PH4606 - Lecture 6

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1 Transmission from one fluid to another: oblique incidence

The boundary is located at $x = 0$ while the incident, reflected and transmitted waves make the respective angles Θ_i, Θ_r, and Θ_t, see Fig. 6.1.

$$p_i = P_i\, e^{i(\omega t - k_1 x \cos\Theta_i - k_1 y \sin\Theta_i)} \tag{6.1}$$

$$p_r = P_r\, e^{i(\omega t + k_1 x \cos\Theta_r - k_1 y \sin\Theta_r)} \tag{6.2}$$

$$p_t = P_t\, e^{i(\omega t - k_2 x \cos\Theta_t - k_2 y \sin\Theta_t)} \tag{6.3}$$

Here we use real values for the incident and the reflected angle, while the transmitted angle is allowed to be complex valued.

Figure 6.1: Reflection and transmission of a plane wave obliquely incident on the planar boundary between fluids with different characteristic impedances.

As the boundary $x = 0$ and the pressure is continuous, thus we can write:

$$P_i e^{-ik_1 y \sin\Theta_i} + P_r e^{-ik_1 y \sin\Theta_r} = P_t e^{-ik_2 y \sin\Theta_t} \tag{6.4}$$

Equation (6.4) holds for all y, thus we can set $y = 0$ and obtain $P_i + P_r = P_t$. Inserting this expression into Eq. (6.4) we obtain

$$P_i e^{-ik_1 y \sin\Theta_i} + P_r e^{-ik_1 y \sin\Theta_r} = P_i e^{-ik_2 y \sin\Theta_t} + P_r e^{-ik_2 y \sin\Theta_t} \quad . \tag{6.5}$$

Again, as Eq. (6.5) holds for all y the exponents must be the same, which demands

$$k_1 \sin\Theta_i = k_2 \sin\Theta_t \tag{6.6}$$

$$k_1 \sin\Theta_r = k_2 \sin\Theta_t \quad . \tag{6.7}$$

Subtracting Eq. (6.7) from (6.6) leads to the expression that the incident angle is equal to the reflected angle or

$$\sin\Theta_i = \sin\Theta_r \quad . \tag{6.8}$$

While rearranging Eq. (6.6) gives Snell's law

$$\frac{\sin\Theta_i}{c_1} = \frac{\sin\Theta_t}{c_2} \quad . \tag{6.9}$$

Next we want to determine the reflection and transmission coefficients as a function of the incidence angle.

We start wih the second boundary condition, that the normal particle veloicty is continuous at the boundary. This can be written as

$$u_i \cos\Theta_i + u_r \cos\Theta_r = u_t \cos\Theta_t \quad . \tag{6.10}$$

We can replace the velocities with pressures using the specific acoustic impedance (and taking care of the sign), e.g. $r_1 = p_i/u_i$ and $r_1 = -p_r/u_r$. Then we obtain

$$\frac{1}{r_1}\cos\Theta_i - \frac{P_r}{P_i}\frac{\cos\Theta_r}{r_1} = \frac{P_t}{P_i}\frac{\cos\Theta_t}{r_2} \quad , \tag{6.11}$$

and see that the reflecion coefficient is appearing at the R.H.S. and the transmission coefficient on the L.H.S. of Eq. (6.11):

$$\frac{1}{r_1}\cos\Theta_i - \frac{P_r}{P_i}\frac{\cos\Theta_r}{r_1} = \frac{P_t}{P_i}\frac{\cos\Theta_t}{r_2} \quad , \tag{6.12}$$

Inserting R and T into Eq. (6.12) we obtain an expression relating the reflection and the transmission coefficient:

$$1 - R = \frac{r_1}{r_2}\frac{\cos\Theta_t}{\cos\Theta_i}T \tag{6.13}$$

We can get rid of the transmission coefficient T using the relationship

$$1 + R = T \tag{6.14}$$

and find the relation named Rayleigh reflection coefficient:

$$R = \frac{r_2\cos\Theta_i - r_1\cos\cos\Theta_t}{r_2\cos\Theta_i + r_1\cos\Theta_t} = \frac{\frac{r_2}{\cos\Theta_t} - \frac{r_1}{\cos\Theta_i}}{\frac{r_2}{\cos\Theta_t} + \frac{r_1}{\cos\Theta_i}} \quad . \tag{6.15}$$

We can use Snell's law to express the cosine of the angle of the transmitted sound Θ_t as a function of the incidence angle Θ_i such as

$$\cos\Theta_t = \sqrt{1-\sin^2\Theta_t} = \sqrt{1-\left(\frac{c_2}{c_1}\right)^2\sin^2\Theta_i} \quad . \tag{6.16}$$

