Fluid dynamics: week 5

streamlines, pathlines…
Euler method
flow across a line

observe and measure flows

material derivative
conservation of mass

basic laws of liquids

today's lecture

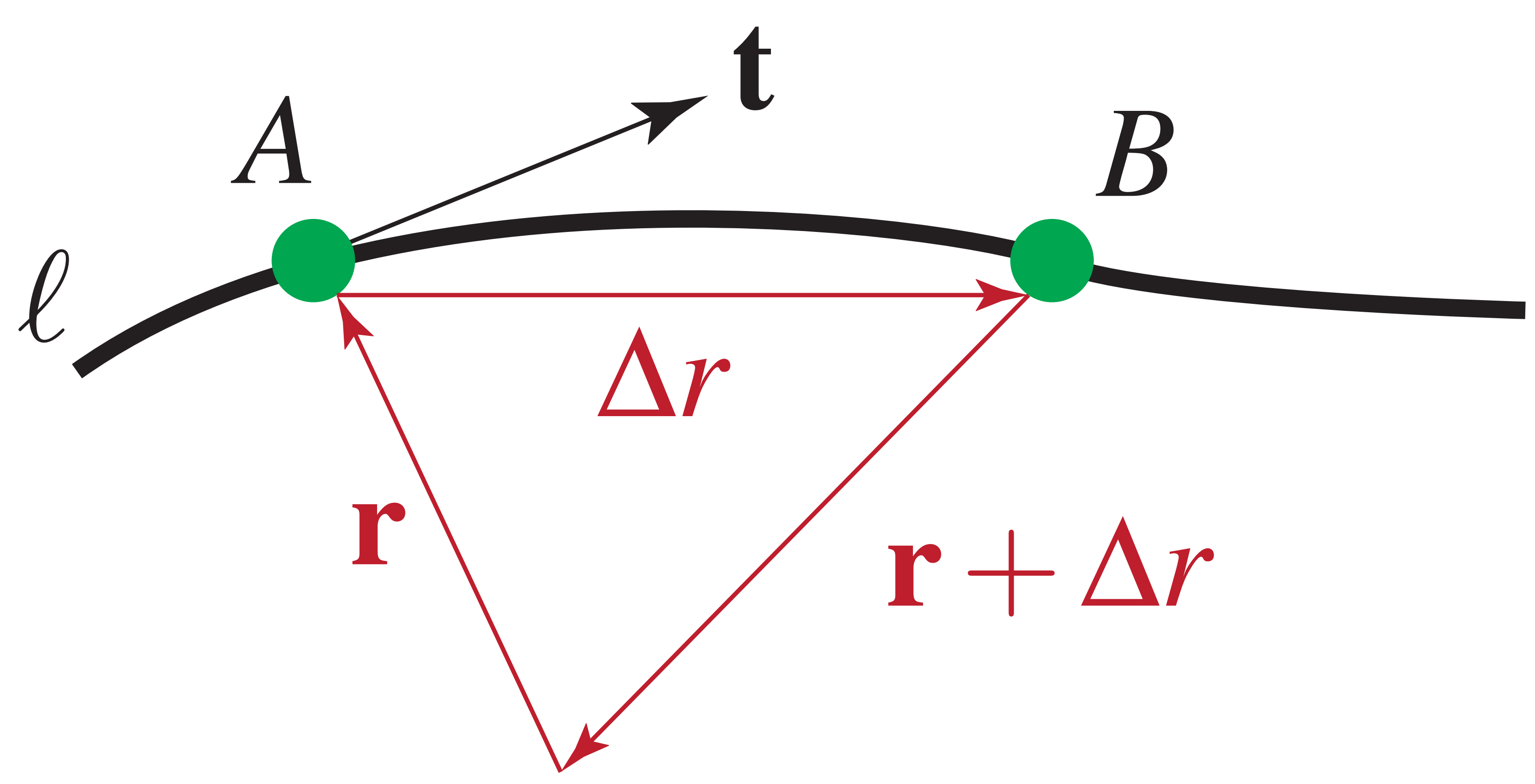
stream function
velocity potential
Laplace equation

using the basic laws to predict flows

(subject to ideal conditions)

Measuring flows

Tangent vector



If dr is small enough,
dr and t will be the same thing

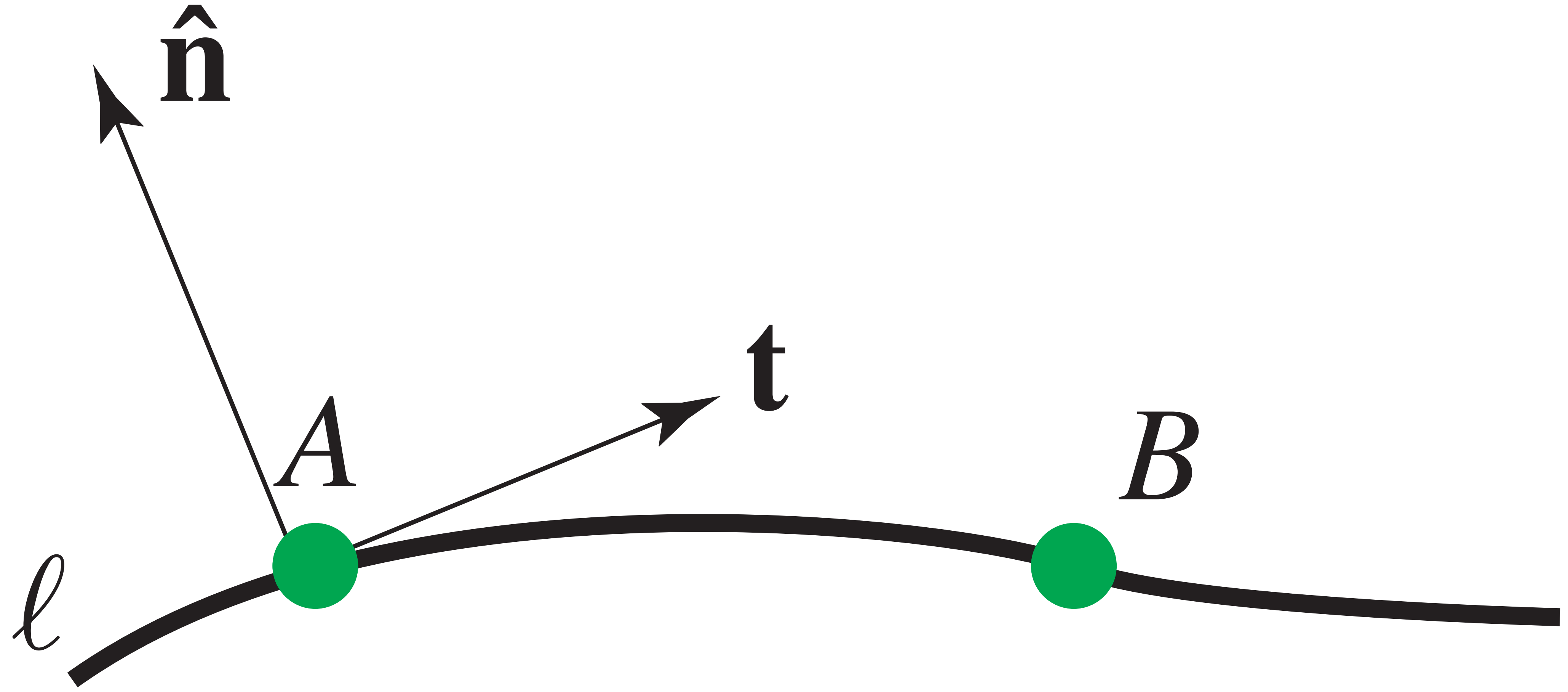
$$\Delta\mathbf{r} = \Delta\ell\,\hat{\mathbf{t}}$$

$$\lim_{\ell\to 0}\frac{\Delta r}{\Delta\ell} = \hat{\mathbf{t}}$$

$$\hat{\mathbf{t}}_x = \frac{dx}{d\ell} \qquad \hat{\mathbf{t}}_y = \frac{dy}{d\ell}$$

Normal vector

The normal vector is perpendicular* to the tangent.
(*conventionally π/2 counter-clockwise)

