

# PH4606 - Lecture 6

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## 1 Transmission from one fluid to another: oblique incidence

The boundary is located at  $x = 0$  while the incident, reflected and transmitted waves make the respective angles  $\Theta_i$ ,  $\Theta_r$ , and  $\Theta_t$ , see Fig. 6.1.

$$p_i = P_i e^{i(\omega t - k_1 x \cos \Theta_i - k_1 y \sin \Theta_i)} \quad (6.1)$$

$$p_r = P_r e^{i(\omega t + k_1 x \cos \Theta_r - k_1 y \sin \Theta_r)} \quad (6.2)$$

$$p_t = P_t e^{i(\omega t - k_2 x \cos \Theta_t - k_2 y \sin \Theta_t)} \quad (6.3)$$

Here we use real values for the incident and the reflected angle, while the transmitted angle is allowed to be complex valued.

Figure 6.1: Reflection and transmission of a plane wave obliquely incident on the planar boundary between fluids with different characteristic impedances.

As the boundary  $x = 0$  and the pressure is continuous, thus we can write:

$$P_i e^{-ik_1 y \sin \Theta_i} + P_r e^{-ik_1 y \sin \Theta_r} = P_t e^{-ik_2 y \sin \Theta_t} \quad (6.4)$$

Equation (6.4) holds for all  $y$ , thus we can set  $y = 0$  and obtain  $P_i + P_r = P_t$ . Inserting this expression into Eq. (6.4) we obtain

$$P_i e^{-ik_1 y \sin \Theta_i} + P_r e^{-ik_1 y \sin \Theta_r} = P_i e^{-ik_2 y \sin \Theta_t} + P_r e^{-ik_2 y \sin \Theta_t} \quad (6.5)$$

Again, as Eq. (6.5) holds for all  $y$  the exponents must be the same, which demands

$$k_1 \sin \Theta_i = k_2 \sin \Theta_t \quad (6.6)$$

$$k_1 \sin \Theta_r = k_2 \sin \Theta_t \quad (6.7)$$

Subtracting Eq. (6.7) from (6.6) leads to the expression that the incident angle is equal to the reflected angle or

$$\sin \Theta_i = \sin \Theta_r \quad (6.8)$$

While rearranging Eq. (6.6) gives Snell's law

$$\frac{\sin \Theta_i}{c_1} = \frac{\sin \Theta_t}{c_2} \quad (6.9)$$

Next we want to determine the reflection and transmission coefficients as a function of the incidence angle. We start with the second boundary condition, that the normal particle velocity is continuous at the boundary. This can be written as

$$u_i \cos \Theta_i + u_r \cos \Theta_r = u_t \cos \Theta_t \quad . \quad (6.10)$$

We can replace the velocities with pressures using the specific acoustic impedance (and taking care of the sign), e.g.  $r_1 = p_i/u_i$  and  $r_1 = -p_r/u_r$ . Then we obtain

$$\frac{1}{r_1} \cos \Theta_i - \frac{P_r \cos \Theta_r}{P_i r_1} = \frac{P_t \cos \Theta_t}{P_i r_2} \quad , \quad (6.11)$$

and see that the reflection coefficient is appearing at the R.H.S. and the transmission coefficient on the L.H.S. of Eq. (6.11):

$$\frac{1}{r_1} \cos \Theta_i - \frac{P_r \cos \Theta_r}{P_i r_1} = \frac{P_t \cos \Theta_t}{P_i r_2} \quad , \quad (6.12)$$

Inserting  $R$  and  $T$  into Eq. (6.12) we obtain an expression relating the reflection and the transmission coefficient:

$$1 - R = \frac{r_1 \cos \Theta_t}{r_2 \cos \Theta_i} T \quad (6.13)$$

We can get rid of the transmission coefficient  $T$  using the relationship

$$1 + R = T \quad (6.14)$$

and find the relation named Rayleigh reflection coefficient:

$$R = \frac{r_2 \cos \Theta_i - r_1 \cos \Theta_t}{r_2 \cos \Theta_i + r_1 \cos \Theta_t} = \frac{\frac{r_2}{\cos \Theta_t} - \frac{r_1}{\cos \Theta_i}}{\frac{r_2}{\cos \Theta_t} + \frac{r_1}{\cos \Theta_i}} \quad . \quad (6.15)$$

We can use Snell's law to express the cosine of the angle of the transmitted sound  $\Theta_t$  as a function of the incidence angle  $\Theta_i$  such as

$$\cos \Theta_t = \sqrt{1 - \sin^2 \Theta_t} = \sqrt{1 - \left(\frac{c_2}{c_1}\right)^2 \sin^2 \Theta_i} \quad . \quad (6.16)$$

## 1.1 Transmission and reflection scenarios

### 1.1.1 Scenario 1: $c_1 > c_2$

Snell's law Eq. (6.9) can be rewritten as

$$\sin \Theta_t = \frac{c_2}{c_1} \sin \Theta_i \quad (6.17)$$

We always find a real-valued solution to Eq. (6.17) if  $c_2 > c_1$ . Then the L.H.S. of Eq. (6.17) is smaller than the R.H.S., thus the acoustic wave is refracted towards the surface normal, or in other words  $\Theta_t < \Theta_i$ .

