

Constructing QITs from Quotients

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0.1 Introduction and Motivation

⁷ Quotient Inductive Types (QITs) are a central mechanism for specifying datatypes equipped
⁸ with identifications in set-truncated Homotopy Type Theory [7]. They support a wide
⁹ range of fundamental constructions, including the HoTT reals [7], partiality monads [3], and
¹⁰ ordinal-like structures.

¹¹ Despite their expressive power, QITs present significant foundational difficulties. Shulman
¹² and Lumsdaine established a no-go theorem showing that certain QITs, including standard
¹³ definitions of the countable ordinals, cannot be constructed using quotients alone [5]. This
¹⁴ suggests that some form of infinitary principle or choice is unavoidable in a constructive
¹⁵ metatheory.

¹⁶ Fiore, Pitts, and Steenkamp subsequently showed that a broad class of QITs, which
¹⁷ they call Quotient-W Types (QW types), admit an Initial Algebra semantics assuming the
¹⁸ Weak Initial Set of Covers (WISC) principle [4]. Their result recovers many important
¹⁹ examples, including the extensional countable ordinals. However, WISC remains a nontrivial
²⁰ choice principle [9], and its necessity is poorly understood. In particular, it is unclear which
²¹ features of a QIT genuinely require choice, and which arise from limitations of existing proof
²² techniques.

²³ In this work we isolate two structural properties of quotient systems that explains this
²⁴ distinction: *locality* and the tighter property of *depth-delta*. Intuitively, locality measures
²⁵ how deeply one must inspect the inductive structure of terms in order to witness that two
²⁶ elements are equal while depth-delta measures the maximum depth change an equation.
²⁷ In pathological examples, such as tree ordinals, the principle of extensionality quantifies
²⁸ over *all* ordinals less than a pair of ordinals, unbounded descent into subtrees, forcing the
²⁹ use of global choice principles. By contrast, in examples such as infinite mobiles—infinitely
³⁰ branching trees quotiented by permutation—equality preserves depth and never requires
³¹ inspecting substructure beyond a fixed finite level. In fact because permutation doesn't
³² change the subtree relation, we can say that it is in a class of depth-preserving QW types,
³³ along with the classical definition of multi-sets as a QW type over a list.

³⁴ Our key observation is that this boundedness of inspection depth is exactly what is needed
³⁵ to recover the Fiore–Pitts–Steenkamp construction constructively. We sketch a proof that
³⁶ for QW signatures whose equations are bounded by a fixed ordinal α , cocontinuity of the
³⁷ associated polynomial functor can be proved without any appeal to any choice principle. As
³⁸ a consequence, we obtain a fully constructive construction of infinite mobiles as an initial
³⁹ QIT, contrary to previous expectations, which we have formalized in Agda [10].

⁴⁰ This leads to our central research question:

⁴¹ Which quotient-W types can be constructed fully constructively, without assuming
⁴² any form of choice?

⁴³ We propose bounded locality as a criterion separating constructible QITs from genuinely
⁴⁴ non-constructive ones, and we develop a framework for expressing and exploiting this
⁴⁵ boundedness in the construction of initial QW-algebras.



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XX:2 Constructing QITs from Quotients

46 0.2 Background

47 We briefly recall the framework of quotient-W types introduced by Fiore, Pitts, and Steenkamp
48 [4]. A QW signature consists of:

- 49 ■ A container $\Sigma = S \triangleleft P$ [1], specifying the shape and positions of a Polynomial Functor
50 whose initial algebra is a W-Type [6].
51 ■ A system of equations (E, V, l, r) , where each equation $e : E$ has a set of variables $V(e)$
52 and left- and right-hand sides given by terms in the free Σ -algebra over those variables.

53 From such a signature, Fiore et al. construct a diagram of approximations indexed by
54 a size structure [4]. Intuitively, this diagram represents successive stages of the quotient
55 construction, where each stage gives the proofs available within a bounded depth. The
56 desired QW type is obtained as the Colimit of this diagram.

57 To show that this colimit yields an initial algebra, two key properties are required:

- 58 1. The existence of suitable covers for the underlying W-type and the equation contexts.
59 2. *Cocontinuity* of the associated polynomial functor, i.e. preservation of the colimit.