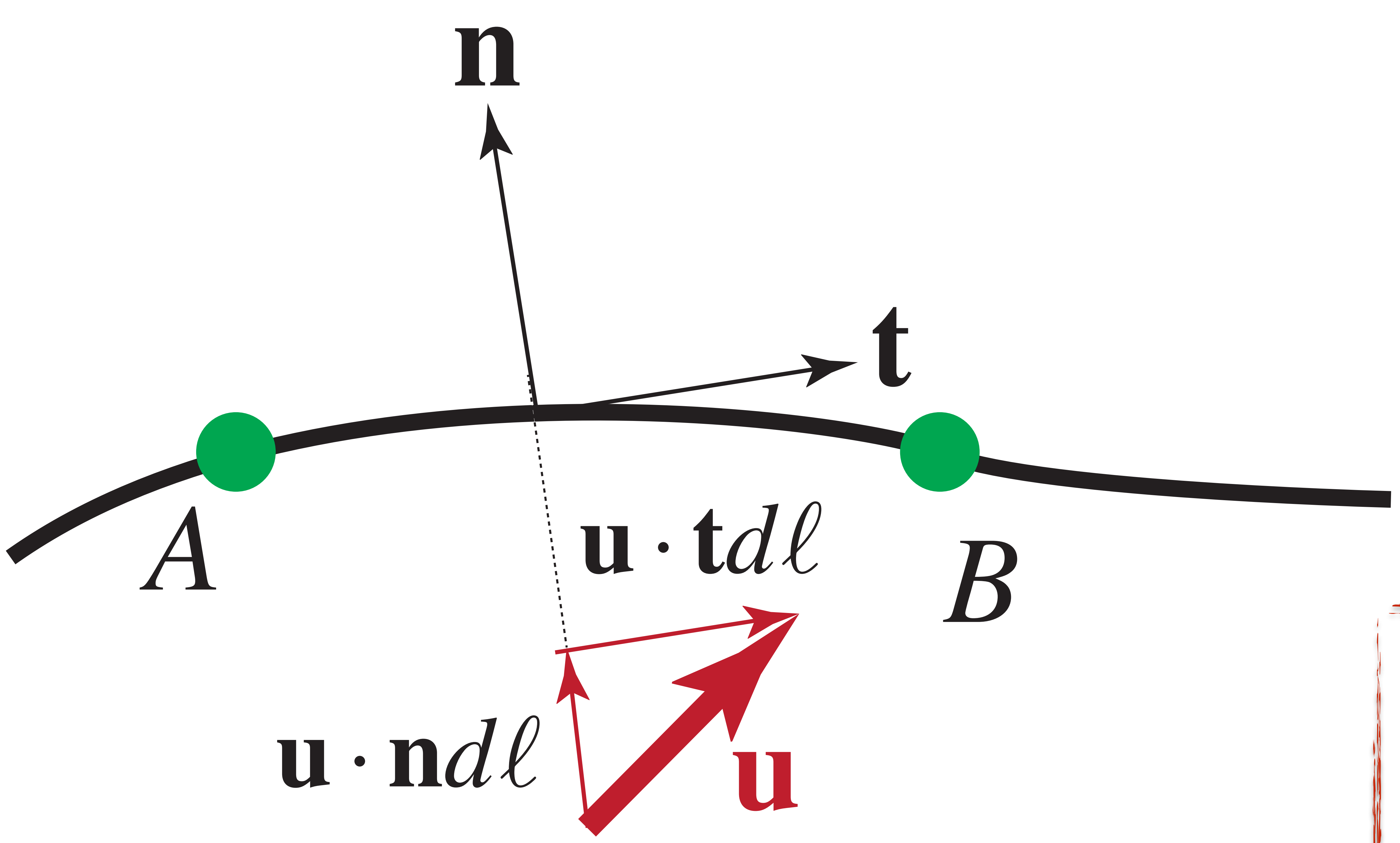


$$\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} dx/dl \\ dy/dl \end{bmatrix}$$

$$\hat{\mathbf{n}}_x = -\frac{dy}{d\ell} \qquad \hat{\mathbf{n}}_y = \frac{dx}{d\ell}$$

Flow across a line: "areal flux"



What's the flux through AB?

$$\boxed{Q_m = \int_A^B \mathbf{u}\cdot\mathbf{n}\,d\ell = \int_A^B (u\,dy - v\,dx)}$$

$$\hat{\mathbf{n}}_x = -\frac{dy}{d\ell} \qquad \hat{\mathbf{n}}_y = \frac{dx}{d\ell}$$

The basic law of liquids: conserving mass

Material derivative

One of the key ideas of this course: if you want to know how a quantity **A** varies at (x_0, y_0, t_0), perform a Taylor expansion…

$$\rho(t,x,y) = \rho(t_0,x_0,y_0) + (t-t_0)\frac{\partial\rho}{\partial t}\bigg|_{t_0} + (x-x_0)\frac{\partial\rho}{\partial x}\bigg|_{x_0} + \ldots$$

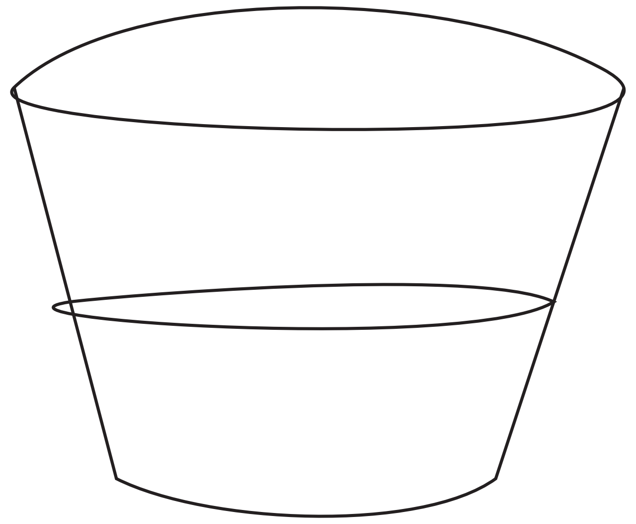
rearrange terms, take limits…

$$\frac{D\rho}{Dt} = \lim_{t\to t_0}\frac{\rho(t,x,y)-\rho(t_0,x_0,y_0)}{t-t_0} = \frac{\partial\rho}{\partial t}\bigg|_{t_0} + \mathbf{u}\cdot\nabla\rho$$

particle view
a.k.a. Lagrangian

field view
a.k.a. Eulerian

The theorem of squeezing a sponge



mass flux through D — rate of change of mass

$$\iint \rho \mathbf{u} \cdot \mathbf{n} dA = -\frac{d}{dt} \iiint \rho dV$$

divergence theorem

$$\iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

the continuity equation

Continuity for an incompressible fluid

$$\frac{\partial \rho}{\partial t} + \boxed{\nabla \cdot (\rho \mathbf{u})} = 0$$

vector identity

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho} + \rho (\nabla \cdot \mathbf{u}) = 0$$

material derivative

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0$$

If the fluid parcel is incompressible

$$\nabla \cdot \mathbf{u} = 0$$

continuity equation for an incompressible liquid

2D streamfunction

Construct a function ψ(x,y) that satisfies mass conservation and allows the flow to be described by 1 variable instead of 2.

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Mixed partial derivatives
(Schwarz-Clairaut theorem)

$$\frac{\partial \psi}{\partial x \partial y} = \frac{\partial \psi}{\partial y \partial x}$$

By inspection,

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

What is ψ physically?

Work backwards from the velocities we defined…

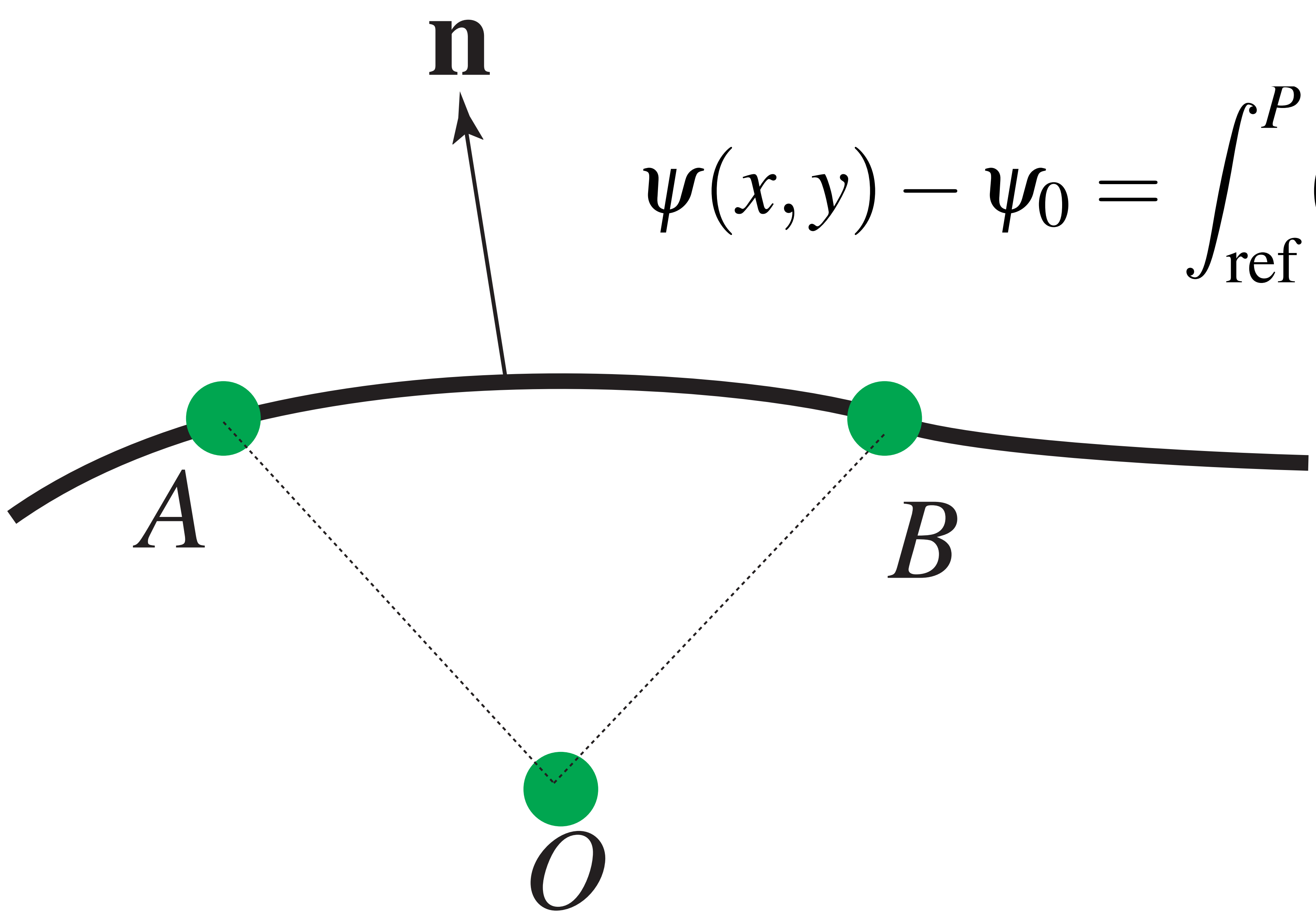
$$u = \frac{\partial \psi}{\partial y} \qquad \psi = \int u dy + f(x) + h_1$$