1.1 Transmission and reflection scenarios

1.1.1 Scenario 1: $c_1 > c_2$

Snell's law Eq. (6.9) can be rewritten as

$$\sin\Theta_t = \frac{c_2}{c_1}\sin\Theta_i \tag{6.17}$$

We always find a real-valued solution to Eq. (6.17) if $c_2 > c_1$. Then the L.H.S. of Eq. (6.17) is smaller than the R.H.S., thus the acoustic wave is refracted towards the surface normal, or in other words $\Theta_t < \Theta_i$.

1.1.2 Scenario 2: $c_1 < c_2$ and $\Theta_i < \Theta_c$

If $c_1 < c_2$ we still find a real-valued solution as long as the incidence angle is $\Theta_i < \Theta_c$, with

$$\sin\Theta_c = c_1/c_2 \tag{6.18}$$

Θ_c is the critical angle under which transmission into medium 2 occurs. From Snell's law it follows that $\sin\Theta_t = c_2/c_1 \sin\Theta_i$, thus Θ_t is always larger than Θ_i and the beam is refracted away from the normal.

1.1.3 Scenario 3: $c_1 < c_2$ and $\Theta_i > \Theta_c$

For this scenario the argument of the squareroot in Eq. (6.16) is negative and we obtain a complex result.

$$\cos\Theta_t = -i\sqrt{\left(\frac{c_2}{c_1}\right)^2 \sin^2\Theta_i - 1} \tag{6.19}$$

Inserting this expression into the pressure of the transmitted sound, Eq. (6.3), we obtain

$$p_t = P_t e^{-\gamma x} e^{i(\omega t - k_1 y \sin\Theta_i)} \tag{6.20}$$

with

$$\gamma = k_2\sqrt{\left(\frac{c_2}{c_1}\right) \sin^2\Theta_i - 1} \tag{6.21}$$

Note that the transmitted wave propagates along the interface, in positive y-direction, and is exponentially damped in the second medium. This waves are called evanescent waves, they do not transport energy into the second medium.

Let's have a look at the reflection coefficient R. It is purely complex and can be written as (after some calculations)

$$R = e^{i\phi} = \cos\phi + i\sin\phi \tag{6.22}$$

with

$$\phi = 2\tan^{-1}\left[\frac{\rho_1}{\rho_2}\sqrt{\left(\frac{\cos\Theta_c}{\cos\Theta_i}\right)^2 - 1}\right] \quad . \tag{6.23}$$

If $\Theta_i \approx \Theta_c$ we can obtain for $\phi = 0$ and thus $R = 1$; that means the wave is totally reflected with unchanged phase.

If we have a grazing incidence, that is $\Theta_i \to \pi/2$ the $\phi \to \pi$ and the reflection coefficient is $R = -1$, thus the interface resembles a pressure release surface.

1.2 Angle of intromission Θ_I

Looking at the formula for the reflection coefficient Eq. (6.15) we see that the reflection coefficient becomes 0 for real θ_t and

$$\frac{r_2}{r_1} = \frac{\cos\Theta_t}{\cos\Theta_i} \quad . \tag{6.24}$$

If we combine Eq. (6.24) with Snell's law Eq. (6.9) to eliminate Θ_t we obtain after some calculations

$$\sin\Theta_I = \sqrt{\frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{\left(\frac{r_2}{r_1}\right)^2 - \left(\frac{c_2}{c_1}\right)^2}} = \sqrt{\frac{1 - \left(\frac{r_1}{r_2}\right)^2}{1 - \left(\frac{\rho_1}{\rho_2}\right)^2}} \tag{6.25}$$

This defines the angle of intromission. It can only exists if

1.) $r_1 < r_2$ and $c_2 < c_1$

2.) $r_1 > r_2$ and $c_2 > c_1$

For the second case there is a critical angle. The angle of intromission is smaller than the critical angle.

Tutorialwork:

Plot the magnitude and phase of the reflection coefficient for the following cases

- $c_2/c_1 = 0.9$ and r_2/r_1=0.9
- $c_2/c_1 = 0.9$ and r_2/r_1=1.1, what is the angle of intromission?
- $c_2/c_1 = 1.1$ and r_2/r_1=1.1, what is the critical angle?
- $c_2/c_1 = 1.1$ and r_2/r_1=0.9, what is the angle of intromission, what is the critical angle?

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