60 Both properties are established in Fiore et al. using the indexed WISC principle [4].
61 In particular, WISC is crucial in proving cocontinuity, which amounts to constructing a
62 isomorphism for the functor F determined by the container Σ .

63 Operationally, cocontinuity expresses the fact that applying constructors commutes with
64 taking quotients: forming a node after quotienting is equivalent to quotienting after forming
65 a node. The proof in Fiore et al. relies on WISC to select a cover to size-bound the family of
66 subtrees, enabling the construction of inverse maps witnessing this isomorphism.

67 Our work revisits this step. We observe that the need for WISC arises precisely when the
68 equations allow arbitrarily deep inspection of terms, as in the extensional ordinal example.
69 When all equations are depth-bounded, the required coherence data can instead be constructed
70 by well-founded recursion on a fixed ordinal bound α , avoiding any appeal to choice.

71 0.3 Locality Principle

72 We formalise the notion of locality by assigning a *rank* function

73 $\text{rank} : T_\Sigma X \rightarrow \mathcal{O}$

74 measuring the height of the underlying tree. An equation system is said to be α -local
75 if, for every equation $e : E$, both sides $l(e)$ and $r(e)$ have rank *at most* a fixed ordinal α .
76 Intuitively, this means that every generating equation only inspects structure up to depth
77 $\leq \alpha$.

78 $\text{rank} : T_\Sigma X \rightarrow \mathcal{O}$

79 0.4 Main Result

80 Our main theorem is that α -local QW signatures admit fully constructive initial algebra
81 semantics.

82 *Theorem* Let Σ be a QW signature whose equations are α -local. Then the associated
83 quotient-W type exists as an initial algebra, and its construction does not require WISC or
84 any other choice principle.

85 The proof follows the outline of Fiore–Pitts–Steenkamp [4], but replaces the use of WISC
 86 in the inverse map by offsetting by a bound ordinal α , giving a constructable function
 87 between term bound and maximum proof depth, giving cocontinuity. This construction is
 88 fully constructive and does not require any choice principles.

89 **0.5 Case Study: Infinite Mobiles**

90 Our formalisation in Agda proves that in the case of mobiles, the defining equations are
 91 provably *depth-preserving*, since all variables on the left appear at the same depth on the
 92 right. This is enough to show that depth is preserved in the quotient type, and that if two
 93 terms are equal then they can be proven in the stage bounded by either ordinal.

94 As a result, the main theorem yields a fully constructive construction of infinite mobiles
 95 as an initial QIT. This provides a concrete example of a genuinely infinitary quotient type
 96 that does *not* require choice, contrary to previous assumptions.

97 **0.6 Boundary Phenomena**

98 Our analysis explains why certain examples genuinely require choice. In the case of extensional
 99 ordinals, equality is defined by mutual containment of downward closures, which requires
 100 unbounded descent into subtrees [4]. No ordinal bound α suffices, and the quotient is therefore
 101 non-local. This accounts for the essential use of WISC in the construction of ordinals.

102 More generally, any quotient whose generators quantify over arbitrarily deep substructures
 103 or require unbounded transitive closure will fall outside the α -local fragment.

104 **0.7 Method Overview**

105 Technically, we construct stratified diagrams indexed by ordinals below α , and show that all
 106 cones factor through a bounded stage. Plumpness of the ordinal ensures the existence of
 107 suprema for such cones, yielding cocontinuity without choice.

108 **0.8 Related Work**

109 Our work builds on the QW framework of Fiore, Pitts, and Steenkamp [4], and complements
 110 the no-go results of Shulman and Lumsdaine [5]. It is also related to Pitts and Steenkamp's
 111 notion of infinitary inflation [8], which similarly isolates sources of non-constructivity in
 112 inductive definitions.

113 **0.9 Future Work**

114 Future directions include formalising the locality theorem stated above, and extending the
 115 analysis to indexed QW types [2].

116 **0.10 Conclusion**

117 We identify bounded locality as a structural criterion characterising which quotient-W
 118 types admit fully constructive constructions. This explains when choice principles such as
 119 WISC are genuinely necessary, and when they can be avoided. Our results show that many
 120 infinitary quotients previously thought to require choice are, in fact, constructible in a purely
 121 constructive setting.

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