$$v = -\frac{\partial \psi}{\partial x} \qquad \psi = \int -v dx + g(y) + h_2$$

$$\boxed{\psi(x,y) - \psi_0 = \int_{\text{ref}}^{P} (u dy - v dx)}$$

It's the flow rate across a line defined by a point P(x,y) and a reference

Big deal… what's ψ for?



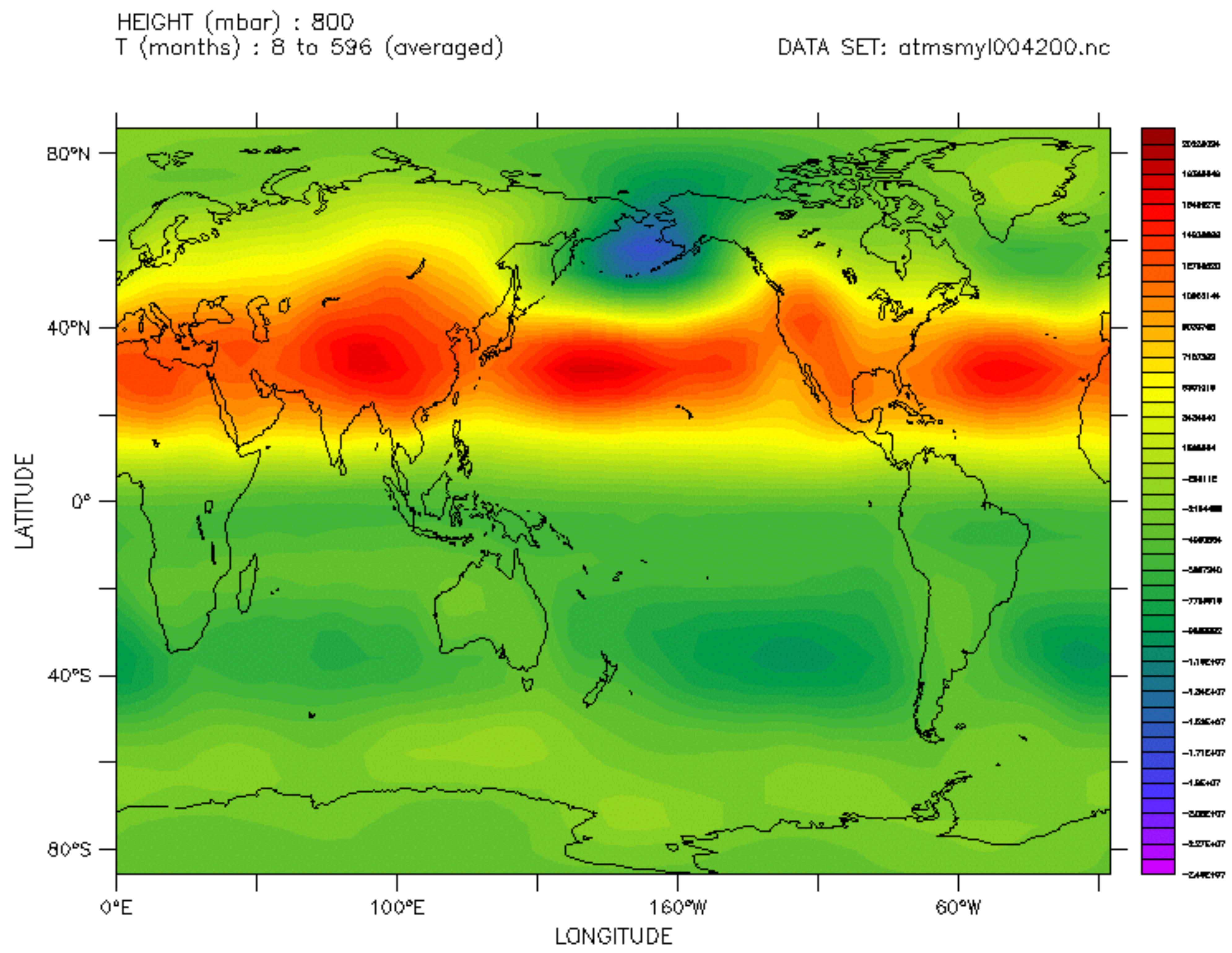
$$\psi(x,y) - \psi_0 = \int_{\text{ref}}^{P} (u\,dy - v\,dx)$$

With the streamfunction we can measure flow rates over a line by any two points

$$Q_{AB} = \overbrace{\psi_A - \psi_O}^{Q_{AO}} - (\overbrace{\psi_B - \psi_O}^{Q_{BO}})$$
$$= \psi_A - \psi_B$$

It's similar to using a ruler. Even though there's a theoretical origin *O*, we don't need to know it to measure distances.

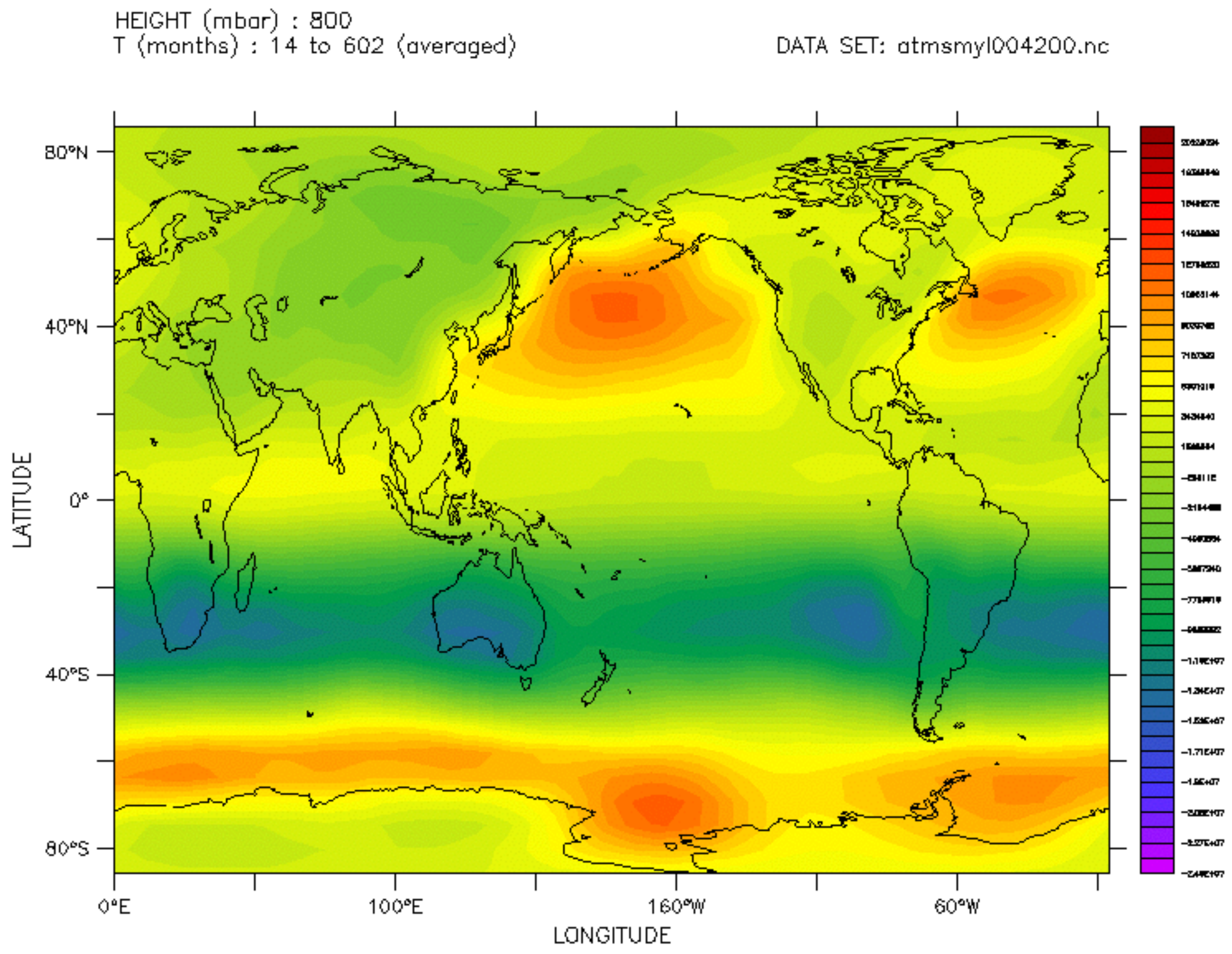
If the two points are arbitrarily close, we can even measure ψ / the flow rate at a 'point'



