Kinematics
Stream/Streak/Pathline
Material Derivative

PH3501

How can we visualize flows in the lab?

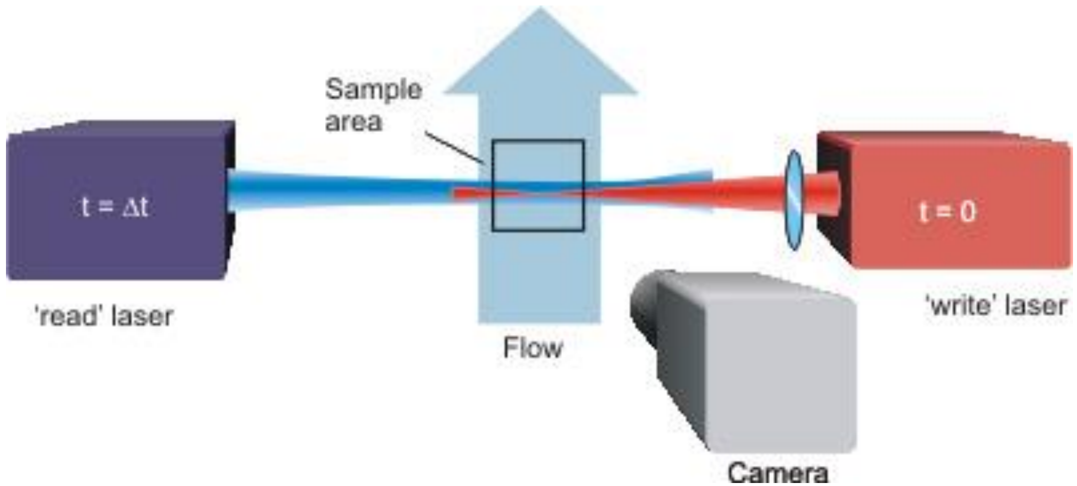
- Add neutrally buoyant “tracer particles” to the flow and take pictures
  - Dye in water…smoke in air
    - Problem 1: over time, dye smoke diffuse due to molecular mixing and therefore do not follow fluid particles
    - Problem 2: this traces out a streakline (next slide)
- Take two snapshots of the flow over a short time interval. Interogate the image and ‘correlate’ particle positions between the two frames in order to determine the change in position
    - Called (Digital) Particle Image Velocimetry (DPIV)
    - Fast computers/good digital cameras have revolutionized in past 20 years
    - Current challenges: holographic 3D, 3 component flows

