi) \\ \cos(\theta_r + \frac{2}{3}\pi) & \cos(\theta_r) & \cos(\theta_r - \frac{2}{3}\pi) \\ \cos(\theta_r - \frac{2}{3}\pi) & \cos(\theta_r + \frac{2}{3}\pi) & \cos(\theta_r) \end{bmatrix} K_{\alpha\beta 2abc} \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{0s} \end{bmatrix}$$

Note this mutual inductance matrix is the transpose of the previous one in stator flux equation,not the same!

Calculate this with matlab

$$\begin{bmatrix} \lambda_{\alpha r} \\ \lambda_{\beta r} \\ \lambda_{0r} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} + \begin{bmatrix} L_{rleak} & 0 & 0 \\ 0 & L_{rleak} & 0 \\ 0 & 0 & L_{rleak} \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{0s} \end{bmatrix}$$

Thus rotor flux equation obtained.

Torque equation

According to the book Analysis of electric machinery and drive system.We know torque is
$T_e = \frac{\partial W_c(\mathbf{i},\theta_{rm})}{\partial \theta_{rm}} = \frac{p}{2}\frac{\partial W_c(\mathbf{i},\theta_r)}{\partial \theta_r}$
$W_c = W_f = \frac{1}{2}(\mathbf{i}_{abcs})^T \mathbf{L}_s \mathbf{i}_{abcs} + (\mathbf{i}_{abcs})^T \mathbf{L}'_{sr} \mathbf{i}'_{abcr} + \frac{1}{2}(\mathbf{i}'_{abcr})^T \mathbf{L}'_r \mathbf{i}'_{abcr}$
since $\mathbf{L}_s$and$\mathbf{L}'_r$ are not functions of θ_r,take of derevertive we have
$\mathbf{T}_e = (\frac{P}{2})(\mathbf{i}_{abcs})^T \frac{\partial}{\partial \theta_r}[\mathbf{L}'_{sr}]\mathbf{i}'_{abcr}$ This is the abc frame,now we change into $\alpha\beta$ reference frame
$\mathbf{T}_e = (\frac{P}{2})(K_{\alpha\beta 2abc}K_{abc2\alpha\beta}\mathbf{i}_{abcs})^T \frac{\partial}{\partial \theta_r}[\mathbf{L}'_{sr}](K_{\alpha\beta 2rabc}K_{rabc2\alpha\beta}\mathbf{i}'_{abcr})$
$\mathbf{T}_e = (\frac{P}{2})\mathbf{i}^T_{\alpha\beta 0}K^T_{\alpha\beta 2abc}\frac{\partial}{\partial \theta_r}[\mathbf{L}'_{sr}]K_{\alpha\beta 2rabc}\mathbf{i}'_{abcr}$
calculate with matlab and we get
$\mathbf{T}_e = -\frac{3}{2}L_m(i_{\alpha s}i_{\beta r} - i_{\alpha r}i_{\beta s})$