Stream Function in DJF (m^2/s)

the stream function averaged over 50 winters

Dec, **J**an, **F**eb



Stream Function in JJA (m^2/s)

the stream function averaged over 50 summers

Jun, **J**ul, **A**ug

Velocity potential

To measure a vector field **u** in a fair way we want to define some scalar **ϕ** that depends only on the points A and B (but not their path).

ϕ

$$\phi(x,y,t) = \phi_0(t) + \int_A^B \mathbf{u} \cdot d\mathbf{x}$$

analogous to work done, electric potential energy…

If A = B

$$\oint_c (udx + vdy) = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = 0$$

Green's theorem

To achieve this magic, the vector field must be **irrotational**, or

$$\nabla \times \mathbf{u} = 0$$

Velocity potential

Define a potential $\phi(x,y)$ from the condition that the flow is irrotational.

Irrotationality
$$\nabla \times \mathbf{u} = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Using the **mixed partials** again…

$$\frac{\partial \phi}{\partial x \partial y} = \frac{\partial \phi}{\partial y \partial x}$$

By inspection

$$\boxed{u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}}$$

Streamfunction

conserves mass by construction

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

impose irrotationality

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Velocity potential

irrotational by construction

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

impose mass conservation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Flows that conserve mass and are irrotational obey the **Laplace equation**

“When a flow is both frictionless and irrotational, pleasant things happen.”

— F. M. White

Discretising the second derivative

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

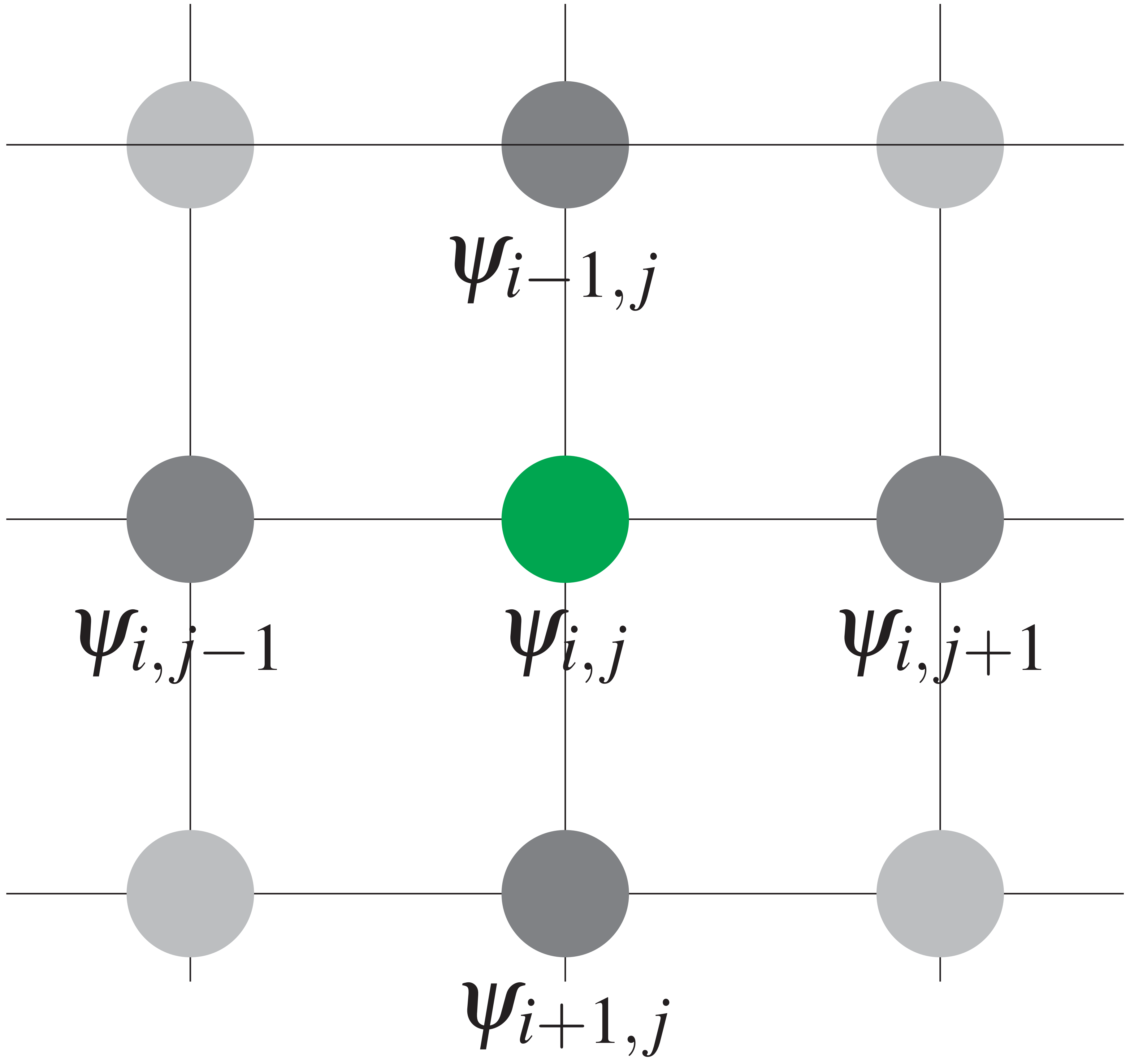
1 $$\psi(x+\Delta x) = \psi(x) + \frac{\partial \psi}{\partial x}\Delta x + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2 \psi}{\partial x^2} + \ldots$$

2 $$\psi(x-\Delta x) = \psi(x) - \frac{\partial \psi}{\partial x}\Delta x + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2 \psi}{\partial x^2} + \ldots$$