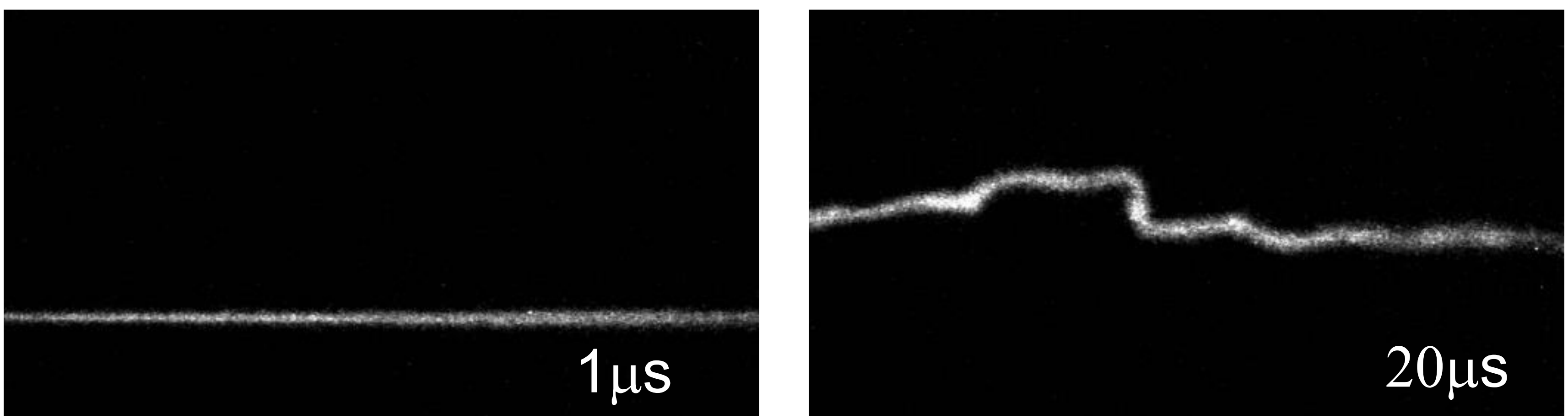
Adding dye

- Suppose we inject dye at a point in the flow over a period of time.
- At any given time (after we start the dye injection), we will see the locus of points in space that are the positions of particles that were at the injection position at previous time
- In unsteady flow, these are neither the pathlines nor streamlines, and we call them *streaklines*
- In steady flow, all three sets of lines coincide
- The discussion neglects the effect of diffusion or non-neutral buoyancy which would cause particles not to follow the fluid particles

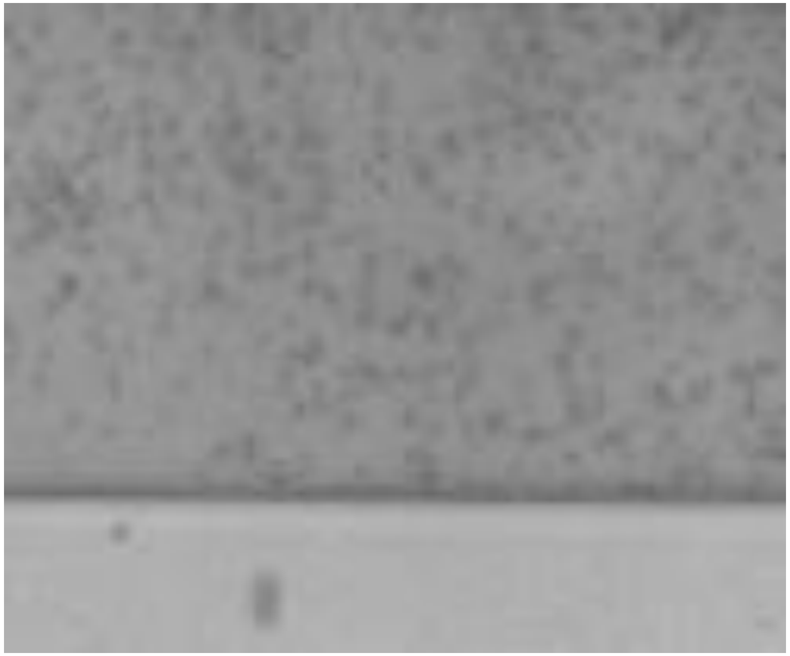
Direct visualization of “fluid particle”

