Fundamentals part 1 (in class)

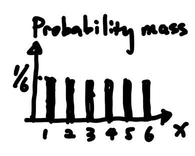
Wednesday, January 14, 2026 9:33 AM

Random Variables

X

$Pr(X=1) = 1/6$
$Pr(X=2) = 1/6$
$\vdots$
$Pr(X=6) = 1/6$

Probability mass function (PMF)



Support. $\Omega = \{1, 2, \dots, 6\}$
$Pr(X=i) = 1/6$ any $i \in \Omega$

Example.

- Bernoulli distribution

$X \sim Bernoulli(p)$: $Pr(X=0) = p$. $Pr(X=1) = 1-p$
$\Omega = \{0,1\}$ Probability of "tail"

- Binomial dist.

$X \sim Bi(n,p)$. Prob. of having x "tails" among n Bernoulli random var.

$$Pr(X=1) = \binom{n}{1} \cdot p \cdot (1-p)^{n-1}$$

$$Pr(X=\underset{\#\text{"tail"}}{x}) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} \quad \text{any } x = \{0, 1, 2, \dots, n\}$$

$\binom{n}{x}$: # combinations if choose x objects out of n objects.

{"1", "2", "3"} $n=3$.

$x=1$. "1" , "2" . "3" 3 $\binom{3}{1}=3$

$x=2$ {"1","2"}. = {"2","1"}, {"1","3"}, {"2","3"}. ~~{"1","1"}~~ $\binom{3}{2}=3$

$x=3$ {"1","2","3"} $\binom{3}{3}=1$ {"1","2"}, {"2","1"}

$$\binom{n}{x} = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-x+1) / x! = \frac{n!}{(n-x)!\,x!}$$

Permutation. : $n!$
different. ordering of n objects.

{"1", "2", "3"}

$P(3) = 3! = 6$. , $P(n) = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 = n!$

1, 2, 3.
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

What's the chance to have at least 2 people in class to have the same birthday?

$$Pr(\text{none have the same birthday}) = \frac{\#\text{ none have the same BD}}{\#\text{ all possible BD}}$$

$\#$ none same BD $= n \cdot (n-1) \cdots (n-x+1) = \frac{n!}{(n-x)!}$

$\#$ all BD $= n^x$

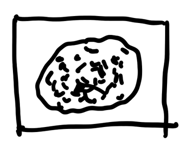
$$Pr(\text{none same BD}) = \frac{n!}{(n-x)!\,n^x} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \dots \cdot \frac{n-x+1}{n} = \prod_{k=0}^{x-1} \left(1 - \frac{k}{n}\right)$$

$$\log Pr(\text{none same BD}) = \log \prod_{k=0}^{x-1} \left(1-\frac{k}{n}\right) = \sum_{k=0}^{x-1} \log\left(1-\frac{k}{n}\right) \approx \sum_{k=0}^{x-1} \left(-\frac{k}{n}\right) = -\frac{(x-1)x}{2n}$$

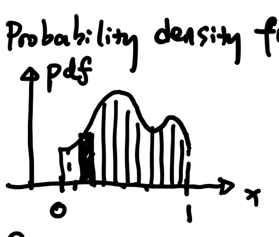
$$Pr(\underset{\text{exist}}{\exists} \text{ same BD}) = 1 - Pr(\text{none same BD}) = 1 - \exp\left(-\frac{(x-1)x}{2n}\right)$$

- Key words so far:

Random variable. Probability Mass Function.
Permutation . Combination . Bernoulli . Binomial dist.

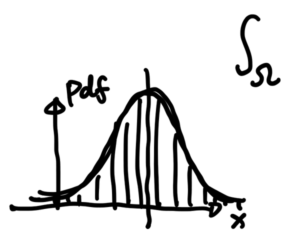
- 

Probability density function.



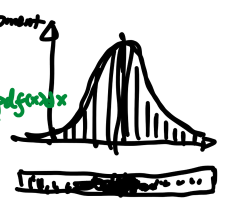
$\Omega = [0,1]$

$$\int_\Omega pdf(x)\,dx = 1 \iff \sum_\Omega pmf(x) = 1$$



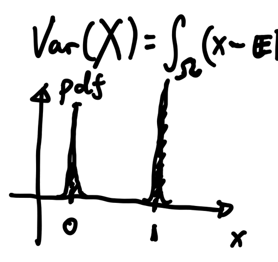
- Expectation (mean) 1-order moment

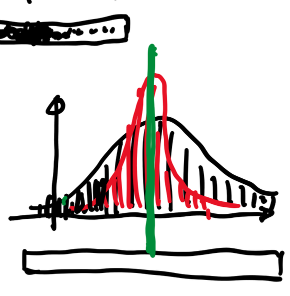
$$\underbrace{\mathbb{E}[X]}_{\text{expectation of } X} = \int_\Omega x \cdot pdf(x)\,dx \Big/ \int_\Omega pdf(x)\,dx$$



- Variance. 2-order moment .

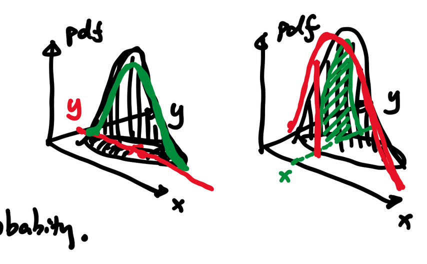
$$Var(X) = \int_\Omega (x - \mathbb{E}[X])^2 \cdot pdf(x)\,dx$$





- Joint distribution

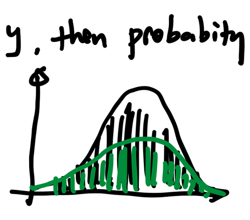
pdf(x, y)



- Conditional probability.