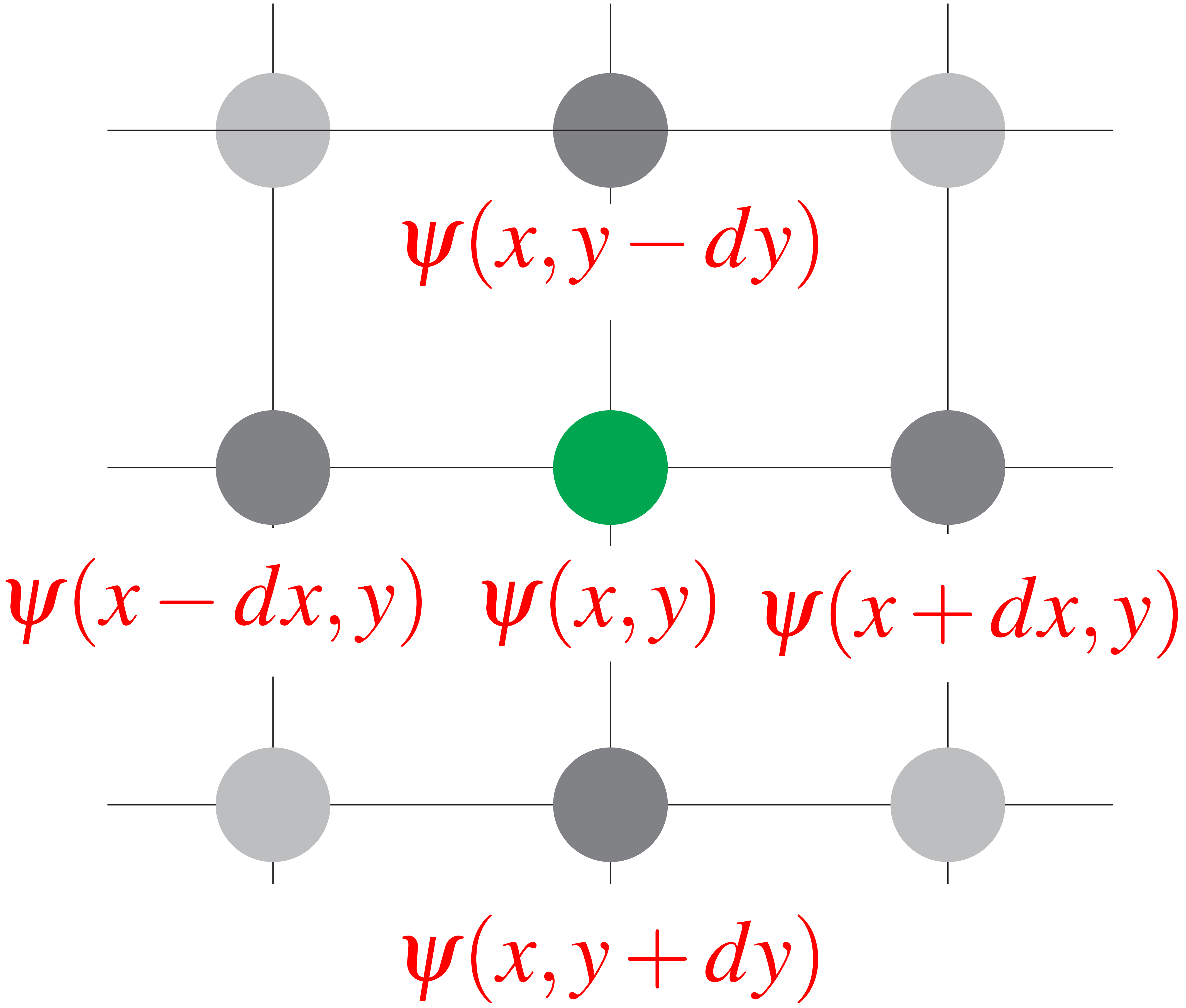
1 + 2 $$\frac{\partial^2 \psi}{\partial x^2} = \frac{\psi(x+\Delta x) - 2\psi(x) + \psi(x-\Delta x)}{(\Delta x)^2}$$

Question: how do we write these quantities in a discrete form that a computer can read?

Discretisation



Matrix representation



Cartesian representation

Discrete Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta x)^2} + \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta y)^2} = 0$$

the update equation

$$\boxed{\psi_{i,j} = \frac{1}{4}\left(\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1}\right)}$$

Each point $\psi_{i,j}$ is updated by taking an average of its neighbours

Boundary conditions

No-slip boundary

Velocity parallel to a wall is zero

for a horizontal wall

$$u_{\text{wall}} = 0 \qquad \frac{\partial \psi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial x} = 0$$

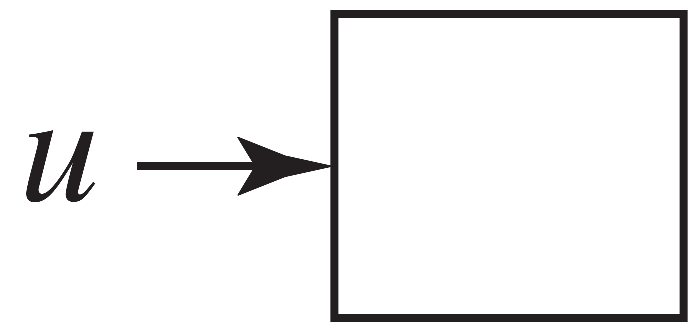
Impermeable boundary

Velocity normal to a wall is zero

for a horizontal wall

$$v_{\text{wall}} = 0 \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = 0$$

Inflow



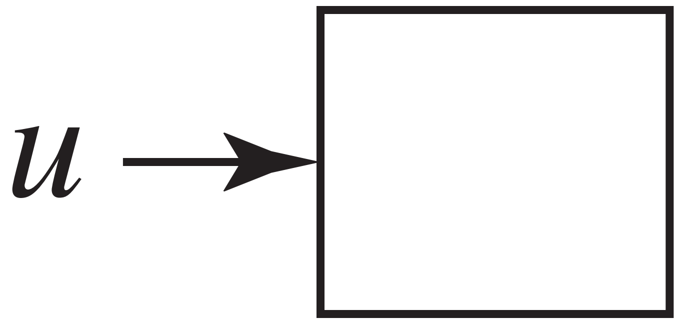
$$\frac{\partial \psi}{\partial y} = u \qquad \frac{\partial \phi}{\partial x} = u$$

For the tutorial you'll have to work out the b.c. for four walls

Calculating the boundary conditions

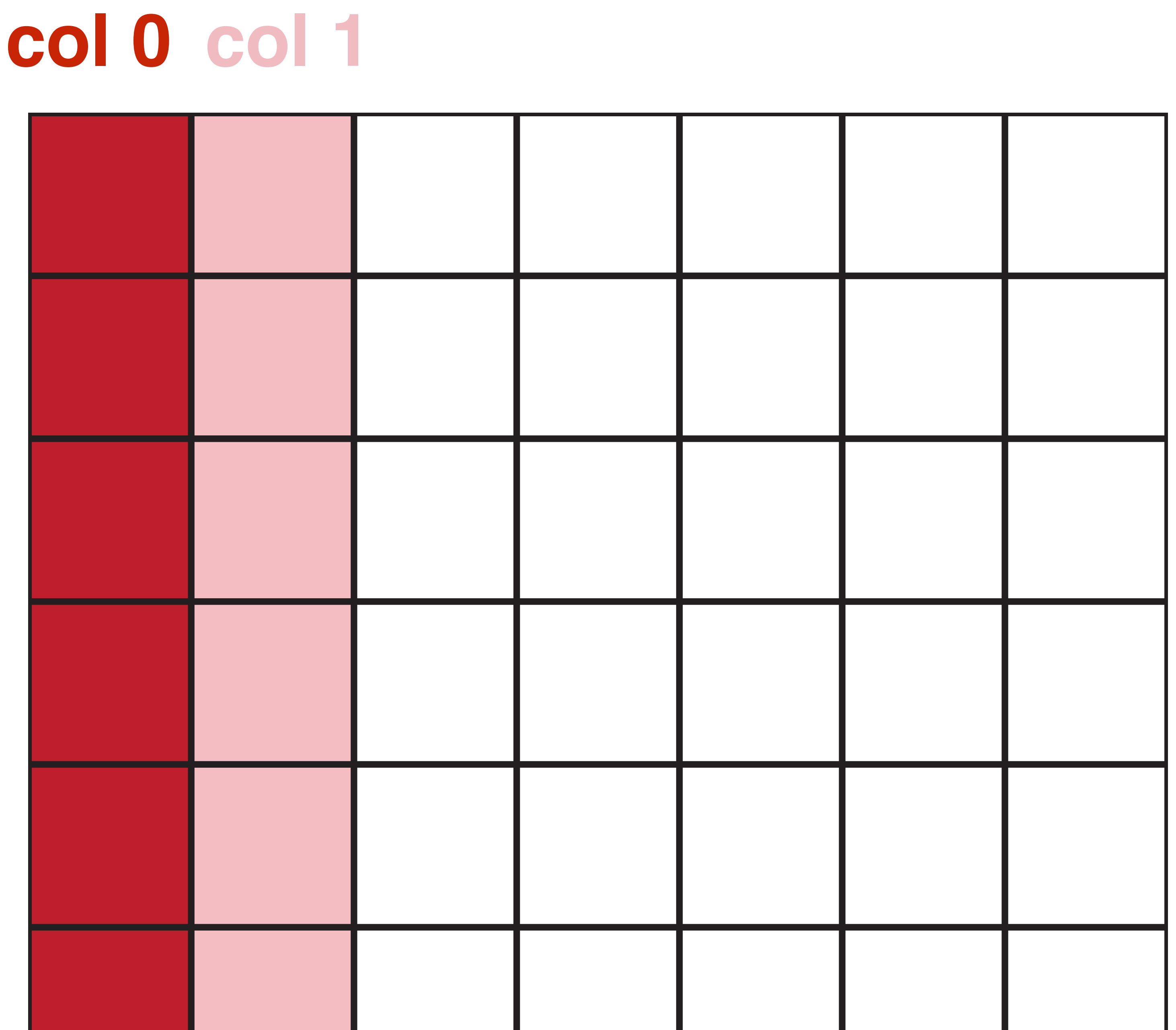
As an example, discretise an inflow **u** from the left (vertical) wall.