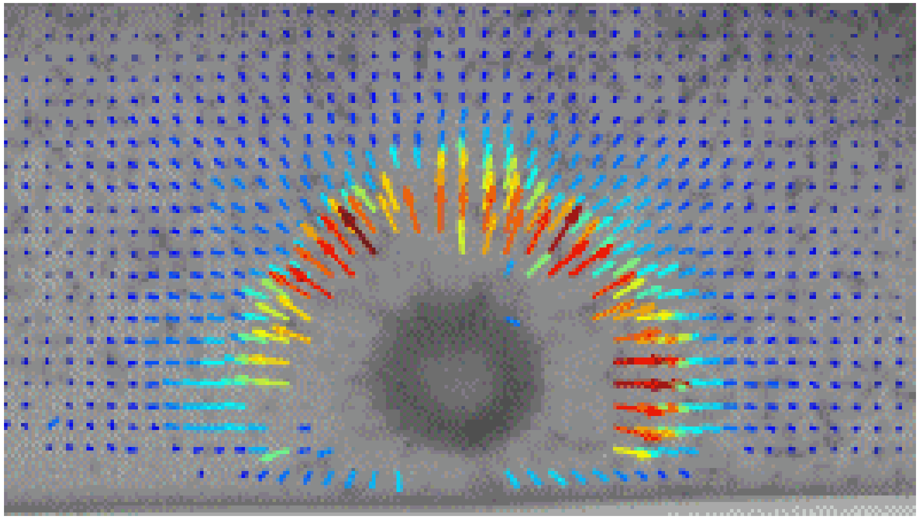
Using Molecular Flow Tagging Technique

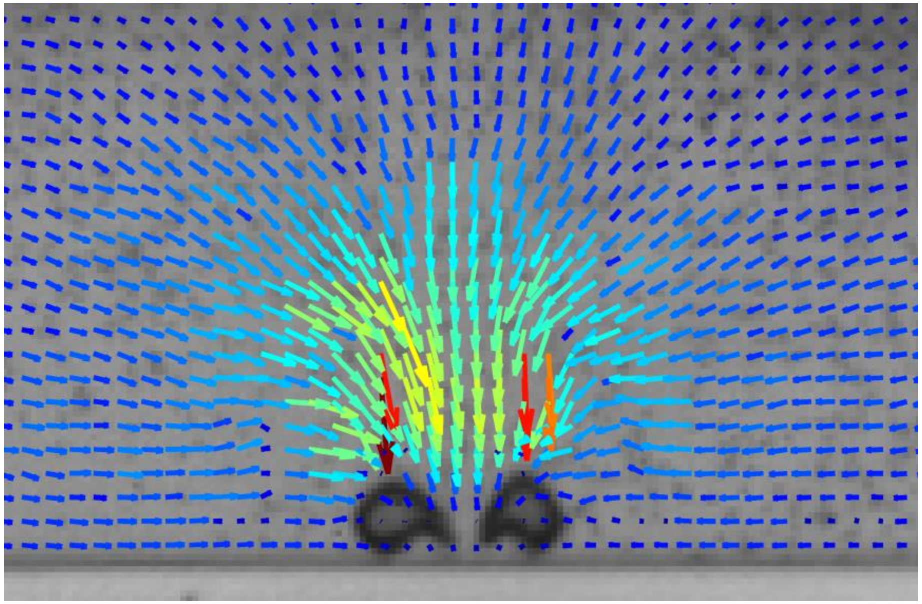




From PhD Thesis (Delft) Thijs Elenbaas







Streamlines

Streamline: A curve tangential to the instantaneous velocity vector; the trajectory of a particle on this "frozen" velocity field.

$$\left.\begin{array}{ll} d\mathbf{x} & \text{streamline} \\ \\ \mathbf{u} & \text{velocity field} \end{array}\right\} \quad d\mathbf{x} \times \mathbf{u} = 0 \longrightarrow \frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}$$

How to solve?

1 **Fix the time, *t=t₀***

$$\frac{dx_1}{u_1(x_1, x_2, x_3; t = t_0)} = \frac{dx_2}{u_2(x_1, x_2, x_3; t = t_0)} = \frac{dx_3}{u_3(x_1, x_2, x_3; t = t_0)} = d\tau$$

2 **Then integrate with respect to τ.**
This is not time.

$$\frac{dx_1}{d\tau} = u_1(x_1, x_2, x_3; t = t_0), \quad x_1(\tau = 0) = x_{10}$$
$$\frac{dx_2}{d\tau} = u_2(x_1, x_2, x_3; t = t_0), \quad x_2(\tau = 0) = x_{20}$$
$$\frac{dx_3}{d\tau} = u_3(x_1, x_2, x_3; t = t_0), \quad x_3(\tau = 0) = x_{30}$$

3 **As result we have a streamline with parameter τ.**

$$x_1(\tau; t_0; x_{10})$$
$$x_2(\tau; t_0; x_{20})$$
$$x_3(\tau; t_0; x_{30})$$

Example Streamline

Find the streamline through $\boldsymbol{x_0}$ = (1,1). at t=1.

$$u_1 = 2x_1 + t$$
$$u_2 = x_2 - 2t$$

$$\begin{aligned} \frac{dx_1}{d\tau} &= 2x_1 + t\big|_{t=1}, \quad x_1(\tau = 0) = 1, \\ \frac{dx_2}{d\tau} &= x_2 - 2t\big|_{t=1}, \quad x_2(\tau = 0) = 1 \end{aligned}$$

$$\begin{aligned} \frac{dx_1}{d\tau} &= 2x_1 + 1, \quad x_1(\tau = 0) = 1, \\ \frac{dx_2}{d\tau} &= x_2 - 2, \quad x_2(\tau = 0) = 1 \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{3}{2}e^{2\tau} - \frac{1}{2}, \\ x_2 &= -e^{\tau} + 2 \end{aligned}$$

Eliminate τ:

$$\tau = \frac{1}{2}\ln\left(\frac{2}{3}\left(x_1 + \frac{1}{2}\right)\right).$$

$$x_2 = 2 - \sqrt{\frac{2}{3}\left(x_1 + \frac{1}{2}\right)}.$$

Pathline

Pathline: Trajectory of a specific fluid particle. We select the particle by choosing the position $\boldsymbol{x_0}$ at at time t_0.