$Pr(X=x \mid Y=y)$: if $Y=y$, then probability of $X=x$

$$= \frac{pdf(x \mid Y=y)}{\int_{\Omega_Y} pdf(x,y)\,dy}$$

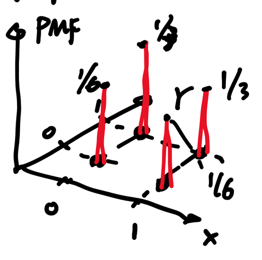


- Marginal probability

$Pr(X=x)$: probability $X=x$ (regardless of Y)

X \ Y	0	1
0	1/6	1/3
1	1/6	1/3

$Pr(X=0, Y=0) = 1/6$
$Pr(X=0, Y=1) = 1/3$
$Pr(X=1, Y=0) = 1/6$
$Pr(X=1, Y=1) = 1/3$



$$Pr(X=0 \mid Y=1) = \frac{Pr(X=0, Y=1)}{Pr(Y=1)} = \frac{1/3}{1/3 + 1/3} = 1/2$$

GG BG
GB BB

X : #G . $\Omega = \{0, 1, 2\}$

$$Pr(X=1 \text{ OR } X=2) = Pr(X=1) + Pr(X=2) = \binom{2}{1} \cdot p^1 \cdot (1-p)^{2-1} + \binom{2}{2} \cdot p^2 (1-p)^{2-2}$$
$$= 1/2 + 1/4 = 3/4 \quad (p=0.5)$$

$Pr(X=2) = 1/4$

$$Pr(X=2 \mid X \geq 1) = \frac{Pr(X=2 \text{ and } X \geq 1)}{Pr(X \geq 1)} = \frac{Pr(X=2)}{Pr(X=1 \text{ OR } X=2)} = \frac{1/4}{3/4} = 1/3$$

- Bayes' Rule.

A, B

$$Pr(B \mid A) = \frac{Pr(A \mid B) \cdot \overset{\text{Prior}}{Pr(B)}}{Pr(A)}$$

B \ A	☀	☁
☺	0.6	0.1
☂	0.1	0.2

$$Pr(B=☂ \mid A=☁) = \frac{Pr(A=☁ \mid B=☂)\,Pr(B=☂)}{Pr(A=☁)} = \frac{Pr(A=☁ \text{ and } B=☂)}{Pr(A=☁, B=☺) + Pr(A=☁, B=☂)} = \frac{0.2}{0.2+0.1} = 2/3$$

$$Pr(B=☂) = Pr(B=☂, A=☀) + Pr(B=☂, A=☁) = 0.1 + 0.2 = 0.3$$

$Pr(a) = \binom{2}{1} \cdot p \cdot (1-p) + \binom{2}{2} \cdot p^2 = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 3/4$
$Pr(b) = \frac{1}{4}$
$Pr(a, b) = Pr(b) \cdot Pr(a \mid b) = \frac{1}{4} \cdot 1$
$= Pr(a) \cdot Pr(b \mid a)$

$Pr(A, B) = Pr(A \mid B) \cdot Pr(B)$
$= Pr(B \mid A) \cdot Pr(A)$

$$\Rightarrow Pr(B \mid A) = \frac{Pr(A \mid B) \cdot Pr(B)}{Pr(A)}$$

B \ A	☀	☁
☺	0.6	0.1
☂	0.1	0.2

$$Pr(B=☂ \mid A=☁) = \frac{Pr(A=☁, B=☂)}{Pr(A=☁, B=☂) + Pr(A=☁, B=☺)} = \frac{0.2}{0.2+0.1} = 2/3$$

$Pr(B=☂) = 0.3$.

- Independence, covariance, correlation

  - Random var. X. Y. independent. if and only if

$$Pr(X=x, Y=y) = Pr(X=x) \cdot Pr(Y=y) \quad \forall x \in \Omega_X,\ y \in \Omega_Y$$

$$Pr(Y=y \mid X=x) = Pr(Y=y)$$

Y \ X	0	1
0	0.16	0.24
1	0.24	0.36.

$Pr(X=0) = Pr(X=0, Y=0) + Pr(X=0, Y=1) = 0.16 + 0.24 = 0.4$
$Pr(Y=0) = 0.16 + 0.24 = 0.4$

$Pr(X=0, Y=0) = 0.16 \overset{\checkmark}{\stackrel{?}{=}} Pr(X=0) \cdot Pr(Y=0) = 0.4 \cdot 0.4 = 0.16$
$Pr(X=0, Y=1) = Pr(X=0) \cdot Pr(Y=1)$
$Pr(X=1, Y=0) = Pr(X=1) \cdot Pr(Y=0)$
$Pr(X=1, Y=1) = Pr(X=1) \cdot Pr(Y=1)$

  - Covariance.

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

$$Var(X) = Cov(X, X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}} \in [-1, 1]$$

If X, Y independent, then $Cov(X,Y) = Corr(X,Y) = 0$
if $Cov(X,Y) = 0$. X, Y are not necessarily independent.

- Property of $\mathbb{E}[\ ]$, $Var(\ )$

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] \quad \forall X, Y$ random var.
$\mathbb{E}[a \cdot X] = a \cdot \mathbb{E}[X]$, a scalar
$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$
$Var(X+Y) = Var(X) + Var(Y)$ if $Cov(X,Y) = 0$ (including X, Y independent)
$Var(a \cdot X) = a^2\,Var(X)$. a scalar.