Inflow



$$\frac{\partial \psi}{\partial y} = u \qquad\qquad \frac{\partial \phi}{\partial x} = u$$

$$\psi_{i,0} - \psi_{i,1} = u\Delta y \qquad \phi_{0,i} - \phi_{1,i} = u\Delta x$$



In English: values of column 0 are copied from column 1

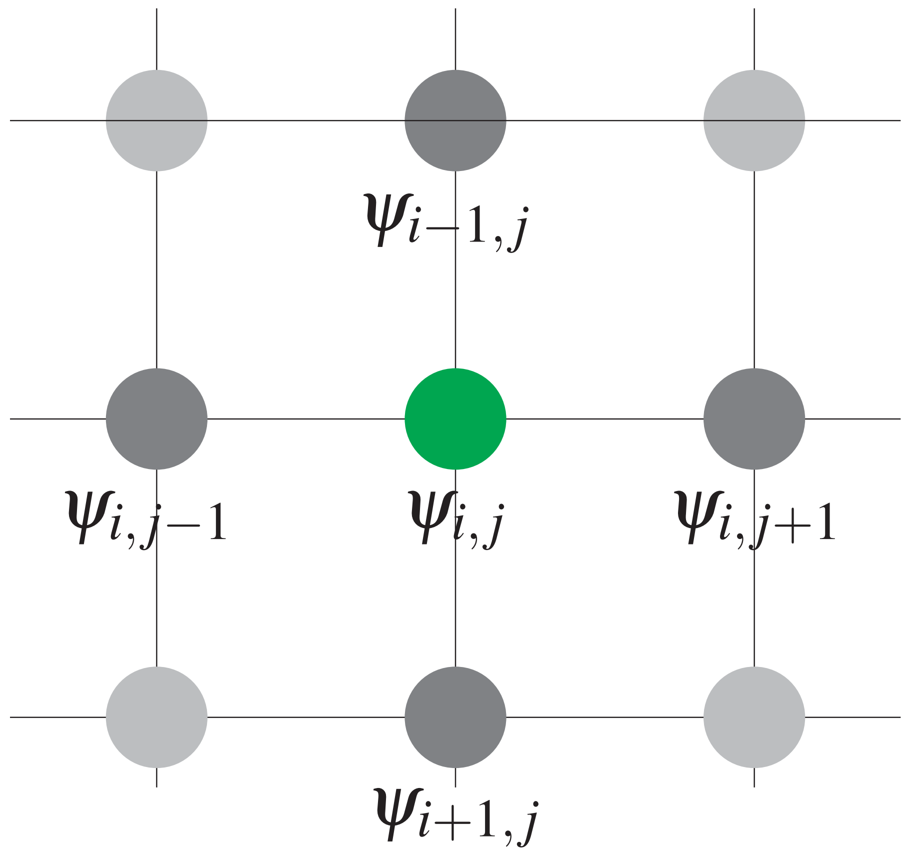
the standard way in Python

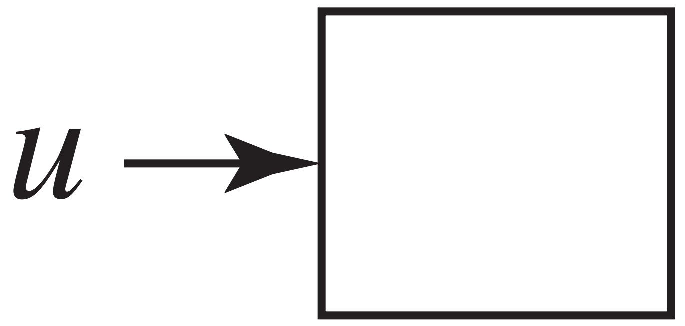
```
for i in range(Ny):
   psi[i,0] = psi[i,1] + u*dy
```

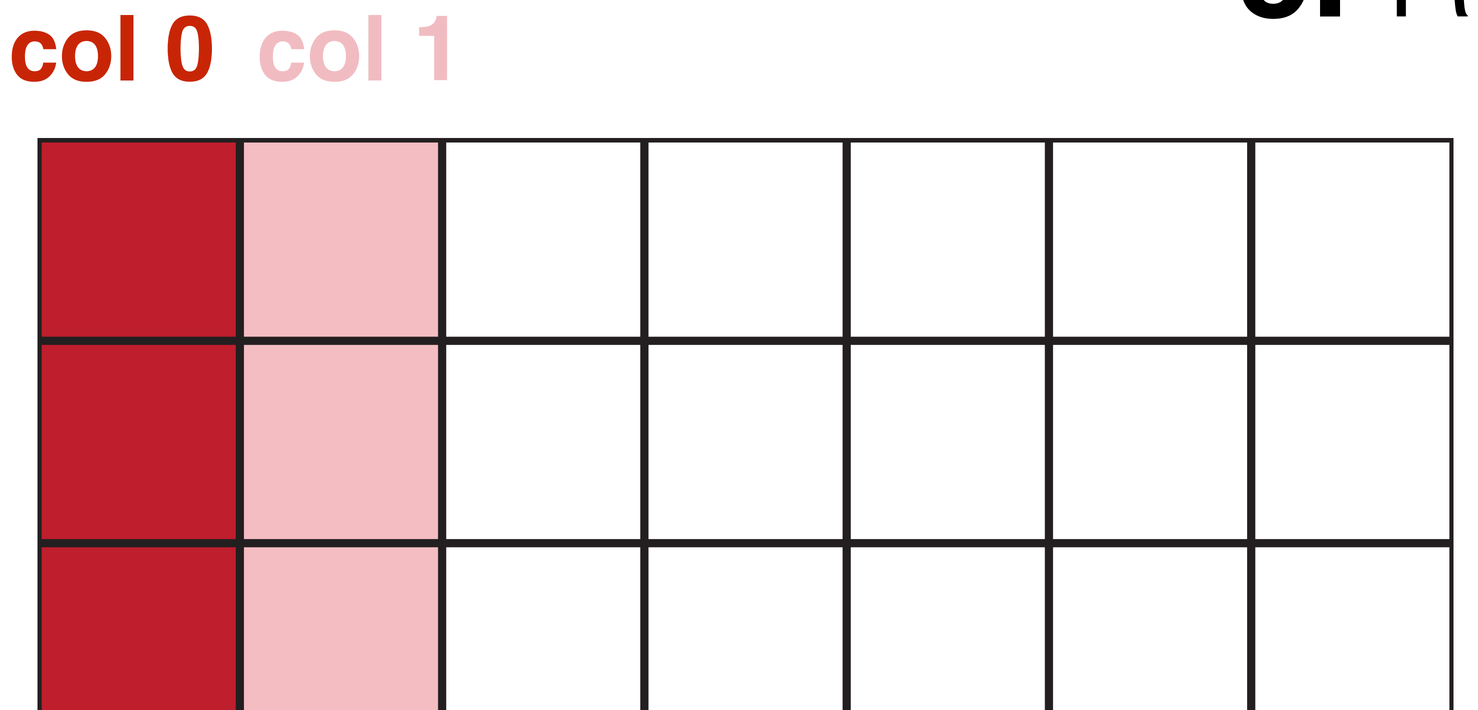
array slicing (needs numpy)

```
psi[:,0] = psi[:,1] + u*dy
```

The relaxation method







1. Set up a simulation box of ψ with size (N_y, N_x)

2. Just guess that every ψ(i,j) is some number (zero is fine)

3. At every time step use the update equation

$$\psi_{i,j} = \frac{1}{4}\left(\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1}\right)$$

4. Impose the boundary conditions

5. Repeat step 2

How do we know when to stop repeating steps 2-5?

Terminating

In the relaxation method the solution "relaxes" to an equilibrium after some time.

That is to say, we stop once the system stops changing much.