By definition:

$$\frac{dx_1}{dt} = u_1(x_1, x_2, x_3, t), \quad x_1(t = t_0) = x_{10}$$
$$\frac{dx_2}{dt} = u_2(x_1, x_2, x_3, t), \quad x_2(t = t_0) = x_{20}$$
$$\frac{dx_3}{dt} = u_3(x_1, x_2, x_3, t), \quad x_3(t = t_0) = x_{30}$$

Integration which most often has to be done numerically gives the pathline:

$$x_1(t, x_{10})$$
$$x_2(t, x_{20})$$
$$x_3(t, x_{30})$$

Example Pathline

Find the pathline for a fluid particle which is at t=1 at $\boldsymbol{x_0}$ = (1,1)..

$$\begin{aligned}
\frac{dx_1}{dt} &= 2x_1 + t, \quad x_1(t=1) = 1, \\
\frac{dx_2}{dt} &= 2x_2 - 2t, \quad x_2(t=1) = 1.
\end{aligned}$$

These have the solution:

$$\begin{aligned}
x_1 &= \frac{7}{4}e^{2(t-1)} - \frac{t}{2} - \frac{1}{4}, \\
x_2 &= -3e^{t-1} + 2t + 2.
\end{aligned}$$

Difficult to eliminate t, thus a parametric representation of the pathline.

Streaklines

Streakline: locus of points at time *t* which have passed through a particular point $\mathbf{x_0}$ at some past time $\hat{t} < t$ **"Dye Streaklines"**

1

Integrate the equation

$$\frac{dx_1}{dt} = u_1(x_1, x_2, x_3, t), \quad x_1(t = \hat{t}) = x_{10}$$
$$\frac{dx_2}{dt} = u_2(x_1, x_2, x_3, t), \quad x_2(t = \hat{t}) = x_{20}$$
$$\frac{dx_3}{dt} = u_3(x_1, x_2, x_3, t), \quad x_3(t = \hat{t}) = x_{30}$$

2

$$x_1(t; \hat{t}, x_{10})$$
$$x_2(t; \hat{t}, x_{20})$$
$$x_3(t; \hat{t}, x_{30})$$

Then fix the time *t* and point $\mathbf{x_0}$.
We obtain a parametric curve:

3

$$x_1(\hat{t})$$
$$x_2(\hat{t})$$
$$x_3(\hat{t})$$

Example Streakline

Find the streakline at time t=1 through the point $\boldsymbol{x_0}$ = (1,1).

$$
\begin{aligned}
\frac{dx_1}{dt} &= 2x_1 + t, \quad x_1(t=\hat{t}) = 1, \\
\frac{dx_2}{dt} &= 2x_2 - 2t, \quad x_2(t=\hat{t}) = 1.
\end{aligned}
$$

The solution is

$$
\begin{aligned}
x_1 &= \frac{5+2\hat{t}}{4} e^{2(t-\hat{t})} - \frac{t}{2} - \frac{1}{4}, \\
x_2 &= -(1+2\hat{t})e^{t-\hat{t}} + 2t + 2.
\end{aligned}
$$

For t=1

$$
\begin{aligned}
x_1 &= \frac{5+2\hat{t}}{4} e^{2(1-\hat{t})} - \frac{3}{4}, \\
x_2 &= -(1+2\hat{t})e^{1-\hat{t}} + 4.
\end{aligned}
$$

Difficult to write explicitly, as $x_2(x_1)$ but can be plotted parametrically..

Examples of Streamlines

Laminar Pipe Flow

http://www.youtube.com/v/KqqtOb30jWs&list=PL605541FDAAC92B45

Turbulent Pipe Flow

http://www.youtube.com/v/NplrDarMDF8&list=PL605541FDAAC92B45

Boundary Layer Flow

http://www.youtube.com/v/cUTkqZeiMow&list=PL605541FDAAC92B45

Summary

- Streamline: lines everywhere tangent to the velocity vectors at an instant in time
- Pathline: trajectory of a fluid particle
- Streakline: the locus of present fluid particles that started at a specific location in the past