### 1.1.2 Scenario 2: $c_1 < c_2$ and $\Theta_i < \Theta_c$

If  $c_1 < c_2$  we still find a real-valued solution as long as the incidence angle is  $\Theta_i < \Theta_c$ , with

$$\sin \Theta_c = c_1/c_2 \quad (6.18)$$

$\Theta_c$  is the critical angle under which transmission into medium 2 occurs. From Snell's law it follows that  $\sin \Theta_t = c_2/c_1 \sin \Theta_i$ , thus  $\Theta_t$  is always larger than  $\Theta_i$  and the beam is refracted away from the normal.

### 1.1.3 Scenario 3: $c_1 < c_2$ and $\Theta_i > \Theta_c$

For this scenario the argument of the squareroot in Eq. (6.16) is negative and we obtain a complex result.

$$\cos \Theta_t = -i \sqrt{\left(\frac{c_2}{c_1}\right)^2 \sin^2 \Theta_i - 1} \quad (6.19)$$

Inserting this expression into the pressure of the transmitted sound, Eq. (6.3), we obtain

$$p_t = P_t e^{-\gamma x} e^{i(\omega t - k_1 y \sin \Theta_i)} \quad (6.20)$$

with

$$\gamma = k_2 \sqrt{\left(\frac{c_2}{c_1}\right)^2 \sin^2 \Theta_i - 1} \quad (6.21)$$

Note that the transmitted wave propagates along the interface, in positive y-direction, and is exponentially damped in the second medium. These waves are called evanescent waves, they do not transport energy into the second medium.

Let's have a look at the reflection coefficient  $R$ . It is purely complex and can be written as (after some calculations)

$$R = e^{i\phi} = \cos \phi + i \sin \phi \quad (6.22)$$

with

$$\phi = 2 \tan^{-1} \left[ \frac{\rho_1}{\rho_2} \sqrt{\left(\frac{\cos \Theta_c}{\cos \Theta_i}\right)^2 - 1} \right] . \quad (6.23)$$

If  $\Theta_i \approx \Theta_c$  we can obtain for  $\phi = 0$  and thus  $R = 1$ ; that means the wave is totally reflected with unchanged phase.

If we have a grazing incidence, that is  $\Theta_i \rightarrow \pi/2$  the  $\phi \rightarrow \pi$  and the reflection coefficient is  $R = -1$ , thus the interface resembles a pressure release surface.

## 1.2 Angle of intromission $\Theta_I$

Looking at the formula for the reflection coefficient Eq. (6.15) we see that the reflection coefficient becomes 0 for real  $\theta_t$  and

$$\frac{r_2}{r_1} = \frac{\cos \Theta_t}{\cos \Theta_i} \quad . \quad (6.24)$$

If we combine Eq. (6.24) with Snell's law Eq. (6.9) to eliminate  $\Theta_t$  we obtain after some calculations

$$\sin \Theta_I = \sqrt{\frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{\left(\frac{r_2}{r_1}\right)^2 - \left(\frac{c_2}{c_1}\right)^2}} = \sqrt{\frac{1 - \left(\frac{r_1}{r_2}\right)^2}{1 - \left(\frac{\rho_1}{\rho_2}\right)^2}} \quad (6.25)$$

This defines the angle of intromission. It can only exist if

- 1.)  $r_1 < r_2$  and  $c_2 < c_1$
- 2.)  $r_1 > r_2$  and  $c_2 > c_1$

For the second case there is a critical angle. The angle of intromission is smaller than the critical angle.

Tutorialwork:

Plot the magnitude and phase of the reflection coefficient for the following cases

- $c_2/c_1 = 0.9$  and  $r_2/r_1=0.9$
- $c_2/c_1 = 0.9$  and  $r_2/r_1=1.1$ , what is the angle of intromission?
- $c_2/c_1 = 1.1$  and  $r_2/r_1=1.1$ , what is the critical angle?
- $c_2/c_1 = 1.1$  and  $r_2/r_1=0.9$ , what is the angle of intromission, what is the critical angle?