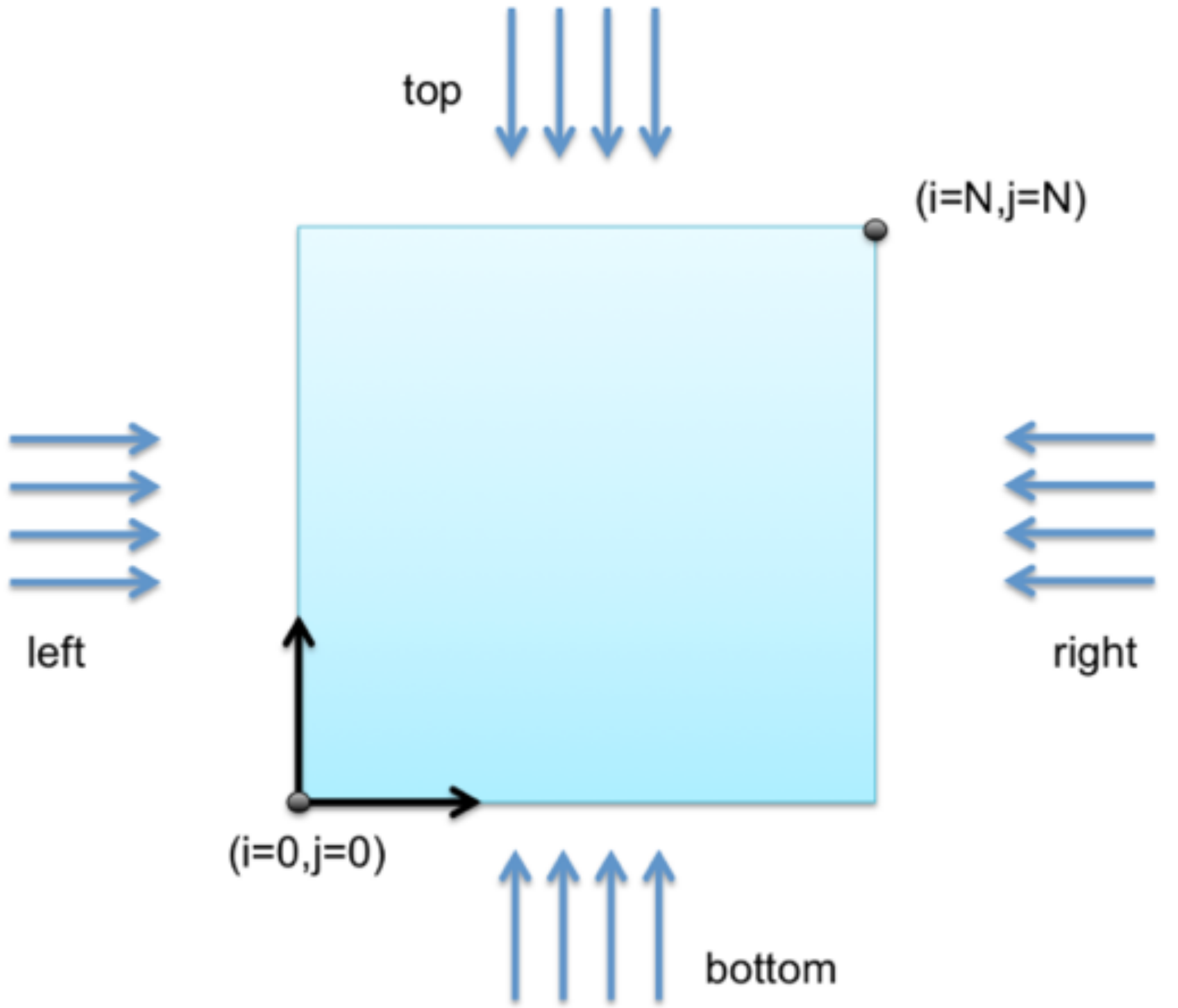
There's no set way to terminate. Perhaps you can calculate

$$\varepsilon = \sum_{i,j} \psi_{i,j}^{n} - \psi_{i,j}^{n-1} \qquad \text{time coordinate } n$$

and terminate when ε < 0.001, for example.

Code demo

- See professor Ohl's notebook on pulasan, or on the Nbviewer website http://nbviewer.jupyter.org/github/cdohl/ph3501/blob/master/notebooks/07_Solving%20the%20Laplace%20Equation%20numerically.ipynb.



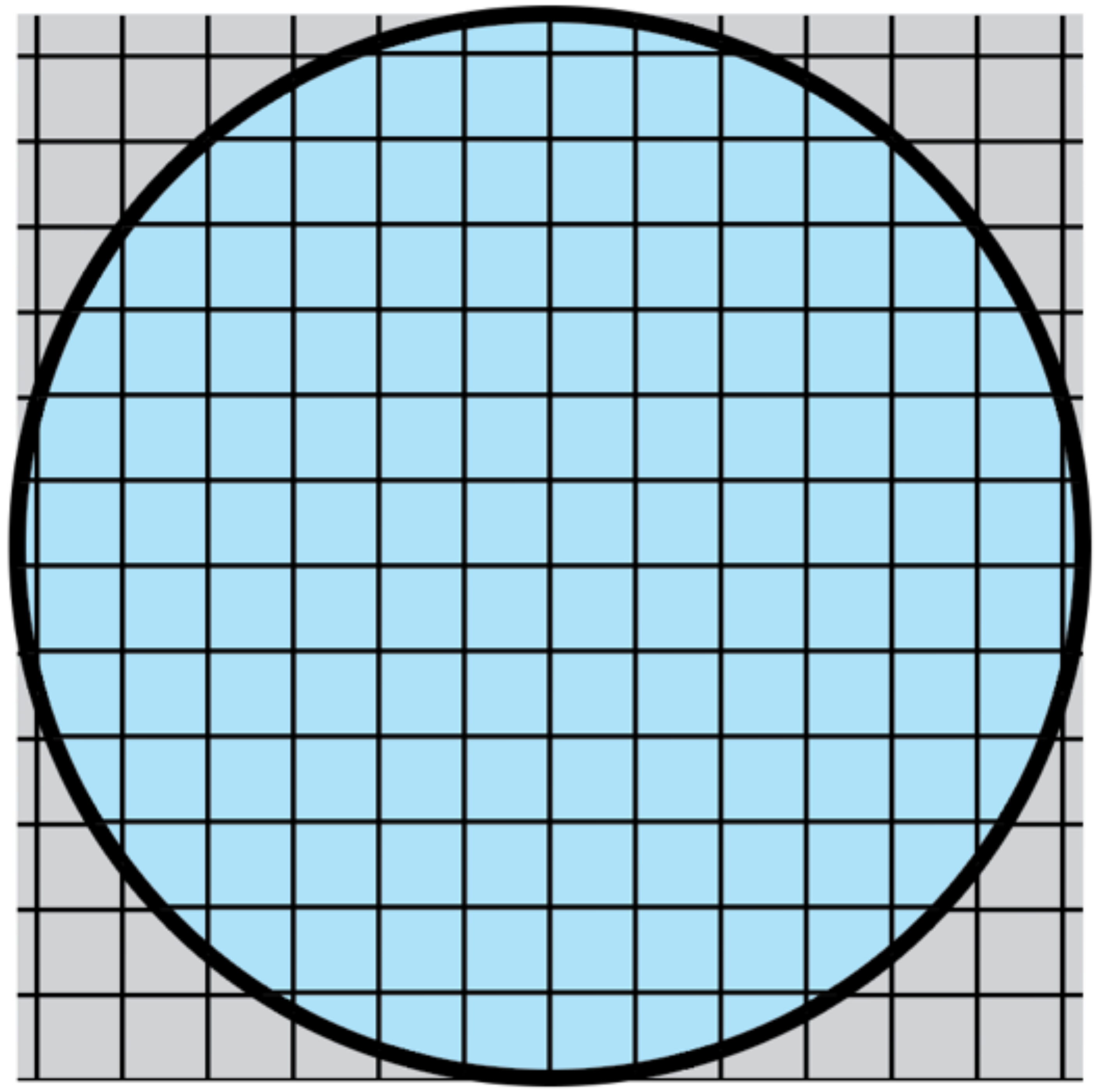

```python
def solveLaplace_Jacobi(phi,deltax):
    phin = np.empty_like(phi)
    for iter in range (maxiter):
        phin=phi.copy()

        #boundary conditions
        #extensional flow
        phin[0,1:-1] = phin[1,1:-1]-Uwall*deltax #right
        phin[-1,1:-1] = phin[-2,1:-1]-Uwall*deltax #left
        phin[1:-1,0] = phin[1:-1,1]+Uwall*deltax #bottom
        phin[1:-1,-1] = phin[1:-1,-2]+Uwall*deltax #top

        #finite difference scheme
        for i in range(1,phin.shape[0]-1):
            for j in range (1,phin.shape[1]-1):
                phi[i,j]=0.25*(phin[i-1,j]+phin[i+1,j]+phin[i,j-1]+phin[i,j+1])
    return phi
```

inside a time loop

boundary conditions

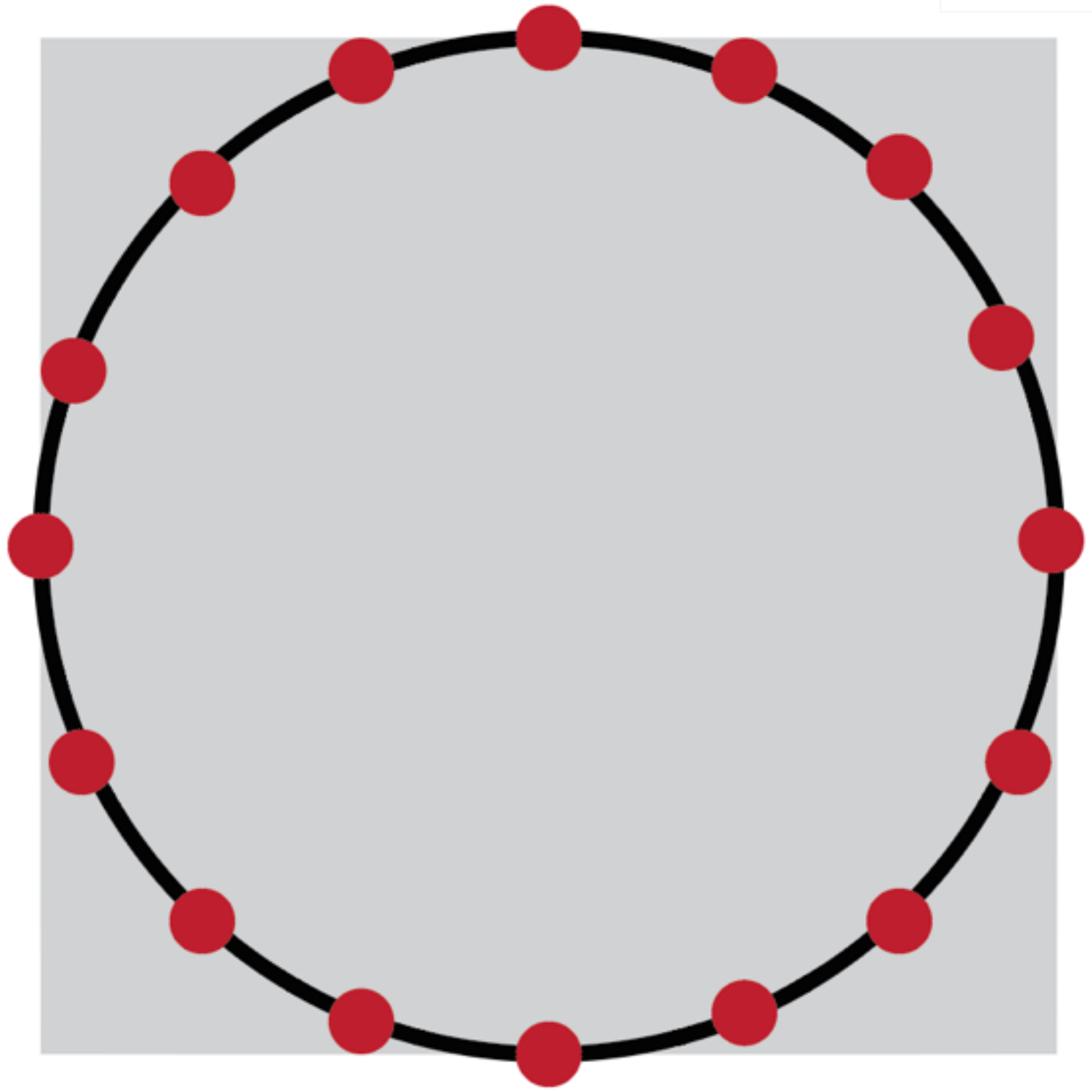
update equation

Advanced methods of solving the Laplace equation



finite elements

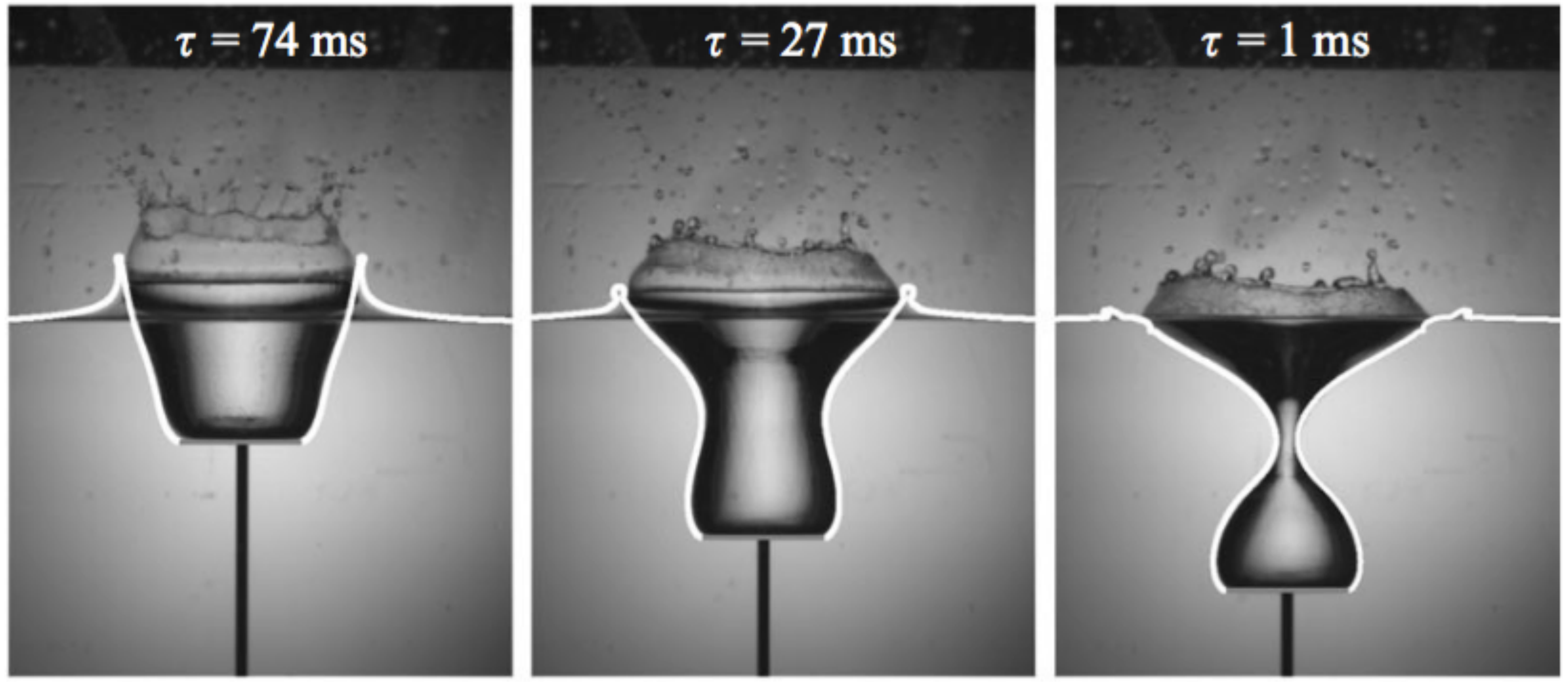
computing time $O(N^2)$ - $O(N^3)$



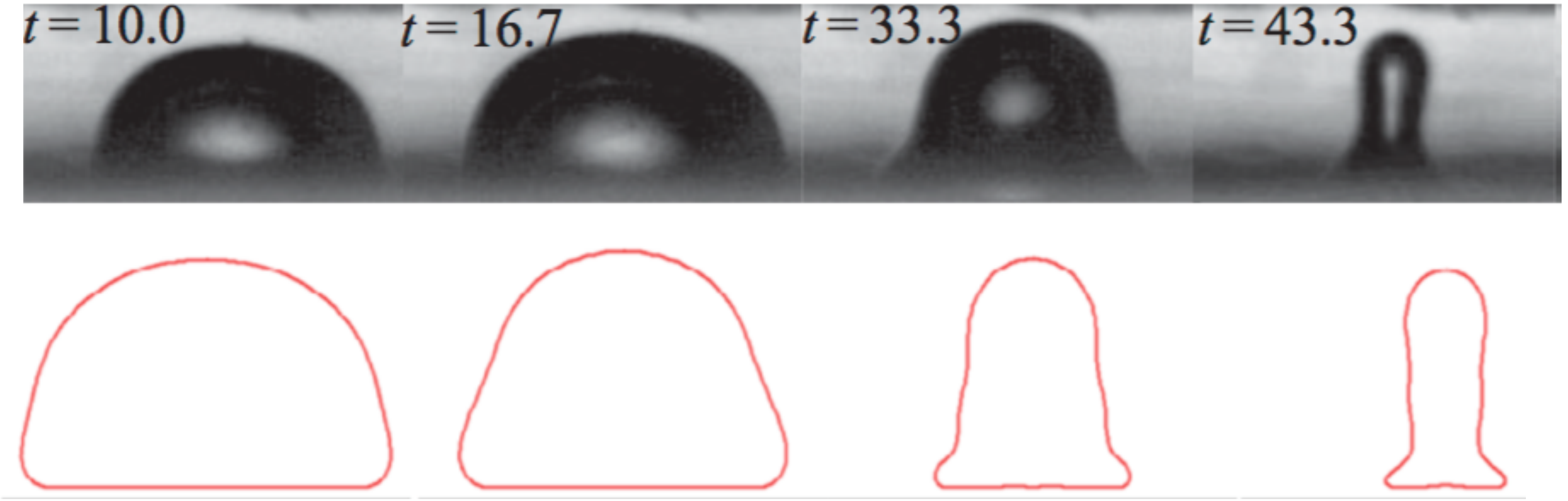
boundary elements

computing time $O(N)$

A surprisingly wide class of fluid phenomena can be fully modelled using only the Laplace equation



A plunger being pulled through a tank of water



The collapse of a cavitation bubble formed when a laser beam evaporates the water